TEORIA DEI SISTEMI (Systems Theory)

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Some solutions of the written exam of February 10th, 2014

Problem 1. Consider a feedback control system with unit feedback gain, with the following transfer function in open-loop

$$W(s) = K\frac{18}{s(s-1)(s-9)}.$$

1. Draw the amplitude and phase Bode diagrams, and the polar diagram for $K = 1$;
2. Compute the denominator of the closed-loop transfer function;
3. Compute the number of poles with negative real part of the closed loop transfer function as a function of the gain $K \in (-\infty, \infty)$, using both the Nyquist criterion and the Routh criterion.

Solution of problem 1.

Let $\widetilde{W}(s)$ denote $W(s)$ for $K = 1$:

$$\widetilde{W}(s) = \frac{18}{s(s-1)(s-9)}$$

We consider the Bode plots and the polar plot of $\widetilde{W}(s)$. In Bode form we have

$$W(s) = K\,\widetilde{W}(s) = K\,\frac{2}{s(1-s)\left(1-\frac{s}{9}\right)}$$

The low frequency gain is

$$K_0 = \lim_{s\to 0} s\widetilde{W}(s) = 2 \quad \Rightarrow \quad |K_0|_{dB} = 20\log_{10}|K_0| = 20\log_{10} 2 = 6 \text{ dB}.$$

Thus

$$|\widetilde{W}(j\omega)| = \frac{2}{\omega|1-j\omega|\,|1-j\omega/9|}$$
$$\langle \widetilde{W}(j\omega)\rangle = -\frac{\pi}{2} - \langle 1-j\omega\rangle - \langle 1-j\omega/9\rangle.$$

(the Bode plots and the Nyquist plot of the open loop transfer function are in the enclosed file).

The closed-loop transfer function is

$$W_{CL}(s) = \frac{K\,\widetilde{W}(s)}{1+K\,\widetilde{W}(s)} = \frac{K\frac{18}{s(s-1)(s-9)}}{1+K\frac{18}{s(s-1)(s-9)}} = \frac{18K}{s(s-1)(s-9)+18K}$$

and the denominator of $W_{CL}(s)$ is:

$$d_{CL}(s) = s^3 - 10\,s^2 + 9\,s + 18\,K$$

NYQUIST ANALYSIS

For the Nyquist analysis it is important to compute the intersections of the graph of $W(j\omega)$ with the real axis. We see from the polar plot that $W(j\omega)$ intersects the real axis at a pair of frequencies $\pm\omega^*$. By writing $\widetilde{W}(j\omega)$ as $\widetilde{W}_r(\omega) + j\,\widetilde{W}_i(\omega)$, we see that at the intersection it must be $\widetilde{W}_i(\omega) = 0$. Solving $\widetilde{W}_i(\omega) = 0$ we find the roots $\omega^* = \pm 3$, and we see that $\widetilde{W}_r(3) = 0.2$.

Thus, the graph of $W(j\omega)$ intersects the real axis at the point $K\,0.2$ at the frequency $\omega^* = \pm 3$, i.e. $W(j\,3) = K\,0.2$.

Let N be the number of times that the Nyquist plot of $W(j\omega)$ encircles the -1 point in the counter-clockwise direction. Analyzing the polar the plot and using the conditions $K\,0.2 > -1$ or $K\,0.2 < -1$, it follows that

- For $K > 0$ we have $N = 0$: the Nyquist plot does not encircle the point -1;
- For $K \in (-5, 0)$ we have $N = -1$: the Nyquist plot encircles one time the point -1 in the clockwise (negative) direction;

- For $K < -5$ we have $N = 1$: the Nyquist plot encircles one time the point -1 in the counter-clockwise (positive) direction;

Recall the Nyquist formula in the form

$$p_{CL} = p_{OL} - N$$

where, p_{CL} is the number of poles with positive real part of the Closed Loop (CL) system, and p_{OL} is the number of poles with positive real part of the Open Loop (OL) system. Since for the given $W(s)$ we have $p_{OL} = 2$ ($p_1 = 1$ and $p_2 = 9$), we have $p_{CL} = 2 - N$, and therefore

- For $K > 0$ we have $N = 0$, and thus $p_{CL} = 2$ (unstable closed loop system);
- For $K \in (-5, 0)$ we have $N = -1$:, and thus $p_{CL} = 3$ (unstable closed loop system);
- For $K < -5$ we have $N = 1$, and thus $p_{CL} = 1$ (unstable closed loop system).

Note that for $K = -5$ the denominator of $W_{CL}(s)$ is 0 for $s = \pm j\,3$, and thus $\pm j\,3$ is a pair of imaginary poles of $W_{CL}(s)$.

ROUTH ANALYSIS

The case $K = 0$ will be not analyzed because it corresponds to the trivial case where the open-loop transfer function is zero.

The characteristic polynomial of the closed-loop system is the denominator of $W_{CL}(s)$:

$$d_{CL}(s) = s^3 - 10\,s^2 + 9\,s + 18\,K$$

The first two rows (rows 3 and 2) of the Routh table are:

3	1	9
2	-10	$18\,K$

The computation of the element in the third row (row number 1) gives

$$a_{1,1} = \frac{1}{-(-10)} \begin{vmatrix} 1 & 9 \\ -10 & 18\,K \end{vmatrix} = \frac{18\,K + 90}{10} = \frac{9}{10}(2\,K + 10)$$

Thus we have

3	1	9
2	-10	$18\,K$
1	$\frac{9}{10}(2\,K + 10)$	
0	$8\,K$	

Analyzing the signs of the first column, and noting that $2\,K + 10 < 0$ is equivalent to $K < -5$, we have

- For $K > 0$ we have two sign variation ($3 \to 2$ and $2 \to 1$), so that $p_{CL} = 2$;
- For $-5 < K < 0$ we have three sign variations ($3 \to 2$, $2 \to 1$,and $1 \to 0$), so that $p_{CL} = 3$;
- For $K < -5$ we have one sign variation ($3 \to 2$) so that $p_{CL} = 1$.

For the particular case of $K = 5$, the characteristic polynomial of the closed-loop system is

$$d_{CL}(s) = s^3 - 10\,s^2 + 9\,s - 90$$

and the Routh table terminates at the third row, which is zero

3	1	9
2	-10	-90
1	0	

We know that in this case the polynomial $d_{CL}(s)$ has a pair of conjugate imaginary roots, which are the roots of the polynomial $-10\,s^2 - 90$. Their computation gives $p_{1,2} = \pm j\,3$. The sign variations of the first column of the table obtained until now, tells the number of roots with positive real part: there is one variation ($3 \to 2$) and therefore $p_{OL} = 1$.

These results coincide with those obtained using the Nyquist analysis.

Problem 2. Siano dati due sistemi lineari autonomi (con ingresso nullo), uno a tempo continuo, $\dot{x}_c(t) = Ax_c(t)$, ed uno a tempo-discreto, $x_d(t+1) = Ax_d(t)$, caratterizzati dalla stessa matrice $A \in \mathbb{R}^{3\times 3}$. Gli autovalori λ_k e gli autovettori destri r_k e sinistri ℓ_k della matrice A sono i seguenti:

$$
\begin{array}{llll}
\lambda_1 = 0.5 + j\,0.5 & & & \ell_1 = \tfrac{1}{2}[-j \quad 1 \quad -1\,] \\
\lambda_2 = 0.5 - j\,0.5 & r_1 = \begin{bmatrix} j \\ 0 \\ -1 \end{bmatrix} \; r_2 = \begin{bmatrix} -j \\ 0 \\ -1 \end{bmatrix} & r_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \ell_2 = \tfrac{1}{2}[\;j \quad 1 \quad -1\,] \\
\lambda_3 = -2 & & & \ell_3 = \;[\;0 \quad 1 \quad 0\,]
\end{array}
$$

Si calcolino le evoluzioni libere dello stato di entrambi i sistemi ($x_c(t)$ e $x_d(t)$) a partire dallo stesso stato iniziale $x_c(0) = x_d(0) = [\,1\ 0\ 0]^T$.

(Suggerimento: per limitare la quantitÃ di calcoli occorre sfruttare la decomposizione spettrale delle matrici di transizione evitandone il calcolo esplicito.)

Problem 3. Given the following discrete-time system, where $x(t)$ and $u(t)$ are scalar variables:

$$x(t+1) = 0.8\,x(t) + u(t)$$

- Compute the unit step response of the state (input $u(t) = 1$ for $t \geq 0$).
- Compute the harmonic response of the state to the input: $u(t) = \cos\big((\pi/2)t\big)$.

Solution of problem 3.
The computation of the forced response and of the harmonic response of discrete-time linear systems is preferably made using the Z-transform. Recall that the input-state transfer function of discrete-time linear systems is $H(z) = (z\,I - A)^{-1}B$, and the impulse response is $h(0) = 0$ and $h(t) = \sum_{\tau=0}^{t-1} A^{t-\tau-1}B$ for $t > 0$. In our problem we have $A = 0.8$ and $B = 1$, so that

$$H(z) = \frac{1}{z - 0.8}.$$

(Note that for the given problem we could equally consider the output $y(t)$ coinciding with the state $x(t)$, so that $y(t) = Cx(t) + Du(t)$ with $C = 1$ and $D = 0$. In this case $W(z) = C(z\,I - A)^{-1}B = H(z)$.)

Using the Z-transform, the forced state response can be computed by the inverse Z-transform of the product $H(z)U(z)$, where $U(z)$ is the Z-transform of the input. Thus, for computing the unit step response we need the transform of the unit step:

$$U(z) = \frac{z}{z-1}$$

The step response of the state in the Z-transform domain is

$$X(z) = H(z)U(z) = \frac{z}{(z-0.8)(z-1)}.$$

In order to exploit the basic Z-transform $Z(a^t) = z/(z-a)$ and inverse transform $Z^{-1}\big(z/(z-a)\big) = a^t$, $t \geq 0$, it is useful to compute the partial fraction expansion of $X(z)/z$ instead of $X(z)$:

$$\frac{X(z)}{z} = \frac{1}{(z-0.8)(z-1)} = \frac{R_1}{z-0.8} + \frac{R_2}{z-1},$$

from which

$$X(z) = R_1\frac{z}{z-0.8} + R_2\frac{z}{z-1} \qquad \Rightarrow \qquad x(t) = R_1 0.8^t + R_2, \quad t \geq 0.$$

The computation of the *residuals* R_1 and R_2 is as follows

$$R_1 = \lim_{z\to 0.8}(z-0.8)\frac{X(z)}{z} = \lim_{z\to 0.8}\frac{1}{z-1} = \frac{1}{0.8-1} = -\frac{1}{0.2} = -5$$
$$R_2 = \lim_{z\to 1}(z-1)\frac{X(z)}{z} = \lim_{z\to 1}\frac{1}{z-0.8} = \frac{1}{1-0.8} = \frac{1}{0.2} = 5$$

and thus the step response of the state is

$$X(z) = (-5)\frac{z}{z-0.8} + 5\frac{z}{z-1} \qquad \Rightarrow \qquad x(t) = 5\,(1 - 0.8^t), \quad t \geq 0.$$

For the computation of the harmonic response, we recall the general formula: if the generic sinusoidal input is $u(t) = M\,\cos(\omega\,t + \varphi)$ and $H(z)$ is the transfer function, then the harmonic response $x_h(t)$ is:

$$x_h(t) = \big|H(e^{j\omega})\big|\,M\,\cos(\omega\,t + \varphi + \langle H(e^{j\omega})\rangle).$$

In our problem $u(t) = \cos\big((\pi/2)t\big)$, so that $M = 1$, $\varphi = 0$ and $\omega = \pi/2$. Thus $e^{j\omega} = e^{j\pi/2} = j$, and the harmonic response is

$$x_h(t) = \big|H(j)\big|\,\cos\big((\pi/2)\,t + \langle H(j)\rangle\big), \quad \text{where} \quad H(j) = \frac{1}{j-0.8}.$$

The computation of the magnitude and phase of $H(j)$ is:

$$\big|H(j)\big| = \frac{1}{|j-0.8|} = \frac{1}{\sqrt{1+0.64}} = 0.781,$$
$$\langle H(j)\rangle = \left\langle \frac{1}{j-0.8}\right\rangle = \left\langle \frac{-1}{0.8-j}\right\rangle = \langle -1\rangle - \langle 0.8-j\rangle = \pi - \arctan(-1/0.8) = \pi + \arctan(1/0.8) = 4.0376\ \textit{rad}.$$

(equivalently, $\langle -1\rangle = -\pi$, so that $\langle H(j)\rangle = -\pi + \arctan(1/0.8) = -2.2455$ *rad*.)

Problem 4.
Given the system $x(t+1) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with matrices

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad C \ = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

1. Find a basis for the space of reachable states, and a basis for the space of unobservable states;
2. Find the 4 subspaces $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$ of the Kalman structural decomposition;
3. Compute an initial state $x(0) \in \mathbb{R}^3$ such that the output sequence of the unforced system is constant, equal to $y(t) = 5$, $\forall t \geq 0$.

Solution of problem 4.

The computation of the reachability matrix gives

$$P_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{P} = \mathcal{R}(P_3)$ (the range of P_3) is the space of reachable states. Note that the second and third columns of P_3 are linearly dependent on the first one, so the rank of P_3 is 1, and 1 is the dimension of $\mathcal{P}$. The first column of P_3 is an admissible basis vector:

$$\mathcal{P} = \mathcal{R}([B\ AB\ A^2B]) = \mathcal{R}(B) = span\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right), \qquad v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The computation of the observability matrix gives

$$Q_3 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathcal{Q} = \mathcal{N}(Q_3)$ (the null-space of Q_3) is the space of unobservable states. Note that the second and third rows of Q_3 are equal to the first row, so the rank of Q_3 is 1, and 2 is the dimension of $\mathcal{Q}$ (2 is the rank drop of Q_3). Thus, we need to find two basis vectors for $\mathcal{Q}$, that is two independent solutions of the homogeneous system $Q_3\, b = 0_{3\times1}$. One solution is easily found because the last column of Q_3 is zero, and thus $Q_3\, b_1 = 0_{3\times1}$ with $b_1 = [0\ 0\ 1]^T$. Then, observing that the first two columns of Q_3 are equal, we have the following solution $b_2 = [1\ -1\ 0]^T$, which is independent of b_1. Thus b_1, b_2 make up a basis for $\mathcal{Q}$:

$$\mathcal{Q} = \mathcal{N}\left(\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}\right) = \mathcal{N}(C) = span\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right), \qquad b_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\mathcal{P} = span(v)$ and $\mathcal{Q} = span(b_1, b_2)$, and we are ready to find the four subspaces $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$.

Let's start with $\mathcal{X}_1 = \mathcal{Q} \cap \mathcal{P}$. Being $\mathcal{Q}$ a two-dimensional subspace of $\mathbb{R}^3$, and $\mathcal{P}$ a one-dimensional subspace of $\mathbb{R}^3$, the intersections is either 0 or $\mathcal{P}$ itself. Thus, we must only check whether $v \in span(b_1, b_2)$ or not. With the chosen bases this test is immediate, because $v = b_1$, so that $\mathcal{X}_1 = \mathcal{Q} \cap \mathcal{P} = \mathcal{P} = span(v)$.

The space $\mathcal{X}_2$ is any space such that $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{P}$, and therefore $\mathcal{X}_2 = \emptyset$.

The space $\mathcal{X}_3$ is any space such that $\mathcal{X}_1 \oplus \mathcal{X}_3 = \mathcal{Q}$. Since $\mathcal{Q} = span(b_1, b_2)$ and $\mathcal{X}_1 = span(v) = span(b_1)$, we can choose $\mathcal{X}_3 = span(b_2)$.

The space $\mathcal{X}_4$ is any space such that $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 = \mathbb{R}^3$. Then, we only need to find a vector linearly independent of b,v_1. Let's choose $\mathcal{X}_4 = span(d)$, with $d = [1\ 0\ 0]^T$, such that

$$rank\left(\begin{bmatrix} b_1 & b_2 & d \end{bmatrix}\right) = rank\left(\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = 3, \quad \text{and then} \quad \begin{cases} \mathcal{X}_1 = span(b_1), \\ \mathcal{X}_2 = \emptyset, \\ \mathcal{X}_3 = span(b_2), \\ \mathcal{X}_4 = span(d). \end{cases}$$

Note that other choices for $\mathcal{X}_3$ and $\mathcal{X}_4$ are possible.

Question 4.3:
- Compute an initial state $x(0) \in \mathbb{R}^3$ such that the output sequence of the unforced system is constant, equal to $y(t) = 5$, $\forall t \geq 0$.

Answer to Question 4.3:

The output of the unforced system is $y(t) = CA^t x(0)$, $t \geq 0$. The Question 4.3 asks to find $x(0)$ such that $CA^t x(0) = 5$, $\forall t \geq 0$.

Note that for the given pair (A, C) we have $CA^t = C$, $\forall t \geq 0$ (look at the rows CA and CA^2 of matrix Q_3). It follows that any $x(0)$ such that $Cx(0) = 5$ is such that $CA^t x(0) = 5$, $\forall t \geq 0$.

For instance, we can choose $x(0) = [5\ 0\ 0]^T$.

Problem 5. Given the system

$$\begin{cases} \dot{x}_1(t) = -x_1^3(t)\big(x_2(t)+1\big)^2 + \alpha\, x_1(t) \\ \dot{x}_2(t) = \big(x_2(t)+1\big)\big(\alpha - 1 - x_1^2(t)\big) \end{cases}$$

study the stability of the equilibrium point $x_e = (0, -1)$ for all the values of the parameter $\alpha \in (-\infty, \infty)$, using the method of linear approximation at the equilibrium point, and the Lyapunov function method (*Suggestion for the Lyapunov function:* $V(x) = (x_1 - x_{e,1})^2 + \beta(x_2 - x_{e,2})^2$, with suitable $\beta > 0$.)

Solution of problem 5.
The system considered is of the form $\dot{x}(t) = f\big(x(t);\alpha\big)$, where $x(t)$ is the state and α is a constant parameter. The vector function $f(x;\alpha)$ is as follows

$$\begin{cases} f_1(x;\alpha) = -x_1^3(x_2+1)^2 + \alpha\, x_1 \\ f_2(x;\alpha) = (x_2+1)(\alpha - 1 - x_1^2) \end{cases}$$

The Jacobian is

$$J(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix} = \begin{bmatrix} -3\, x_1(x_2+1)^2 + \alpha & -x_1^3\, 2\,(x_2+1) \\ -2x_1(x_2+1) & \alpha - 1 \end{bmatrix}$$

The Jacobian computed at $x_e = (0, -1)$ is

$$J(x_e) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha - 1 \end{bmatrix}$$

Since $J(x_e)$ is diagonal, the two eigenvalues are the diagonal terms: $\lambda_1 = \alpha, \qquad \lambda_2 = \alpha - 1.$
Thus:

- $(0, -1)$ is asymptotically stable when *both* eigenvalues are strictly negative, that is when $(\alpha < 0)$ **and** $(\alpha - 1 < 0)$. That means that $(0, -1)$ is asymptotically stable for $\alpha < 0$;
- $(0, -1)$ is unstable when *at least one* eigenvalue is strictly positive i.e. when $(\alpha < 0)$ **or** $(\alpha - 1 > 0)$. That means that $(0, -1)$ is unstable for $\alpha > 0$.

The only case where we can not conclude anything about the stability of $(0, -1)$ is when $\alpha = 0$ (in this case the origin is a simply stable equilibrium point *of the linear approximation* of the nonlinear system, but nothing can be concluded about the nonlinear system). In this case we must study the stability of the point $x_e = (0, -1)$ by using a suitable Lyapunov function.

For $\alpha = 1$ the function $f(x;\alpha)$ is

$$\begin{cases} f_1(x;0) = -x_1^3(x_2+1)^2 \\ f_2(x;0) = -(x_2+1)(1 + x_1^2) \end{cases}$$

Following the suggestion we consider the Lyapunov function

$$V(x) = x_1^2 + \beta(x_2+1)^2,$$

which is positive definite for any $\beta > 0$. According to the Lyapunov theorem, if the derivative $\dot{V}(x) = (dV/dx)f(x;1)$ is semidefinite negative, then the equilibrium x_e is (simply) stable, while if $\dot{V}(x)$ is definite negative, then the equilibrium is asymptotically stable. The computation of $\dot{V}(x)$ gives

$$\begin{aligned} \dot{V}(x) &= \frac{dV}{dx} f(x;0) = (\partial_{x_1} V) f_1(x;0) + (\partial_{x_2} V) f_2(x;0) = 2\, x_1\, f_1(x;0) + 2\beta\, x_2\, f_2(x;0) \\ &= -2\, x_1^4(x_2+1)^2 - 2\beta\,(x_2+1)^2(x_1^2+1). \end{aligned}$$

For any $\beta > 0$, $\dot{V}(x)$ is semidefinite negative (it is never positive, but it is zero for any pair $(x_1, -1)$, and not only at the equilibrium point $x_e = (0, -1)$). As a consequence, when $\alpha = 0$ the equilibrium $(0, -1)$ is *simply* stable (stable, but not asymptotically stable).