TEORIA DEI SISTEMI (Systems Theory)

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Some solutions of the written exam of January 27th, 2014

Problem 1. Consider a feedback control system with unit feedback gain, with the following transfer function in open-loop

$$W(s) = K\frac{-2}{(s+1)^2(s^2+4)}.$$

1. Draw the amplitude and phase Bode diagrams, and the polar diagram for $K = 1$;
2. Compute the denominator of the closed-loop transfer function;
3. Compute the number of poles with negative real part of the closed loop transfer function as a function of the gain $K \in (-\infty, \infty)$, using both the Nyquist criterion and the Routh criterion.

Solution of problem 1.

Let $\widetilde{W}(s)$ denote $W(s)$ for $K = 1$:

$$\widetilde{W}(s) = \frac{-2}{(s+1)^2(s^2+4)}$$

We consider the Bode plots and the polar plot of $\widetilde{W}(s)$. In Bode form we have

$$W(s) = K\,\widetilde{W}(s) = K\left(-\frac{1}{2}\right)\frac{1}{(1+s)^2\left(1+\dfrac{s^2}{2^2}\right)}$$

The low frequency gain is

$$K_0 = \lim_{s\to 0}\widetilde{W}(s) = -\frac{1}{2} \quad\Rightarrow\quad |K_0|_{dB} = 20\log_{10}|K_0| = 20\log_{10}\left|-\frac{1}{2}\right| = -20\log_{10}2 = -6 \text{ dB}.$$

Thus

$$|\widetilde{W}(j\omega)| = \frac{1}{2}\frac{1}{|1+j\omega|^2\,\left|1+(j\omega/2)^2\right|} = \frac{1}{2}\frac{1}{|1+j\omega|^2\,\left|1-\omega^2/2^2\right|}$$
$$\langle\widetilde{W}(j\omega)\rangle = -2\langle 1+j\omega\rangle - \langle 1-\omega^2/2^2\rangle.$$

Note that the term $1-\omega^2/2^2$ is real, and therefore its phase is 0 when $\omega < 2$ (because $1-\omega^2/2^2 > 0$) and is π when $\omega > 2$ (because $1-\omega^2/2^2 < 0$). (the Bode plots and the Nyquist plot of the open loop transfer function are in the enclosed file).

The closed-loop transfer function is

$$W_{CL}(s) = \frac{K\,\widetilde{W}(s)}{1+K\,\widetilde{W}(s)} = \frac{K\frac{-2}{(s+1)^2(s^2+4)}}{1+K\frac{-2}{(s+1)^2(s^2+4)}} = \frac{-2K}{(s+1)^2(s^2+4)-2K}$$

and the denominator of $W_{CL}(s)$ is:

$$\begin{aligned} d_{CL}(s) &= (s+1)^2(s^2+4)-2K = (s^2+2s+1)(s^2+4)-2K \\ &= s^4+2s^3+5s^2+8s+4-2K \end{aligned}$$

NYQUIST ANALYSIS

Let N be the number of times that the Nyquist plot of $W(j\omega)$ encircles the -1 point in the counterclockwise direction. From the plot it is clear that

- For $K \in (0, 2)$ we have $N = 0$: the Nyquist plot does not encircle the point -1;
- For $K > 2$ we have $N = -1$: the Nyquist plot encircles one time the point -1 in the clockwise (negative) direction;
- For $K < 0$ we have $N = -2$: the Nyquist plot encircles two times the point -1 in the clockwise (negative) direction;

Recall the Nyquist formula in the form

$$p_{CL} = p_{OL} - N$$

where, p_{CL} is the number of poles with positive real part of the Closed Loop (CL) system, and p_{OL} is the number of poles with positive real part of the Open Loop (OL) system. Since for the given $W(s)$ we have $p_{OL} = 0$ (there isn't any unstable pole in the open-loop system) we have $p_{CL} = -N$, and therefore

- For $K \in (0, 2)$ we have $N = 0$, and thus $p_{CL} = 0$ (stable closed loop system);
- For $K > 2$ we have $N = -1$, and thus $p_{CL} = 1$ (unstable closed loop system);
- For $K < 0$ we have $N = -2$, and thus $p_{CL} = 2$ (unstable closed loop system).

Note that for $K = 2$ the denominator of $W_{CL}(s)$ is 0 for $s = 0$, and thus $s = 0$ is a pole of $W_{CL}(s)$.

ROUTH ANALYSIS

The case $K = 0$ will be not analyzed because it corresponds to the trivial case where the open-loop transfer function is zero.

The characteristic polynomial of the closed-loop system is the denominator of $W_{CL}(s)$:

$$d_{CL}(s) = s^4 + 2s^3 + 5s^2 + 8s + 4 - 2K$$

The first two rows (rows 4 and 3) of the Routh table are:

$$\begin{array}{c|ccc} 4 & 1 & 5 & 4-2K \\ 3 & 2 & 8 & \end{array}$$

The computation of the elements in the third row (row number 2) gives

$$a_{2,1} = \frac{1}{-2}\begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix} = \frac{8-10}{-2} = 1, \qquad a_{2,2} = \frac{1}{-2}\begin{vmatrix} 1 & \text{4-2K} \\ 2 & 0 \end{vmatrix} = \frac{-2(4-2K)}{-2} = 4 - 2K$$

Thus we have

$$\begin{array}{c|ccc} 4 & 1 & 5 & 4-2K \\ 3 & 2 & 8 & \\ 2 & 1 & 4-2K & \end{array}$$

The computation of the element in the fourth row (row number 1) gives

$$a_{1,1} = \frac{1}{-1}\begin{vmatrix} 2 & 8 \\ 1 & 4-2K \end{vmatrix} = \frac{2(4-2K)-8}{-1} = 4K,$$

and the last element, $a_{0,1}$ is $4 - 2K$

$$\begin{array}{c|ccc} 4 & 1 & 5 & 4-2K \\ 3 & 2 & 8 & \\ 2 & 1 & 4-2K & \\ 1 & \text{4K} & & \\ 0 & 4-2K & & \end{array}$$

Analyzing the signs of the first column we have:

- For $K \in (0, 2)$ we have no sign variation, so that $p_{CL} = 0$;
- For $K > 2$ we have one sign variation $(1 \to 0)$, so that $p_{CL} = 1$;
- For $K < 2$ we have two sign variation $(2 \to 1$ and $1 \to 0)$ so that $p_{CL} = 2$.

For the particular case of $K = 2$, the characteristic polynomial of the closed-loop system is

$$d_{CL}(s) == s^4 + 2s^3 + 5s^2 + 8s = s(s^3 + 2s^2 + 5s + 8),$$

thus, we have a pole in the origin. The sign of the remaining poles can be studied again by means of the Routh criterion:

3	1	5
2	2	8
1	1	
0	8	

There is no sign variation in the first column, and therefore the remaining poles have all negative real part (simple stability of the origin of the state space).

These results coincide with those obtained using the Nyquist analysis.

Problem 2. Given the following discrete-time system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \qquad \text{dove} \qquad A = \begin{bmatrix} 0.25 & -0.25 \\ 0.25 & 0.25 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

1. discuss the properties of the natural modes;
2. compute the state-transition matrix A^t;
3. compute the input-output impulse response and transfer function

Solution of problem 2.
Let's start computing the eigenvalues of A, which are the roots of the characteristic polynomial of A. The computation of the characteristic polynomial gives

$$A = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad \Rightarrow \quad |\lambda I_2 - A| = \left(\lambda - \frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2$$

The eigenvalues of A are computed solving $|\lambda I_2 - A| = 0$. Instead of using the classical formula for second-degree equations, we can proceed as follows

$$\left(\lambda - \frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = 0 \quad \Rightarrow \quad \left(\lambda - \frac{1}{4}\right)^2 = -\left(\frac{1}{4}\right)^2 \quad \Rightarrow \quad \lambda - \frac{1}{4} = \pm\sqrt{-\frac{1}{4}}.$$

From this we have the two complex-conjugates eigenvalues:

$$\lambda_1 = \frac{1}{4} + j\,\frac{1}{4}, \quad \lambda_2 = \frac{1}{4} - j\,\frac{1}{4}$$

We denote $r_1 \in \mathbb{C}^2$ and $\ell_1 \in \mathbb{C}^{1\times 2}$ the right and left eigenvalues associated to λ_1. Their computation is made by solving the two systems of equation

$$(\lambda_1 I_2 - A) r_1 = 0_{2\times 1}, \quad \ell_1(\lambda_1 I_2 - A) = 0_{1\times 2},$$

and taking, among the infinite solutions, one solution such that $\ell_1\, r_1 = 1$. The right and left eigenvalues associated to λ_2 are the conjugates of r_1 and ℓ_1 (so that $\lambda_2 = \lambda_1^*$, $r_2 = r_1^*$ and $\ell_2 = \ell_1^*$). We choose the following set of eigenvalues

$$\lambda_1 = \frac{1}{4} + j\,\frac{1}{4} \quad r_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix}, \quad \ell_1 = \frac{1}{2}\begin{bmatrix} 1 & j \end{bmatrix}, \qquad \lambda_2 = \frac{1}{4} - j\,\frac{1}{4} \quad r_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}, \quad \ell_1 = \frac{1}{2}\begin{bmatrix} 1 & -j \end{bmatrix}.$$

(check that $\ell_1\, r_1 = \ell_2\, r_2 = 1$ and $\ell_2\, r_1 = \ell_1\, r_2 = 0$).

The properties of the natural modes to be discussed are the following: - Stability, Eccitability, Observability.

Both modes are asymptotically stable, because they satisfy the stability condition $|\lambda_k| < 1$ (note that the system is a discrete-time system). In particular

$$|\lambda_1| = |\lambda_2| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4} < 1$$

Moreover, both natural modes are eccitable because $\ell_1 B = j/2 \neq 0$ and $\ell_2 B = -j/2 \neq 0$, and are observable because $C\, r_1 = C\, r_2 = 1 \neq 0$.

The transition matrix can be computed using the spectral decomposition

$$A^t = \lambda_1^t\, r_1\, \ell_1 + \lambda_2^t\, r_2\, \ell_2 = 2\Re\big(\lambda_1^t\, r_1\, \ell_1\big)$$

The expression

$$A^t = 2\Re\left(\left(\tfrac{1}{4} + j\,\tfrac{1}{4}\right)^t \tfrac{1}{2}\begin{bmatrix} 1 & j \end{bmatrix}\begin{bmatrix} 1 \\ -j \end{bmatrix}\right) = 2\Re\left(\left(\tfrac{1}{4} + j\,\tfrac{1}{4}\right)^t \tfrac{1}{2}\begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}\right)$$

is not useful if we can not expand the power $\left(\frac{1}{4} + j\,\frac{1}{4}\right)^t$. Thus, it is important to write λ_1 using the magnitude and phase form (Euler formula)

$$\lambda_1 = |\lambda_1| e^{j\langle\lambda_1\rangle} = |\lambda_1|\left(\cos(\langle\lambda_1\rangle) + j\,\sin(\langle\lambda_1\rangle)\right)$$

so that

$$\lambda_1^t = |\lambda_1|^t e^{j\langle\lambda_1\rangle t} = |\lambda_1|^t\,(\cos(\langle\lambda_1\rangle t) + j\,\sin(\langle\lambda_1\rangle t))$$

The magnitude and phase of $\lambda_1 = \frac{1}{4} + j\,\frac{1}{4}$ are $|\lambda_1| = 1/\sqrt{8}$ and $\langle\lambda_1\rangle = \arctan(1) = \frac{\pi}{4}$. Thus

$$\lambda_1^t = \left(\frac{1}{\sqrt{8}}\right)^t e^{j\frac{\pi}{4}t} = \left(\frac{1}{\sqrt{8}}\right)^t \left(\cos(\frac{\pi}{4}t) + j\,\sin(\frac{\pi}{4}t)\right)$$

Thus the transition matrix A^t becomes

$$\begin{aligned} A^t &= 2\Re\left(\left(\frac{1}{\sqrt{8}}\right)^t \left(\cos(\frac{\pi}{4}t) + j\,\sin(\frac{\pi}{4}t)\right)\frac{1}{2}\begin{bmatrix}1 & j\\ -j & 1\end{bmatrix}\right) \\ &= \left(\frac{1}{\sqrt{8}}\right)^t \Re\left(\left(\cos(\frac{\pi}{4}t) + j\,\sin(\frac{\pi}{4}t)\right)\begin{bmatrix}1 & j\\ -j & 1\end{bmatrix}\right). \end{aligned}$$

Taking the real part we have

$$A^t = \left(\frac{1}{\sqrt{8}}\right)^t \begin{bmatrix}\cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right)\\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right)\end{bmatrix}$$

The impulse response is $w(0) = D$ and $w(t) = CA^{t-1}B$ for $t > 0$. Then we have $w(0) = 0$ and for$t > 0$

$$w(t) = \begin{bmatrix}1 & 0\end{bmatrix}\left(\frac{1}{\sqrt{8}}\right)^{t-1}\begin{bmatrix}\cos\left(\frac{\pi}{4}(t-1)\right) & -\sin\left(\frac{\pi}{4}(t-1)\right)\\ \sin\left(\frac{\pi}{4}(t-1)\right) & \cos\left(\frac{\pi}{4}(t-1)\right)\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = -\left(\frac{1}{\sqrt{8}}\right)^{t-1}\sin\left(\frac{\pi}{4}(t-1)\right).$$

The transfer function $W(z)$ can be computed in two ways:
1) by taking the Z-transform of the impulse response $w(t)$;
2) by using the formula $W(z) = C(zI_2 - A)^{-1}B + D$, with $D = 0$.

Using the second method we easily get

$$\begin{aligned} W(z) &= \frac{1}{\left(z-\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z-\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{4} & z-\frac{1}{4}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \\ &= -\frac{1}{4}\frac{1}{\left(z-\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} \end{aligned}$$

As an exercise you can compute $W(z)$ by transforming $w(t)$. Of course, you should obtain the same result!

I suggest to develop the Z-transform of a generic function $f(t) = c\,a^t\sin\left(b\,t\right)$ and then to apply the *delay theorem*:

Consider a function $f(t)$, $t \geq 0$, and its Z-transform $F(z) = Z(f(t))$. Let $g(t) = f(t-1)$. Then, the Z-transform of $g(t)$ is such that

$$G(z) = \frac{1}{z}F(z).$$

When $f(t) = c\,a^t\sin\left(b\,t\right)$, then $g(t) = c\,a^{t-1}\sin\left(b\,(t-1)\right)$.

Notice that the impulse response $w(t)$ has the same structure of $g(t)$ when $c = -1$, $a = 1/\sqrt{8}$ and $b = \pi/4$. Thus, the computation of $W(z)$ from $G(z)$ and from $F(z)$ is straightforward. The computation of the Z-transform $F(z)$ is easy if we rewrite the sinus in the exponential form: $f(t) = c\,a^t(e^{jbt} - e^{-jbt})/2j$, so that $f(t) = c\,\left((a\,e^{jb})^t - (a\,e^{-jb})^t\right)/2j$. The only transformation we need to remember is $Z(p^t) = z/(z-p)$. From this,

$$Z\big(f(t)\big) = c\frac{1}{2j}\left(\frac{z}{z - a\,e^{jb}} - \frac{z}{z - a\,e^{-jb}}\right)$$

Replacing $c = -1$, $a = 1/\sqrt{8}$ and $b = \pi/4$, and noting that $a\,e^{jb} = (1/\sqrt{8})e^{j\pi/4} = (1/\sqrt{8})(1+j)/\sqrt{2} = (1+j)/4$, after few computation we get the expression for $W(z)$.

Problem 3. Consider a continuous-time system characterized by the following impulse response:

$$w(t) = 3\,e^{-2t}.$$

Compute the forced response and the harmonic response to the input $u(t) = 5\,\sin(2\,t)$.

Solution of problem 3.
Both responses are easily computed in the Laplace domain. First of all we need to compute the transfer function of the system by transforming the impulse response: $W(s) = \mathcal{L}\big(w(t)\big)$. Recalling that $\mathcal{L}\big(e^{a\,t}\big) = 1/(s-a)$ we get

$$W(s) = \frac{3}{s+2}.$$

Forced response.
The forced response in the Laplace domain is easily computed as $Y(s) = W(s)U(s)$, where $U(s)$ is the Laplace transform of the input function. Recalling that $\mathcal{L}\big(c\,\sin(\omega\,t)\big) = c\,\omega/(s^2+\omega^2)$, we have

$$U(s) = \frac{10}{s^2+4}$$

and then

$$Y(s) = W(s)U(s) = \frac{30}{(s+2)(s^2+4)} = \frac{30}{(s+2)(s-j2)(s+j2)}$$

Now must compute the partial fraction decomposition of $Y(s)$, and we can choose one of the two forms:

$$Y(s) = \frac{R_1}{(s+2)} + \frac{R_2}{(s-j2)} + \frac{R_3}{(s+j2)} \qquad \text{(recall that} \quad R_3 = R_2^*)$$

or

$$Y(s) = \frac{R_1}{(s+2)} + \frac{a\,s+b}{(s^2+4)}$$

...to be continued...

the back-transformation of $Y(s)$ gives the forced response $y(t) =$ ....to be computed....

Harmonic response.
The harmonic response exists if an only if the system is asymptotically stable (or at least the controllable and observable subsystem should be asymptotically stable). This means that all poles of the transfer function (that are also the eigenvalues of the controllable and observable subsystem), must have negative part. Since this condition is verified in this problem (the pole is -2), then the harmonic response exists and we can proceed in its computation.

The general formula for the harmonic response of continuous-time systems to an input of the form $u(t) = c\,\sin(\omega\,t)$ is

$$y_h(t) = c\,|W(j\omega)|\,\sin(\omega\,t + \langle W(j\omega)\rangle).$$

In our problem $W(s) = 3/(s+2)$, and $\omega = 2$. Thus we need to compute $|W(j2)|$ and $\langle W(j2)\rangle$):

$$|W(j\omega)| = \frac{3}{|j\omega+2|} = \frac{3}{\sqrt{\omega^2+4}}$$

$$\langle W(j\omega)\rangle = \left\langle \frac{3}{j\omega+2} \right\rangle = \langle 3\rangle - \langle j\omega+2\rangle = -\langle j\omega+2\rangle = -\arctan\left(\frac{\omega}{2}\right).$$

From these, when $\omega = 2$ we have

$$|W(j2)| = \frac{3}{\sqrt{4+4}} = \frac{3}{2\sqrt{2}}, \qquad \langle W(j2)\rangle = -\arctan\left(\frac{2}{2}\right) = -\frac{\pi}{4}.$$

Thus, the harmonic response to the input $u(t) = 5\,\sin(2\,t)$ is

$$y_h(t) = 5\,|W(j2)|\,\sin(2\,t + \langle W(j2)\rangle) = \frac{15}{2\sqrt{2}}\sin(2\,t - \pi/4)$$

Problem 4.

Given the system $x(t+1) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with matrices

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

1. Find a basis for the space of reachable states, and a basis for the space of unobservable states;
2. Find the 4 subspaces $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$ of the Kalman structural decomposition;
3. Find an input sequence that brings the state from $x(0) = [0\ 0\ 0]^T$ to $x_a = [2 - 2\ \ 0]^T$.
4. Find an input sequence that brings the state from $x(0) = [0\ 0\ 0]^T$ to $x_b = [2 \quad 0\ 2]^T$.

Solution of problem 4.

The computation of the reachability matrix gives

$$P_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$\mathcal{P} = \mathcal{R}(P_3)$ (the range of P_3) is the space of reachable states. Note that only the first two columns of P_3 are independent, so the rank of P_3 is 2, and 2 is the dimension of $\mathcal{P}$. The first two columns of P_3 are two admissible basis vectors:

$$\mathcal{P} = \mathcal{R}([B\ AB\ A^2B]) = \mathcal{R}([B\ AB]) = span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right), \qquad v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

The computation of the observability matrix gives

$$Q_3 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{Q} = \mathcal{N}(Q_3)$ (the null-space of Q_3) is the space of unobservable states. Note that only the first two rows of Q_3 are independent, so the rank of Q_3 is 2, and 1 is the dimension of $\mathcal{Q}$ (1 is the rank drop of Q_3). Thus, we need to find one basis vector for $\mathcal{Q}$, that is a solution of the homogeneous system $Q_3\, b = 0_{3\times 1}$. Observing that the first two columns of Q_3 are equal, we have the following solution $b = [-1\ \ -1\ 0]^T$, which is a basis for $\mathcal{Q}$. Thus

$$\mathcal{Q} = \mathcal{N}\left(\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} C \\ CA \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right), \qquad b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Now we are ready to find the four subspaces $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$.

Let's start with $\mathcal{X}_1 = \mathcal{Q} \cap \mathcal{P}$. Being $\mathcal{Q}$ a one-dimensional space and $\mathcal{P}$ a two-dimensional space, the intersections is either 0 or $\mathcal{Q}$ itself. Thus, we must only check if $b \in span(v_1, v_2)$ or not.
This can be made by checking whether $rank([v_1\ v_2]) = rank(([v_1\ v_2\ b])$ or not. It is easy to check that

$$rank\left(\begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}\right) = rank\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\right)$$

In other words, b is linearly dependent on v_1, v_2 (it is clear that $b = v_1 + v_2$).

Thus, b is a basis for $\mathcal{X}_1$.

The space $\mathcal{X}_2$ is any space such that $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{P}$. We can choose either $\mathcal{X}_2 = span(v_1)$ or $\mathcal{X}_2 = span(v_2)$, because we have $\mathcal{P} = span([b_1\ v_1)]$, and also $\mathcal{P} = span([b_1\ v_2)]$. Let's choose $\mathcal{X}_2 = span(v_1)$, so that a basis for $\mathcal{X}_2$ is $v_1 = [0\ 0\ 1]^T$.

The space $\mathcal{X}_3$ is any space such that $\mathcal{X}_1 \oplus \mathcal{X}_3 = \mathcal{Q}$. However, since we have $\mathcal{X}_1 = \mathcal{Q}$, then $\mathcal{X}_3 = \emptyset$.

The space $\mathcal{X}_4$ is any space such that $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 = \mathbb{C}^3$. Then, we only need to find a vector linearly independent of b,v_1. Let's choose $\mathcal{X}_4 = span(d)$, with $d = [1\ 0\ 0]^T$, such that

$$rank\left(\begin{bmatrix} b & v_1 & d \end{bmatrix}\right) = rank\left(\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = 3, \quad \text{and then} \quad \begin{cases} \mathcal{X}_1 = span(b), \\ \mathcal{X}_2 = span(v_1), \\ \mathcal{X}_3 = \emptyset, \\ \mathcal{X}_4 = span(d). \end{cases}$$

Note that other choices for $\mathcal{X}_2$ and $\mathcal{X}_4$ are possible.

Question 4.1 and 4.2:
- Find an input sequence that brings the state from $x(0) = [0\ 0\ 0]^T$ to $x_a = [2 - 2\ \ 0]^T$;
- Find an input sequence that brings the state from $x(0) = [0\ 0\ 0]^T$ to $x_b = [2\quad 0\ \ 2]^T$.

Note that $x_a \in \mathcal{P}$, and therefore we know that there exists an input sequence that brings $x(0) = 0_{3\times 1}$ to x_a. From the general explicit form of a discrete-time system:

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau) = A^t x(0) + \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix} \begin{bmatrix} u(t-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}.$$

We know that the matrix P_3 transforms an input sequence $[u(0)\ u(1)\ u(2)]^T$ to $x(3)$, starting with $x(0)$:

$$x(3) = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}.$$

Thus, if a vector $\bar{x} \in P_3$, then there exists a triple $\big(u(0), u(1), u(2)\big)$ that transfers $x(0) = 0$ to $\bar{x}$. However, since only the first two columns of P_3 are independent, when $\bar{x} \in \mathcal{P}$, then we can also look for a pair $\big(u(0), u(1)\big)$ that transfers $x(0) = 0$ to $\bar{x}$, by solving

$$\bar{x} = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}.$$

We see that $x_a \in \mathcal{P}$ (x_a is reachable) while $x_b \notin \mathcal{P}$ (x_b is not reachable).

Thus, for $x_a = [2\ \ -2\ 0]^T$ we can find a pair $\big(u(0), u(1)\big)$ that transfers $x(0) = 0$ to x_a. We only have to solve:

$$x_a = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}.$$

Solving this vector equation it is easy to see that $u(0) = 2$ and $u(1) = 2$ bring the state $x(0) = 0$ to the state $x(2) = x_a$.

Problem 5. Given the system

$$\begin{cases} \dot{x}_1(t) = (1-\alpha)x_1(t) + \alpha\, x_1(t)x_2(t) \\ \dot{x}_2(t) = -\alpha^2 x_2(t) - x_1^2(t) \end{cases}$$

1. study the stability of the equilibrium point $x_e = (0,0)$ for all the values of the parameter $\alpha \in (-\infty, \infty)$, using the method of linear approximation at the equilibrium point, and the Lyapunov function method;
2. study for what values of α the system admits other equilibrium points, and study their stability properties.

(*Suggestion for the Lyapunov function:* $V(x) = (x_1 - x_{e,1})^2 + \beta(x_2 - x_{e,2})^2$, with suitable $\beta > 0$.)

Solution of problem 5.
The system considered is of the form $\dot{x}(t) = f\big(x(t);\alpha\big)$, where $x(t)$ is the state and α is a constant parameter. The vector function $f(x;\alpha)$ is as follows

$$\begin{cases} f_1(x;\alpha) = (1-\alpha)x_1 + \alpha\, x_1 x_2 \\ f_2(x;\alpha) = -\alpha^2 x_2 - x_1^2 \end{cases}$$

The Jacobian is

$$J(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix} = \begin{bmatrix} 1-\alpha+\alpha\, x_2 & \alpha\, x_1 \\ -2x_1 & -\alpha^2 \end{bmatrix}$$

The Jacobian computed at $x_e = (0,0)$ is

$$J(x_e) = \begin{bmatrix} 1-\alpha & 0 \\ -0 & -\alpha^2 \end{bmatrix}$$

Since $J(x_e)$ is diagonal, the two eigenvalues are the diagonal terms:

$$\lambda_1 = 1-\alpha, \qquad \lambda_2 = -\alpha^2.$$

Thus:

- $(0,0)$ is asymptotically stable when *both* eigenvalues are strictly negative, that is when $(1-\alpha < 0)$ **and** $(-\alpha^2 < 0)$. That means that $(0,0)$ is asymptotically stable for $\alpha > 1$;
- $(0,0)$ is unstable when *at least one* eigenvalue is strictly positive i.e. when $(1-\alpha > 0)$ **or** $(-\alpha^2 > 0)$. That means that $(0,0)$ is unstable for $\alpha < 1$.

The only case where we can not conclude anything about the stability of $(0,0)$ is when $\alpha = 1$ (in this case the origin is a simply stable equilibrium point *of the linear approximation* of the nonlinear system, but nothing can be concluded about the nonlinear system). In this case we must study the stability of the point $x_e = (0,0)$ by using a suitable Lyapunov function.

For $\alpha = 1$ the function $f(x;\alpha)$ is

$$\begin{cases} f_1(x;1) = x_1 x_2 \\ f_2(x;1) = -x_2 - x_1^2. \end{cases}$$

Following the suggestion we consider the Lyapunov function

$$V(x) = x_1^2 + \beta x_2^2,$$

which is positive definite for any $\beta > 0$. According to the Lyapunov theorem, if the derivative $\dot{V}(x) = (dV/dx)f(x;1)$ is semidefinite negative, then the equilibrium x_e is (simply) stable, while if $\dot{V}(x)$ is definite negative, then the equilibrium is asymptotically stable. The computation of $\dot{V}(x)$ gives

$$\begin{aligned} \dot{V}(x) &= \frac{dV}{dx} f(x;1) = (\partial_{x_1} V) f_1(x;1) + (\partial_{x_2} V) f_2(x;1) = 2\, x_1\, f_1(x;1) + 2\beta\, x_2\, f_2(x;1) \\ &= 2\, x_1^2 x_2 - 2\beta\, x_2^2 - 2\beta\, x_2 x_1^2. \end{aligned}$$

By choosing $\beta = 1$ we can eliminate the term $x_1^2 x_2$ which is sign-indefinite, so that we get

$$\dot{V}(x) = -2\,x_2^2.$$

which is semidefinite negative (it is never positive, but it is zero for any pair $(x_1, 0)$, and not only at the equilibrium point $x_e = (0,0)$). As a consequence, when $\alpha = 1$ the equilibrium $(0,0)$ is *simply* stable (stable, but not asymptotically stable).

All other equilibrium points can be found by solving the system of equations $f(x;\alpha) = 0$, i.e.

$$\begin{cases} (1-\alpha)x_1 + \alpha\, x_1 x_2 = 0 \\ -\alpha^2 x_2 - x_1^2 = 0. \end{cases}$$

From the first equation we have

$$x_1\left((1-\alpha) + \alpha\, x_2\right) = 0 \quad \longrightarrow \quad 1 - \alpha + \alpha\, x_2 = 0 \quad \rightarrow \quad x_2 = \frac{\alpha - 1}{\alpha}, \quad \alpha \neq 0$$

(the other solution is $x_1 = 0$, and leads to the equilibrium point $(0,0)$ already studied in the previous question). From the second equation we have

$$-\alpha^2 x_2 - x_1^2 = 0 \quad \rightarrow \quad -\alpha^2 \frac{\alpha - 1}{\alpha} - x_1^2 = 0 \quad \rightarrow \quad x_1^2 = \alpha(1-\alpha)$$

Thus, there exists other equilibrium points in $\mathbb{R}^2$ only if $\alpha \neq 0$ and $\alpha(1-\alpha) > 0$, that is for $\alpha \in (0,1)$. In this case we have two more equilibrium points:

$$x_{e,1} = \left(\sqrt{\alpha(1-\alpha)}, \frac{\alpha-1}{\alpha}\right), \qquad x_{e,2} = \left(-\sqrt{\alpha(1-\alpha)}, \frac{\alpha-1}{\alpha}\right).$$

Note that when $\alpha = 1$ all the equilibrium points collapse to $(0,0)$. The Jacobian computed in the two equilibrium points is

$$J(x_{e,1,2}) = \begin{bmatrix} 0 & \pm\alpha\sqrt{\alpha(1-\alpha)} \\ \mp 2\sqrt{\alpha(1-\alpha)} & -\alpha^2 \end{bmatrix} \qquad x_{e,1,2} = \left(\pm\sqrt{\alpha(1-\alpha)}, \frac{\alpha-1}{\alpha}\right).$$

The characteristic polynomial is

$$\left|\lambda I - J(x_{e,1,2})\right| = \left|\begin{bmatrix} \lambda & \mp\alpha\sqrt{\alpha(1-\alpha)} \\ \pm 2\sqrt{\alpha(1-\alpha)} & \lambda + \alpha^2 \end{bmatrix}\right| = \lambda^2 + \alpha^2\lambda + 2\alpha^2(1-\alpha).$$

Note that the two Jacobians have the same characteristic polynomial. Recall that the roots of a second order polynomial all have negative real part if and only if the three coefficients have the same sign (check this property using the Routh table). Thus, both equilibrium points are asymptotically stable, because they exists if and only if $\alpha \neq 0$ and $\alpha \in (0,1)$, which imply that $\alpha^2 > 0$ and $\alpha^2(1-\alpha) > 0$.