The Keller–Segel model for chemotaxis with prevention of overcrowding: linear vs. nonlinear diffusion

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General formulation of the model

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (M(\rho) \nabla \mu(\rho, S)) \\ -\Delta S + S = \rho, \end{cases} \tag{1}$$

Nonlinear *mobility* term $M(\rho) = \rho(1-\rho)$ prevents overcrowding.

Potential $\mu(\rho, S) = \frac{\delta E}{\delta \rho}(\rho, S)$ is functional derivative of some *energy functional* with respect to ρ.

Energy: combination of a local *repulsive* part (internal energy) $\mathcal{I}$ and of a nonlocal *aggregation* part

$$E(\rho, S) = \mathcal{I}(\rho) - \int_{\mathbb{R}^d} \rho S \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \left(|\nabla S|^2 + S^2 \right) \, dx.$$

Initial datum: $\rho|_{t=0} = \rho_0 \in L^1(\mathbb{R})$ and $0 \leq \rho_0 \leq 1$.

We consider two internal energies:

(1) $\mathcal{I}(\rho) = \varepsilon \int_{\mathbb{R}^d} (\rho \log \rho + (1-\rho)\log(1-\rho)) \quad dx$, yielding the linear diffusion case (LD)

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\varepsilon \nabla \rho - \rho(1-\rho)\nabla S),$$

(2) $\frac{\varepsilon}{2}\int_{\mathbb{R}^d} \rho^2 \ dx$, yielding the quadratic nonlinear diffusion case (NLD)

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho(1-\rho)\nabla(\varepsilon\rho - S)).$$

Aim: to understand the interplay between the size of the diffusivity constant $\varepsilon > 0$ and the asymptotic behavior in both cases.

A bit of existence theory

The condition $0 \leq \rho \leq 1$ is preserved by the flow $\Rightarrow$ Global uniform bound for ρ in $L^1 \cap L^\infty$. S can be computed via convolution with the Bessel potential

$$S(x,t) = \mathcal{B} * \rho(t)(x), \quad \mathcal{B}(x) = \frac{1}{(4\pi)^{d/2}} \int_0^{+\infty} \frac{e^{-t-\frac{|x|^2}{4t}}}{t^{d/2}} dt.$$

In (NLD), uniqueness can be pursued via the notion of entropy solution (Cf. Carrillo, ARMA 1999). Finite speed of propagation of the support in (NLD) (by estimate of the p–Wasserstein distances).

Main question: ‘Stationary solutions or decaying solutions’?

`Case 1`: **Linear diffusion**

Similarly to the model with $M(\rho) = \rho$ and $-\Delta S = \rho$ as equation for S,

Proposition 1. *There exists a constant $C(d)$ depending only on the dimension d such that, for total mass satisfying*

$$\int_{\mathbb{R}^d} \rho_0 dx < \left(\frac{4\varepsilon}{C(d)} \right)^{1/\beta} \qquad \beta = \min\{1, 2/d\}$$

the solution $\rho(x,t)$ to (LD) satisfies the decay estimates

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leq C(t+1)^{-\frac{d(p-1)}{2p}} \quad \textit{if} \;\; 2 \leq p < +\infty,$$

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(t+1)^{-\frac{d}{2}}.$$

`Case 1`: **Linear diffusion**

Moreover, in case $d = 1$ and $\varepsilon > 1/4$, we have long time decay no matter how large the initial mass is.

Proposition 2. *Let* $\varepsilon > 1/4$ *and* $d = 1$. *Then, the solution* $\rho(x,t)$ *to (LD) satisfies the decay estimates*

$$\|\rho(t)\|_{L^p(\mathbb{R})} \leq C(t+1)^{-\frac{(p-1)}{2p}}, \quad 2 \leq p \leq \infty.$$

Open problem: do the solutions decay for any ε and for any initial mass?

Nonexistence of stationary states in the linear case

There exist no nonzero stationary solutions to (LD) in $L^1(\mathbb{R}^d)$. Energy dissipation:

$$\frac{dE}{dt} = \int \nabla \cdot \left(\rho(1-\rho) \nabla \frac{\delta E}{\delta \rho} \right) \frac{\delta E}{\delta \rho} dx = - \int \rho(1-\rho) \left| \nabla \frac{\delta E}{\delta \rho} \right|^2 dx \leq 0.$$

Stationary state: $\frac{dE}{dt} = 0$, i. e. $\rho = 0$, $\rho = 1$ or $\frac{\delta E}{\delta \rho} = const \Rightarrow$
the stationary solution (ρ, S) should satisfy, for some constant C,

$$\frac{\rho}{1-\rho} = e^{\frac{S+C}{\varepsilon}},$$

in some open set $\Omega \subset \mathbb{R}^d$, with $\rho = 0$ at some point of $\partial\Omega$, incompatible with S being bounded, because of the continuity of ρ (regularizing effect).

Self–similar large time behavior

Time dependent scaling

$$\rho(x,t) = R(t)^{-\frac{d}{2}} v(y,s), \quad y = R(t)^{-\frac{1}{2}} x, \quad s = \frac{1}{2}\log R(t), \quad R(t) = 2t+1.$$

$$\begin{cases} \frac{\partial v}{\partial s} = \varepsilon \nabla \cdot \left(v \nabla \left(\log v + \frac{|y|^2}{2\varepsilon} \right) \right) - e^{-ds} \nabla \cdot (v(1-e^{-s}v)\mathcal{B}_s * \nabla v). \\ v(y,0) = \rho_0(y) \end{cases}$$

where $\mathcal{B}_s(y) = e^s \mathcal{B}(e^s y)$.

Under the assumption $\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(t+1)^{-\frac{d}{2}}$ the last term in the r.h.s. can be treated as a higher order perturbation of the linear Fokker–Planck equation $\Rightarrow$ We can apply the relative entropy method and prove algebraic decay towards Gaussian profiles (Cf. Carrillo–Toscani, IUMJ 2000).

`Case 2`: Nonlinear diffusion

Energy functional

$$\tilde{E}(\rho) := \int_{\mathbb{R}^d} \rho(\varepsilon\rho - S(\rho))\ dx,$$

admissible set

$$\mathcal{K} := \{\rho \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \mid 0 \leq \rho \leq 1 \text{ a.e. } \}.$$

Lemma 1. *The functional $\tilde{E} : \mathcal{K} \to \mathbb{R}$ is bounded below by $-\int_{\mathbb{R}^d} \rho\ dx$ for $\varepsilon > 0$. Moreover, $\tilde{E}$ is positive and strictly convex for $\varepsilon > 1$.*

Energy dissipation:

$$\frac{d}{dt}\tilde{E}(\rho(t)) = -2\int_{\mathbb{R}^d} \rho(1-\rho)\left|\nabla(\varepsilon\rho - \mathcal{B} * \rho)\right|^2\ dx := -2I(\rho, S) < 0.$$

`Case 2`: Nonlinear diffusion. Attractors of the semigroup in 1–d.

Stationary solutions:

$$\rho(1-\rho)\nabla(\varepsilon\rho - S(\rho)) = 0 \qquad \text{a.e. in } \mathbb{R}\times[0,+\infty), \tag{2}$$

Theorem 1. *Any sequence of times admits a subsequence t_k such that $\rho(t_k)\to\rho^\infty$ almost everywhere and ρ^∞ is a solution to (2).*

Sketch of the proof: The finiteness of the term $\int_0^{+\infty} I(\rho(t),S(t))dt$ implies $I(t_k)\to 0$ up to subsequences. Expansion of I and standard estimates imply a uniform bound for $\int_{\mathbb{R}} B(\rho(t_k))_x^2 dx$ where $B(\rho)=\int_0^\rho \sqrt{r(1-r)}dr$, which implies strong compactness. A standard weak l.s.c. argument yields consistency of the limit.

`Case 2`: **Nonlinear diffusion**

Proposition 3. [Initial data with negative entropy] *Let $d = 1$ and $\varepsilon < 1$, then for each $m > 0$ there exists $\rho \in \mathcal{K}$ satisfying*

$$\tilde{E}(\rho) < 0, \qquad \textit{and} \qquad \int_{\mathbb{R}^d} \rho \, dx = m. \tag{3}$$

Sketch of the proof: 'construct' $S(\rho)$ such that $S(\rho) = \beta\rho$ for some $\varepsilon < \beta < 1 \Rightarrow \int \rho(\varepsilon\rho - S) < -(\beta - \varepsilon) \int \rho^2$. More precisely, such construction is performed on an interval $[a, b]$ whereas $\rho \equiv 1$ on $[0, a]$ and $\rho \equiv 0$ on $[b, +\infty)$ and ρ is extended symmetrically. In particular, we have to solve

$$-S'' - (1 - \beta)S = 0 \quad \text{on } [a, b].$$

This fact has an important consequence:

Theorem 2. *If $\varepsilon < 1$, a non–decaying solution $\rho(t)$ exist. Moreover, $\rho(t)$ converges almost everywhere to a nontrivial stationary solution ρ^∞ up to time subsequences.*

Sketch of the proof: choose ρ_0 as in the previous proposition,

$$\tilde{E}(\rho^\infty) \leq \liminf \tilde{E}(\rho(t_k)) \leq \tilde{E}(\rho_0) < 0,$$

therefore ρ^∞ is not identically zero $\Rightarrow$ nontrivial stationary solutions exist.

Open problem: is it possible to apply a relative entropy argument in order to have more info's about the convergence? Problems: existence of several stationary states for fixed mass ('multi peak' solutions) and the entropy functional is not convex.

`Case 2`: Nonlinear diffusion. Decay for $\epsilon > 1$.

Additional logarithmic functional

$$L(\rho) = \int_{-\infty}^{+\infty} [\rho \log \rho + (1-\rho)\log(1-\rho)]dx.$$

$$L(\rho(t)) - L(\rho(0)) = \int_0^t \int (\log \rho - \log(1-\rho))\rho_t dx d\tau$$

$$= -\varepsilon \int_0^t \int \rho_x^2 dx d\tau + \int_0^t \int \rho_x S_x dx d\tau \le -(\varepsilon - 1) \int_0^t \int \rho_x^2 dx d\tau.$$

Unfortunately, $L(\rho)$ is not bounded from below. Therefore the above estimate cannot be used directly to discern the asymptotic behavior of $\rho(t)$. However, we can use the above estimate in the following theorem.

`Case 2`: **Nonlinear diffusion. Decay for $\epsilon > 1$.**

Theorem 3. [Decay for large diffusivity in 1–d] *Let ρ, S be a solution to (NLD) with $\varepsilon > 1$ such that ρ has finite support at any time. Then, the support of ρ is not uniformly bounded with respect to t. Consequently, there exist no compactly supported stationary solutions ρ, S to (NLD) if $\varepsilon > 1$ different from zero and $\rho(t)$ tends to zero a.e. as $t \to +\infty$.*

Sketch of the proof: Suppose $\rho(t)$ has uniformly bounded support. Since the function $[0,1] \ni \rho \mapsto \rho \log \rho + (1-\rho)\log(1-\rho)$ is bounded, $L(\rho(t))$ is uniformly bounded in time. A strong compactness argument as above implies that there exists a divergent sequence of times t_k such that $\rho(t_k)$ converges to some ρ^∞ almost everywhere. The additional logarithmic estimate implies $\rho^\infty \equiv 0$. Sobolev interpolation lemma implies $\rho(t_k) \to 0$ uniformly, and this contradicts $\rho(t)$ having uniformly bounded support because of the conservation of the mass.

Conclusions and open problems

- The nonlinear diffusion seems to be more appropriate because it induces a fair competition between diffusion and aggregation.
- The diffusivity constant ε plays a decisive role in the asymptotics, unlike in the classical models where the mass is the key ingredient (Cf. Dolbeault et al.). What happens in the critical case $\varepsilon = 1$?
- Open problem: decay towards Barenblatt type profiles for large diffusivity.
- Question: is the nonlinear diffusion model consistent with the inviscid case $\varepsilon = 0$ (unlike the linear diffusion case, cf. Dolak – Schmeiser)?
- General question: is it possible to develop a *gradient flow* approach in this case (i.e. with a nonlinear mobility $M(\rho)$)?
- In a forthcoming paper with M. Burger, existence of nontrivial stationary states for general diffusion vs. aggregation models with linear mobility and *bounded* interaction kernels.