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PhD Thesis

Diffusive behavior and asymptotic self similarity for fluid models

Author: Marco Di Francesco

Advisor: Pierangelo Marcati

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Abstract

The present PhD thesis deals with diffusive relaxation limits and long time asymptotics for several partial differential equations or systems of equations of hyperbolic or parabolic type. Most of the models considered arise from compressible gas dynamics; some of them take into account of heat radiation phenomena, some others involve the presence of porous media. We make use of classical techniques, such as Friedrichs symmetrization for quasi linear hyperbolic systems and energy estimates, as well as of more recently developed tools, such as the entropy dissipation method and the optimal transportation approach.

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Contents

Chapter 1

Introduction

1.1 General outline

The present PhD thesis is concerned with the asymptotic analysis of hyperbolic and parabolic partial differential equations and systems of equations, mainly models in continuum mechanics and physics such as compressible gas–dynamics, flow through porous media, motion of radiating gases, viscoelastic materials, general nonlinear diffusion phenomena. The present thesis is devoted to investigate qualitative properties of the solutions of initial value problems for the previously mentioned models, in one or several space dimension. More precisely, we focus our attention mainly on two kinds of problems:

(i) The former regards the analysis of the limiting behavior of solutions with respect to some singular *parameter* appearing in the model. This parameter may arise either from *relaxation* phenomena or as a result of an internal *rescaling*. In both the cases, our strategy is to determine an *equilibrium* or limit problem, then to justify rigorously the convergence process.

(ii) The latter concerns with the analysis of the *long time* behavior of the solutions and the appearance of intermediate asymptotic states enjoying a *self–similar* structure. In most of the models studied, we will be also interested in determining the *optimal* rate of convergence in some L^p or Sobolev spaces.

The link between (i) and (ii) can be intuitively understood by means of the following example, representing a simplified model for radiation gas dynamics (we will analyze this model in Chapters 4 and 8)

$$\begin{cases} u_t + uu_x = -q_x \\ -q_{xx} + q = -u_x. \end{cases} \tag{1.1}$$

In this case (as in many more cases throughout our work), the rescaling process we perform in order to detect the an asymptotic approximation is of the form

$$u^\varepsilon(x,t) = \frac{1}{\varepsilon} u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), \quad q^\varepsilon(x,t) = \frac{1}{\varepsilon^2} u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right). \tag{1.2}$$

Here the formal limit is represented by the viscous Burgers' equation for the variable u (we skip the details, see Chapter 4). Hence, since the scaling factor for the time variable t grows faster than the scaling factor for the space variable x as ε approaches zero, one can pose the problem of detecting a diffusive limiting behavior also in a framework of pure long time asymptotics.

This introductive chapter is organized as follows. In the section 1.2 we explain some basic ideas of relaxation limits for hyperbolic problems, by focusing in particular on the diffusive relaxation limits. In the section 1.3 we shall discuss one of the most classical models which motivates the study of diffusive relaxation hyperbolic systems, namely the Euler equations of compressible fluids thorugh of porous media. In the section 1.4 we introduce a relaxation model for viscoelasticity, and in the section 1.5 we shall give a brief introduction to the mathematical theory for the models in radiation gas dynamics. The section 1.6 is devoted to the porous medium equation, while the section 1.7 will provide an outline of the entropy methods recently used to analyze the long time asymptotics for nonlinear diffusion equations. The section 1.8 contains an introduction to another simplified model for compressible gas dynamics, i.e. the viscous Burgers' equation. Finally, the section 1.9 is devoted to another tool used recently in the study nonlinear diffusion, i.e. the Wasserstein distance. In our introduction, the most important results contained in the present thesis will be presented and framed into their natural context. We remark that the references reported here are obtained as the result of a selection done in relation with the topics and with the results contained in this thesis. Hence, the literature presented later will be not complete and will not appropriately describe the general subject.

1.2 Relaxation limits for hyperbolic models.

Relaxation phenomena arise typically in cases of perturbations of an equilibrium state for a given physical model. The simplest cases is the following linear hyperbolic system

$$\begin{cases} u_t + v_x = 0 \\ v_t + \lambda u_x = \frac{1}{\varepsilon}\left(\alpha u - v\right), \end{cases}$$

where the equilibrium is represented by the equation

$$u_t + \alpha u_x = 0$$

and by the relation

$$\alpha u - v = 0.$$

In this simple case a straightforward computation shows that the convergence towards equilibrium occurs if and only if $\lambda^2 \geq \alpha^2$ (this is known as the *sub-characteristic* condition). The typical general form for a hyperbolic systems with relaxation in the framework of smooth solutions is the following,

$$\partial_t U + \sum_{j=1}^{d} A_j(U)\partial_{x_j} U = \frac{1}{\varepsilon} Q(U) \tag{1.1}$$

where the unknown U is a function of $x \in \mathbb{R}^d$ and $t \geq 0$ with values in an open subset A of $\mathbb{R}^k$, A_j and Q are smooth functions of U. The quasi linear system above is usually supposed to be *symmetrizable* hyperbolic in the sense of Friedrichs (see [Fri54, Maj84]), i.e. there exists a symmetric positive definite matrix valued function $A_0(U)$ that symmetrizes simultaneously $A_j(U)$ for all $j = 1, \ldots, d$ (see [Fri54, Lax57, Kat75, Maj84]).

The source term $\frac{1}{\varepsilon}Q(U)$ is often referred to as the *relaxation term.* This term is supposed to be endowed with two operators. The first one, which we denote by P, is called the *projection on the momenta.* In a simplified situation, P can be assumed to be a constant linear operator from $\mathbb{R}^k$ to $\mathbb{R}^n$ with $n < k$, such that $PQ(U) = 0$ for all $U \in A$. This structural assumption express the fact that n equations in system (1.1) can be rewritten in the homogeneous form

$$\partial_t PU + \sum_{j=1}^{d} PA_j(U)\partial_{x_j} U = 0.$$

The second operator is a map M defined on an open set of $\mathbb{R}^n$ with values in $\mathbb{R}^k$ (often called the *Maxwellian operator*) such that $QM(u) = 0$ and $PM(u) = u$. Under certain hypothesis (see [Nat99] for further details), one may expect that the unknown U converges in some sense, as $\varepsilon \to 0$, to the *equilibrium* $M(u)$ where the vector valued function u satisfies the reduced system

$$\partial_t u + \sum_{j=1}^{d} PA_j(M(u))\partial_{x_j} M(u) = 0.$$

A quite general theory for these kinds of systems has been recently developed by W.A. Yong (see [Yon99]). There exist then a large variety of interesting physical examples where relaxation schemes can be applied, such as singular perturbations of the wave equation, traffic flow models, and the kinetic Broadwell model. For a detailed explanations of these models we refer to the survey paper by Natalini [Nat99], where also a huge list of references can be found, together with detailed proofs of more recent and advanced results on these topics. One of the main scopes of relaxation schemes is to construct admissible weak solutions to the reduced hyperbolic equations or systems. A typical case is the relaxation towards a scalar conservation laws (see [Nat99, Daf00] and the references therein). In this sense, this approach is quite similar to the kinetic formulation of a conservation law (see [Daf00]). For the basic theory of linear relaxation schemes we refer to the pioneering book by Whitham [Whi74]. The corner stone for the nonlinear theory is the fundamental paper by Tai Ping Liu [Liu87] (see also [CLL94]).

In the recent years, hyperbolic systems relaxing towards *diffusive* models have been the subject of a wide investigation. A very simple (linear) example in this sense is provided by the dissipative wave equation

$$u_{tt} - \Delta u + \frac{1}{\varepsilon} u_t = 0.$$

By rescaling a solution to that equation as follows

$$u(x,t) = \frac{1}{\varepsilon} v\left(x, \frac{t}{\varepsilon}\right),$$

it can be proven that v converges (e.g. in H^s) to the solutions of the linear heat equation $v_t = \Delta v$. We explain hereafter the basic ideas of diffusive relaxation processes, without taking care of all the structural assumptions and the technical details, for which we refer to the paper by Marcati and Rubino [MR00]. Let us consider a quasi linear hyperbolic system in vector form

$$\begin{cases} U_t + F(U,V)_x = 0 \\ V_t + G(U,V)_x = H(U,V) \end{cases}$$

We then perform the parabolic scaling

$$U^\varepsilon(x,t) = U\left(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}\right) \quad V^\varepsilon(x,t) = \frac{1}{\sqrt{\varepsilon}} V\left(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}\right) \tag{1.2}$$

for any $\varepsilon > 0$. Then, by substituting the new variables into the original system and by taking the formal limit as $\varepsilon \to 0$ (under more structural assumptions), see [MR00]), one recovers the reduced system

$$\begin{cases} U_t + (F_V(U,0)V)_x = 0 \\ G(U,0)_x = H_V(U,0)V \end{cases}$$

Under the extra assumption $\det H_V(U,0) \neq 0$, one can rewrite the above reduced system as

$$U_t + \left(F_V(U,0)\,(H_V(U,0))^{-1}\,G(U,0)_x\right)_x = 0,$$

which is a parabolic system in the sense of Petrowski (see the book of Kreiss and Lorentz [KL89]).

The parabolic scaling we use here is the classical transformation one needs to apply to a nonhomogeneous hyperbolic system to detect its parabolic behavior time–asymptotically. Among all, in this framework, we recall the papers of Kurtz [Kur73] and McKean [McK75], where for the first time this feature for hyperbolic systems has been put into evidence. Afterwards, we recall the papers of Marcati with various collaborators (see [MR00, DM00] and the references therein), where the above scaling has been used for several systems and the convergence has been obtained for weak solutions with the aid of the compensated compactness. Moreover, we recall the paper of Lions and Toscani [LT97], where the same parabolic behavior has been pointed out for Boltzmann kinetic models with a finite number of velocities, proving in particular the convergence towards the porous media equation. In [LN02], the authors proposed a BGK approximation for strongly parabolic systems verifying certain conditions and they proved the convergence of weak solutions, again using compensated compactness. The study of H^s solutions, with a detailed analysis of the initial layer phenomenon, has been carried out in [LY01], for hyperbolic relaxation systems with a strongly parabolic equilibrium system.

1.3 The Euler equations for compressible gas dynamics

The inviscid flow of a compressible fluid is described by the Euler equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u + p\mathbb{I}) = 0 \\ \left\{\rho\left(e + \frac{|u|^2}{2}\right)\right\}_t + \operatorname{div}\left\{\rho u\left(e + \frac{|u|^2}{2}\right) + pu\right\} = 0, \end{cases} \tag{1.1}$$

where the three equations express conservation of mass, momentum and energy respectively. We refer to the book of Courant and Friedrichs [CF48] for a detailed derivation of several models in compressible gas–dynamics. The study of system (1.1) is a classical topic. Such system enjoys a quasi–linear symmetric hyperbolic structure, in such a way that the local–in–time existence of H^s solutions is guaranteed by the existence of a Friedrichs type symmetrizer (see Friedrichs [Fri54] and the book of Majda [Maj84]). Then, the formation of shock waves in finite time may occur. This phenomenon is best understood in one space dimension, where the method of characteristics can be employed (see Courant–Friedrichs [CF48] and Whitham [Whi74]). The formation of singularities in several space variables has been the topic of more recent investigations, see [Sid85, Sid91, Sid97, MUK86, Ali93, Ali95].

In Chapter 2 we will concern with a class of smooth solutions to the onedimensional isentropic compressible inviscid Euler equations through porous media. In this case, a dissipative nonhomogeneous term in the balance law of the momentum appears, because of the friction due to the presence of a porous medium. The model then reads

$$\begin{cases} \rho_\tau + (\rho u)_x = 0 \\ u_\tau + u u_x + \dfrac{p(\rho)_x}{\rho} = -\dfrac{u}{\varepsilon}. \end{cases} \tag{1.2}$$

As was first proven by Nishida (see [Nis68]), the presence of the *damping term* $-\frac{u}{\varepsilon}$ prevents the formation of singularities for *small* initial data belonging in certain Sobolev spaces (of course the *smallness* needed to preserve regularity depends on the amplitude of the positive parameter ε). This result has been only recently extended to the three dimensional case in the paper [STW03]. In both cases, to ensure global existence and smoothness of the solutions, an abstract continuation principle must be satisfied (see [Maj84] for its precise formulation), which is basically a global (w.r.t. to time) energy estimate for the space derivatives of the solutions up to a certain order. In Chapter 2 we will show, in particular, that the global–in–time existence of smooth solutions for system (1.2) can be extended to small perturbations of a special class of *diffusion waves* (see Remark 2.1.3).

After the scaling

$$\rho^\varepsilon(x,t) = \rho(x, t/\varepsilon) \qquad u^\varepsilon(x,t) = \frac{1}{\varepsilon} u(x, t/\varepsilon),$$

system (1.2) can be rewritten as follows

$$\begin{cases} \rho^\varepsilon_t + (\rho^\varepsilon u^\varepsilon)_y = 0 \\ u^\varepsilon_t + u^\varepsilon u^\varepsilon_y + \dfrac{p(\rho^\varepsilon)_y}{\varepsilon^2 \rho^\varepsilon} = -\dfrac{u^\varepsilon}{\varepsilon^2}. \end{cases} \tag{1.3}$$

Hence, the density ρ^ε formally converges as $\varepsilon \to 0$ to a solution of the non-linear diffusion equation $\rho_t = p(\rho)_{xx}$. This was first proven by Marcati and Milani in [MM90] in a framework of weak solutions, by means of techniques based upon compensated compactness tools. This paper provided a contribution in understanding the hyperbolic nature of porous media flows. Let us remark that the previous scaling, even though apparently different from the usual parabolic scaling (1.2), gives raise to the same reduced system even in the general framework of diffusive relaxation limits (see the introduction to [MR00]).

The study developed in Chapter 2 goes in the same direction as [MM90], but in a context of smooth solutions, as already pointed out before. In the present case, we have to do with solutions *far from vacuum*, i.e. such that $\rho \geq \bar{\rho} > 0$. In particular, the initial data must satisfy certain limiting conditions at $\pm\infty$ in order to ensure absence of vacuum for large x. We will study the behavior of the solutions as the parameter ε goes to zero, and show that, under some assumption on the initial data and on the limiting states at infinity, the density in (1.3) converges in a suitable norm to a caloric self similar solution to the generalized Porous Medium equation (see also Section 1.6)

$$\widetilde{\rho}_t = p(\widetilde{\rho})_{yy}. \tag{1.4}$$

Such solutions depend only on the similarity variable $x/t^{1/2}$, and satisfy the same limiting conditions as the density in (1.3). The existence of such caloric self–similar solutions has been the subject of many papers starting from the seventies, see [AP71, AP74, vDP77, vDP77]. We will show in Chapter 6 that these caloric profiles are asymptotically stable states for equation (1.4) if the initial datum is a small perturbation of them.

We remark that our result holds in a *well–prepared initial data* regime, i.e. the initial data satisfy the equilibrium relation

$$p(\rho)_x = -\rho u,$$

which in this case is known as *Darcy's law*. In this sense, we do not deal with the so called *initial layer* problem, which is one of the main issues of some other papers in the literature (see Lattanzio and Yong [LY01] e.g.).

Besides what mentioned above for the compressible Euler system, a parallel study of the asymptotic behavior for large times for the damped compressible Euler flow, both in Lagrangian and in Eulerian coordinates, has been developed by Hsiao and Liu [HL92], [HL93] and by Nishihara [Nis96], where the above mentioned caloric self similar profiles were recovered to be asymptotically stable states under small perturbations for the damped onedimensional p–system

$$\begin{cases} u_t - v_x = 0 \\ v_t - p(u)_x = -v \end{cases} \tag{1.5}$$

Moreover, a result concerning the 2-D perturbation of this problem was proved by Lattanzio and Marcati in [LM99]. Recently in [NWY00], Nishihara, Wang and Yang proved a sharper result on the L_p–convergence ($2 \leq p \leq \infty$) by means of a Green function technique. We also mention [MP01], in which an analogous result is proven for a compressible adiabatic flow, and [MN03], where the authors recover sharp L^p–L^q estimates for the damped wave equation and apply them to the asymptotic study of the damped p–system (1.5). The book by Hsiao [Hsi97] collects many results in these directions, most of them are based on energy estimate and Green functions techniques and hold in a small perturbation setting. In Eulerian coordinates, the first convergence result to self similar solutions including vacuum has been carried out in [HMP]. We remark that although in both the Eulerian and the Lagrangian case the limiting profiles satisfy the Porous Media equation, the two cases cover different physical situations.

The results in Chapter 2 are contained in the author's paper (joint with Marcati) [DFM02]. In particular, a convergence result in Sobolev norm has been proven under the assumption of well prepared initial data. The technique used is based on the standard symmetric hyperbolic systems framework developed by Friedrichs, Lax, Majda and many more in [CF48, Lax57, KM81, KM82, Maj84].

1.4 Diffusive relaxation model for a system of viscoelasticity

We analyze here the following semilinear hyperbolic system with relaxation term

$$\begin{cases} U_s - V_y = 0 \\ V_s - Z_y = 0 \\ Z_s - \mu V_y = -Z + \varepsilon^2 \sigma(U), \end{cases} \tag{1.1}$$

where U, V, $Z \in \mathbb{R}$, $y \in \mathbb{R}$, $s > 0$, μ is a strictly positive parameter and $\varepsilon > 0$ denotes the relaxation time. Our goal is to detect the diffusive relaxation limit for the system (1.1) as $\varepsilon \downarrow 0$. To perform this task, we scale both the dependent and the independent variables with respect to the relaxation time

ε in the following way

$$u(x,t) = U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \quad v(x,t) = \frac{1}{\varepsilon}V\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \quad z(x,t) = \frac{1}{\varepsilon^2}Z\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right).$$

With the above diffusive scaling, the system (1.1) becomes

$$\begin{cases} u_t - v_x = 0 \\ v_t - z_x = 0 \\ \varepsilon^2 z_t - \mu v_x = -z + \sigma(u), \end{cases} \tag{1.2}$$

which, with $\varepsilon = 0$, formally reduces to

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx}. \end{cases} \tag{1.3}$$

In Chapter 3 we will prove that solutions of (1.2) converge to solutions of (1.3), giving in this way a rigorous justification of the relaxation limit.

We shall assume the function σ in the relaxation term of (1.2) to be globally Lipschitz, namely

$$\sup_{u\in\mathbb{R}} |\sigma'(u)| < +\infty. \tag{1.4}$$

We recall that condition (1.4) represents the soft hardening condition for the stress–strain function σ in system (1.3). Moreover, at this stage we do not need to require the positivity of σ', namely we can approximate also incompletely parabolic systems where the corresponding inviscid first order system is not necessarily hyperbolic. We emphasize this is the first rigorous proof of a relaxation limit from a hyperbolic toward an hyperbolic parabolic system.

This semilinear relaxation approximation has another physical interpretation in terms of mathematical models in the study of viscoelastic materials [RHN87]. Indeed, the system (1.2) can be rewritten as follows

$$\begin{cases} u_t - v_x = 0 \\ v_t - z_x = 0 \\ \left(z - \frac{\mu}{\varepsilon^2}u\right)_t = -\frac{1}{\varepsilon^2}\left(z - \sigma(u)\right), \end{cases} \tag{1.5}$$

that is, a system of viscoelasticity with memory. In the system (1.5), the stress function z is given by the following relation

$$z = \frac{\mu}{\varepsilon^2}u - \int_{-\infty}^{t} \frac{1}{\varepsilon^2} e^{-\frac{t-\tau}{\varepsilon^2}} \left(\frac{\mu}{\varepsilon^2}u - \sigma(u)\right)(\tau) d\tau,$$

while in the limit (1.3), the stress is given by the relation

$$z = \sigma(u) + \mu v_x,$$

which is the case of viscoelasticity of the rate type. Therefore, our relaxation limit can be viewed as the passage from the viscosity of the memory type to the viscosity of the rate type in the study of viscoelastic materials. To detect this phenomenon, in the definition of the stress z, we scale either the kernel of the memory, and the (linear) elastic response of the material at initial time. Here we recall that the case of a fixed response of the material, which corresponds to a hyperbolic scaling of the relaxation approximation, has been studied in [Tza99], where the convergence toward the 2×2 hyperbolic system of elasticity has been proved.

The results collected in Chapter 3 are the subject of the author's paper (joint with Lattanzio) [DFL].

1.5 The Hamer model for radiating gases

A quite general model for compressible gas dynamics where heat radiative transfer phenomena are taken into account is given by the hyperbolic elliptic coupled model

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u + p\mathbb{I}) = 0 \\ \left\{\rho\left(e + \frac{|u|^2}{2}\right)\right\}_t + \operatorname{div}\left\{\rho u\left(e + \frac{|u|^2}{2}\right) + pu + q\right\} = 0 \\ -\nabla \operatorname{div} q + aq + b\nabla T^4 = 0. \end{cases} \tag{1.1}$$

As usual, in (1.1), ρ, u, p, e and T are respectively the mass density, velocity, pressure, internal energy and absolute temperature of the gas, while q is the radiative heat flux and a and b are given positive constants depending on the gas itself. We give here a sketch of the physical motivation of the fourth equation in (1.1) (see the books by Vincenti and Kruger [VK65] and by Zel'dovich and Raizer [ZR66] for a detailed explanation), the first three equations being motivated as for the usual Euler system (see [CF48]).

We start from the physical observation that high temperature gases emit energy in the form of electromagnetic radiation. This is due both to transitions from upper to lower energy levels of the atoms or molecules of the gas and from transitions that involve free electrons. We consider radiation fields as composed by photons, each one of them has energy and moves with light speed c. The spectral radiation intensity, then, can be expressed as

$$I_\nu(r, \Omega, t)d\nu d\Omega = h\nu c f(\nu, r, \Omega, t)d\nu d\Omega,$$

where $f(\nu, r, \Omega, t)$ is the number of photons in the frequency interval $[\nu, \nu + d\nu]$, in the unit volume dr, within the solid angle $d\Omega$, at time t, and h is the Planck constant. The spectral energy flux reads

$$q_\nu(r,t) = \int I_\nu \Omega d\Omega.$$

Hence, by writing down the kinetic equation for I_ν one recovers the following *radiative heat transfer equation*

$$\frac{1}{c}\left[\frac{\partial I_\nu}{\partial t} + c\Omega \cdot \nabla I_\nu\right] = j_\nu \left(1 + \frac{c^2}{2h\nu^3} I_\nu\right) - k_\nu I_\nu, \tag{1.2}$$

where the gradient is taken with respect to r and where the right hand side is the difference between the emitted radiation and the absorbed radiation. Since the photons obey to Bose–Einstein statistics at the equilibrium, the stationary distributions in the equation (1.2) are represented by

$$I_\nu^* = \frac{2h\nu^3}{c^2}\left(\frac{1}{e^{h\nu/kT} - 1}\right),$$

where k is the Boltzmann constant and T the temperature. Under the assumption of *quasi equilibrium* approximation, equation (1.2) reads

$$\operatorname{div}(\Omega I_\nu) = \alpha_\nu \left[I_\nu^* - I_\nu\right],$$

with $\alpha = \rho k_\nu \left(1 - e^{-h\nu/kT}\right)$. After integration with respect to $\Omega \in \mathrm{S}^2$ and with respect to $\nu \in [0, +\infty)$ we obtain

$$\operatorname{div} q = -\alpha\left(I - 4\sigma T^2\right),$$

where we have set

$$I = \int\int I_\nu(\Omega) d\nu d\Omega, \qquad q = \int\int \Omega I_\nu(\Omega) d\nu d\Omega,$$

for suitable constants α and σ. Under another extra assumption on the model, namely the so called *Milne–Eddington* approximation

$$\int\int \Omega_i \Omega_j I_\nu(\Omega) d\nu d\Omega = \frac{1}{3}\delta_{ij} \int\int I_\nu d\Omega$$

(see [VK65, ZR66]), we finally obtain

$$-\nabla \operatorname{div} q + 3\alpha^2 q + 4\sigma\alpha\nabla T^4 = 0, \tag{1.3}$$

which is exactly the fourth equation in the system (1.1).

The model (1.1) has a simplified version, namely

$$\begin{cases} u_t + a \cdot \nabla u^2 = -\operatorname{div} q \\ -\nabla \operatorname{div} q + q = -\nabla u \end{cases} \tag{1.4}$$

where $u = u(x,t) \in \mathbb{R}$, $q = q(x,t) \in \mathbb{R}^3$ $x \in \mathbb{R}^3$, $t \geq 0$ and a is a constant vector. The simplified model (1.4) was first recovered by Hamer (see [Ham71]). The most convenient approach to such system is to solve the elliptic equation satisfied by the term $\operatorname{div} q$ in terms of u and substitute it into the first equation in order to get the scalar balance law

$$u_t + a \cdot \nabla u^2 = -u + K * u, \tag{1.5}$$

where the kernel K is given by the *Bessel potential*

$$K(x) = \frac{1}{(4\pi)^{d/2}} \int_0^{+\infty} \frac{e^{-s-\frac{|x|^2}{4s}}}{s^{d/2}} ds$$

In one space dimension, equation (1.5) was first studied in [ST92], where the authors referred to it as the *Rosenau–Chapman–Enskog* equation, and successively in [LM03]. In particular, it has been proven that this equation induces a contraction semigroup in L^p, $p \in [1, \infty]$ for all positive times, and it has a critical threshold (in terms of Sobolev norms) below which it preserves the regularity of the initial datum. This is essentially proven by taking advantage of the dissipative nature of the inhomogeneous term $-u + K * u$. Concerning the general model (1.1) and its simplified version (1.4), the study of the existence of smooth solutions, together with the existence and stability of shock profiles in one space dimension has been perfomed by Kawashima and various collaborators in the general context of quasi linear hyperbolic elliptic coupled systems (see [KN98, KN99a, KN99b, KNN99, KN02, KNN03]), and by Serre (see [Ser, Ser03]). In Chapter 4 we extend the global well–posedness of the model to the case of several space variables, and we provide an abstract proof of the global existence in $L^1 \cap L^\infty$ by means of the Crandall–Liggett theory of nonlinear semigroups (see [CL71, Cra72, Daf00]).

As already pointed out in the first section, this model represents a good example of diffusive behavior both in a relaxation sense and in a pure time–asymptotic framework. Concerning the long time asymptotic diffusive behavior (in one space dimension), this feature has been one of the subjects of the research by Kawashima and collaborators (see [KNN99, KN02]). The main tool used to analyze this kind of models in those papers are the classical energy estimates. Hence, under suitable stability conditions, it is proved global existence in H^s for solutions to those systems with initial data that are small perturbation od constant states, and the convergence of such solutions towards the superposition of diffusion waves of the reduced parabolic limit system. This theory has been significatively improved in [IK02], by considering the pointwise estimates for the Green function of the linearized system together with the classical energy estimates. This technique yields L^p convergence results for $p \in [1, +\infty]$, for initial data in $H^s \cap L^1$ but with a smaller regularity than in the papers quoted before. In the particular case of our scalar model (1.5) in one space dimension, the results in [IK02] provide the convergence for large times towards self similar solutions to the viscous Burgers' equation

$$u_t + uu_x = u_{xx},$$

with a non optimal rate. More recently, the relaxation limit of the rescaled solutions of (1.5) in one space dimension towards the solutions of the viscous Burgers' equation has been improved in [Lau] to include also sequences of initial data approaching to Dirac masses. This allows to interpret the relaxation limit as an asymptotic convergence toward the diffusion wave of the viscous Burgers' equation, thanks to a rescaling method and thanks to the self–similarity of the limit. The proof is performed for large initial data in $L^\infty \cap L^1$ and it doesn't give rate of convergence in L^1. In Chapter 8 we fill the gap in the optimality of the rate by means of relative entropy methods, under regularity assumption (namely, small data in Sobolev norms) required to employ the decay properties results in [IK02].

Concerning the aspect of relaxation limits, the scalar equation (1.5) can be rescaled in two different ways, in order to get as a formal limit an inviscid and a viscous conservation law respectively (hyperbolic–hyperbolic and hyperbolic–parabolic relaxation respectively). More precisely, let us consider again our model with a general flux $f(u)$, namely

$$\begin{cases} u_t + \operatorname{div} f(u) = -\operatorname{div} q \\ -\nabla \operatorname{div} q + q = -\nabla u \\ u(\cdot, 0) = u_0, \end{cases} \tag{1.6}$$

After the hyperbolic scaling $(x,t) \to \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$ of the independent variables, we get formally in the limit as $\varepsilon \to 0$ the inviscid scalar conservation law

$$u_t + \operatorname{div} f(u) = 0,$$

while, under the scaling

$$u^\varepsilon(x,t) = \frac{1}{\varepsilon} u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$$
$$q^\varepsilon(x,t) = \frac{1}{\varepsilon^2} q\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

the sequence u^ε converges to the solution of the viscous equation

$$u_t + \frac{1}{2} f''(0) \cdot \nabla u^2 = \Delta u.$$

Both these relaxation limits are analyzed and rigorously justified in Chapter 4 by means of standard compactness tools, in a similar fashion to [LM03].

The content of Chapter 4 is the subject of the paper [DF], while the asymptotic stability result of diffusive waves contained in Chapter 8 is part of a work in preparation by the author in collaboration with Lattanzio.

1.6 Nonlinear diffusion phenomena. The porous medium equation.

In this section we change our point of view and we start treating the asymptotic behavior as $t \to \infty$ of solutions to some nonlinear parabolic equations. Our point of departure is the famous *Porous medium equation* (PME in short)

$$u_t = \Delta u^m, \tag{1.1}$$

where $u \geq 0$ is a scalar function of $x \in \mathbb{R}^d$ and $t \geq 0$, and m is a constant larger than 1. By rewriting the PME in the divergence form

$$u_t = \mathrm{div}(m u^{m-1} \nabla u),$$

one realizes that it is a parabolic equation only in the region where $u > 0$. In this sense, the PME is said to be a *degenerate* parabolic equation.

There exist a variety of physical situation which may be described properly by the PME. They are described very carefully in the survey paper by Vazquez [Váz90]. Here we put in evidence two of them. The first one is (as suggested by the name of the equation) the flow of a polytropic gas through a porous medium, for which we refer to [Mus37]. The variables describing this phenomenon are the density of the gas u, its pressure v and its velocity V. They are related by the mass balance

$$\rho u_t + \nabla \cdot (uV) = 0,$$

the *Darcy's law*

$$\mu V = -k \nabla v,$$

and the equation of state

$$v = v_0 u^\gamma.$$

The constants ρ, μ, k and v_0 are supposed to be positive, while the *adiabatic* exponent γ is larger than one. Easy calculations then yield

$$u_t = c \Delta u^m$$

with $m = 1 + \gamma$, and c a suitable positive constant. We observe that in the above application the exponent m is larger than 2.

The second physical application of the PME occurs in the theory of heat propagation. The general equation for this phenomenon, in absence of sources, is

$$c\rho \frac{\partial T}{\partial t} = \operatorname{div}(k \nabla T),$$

where T is the absolute temperature, c the specific heat at constant temperature, ρ the density of the medium (solid, fluid or plasma) and k the thermal conductivity. In case where the variation of c, ρ and k are small, one obtains the linear heat equation. However, in case of large perturbations of the temperature, this model is not any longer reasonable. In the simplest cases, the conductivity k becomes a function of the temperature

$$k = \phi(T).$$

Thus, we obtain the equation

$$T_t = \Delta \Phi(T), \tag{1.2}$$

where

$$\Phi(T) = \frac{1}{c\rho} \int_0^T \phi(s) ds.$$

Equation (1.2) is often referred to as the *Nonlinear filtration equation.* In case $\phi(T)$ is a power law, we obtain again the PME.

The mathematical theory of the PME started around the fifties, when Zel'do– vich and Kompaneets [ZK66] and Barenblatt [Bar52] found the special source type solutions (successively called *Barenblatt* solutions)

$$U(x,t) = t^{-\lambda} \left[C - k \frac{|x|^2}{t^{2\mu}} \right]_+^{\frac{1}{m-1}},$$

where

$$\lambda = \frac{d}{d(m-1)+2} \quad \mu = \frac{\lambda}{d} \quad k = \frac{\lambda(m-1)}{2md}$$

and $C > 0$ is an arbitrary constant. These solutions approach a Dirac delta in the sense of distributions as $t \to 0$.

It is immediately seen that the Barenblatt solutions are compactly supported for all times. This fact, together with more properties of the PME such as local comparison and maximum principle, leads to detect a basic general property for this equation, i.e. the *finite speed of propagation*, which is in contrast with the typical infinite speed of propagation property of the heat equation. This phenomenon gives raise to the appearance of a *free boundary*, which is the $d-1$ surface where the solution attains the value zero. At the free boundary the solution ceases to be smooth.

The systematic theory of existence of weak solutions for the PME was begun by Oleinik and her collaborators around 1958, and continued by Sabinina and Kamin (see [Kam76, Kam76]) who studied the asymptotic behavior. Starting from 1970, PME has inspired the interest of many mathematicians. We mention among the others Kalashnikov, Friedman, Aronson, Peletier, Benilan, Crandall, Caffarelli, Vazquez, Kenig. There exist nowadays a quite complete theory about the existence and regularity theory of the solutions, together with a large variety of results on the regularity and the asymptotic behavior of the free boundary (see [Bén76, AB79, Vér, Váz83, Váz84b, Váz84a, BV87, CVW87, LV03] the surveys by Kalashnikov [Kal87], Aronson [Aro86] and Vazquez [Váz90]).

1.7 The entropy dissipation method

One of the classical problems related with the PME is the L^1 convergence of integrable solutions for large times towards the Barenblatt profiles. The problem of determining the optimal rate of convergence for general L^1 data has found only recently a natural solution via the so called *entropy dissipation method*. This result is due to Carrillo and Toscani [CT00], Felix Otto [Ott01] and Del Pino and Dolbeault [DPD02a] at the same time. Here we explain this procedure in a simpler case, that is the linear heat equation, where of course the Barenblatt profiles are replaced by Gaussian kernels and the solution can be explicitly represented via convolution. Hence, let us consider the Cauchy problem for the heat equation on $\mathbb{R}^d$

$$\begin{cases} u_t = \Delta u \\ u(x,0) = u_0(x) \geq 0. \end{cases}$$

The heat equation has the fundamental self similar solutions

$$G(x,t) = \frac{M}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

We perform the time dependent scaling

$$\begin{cases} u(x,t) = (2t+1)^{-d/2} v(y,s) \\ y = \frac{x}{\sqrt{2t+1}} \\ s = \frac{1}{2}\log(2t+1). \end{cases} \tag{1.1}$$

Then, a straightforward computation shows that the new variable $v(y,s)$ satisfies the linear *Fokker–Planck* equation

$$v_s = \operatorname{div}\left(yv + \nabla v\right).$$

One of the motivations for the above change of variable is that, in the new variables, the fundamental solutions $G(x,t)$ have turned into the *stationary* solutions of the Fokker–Planck equation

$$G(y) = M e^{-\frac{|y|^2}{2}}.$$

One then analyzes the problem of the asymptotic stability in L^1 of the stationary solutions to the Fokker–Planck equation and finally converts the result in terms of the original variables. The expression for the time dependent scaling (1.1) shows that exponential decay w.r.t. the new time variable s is converted into a polynomial decay w.r.t. t. The entropy dissipation approach consists in the following. We rewrite the Fokker–Planck equation as follows

$$v_s = \operatorname{div}\left(v\nabla\left(\frac{|y|^2}{2} + \log v\right)\right). \tag{1.2}$$

We now multiply equation (1.2) by $\left(\frac{|y|^2}{2} + \log v\right)$ and we obtain (by conservation of the mass) the following identity

$$\begin{aligned} &\frac{d}{ds}\int v(y,s)\left(\log v(y,s) + \frac{|y|^2}{2}\right)dy \\ &= -\int v(y,s)\left|\nabla\left(\frac{|y|^2}{2} + \log v(y,s)\right)\right|^2 dy. \end{aligned} \tag{1.3}$$

The logarithmic functional

$$H(v(s)) = \int v\left(\log v + \frac{|y|^2}{2}\right)$$

is called *relative entropy* (because of its evident relation with the classical kinetic entropy functional). The Dirichlet type integral on the right hand side of (1.3) is called *Fisher information* or *entropy production*. It is related to the relative entropy via the celebrated *logarithmic Sobolev inequality* (see [Gro75, AMTU01])

$$\begin{aligned}
&\int v(y,s)\left(\log v(y,s)+\frac{|y|^2}{2}\right)dy \\
&\leq \frac{1}{2}\int v(y,s)\left|\nabla\left(\frac{|y|^2}{2}+\log v(y,s)\right)\right|^2 dy. \qquad (1.4)
\end{aligned}$$

Hence, (1.4) and (1.3) yield the exponential decay for the relative entropy

$$\begin{aligned}
&\frac{d}{ds}\int v(y,s)\left(\log v(y,s)+\frac{|y|^2}{2}\right)dy \\
&\leq e^{-2s}\int v(y,0)\left(\log v(y,0)+\frac{|y|^2}{2}\right)dy. \qquad (1.5)
\end{aligned}$$

The following *Csiszár–Kullback* inequality

$$\|v(s)-G\|_{L^1}^2 \leq H(v(s))$$

provides then the exponential decay of the L^1 difference between the solution v and the gaussian equilibrium G with rate e^{-s}, which in terms of the old variables $u(x,t)$ turns into an L^1 polynomial decay towards the fundamental solutions with the *optimal* rate of $t^{-1/2}$ (the optimality is immediately checked by testing the relative entropy functional with shifted Gaussian kernels). This is done by reasonable extra assumptions on the initial datum, namely $L^1 \log L^1$ and finite second moment. The standard references for the above procedure, together with its generalizations to variable coefficient cases and the use of alternative entropy functionals, are [AMTU00, AMTU01, MV00]. One important remark is that, under few extra technical assumptions, the validity of a logarithmic Sobolev type inequality and the exponential decay in relative entropy are equivalent. Hence, the use of different entropy functionals yields to the proof of a huge variety of generalized Sobolev inequalities (this equivalence is known as the *Bakry–Emery point of view*).

The above idea has been successfully generalized to nonlinear diffusion equations. In particular, the already mentioned results by Carrillo and Toscani [CT00], Felix Otto [Ott01] and Del Pino and Dolbeault [DPD02a], allowed to obtain optimal rates of convergence in L^1 for solutions to the PME towards Barenblatt profiles. These results hold also in cases of $\frac{d-2}{d} < m < 1$, for which the nonlinear diffusion equation (1.1) is called *fast diffusion equation*. After those results, a lot of generalization have been carried out for general parabolic equations and systems (see [CJM$^+$01a]), p–laplacian equation (see [DPD02b]), convection–diffusion equations (see [CF03]), fourth–order equations (see [CCT]).

The present thesis contains two contribution to this theory. The first one concerns with nonlinear diffusion equations of the form

$$u_t = \Delta\phi(u),$$

where the function ϕ behaves like a power at the origin. In this case a result was already present in the literature (namely [BDE02]). In Chapter 5 we used an alternative entropy method in order to relax the hypothesis on the nonlinearity present in [BDE02] in order to obtain the optimal rate of convergence towards Barenblatt profiles. The content of Chapter 5 is the subject of a work in preparation of the author in collaboration with J. A. Carrillo and G. Toscani. In the next section we describe the second result in this context, concerning with the one–dimensional viscous Burgers' equation.

Finally, we remark that the entropy dissipation approach has been recently used outside of the context of diffusion equations. The first result for scalar conservation laws is due to Dolbeault and Escobedo in [DE]. Our result in Chapter 8 for the radiating gas model represents a new step in this direction.

1.8 The viscous Burgers' equation

The simplest model describing both nonlinear convection and diffusive behavior is the viscous Burgers' equation

$$u_t + uu_x = \mu u_{xx}.$$

The study of this equation (introducted by Burgers in 1940), has been the key point for the development of the theory of shock waves and diffusion waves for viscous and non–viscous systems of conservation laws. A first detailed analysis of this phenomena was performed by Hopf ([Hop50]) and successively by Whitam ([Whi74]), who both constructed the typical intermediate asymptotic states for this equation (with summable initial data), namely, the diffusive N-waves and the nonnegative diffusive waves, and studied the trend towards these profiles for solutions having initial datum in L^1.

The study of the viscous Burgers' equation is naturally related to that of the inviscid Burgers' equation

$$u_t + uu_x = 0. \tag{1.1}$$

Indeed, it is well known that one of the most intuitive criterion for the selection of the unique entropy solution of the Cauchy problem for the inviscid case is the $\mu \to 0$ limit of the solutions of the viscous case.

At the stage of time–asymptotics, it is also well–known that the diffusion waves of the viscous system are replaced by the *N–waves* (which are the pointwise limit of a diffusion wave as the viscosity tends to zero, in case of positive data) in the inviscid case (see the important paper by Tai Ping Liu [Liu85] and the paper by Liu and Pierre [LP84]). We refer to the introduction to [Liu85] for a clear explanation of the asymptotic stability of nonlinear waves for viscous conservation laws.

This study is also related to the asymptotic self similar behavior for a general convection–diffusion equation. There are many important results on this subject, see [EZ91, DZ92, Zua93, Zua94, EVZ93, EZ97, EZ99].

In Chapter 7, the long time asymptotics for the viscous Burgers' equation is studied with the help of the entropy dissipation techniques described in the previous section. In particular, the optimal rate of convergence in L^1 towards diffusive waves is obtained. This was already known for this simple example (the solution is obtained via the Hopf Cole formula which turns the viscous Burgers' equation into the linear heat equation). However, this is the first case where entropy dissipation has been employed to obtain optimal rates of convergence to self–similarity for convection–diffusion equations (almost at the same time of [CF03] for the diffusion dominant cases). Moreover, it must be remarked that the description of the evolution of such systems via the entropy functionals is more significant from a thermodynamical point of view than the classical L^p estimates. The results in Chapter 7 are contained into the author's paper [DFM] (joint work with Markowich).

1.9 The Wasserstein distance

The entropy dissipation machinery developed in the above mentioned papers is closely related to the so–called Wasserstein distance. This distance is defined on the space of probability measures with finite second moment and comes as the minimal quadratic cost in the variational Monge–Kantorovich problem of optimal mass tranportation (see [Vil03]). Its relation with the above mentioned entropy functionals has been recently clarified by Felix Otto ([Ott01]) in the context of gradient flows. More precisely, the porous medium equation (and the linear heat equation as a special case) can be viewed as the gradient flow of an entropy–type functional with respect to a Dirichlet integral type metric defined on the space of probability measures, which is endowed with a Riemannian structure. The Wasserstein metric comes as the minimal distance induced by this metric on the infinite–dimension Riemannian manifold of probability densities. As already pointed out before, this approach has the advantage of providing optimal rates of convergence in relative entropy towards stationary solutions for Fokker–Planck type equations via the generalized logarithmic Sobolev inequalities. In this gradient flow context, this feature is equivalent to the convexity of the entropy functional along the geodesics induced by the Wasserstein distance. This concept, known as *displacement convexity*, has been first introduced by McCann in [McC97] (see also the book of Villani [Vil03]).

The analysis of the time–decay for the Wasserstein metric in one space dimension is simplified by the representation of the optimal map (in the variational problem mentioned above) involving the distribution functions of the solutions. More precisely, one can estimate the Wasserstein distance by computing directly the equation for the pseudo–inverse of the distribution function. This technique has been recently applied to general diffusion equations (see the review paper [CT03]). In Chapter 5, this optimal transportation tool has been used to provide an optimal estimate for the speed of propagation of the support (in case of slow diffusion) for the already mentioned general nonlinear diffusion equation. This technique appears to be very natural, and permits to prove results of this type in a much simpler way than in previous papers in the literature for the porous medium case (see [Váz03]). We didn't find an optimal result concerning the cases we threat in our paper (namely, nonlinearity behaving like a power at the origin).

Part I

Relaxation limits

Chapter 2

The compressible Euler equations with damping

In this chapter we will concern with a class of smooth solutions to the one-dimensional isentropic compressible Euler equations through porous media introduced in section 1.3. As already pointed out in the introduction, after a parabolic scaling we analyze the singular convergence towards caloric solutions to the porous medium equation. The machinery developed here holds far from vacuum, and it could be easily generalized to limiting profiles satisfying suitable decay estimates. In section 2.1 we describe the problem in details and we collect the main results in theorems 2.1.1 and 2.1.4. The result in 2.1.4 is closely related to the asymptotic stability result in chapter 6. Section 2.2 is devoted to the proof of the main theorem 2.1.1.

2.1 Statement of the problem and results

Let us consider the one-dimensional, isentropic, compressible Euler equations through a porous medium in eulerian coordinates. In case of smooth solutions, with $\rho > 0$, the system may be written as

$$\begin{cases} \partial_\tau \rho + u\partial_x \rho + \rho \partial_x u = 0 \\ \partial_\tau u + u\partial_x u + \dfrac{p'(\rho)}{\rho}\partial_x \rho = -\dfrac{u}{\varepsilon}. \end{cases} \tag{2.1.1}$$

Here, $\rho > 0$ is the density, u is the velocity, $x \in \mathbb{R}$, $\tau > 0$, $p : \mathbb{R} \to \mathbb{R}_+$ is a smooth function such that $p' > 0$, and $\varepsilon > 0$ is a small parameter. After the time scaling $\tau = \frac{t}{\varepsilon}$, $\rho^\varepsilon(x,t) = \rho(x, \frac{t}{\varepsilon})$, $u^\varepsilon(x,t) = \frac{1}{\varepsilon}u(x, \frac{t}{\varepsilon})$, the system (2.1.1) becomes

$$\begin{cases} \partial_t \rho^\varepsilon + u^\varepsilon \partial_x \rho^\varepsilon + \rho^\varepsilon \partial_x u^\varepsilon = 0 \\ \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \dfrac{p'(\rho^\varepsilon)}{\varepsilon^2 \rho^\varepsilon} \partial_x \rho^\varepsilon = -\dfrac{u^\varepsilon}{\varepsilon^2}. \end{cases} \tag{2.1.2}$$

Thus, as ε goes to 0, we expect the solutions to (2.1.2) to be described by the solutions to the following system

$$\begin{cases} \partial_t \widetilde{\rho} + \partial_x(\widetilde{\rho}\widetilde{u}) = 0 \\ \partial_x p(\widetilde{\rho}) = -\widetilde{\rho}\widetilde{u}, \end{cases} \tag{2.1.3}$$

which is equivalent to the Porous Medium equation

$$\widetilde{\rho}_t = p(\widetilde{\rho})_{xx}, \tag{2.1.4}$$

where the relation between the pressure p and the velocity $\widetilde{u}$ is given by the well known Darcy's law

$$\widetilde{u} = -\frac{p(\widetilde{\rho})_x}{\widetilde{\rho}}. \tag{2.1.5}$$

For the system (2.1.2), we prescribe the following limiting conditions at infinity

$$\begin{aligned} &\rho^\varepsilon(\pm\infty, t) = \rho^\pm \quad \text{for any } t \geq 0 \\ &u^\varepsilon(\pm\infty, 0) = u^\pm, \end{aligned}$$

with $\rho^+, \rho^- > 0$. Since we expect the inertial terms of the second equation in (2.1.2) to decay faster than the others, in addition we require

$$u^\varepsilon(\pm\infty, t) = e^{-t/\varepsilon^2} u^\pm \quad \text{for any } t \geq 0.$$

Therefore, we assume the following behaviour at $x \to \pm\infty$ for the system (2.1.3)

$$\begin{aligned} &\widetilde{\rho}(\pm\infty, t) = \rho^\pm \\ &\widetilde{u}(\pm\infty, t) = 0, \end{aligned}$$

for any $t \geq 0$. The initial datum on the density of the hyperbolic problem (2.1.2) is assumed to be the same of (2.1.4), namely

$$\rho^\varepsilon(x, 0) = \widetilde{\rho}(x, 0) = \widetilde{\rho}_0(x),$$

where $\widetilde{\rho}_0$ is a bounded smooth function (e.g. $\widetilde{\rho}_0 \in H^3(\mathbb{R})$) such that

$$0 < \mu_0 \leq \widetilde{\rho}_0(x) \leq \mu_1.$$

Moreover, we require the initial datum on the velocity u^ε to be given by the initial value of $\widetilde{u}$ in the system (2.1.3) (which is determined by the Darcy's law) plus a small corrector, needed to match the limiting conditions, namely

$$u_0^\varepsilon(x) = -\frac{p'(\widetilde{\rho}_0(x))}{\widetilde{\rho}_0(x)}\widetilde{\rho}_0'(x) + w^\varepsilon(x,0). \tag{2.1.6}$$

The expression for the corrector w^ε is

$$w^\varepsilon(x,t) = e^{-t/\varepsilon^2}\left[u^- + (u^+ - u^-)\psi(x)\right], \tag{2.1.7}$$

where

$$\psi(x) = \frac{\displaystyle\int_{-\infty}^{x}\phi(y)dy}{\displaystyle\int_{-\infty}^{+\infty}\phi(y)dy},$$

for some $\phi \in C_c^\infty(\mathbb{R})$, $\phi \geq 0$. We observe that w^ε satisfies the equation

$$\partial_t w^\varepsilon = -\frac{1}{\varepsilon^2}w^\varepsilon.$$

This corrector doesn't affect the asymptotic analysis since it decays exponentially fast. The well-prepared initial data condition (2.1.6) is prescribed in order to avoid the problem of the initial layer.

As a consequence of the boundedness of $\widetilde{\rho}_0$ and of the comparison principle for the parabolic equation (2.1.4), we have

$$\mu_0 \leq \widetilde{\rho}(x,t) \leq \mu_1. \tag{2.1.8}$$

We will consider solutions $(\widetilde{\rho},\widetilde{u})$ to (2.1.4) satisfying the time-asymptotic estimates

$$\begin{aligned}
\left|\frac{\partial^{\alpha+\beta}\widetilde{\rho}(t)}{\partial x^\alpha\partial t^\beta}\right|_\infty &= O(\delta)\frac{1}{(t+1)^{\frac{\alpha}{2}+\beta}} \quad \alpha,\beta > 0\\
\int_{-\infty}^{+\infty}\left|\frac{\partial^{\alpha+\beta}\widetilde{\rho}(x,t)}{\partial x^\alpha\partial t^\beta}\right|^2 dx &= O(\delta^2)\frac{1}{(t+1)^{\alpha+2\beta-\frac{1}{2}}} \quad \alpha,\beta > 0\\
\left|\frac{\partial^{\alpha+\beta}\widetilde{u}(t)}{\partial x^\alpha\partial t^\beta}\right|_\infty &= O(\delta)\frac{1}{(t+1)^{\frac{\alpha}{2}+\beta+\frac{1}{2}}} \quad \alpha,\beta \geq 0\\
\int_{-\infty}^{+\infty}\left|\frac{\partial^{\alpha+\beta}\widetilde{u}(x,t)}{\partial x^\alpha\partial t^\beta}\right|^2 dx &= O(\delta^2)\frac{1}{(t+1)^{\alpha+2\beta+\frac{1}{2}}}, \quad \alpha,\beta \geq 0
\end{aligned} \tag{2.1.9}$$

where

$$\delta = |\rho^+ - \rho^-| + |u^+ - u^-|. \tag{2.1.10}$$

In particular, these estimates are satisfied both by the caloric self-similar solutions of (2.1.4) described in [HL92][Nis96] and by a small perturbation of these solutions w.r.t. initial datum (as we will show in the Theorem 6.1.1 of Chapter 6).

Our first risult concerns the asymptotic behaviour as $\varepsilon \searrow 0$ of the scaled hyperbolic system (2.1.2) with Sobolev norms. The time interval where the asymptotic analysis is valid, is given by the condition $\varepsilon T^{\alpha} \ll 1$, for some constant $\alpha > 0$, which allows, for small ε, to include the solutions at large time.

Theorem 2.1.1 *Let $0 < \nu < 1/2$ be arbitrary. Suppose $\varepsilon T^{\frac{1+\nu}{2}} \ll 1$, $\varepsilon \ll 1$ and $\delta \ll 1$; then, there exists a fixed constant $\Delta > 0$ such that*

$$\sup_{0\le t\le T}\left\{\frac{1}{(t+1)^{\nu}}\left[\frac{1}{\varepsilon^2}\|\rho^{\varepsilon}(t)-\widetilde{\rho}(t)\|^2_{H^{3\theta}}+\|u^{\varepsilon}(t)-\widetilde{u}(t)-w^{\varepsilon}(t)\|^2_{H^{3\theta}}+\right.\right.$$
$$\left.\left.\frac{1}{\varepsilon^2}\int_0^t\|u^{\varepsilon}(s)-\widetilde{u}(s)-w^{\varepsilon}(s)\|^2_{H^{3\theta}}ds\right]\right\}\le\Delta, \tag{2.1.11}$$

for any $\theta \in (0,1)$.

Corollary 2.1.2 *Let $t > 0$ be arbitrary. Let $\beta > 0$ be arbitrarily small. Then, for small values of δ, we have*

$$\|\rho^{\varepsilon}(t)-\widetilde{\rho}(t)\|^2_{L^\infty}+\|\rho^{\varepsilon}_x(t)-\widetilde{\rho}_x(t)\|^2_{L^\infty}\le O(\varepsilon^{2-\beta}). \tag{2.1.12}$$

Remark 2.1.3 Another simple consequence of Theorem 2.1.1 is the global–in–time existence of smooth solution for system (2.1.2) with fixed $\varepsilon > 0$ when the initial data are chosen to be small perturbations of the initial datum of the caloric profile $\widetilde{\rho}$. This comes from Theorem 6.1.1 in Chapter 6 and from the continuation principle for quasi linear hyperbolic systems (see [Maj84]).

The proof of the Theorem (2.1.1) will be given in the Section 2.2. As a consequence of the stability result in chapter 6, the result in Theorem 2.1.1 is also true when the initial datum for $\widetilde{\rho}$ is replaced by a small perturbation of the initial datum of a caloric self similar profile. Moreover, as a consequence of both Theorem 2.1.1 and Theorem 6.1.1 in chapter 6, we have the following asymptotic result.

Theorem 2.1.4 *Let $\widetilde{\rho}(x,t)$ be the caloric self-similar solution to*

$$\begin{cases}\widetilde{\rho}_t = p(\widetilde{\rho})_{xx}\\ \widetilde{\rho}(x,0)=\widetilde{\rho}_0(x)\\ \widetilde{\rho}(\pm\infty,t)=\rho^{\pm}.\end{cases}$$

Let $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ be the solution to

$$\begin{cases} \rho_t^\varepsilon + (\rho^\varepsilon u^\varepsilon)_x = 0 \\ u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + \dfrac{p(\rho^\varepsilon)_x}{\varepsilon^2 \rho^\varepsilon} = -\dfrac{u^\varepsilon}{\varepsilon^2} \\ \rho^\varepsilon(x,0) = \rho_0(x) = \widetilde{\rho}(x+x_0) + r_0(x) \\ u^\varepsilon(x,0) = -\dfrac{p(\rho_0(x))_x}{\rho_0(x)} + w^\varepsilon(x,0) \\ \rho^\varepsilon(\pm\infty,t) = \rho^\pm \\ u^\varepsilon(\pm\infty,t) = e^{-t/\varepsilon^2} u^\pm, \end{cases}$$

with $w^\varepsilon(x,t)$ given by (2.1.7), x_0 given by (6.1.3). Suppose that $\|R(0)\|_5^2$, δ and ε are sufficiently small ($R(0)$ defined by (6.1.4)). Then, there exists a fixed $\Gamma > 0$ such that

$$\sup_{\gamma T(\varepsilon) \le t \le \Gamma T(\varepsilon)} \|\rho^\varepsilon(t) - \widetilde{\rho}(t)\|_{L^\infty(\mathbb{R})} \le O(\varepsilon^{\frac{1}{1+\nu}}), \tag{2.1.13}$$

where

$$T(\varepsilon) = \varepsilon^{-\frac{2}{1+\nu}},$$

$\nu > 0$ is arbitrary small and γ is an arbitrary constant such that $0 < \gamma < \Gamma$.

The proof of the theorem (2.1.4) is straightforward.

2.2 The Proof of the main Theorem

We prove Theorem 2.1.1 by means of an iteration scheme. Let us define an approximating sequence $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ by setting,

$$\rho_0^\varepsilon = \widetilde{\rho}\ , \qquad u_0^\varepsilon = \widetilde{u} + w^\varepsilon,$$

and let, for any $n > 1$, $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ be the solution to the system

$$\begin{cases} \partial_t \rho^\varepsilon_{(n)} + \rho^\varepsilon_{(n-1)} \partial_x u^\varepsilon_{(n)} + u^\varepsilon_{(n-1)} \partial_x \rho^\varepsilon_{(n)} = 0 \\ \partial_t u^\varepsilon_{(n)} + u^\varepsilon_{(n-1)} \partial_x u^\varepsilon_{(n)} + \dfrac{p'(\rho^\varepsilon_{(n-1)})}{\varepsilon^2 \rho^\varepsilon_{(n-1)}} \partial_x \rho^\varepsilon_{(n)} = -\dfrac{u^\varepsilon_{(n)}}{\varepsilon^2} \\ \rho^\varepsilon_{(n)}(x,0) = \widetilde{\rho}_0(x) \\ u^\varepsilon_{(n)}(x,0) = -\dfrac{p'(\widetilde{\rho}_0(x))}{\widetilde{\rho}_0(x)} \widetilde{\rho}_0'(x) + w^\varepsilon(x,0) \\ \rho^\varepsilon_{(n)}(\pm\infty,t) = \rho^\pm \\ u^\varepsilon_{(n)}(\pm\infty,t) = e^{-t/\varepsilon^2} u^\pm. \end{cases} \tag{2.2.1}$$

We will prove the convergence of the approximating sequence $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ to the solution of the system (2.1.2) via the uniform boundedness of this sequence in some weighted high Sobolev norm (namely $H^3(\mathbb{R})$) and the contraction in some weighted L^2-norm. Thus, we obtain the desired estimate via interpolation. This strategy is used in [KM81][KM82][Maj84].

Denote, for any $T > 0$,

$$\mathcal{E}^n_\varepsilon(T) = \sup_{0\le t\le T}\left\{\frac{1}{(t+1)^\nu}\left[\frac{1}{\varepsilon^2}\|(\rho^\varepsilon_{(n)} - \widetilde{\rho})(t)\|^2_{H^3} + \right.\right.$$
$$\left.\left. + \|(u^\varepsilon_{(n)} - \widetilde{u} - w^\varepsilon)(t)\|^2_{H^3} + \frac{1}{\varepsilon^2}\int_0^t \|(u^\varepsilon_{(n)} - \widetilde{u} - w^\varepsilon)(s)\|^2_{H^3} ds\right]\right\}. \tag{2.2.2}$$

Hence, we have the following result

Proposition 2.2.1 *Let us suppose that $\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \le \lambda$, where $\lambda \ll 1$. Then, there exists a positive constant $\Delta > 0$ such that, for any $n \in \mathbb{N}$,*

$$\mathcal{E}^n_\varepsilon(T) \le \Delta. \tag{2.2.3}$$

Proof. From now on, we denote

$$\begin{array}{llll}
\rho = \rho^\varepsilon_{(n)} & u = u^\varepsilon_{(n)} & \widehat{\rho} = \rho^\varepsilon_{(n-1)} & \widehat{u} = u^\varepsilon_{(n-1)} \\
\bar{\rho} = \rho - \widetilde{\rho} & \bar{u} = u - w^\varepsilon - \widetilde{u} & \widehat{\widehat{\rho}} = \rho^\varepsilon_{(n-2)} & \widehat{\widehat{u}} = u^\varepsilon_{(n-2)} \\
\bar{\bar{\rho}} = \widehat{\rho} - \widetilde{\rho} & \bar{\bar{u}} = \widehat{u} - w^\varepsilon - \widetilde{u} & \bar{\bar{\bar{u}}} = \widehat{\widehat{u}} - \widetilde{u} - w & \pi(z) = \frac{p'(z)}{z}, \text{for any}\, z \in \mathbb{R}_+.
\end{array}$$

The system (2.2.1) becomes

$$\bar{\rho}_t + \widehat{u}\bar{\rho}_x + \widehat{\rho}\bar{u}_x = -\,(\bar{\bar{u}} + w)\widetilde{\rho}_x - \bar{\bar{\rho}}\widetilde{u}_x - \widehat{\rho}w_x \tag{2.2.4}$$
$$\bar{u}_t + \widehat{u}\bar{u}_x + \frac{1}{\varepsilon^2}\pi(\widehat{\rho})\bar{\rho}_x = -\widetilde{u}_t - \widehat{u}(\widetilde{u}_x + w_x) - \frac{1}{\varepsilon^2}(\pi(\widehat{\rho}) - \pi(\widetilde{\rho}))\widetilde{\rho}_x - \frac{\bar{u}}{\varepsilon^2}. \tag{2.2.5}$$

We now assume that the estimate (2.2.3) holds for $(\widehat{\rho}, \widehat{u})$ and show that it is true for (ρ, u). In particular, we assume

$$\sup_{0\le t\le T}\left\{\frac{1}{(t+1)^\nu}\left[\frac{1}{\varepsilon^2}\|\bar{\bar{\rho}}(t)\|^2_{H^3} + \|\bar{\bar{u}}(t)\|^2_{H^3} + \frac{1}{\varepsilon^2}\int_0^t \|\bar{\bar{u}}(s)\|^2_{H^3} ds \le \Delta\right]\right\}, \tag{2.2.6}$$

for any $T > 0$, $\varepsilon > 0$ and $\delta > 0$ (δ defined by (2.1.10)) such that

$$\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \le \lambda\ , \quad \lambda \ll 1. \tag{2.2.7}$$

As usual in this framework, we determine the conditions on the constant Δ in the estimate at the $n-$th step. As we will see, this constant depends only on the constant λ in (2.2.7). Let us multiply (2.2.4) by $(1/\varepsilon^2)\pi(\widehat{\rho})\bar{\rho}$ and (2.2.5) by $\widehat{\rho}\bar{u}$. Then, via standard energy identity (as a consequence of symmetrization), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\left[\frac{1}{\varepsilon^2}\pi(\widehat{\rho})\frac{\bar{\rho}^2}{2}+\widehat{\rho}\frac{\bar{u}^2}{2}\right]dx=\\
&=\int_{-\infty}^{+\infty}(\pi'(\widehat{\rho})(\widehat{\rho}_t+\widehat{\rho}_x\widehat{u})+\pi(\widehat{\rho})\widehat{u}_x)\frac{\bar{\rho}^2}{2\varepsilon^2}dx+\int_{-\infty}^{+\infty}(\widehat{\rho}_t+\widehat{\rho}_x\widehat{u}+\widehat{\rho}\widehat{u}_x)\frac{\bar{u}^2}{2}dx+\\
&+\int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}p''(\widehat{\rho})\widehat{\rho}_x\bar{\rho}\bar{u}dx-\int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\pi(\widehat{\rho})\bar{\rho}(\bar{\bar{u}}+w)\widetilde{\rho}_xdx-\int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\pi(\widehat{\rho})\bar{\rho}\bar{\bar{\rho}}\widetilde{u}_xdx+\\
&-\int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\pi(\widehat{\rho})\bar{\rho}\widehat{\rho}w_xdx-\int_{-\infty}^{+\infty}\widehat{\rho}\widetilde{u}_t\bar{u}dx-\int_{-\infty}^{+\infty}\widehat{\rho}\bar{u}\widehat{u}(\widetilde{u}_x+w_x)dx+\\
&-\int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\widehat{\rho}\bar{u}(\pi(\widehat{\rho})-\pi(\widetilde{\rho}))\widetilde{\rho}_xdx-\int_{-\infty}^{+\infty}\widehat{\rho}\frac{\bar{u}^2}{\varepsilon^2}dx=\\
&=:\sum_{k=1}^{3}\widetilde{J}_k(t)+\sum_{k=1}^{6}I_k(t)-\int_{-\infty}^{+\infty}\widehat{\rho}\frac{\bar{u}^2}{\varepsilon^2}dx.
\end{aligned}
\tag{2.2.8}
$$

Remark 2.2.2 We remark that the function $\pi(z)$ satisfies

$$0<c_0\le\pi(z)\le c_1,\qquad \text{as}\quad z\in(c_2,c_3),$$

for some positive constants c_0,c_1,c_2,c_3. Now, from the assumption (2.2.6) and from (2.1.8), it follows that $\widehat{\rho}$ satisfies

$$0<\frac{\mu_0}{2}\le\widehat{\rho}(x,t)\le\mu_1+\frac{\mu_0}{2},\quad \text{for any}\quad x\in\mathbb{R},\quad 0\le t\le T. \tag{2.2.9}$$

(where μ_0,μ_1 are defined in (2.1.8)) provided that

$$\varepsilon(T+1)^{\nu/2}\le\Delta^{-1/2}\frac{\mu_0}{2},$$

(with Δ as in (2.2.6)). Thus, from the condition (2.2.7) and by requiring $\Delta<1$ and $\lambda^{\frac{\nu}{1+\nu}}\varepsilon^{\frac{1}{1+\nu}}<\frac{\mu_0}{2}$ (i.e. $\varepsilon+\lambda\ll 1$), there exist C_1,C_2 fixed positive constants such that

$$C_1\le\pi(\widehat{\rho})\le C_2. \tag{2.2.10}$$

Moreover, from (2.2.9) it follows that

$$\pi'(\widehat{\rho})+p''(\widehat{\rho})\le C_3, \tag{2.2.11}$$

for some positive fixed C_3.

Now, from (2.2.9), (2.2.10), (2.2.11), and after time integration of (2.2.8) in $[0,t]$, for $0 < t \le T$, T satisfying (2.2.7), it follows that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\|\bar\rho(t)\|^2 + \|\bar u(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t \|\bar u(s)\|^2 ds \le \\
\le &O(1)\int_0^t \left[|\widehat\rho_t|_\infty + |\widehat\rho_x \widehat u|_\infty + |\widehat u_x|_\infty\right]\left[\frac{\|\bar\rho(s)\|^2}{\varepsilon^2} + \|\bar u(s)\|^2\right] ds + \\
+&\frac{O(1)}{\varepsilon^2}\int_0^t |\widehat\rho_x|_\infty \|\bar\rho(s)\|\|\bar u(s)\| ds + \int_0^t \sum_{k=1}^6 I_k(s) ds = \\
=:&\sum_{h=1}^2 J_h(t) + \sum_{k=1}^6 \int_0^t I_k(s) ds. \qquad (2.2.12)
\end{aligned}$$

We devote ourselves to the estimate of the terms $\int_0^t I_k(t)ds$, $k = 1, \dots 6$. In what follows, we exploit (2.2.6), (2.2.7), the time asymptotic estimates (2.1.9) and the estimates (2.1.8), (2.2.10), (2.2.11).

$$\begin{aligned}
&\int_0^t I_1 ds \le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar u(s)\|\|\bar\rho(s)\| \frac{1}{(s+1)^{1/2}} ds + \\
+&\frac{O(1)}{\varepsilon^2}\int_0^t e^{-\frac{s}{\varepsilon^2}}\|\bar\rho(s)\|\|\widetilde\rho_x(s)\| ds \le \\
\le&\frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar u\|^2 ds + \frac{O(\delta)}{\varepsilon^2}\int_0^t \frac{\|\bar\rho(s)\|^2}{s+1} ds + \\
+&\frac{O(1)}{\varepsilon^2}\int_0^t e^{-\frac{s}{\varepsilon^2}}\left[\|\bar\rho(s)\|^2 + \|\widetilde\rho_x(s)\|^2\right] ds \le \\
\le&O(\delta)\left(\Delta(t+1)^\nu + \int_0^t \frac{\mathcal{E}^n(s)}{(s+1)^{1-\nu}} ds + 1\right) + \mathcal{E}^n(t)(t+1)^\nu \varepsilon^2 \le \\
\le&O(\delta)\left(\Delta(t+1)^\nu + 1\right) + \mathcal{E}^n(t)(t+1)^\nu (O(\delta) + O(\varepsilon^2)). \qquad (2.2.13)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t I_2 ds \le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar\rho(s)\|\|\bar{\bar\rho}(s)\| \frac{1}{s+1} ds \le \\
\le&\frac{O(\delta)}{\varepsilon^2}\int_0^t \left[\|\bar\rho(s)\|^2 + \|\bar{\bar\rho}(s)\|^2\right] \frac{1}{s+1} ds \le \\
\le&O(\delta)\int_0^t \frac{\mathcal{E}^n(s)}{(s+1)^{1-\nu}} ds + O(\delta)\int_0^t \Delta \frac{1}{(s+1)^{1-\nu}} ds \le \\
\le&O(\delta)\mathcal{E}^n(t)(t+1)^\nu + O(\delta)\Delta(t+1)^\nu. \qquad (2.2.14)
\end{aligned}$$

$$
\begin{aligned}
&\int_0^t I_3 ds = \int_0^t \int_{-\infty}^{+\infty} \frac{\pi(\widehat{\rho})}{\varepsilon^2} \bar{\rho}\bar{\bar{\rho}} w_x dx ds + \int_0^t \int_{-\infty}^{+\infty} \frac{\pi(\widehat{\rho})}{\varepsilon^2} \widetilde{\rho}\bar{\rho} w_x dx ds \leq \\
\leq &\frac{O(\delta)}{\varepsilon^2} \int_0^t \|\bar{\rho}(s)\| \|\bar{\bar{\rho}}(s)\| e^{-\frac{s}{\varepsilon^2}} ds + \frac{O(\delta)}{\varepsilon^2} \int_0^t \int_{supp(\phi)} \bar{\rho} e^{-\frac{s}{\varepsilon^2}} dx ds \leq \\
\leq &O(\delta)\mathcal{E}^n(t)(t+1)^\nu + O(\delta)\Delta(t+1)^\nu + \frac{O(\delta)}{\varepsilon^2} \int_0^t |\bar{\rho}|_\infty e^{-\frac{s}{\varepsilon^2}} ds \leq \\
\leq &O(\delta) + \mathcal{E}^n(t) O(\delta)(1 + (t+1)^\nu) + O(\delta)\Delta(t+1)^\nu,
\end{aligned}
$$

where in the second inequality we have used the estimate for $\int_0^t I_2$.

$$
\begin{aligned}
&\int_0^t I_4 ds \leq O(1) \int_0^t \left[\|\widetilde{u}_t(s)\|^2 + \|\bar{u}(s)\|^2 \right] ds \leq \\
\leq &O(\delta^2) \int_0^t \frac{1}{(s+1)^{5/2}} ds + O(\varepsilon^2)\mathcal{E}^n(t)(t+1)^\nu \leq \\
\leq &O(\delta^2) + O(\varepsilon^2)\mathcal{E}^n(t)(t+1)^\nu.
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t I_5 ds \leq \int_0^t \int_{-\infty}^{+\infty} \bar{\bar{u}} \widehat{\rho} \bar{u} (\widetilde{u}_x + w_x) dx ds + \int_0^t \int_{-\infty}^{+\infty} (\widetilde{u} + w)(\widetilde{u}_x + w_x) \widehat{\rho} \bar{u} dx ds \leq \\
\leq &O(1) \int_0^t \left[\|\bar{\bar{u}}(s)\|^2 + \|\bar{u}(s)\|^2 \right] ds + O(1) \int_0^t \left[\|\widetilde{u}_x(s)\|^2 + \|\bar{u}(s)\|^2 \right] ds + \\
+ &O(1) \int_0^t \left[\|w_x(s)\|^2 + \|\bar{u}(s)\|^2 \right] ds \leq \\
\leq &O(\varepsilon^2)\big(\Delta(t+1)^\nu + \mathcal{E}^n(t)(t+1)^\nu + O(\delta^2)\big) + O(\delta^2) \int_0^t \frac{1}{(s+1)^{3/2}} ds \leq \\
\leq &O(\delta^2) + O(\varepsilon^2) \left(\Delta(t+1)^\nu + \mathcal{E}^n(t)(t+1)^\nu + O(\delta^2) \right).
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t I_6 ds \leq \frac{O(\delta)}{\varepsilon^2} \int_0^t \|\bar{\bar{\rho}}(s)\| \|\bar{u}(s)\| \frac{1}{(s+1)^{1/2}} ds \leq \\
\leq &\frac{O(\delta)}{\varepsilon^2} \int_0^t \|\bar{\bar{\rho}}(s)\|^2 \frac{1}{s+1} ds + \frac{O(\delta)}{\varepsilon^2} \int_0^t \|\bar{u}(s)\|^2 ds \leq \\
\leq &O(\delta)(t+1)^\nu \left(\Delta + \mathcal{E}^n(t) \right).
\end{aligned}
$$

Thus, by requiring $\Delta < 1$ and $\lambda \ll 1$, the estimates of these terms yields

$$
\sum_{k=1}^6 \int_0^t I_k(t) \leq O(\lambda) \left(\Delta + \mathcal{E}^n(t) \right) (t+1)^\nu + O(\lambda)(t+1)^\nu.
$$

Hence, we compute the integrals denoted by J_h, $h = 1, 2$.

$$J_1(t) \leq O(1) \int_0^t \Big[|\widehat{\rho}(s)|_\infty |\widehat{u}_x(s)|_\infty + |\widehat{\widehat{u}}(s)|_\infty |\widehat{\rho}_x(s)|_\infty +$$

$$+ |\widehat{\rho}_x(s)|_\infty |\widehat{u}(s)|_\infty + |\widehat{u}_x(s)|_\infty \Big] \left[\frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2} + \|\bar{u}(s)\|^2 \right] ds \leq$$

$$\leq O(1) \int_0^t \Big[|\widehat{u}_x(s)|_\infty + |\widehat{\rho}_x(s)|_\infty \Big(|\widehat{u}(s)|_\infty + |\widehat{\widehat{u}}(s)|_\infty \Big) \Big] \left[\frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2} + \|\bar{u}(s)\|^2 \right] ds \leq$$

$$\leq O(1) \mathcal{E}^n(t) \int_0^t \Big[|\widehat{u}_x(s)|_\infty + |\widehat{\rho}_x(s)|_\infty \Big(|\widehat{u}(s)|_\infty + |\widehat{\widehat{u}}(s)|_\infty \Big) \Big] (s+1)^\nu ds \leq$$

$$\leq O(1) \mathcal{E}^n(t) \int_0^t [|\bar{\bar{u}}_x(s)|_\infty + |\widetilde{u}_x(s)|_\infty + |w_x(s)|_\infty + (|\bar{\bar{\rho}}_x(s)|_\infty + |\widetilde{\rho}_x(s)|_\infty) \cdot$$

$$\cdot \Big(|\bar{u}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty + |\widetilde{u}(s)|_\infty + |w(s)|_\infty \Big)] (s+1)^\nu ds. \tag{2.2.15}$$

We now estimate separately the following terms.

$$\int_0^t |\bar{\bar{u}}_x(s)|_\infty (s+1)^\nu ds \leq \int_0^t \left(\lambda \frac{|\bar{\bar{u}}_x(s)|_\infty^2}{\varepsilon^2} + \frac{1}{\lambda} \varepsilon^2 (s+1)^{2\nu} \right) ds \leq$$

$$\leq \lambda \Delta (t+1)^\nu + \frac{1}{\lambda} \varepsilon^2 (t+1)^{1+2\nu} \leq O(\lambda)(t+1)^\nu \tag{2.2.16}$$

where λ is the fixed constant in (2.2.7).

$$\int_0^t (|\bar{\bar{\rho}}_x(s)|_\infty + |\widetilde{\rho}_x(s)|_\infty) \left(|\bar{u}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty \right) (s+1)^\nu ds \leq$$

$$\leq \int_0^t \left(\varepsilon \Delta^{1/2} (s+1)^{\frac{3\nu}{2}} + O(\delta)(s+1)^{-1/2+\nu} \right) \left(|\bar{u}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty \right) \leq$$

$$\leq (O(\lambda) + O(\delta)) \int_0^t \left(|\bar{u}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty \right) \leq (O(\lambda) + O(\delta))(t+1)^\nu \tag{2.2.17}$$

where the last inequality is justified by the preceding estimate (2.2.16), and where we used $\nu < 1/2$ and $\Delta < 1$.

$$\int_0^t (|\bar{\bar{\rho}}_x(s)|_\infty + |\widetilde{\rho}_x(s)|_\infty) (|\widetilde{u}(s)|_\infty + |w(s)|_\infty) (s+1)^\nu ds \leq$$

$$\leq \int_0^t \left(\varepsilon \Delta^{1/2} (s+1)^{\frac{3\nu}{2}} + O(\delta)(s+1)^{-1/2+\nu} \right) O(\delta)(s+1)^{-1/2} ds +$$

$$+ O(\varepsilon) \Delta^{1/2} (t+1)^{\nu/2} + O(\delta) O(\varepsilon^2) \leq (O(\lambda) + O(\delta))(t+1)^\nu. \tag{2.2.18}$$

Now we can complete the estimate of the integral J_1 in (2.2.15), and obtain

$$J_1(t) \le \big(O(\delta) + O(\lambda) + O(\varepsilon^2)\big)\,(t+1)^{\nu}\mathcal{E}^n(t).$$

Let us estimate $J_2(t)$;

$$\begin{aligned}
J_2(t) \le O(1)&\int_0^t \left[|\bar{\bar{\rho}}_x(s)|_\infty + |\widetilde{\rho}_x(s)|_\infty\right]\frac{1}{\varepsilon^2}\|\bar\rho(s)\|\|\bar u(s)\|ds \le\\
\le& O(1)\int_0^t \left[\varepsilon\Delta^{1/2}(s+1)^{\nu/2} + O(\delta)(s+1)^{-1/2}\right]\frac{1}{\varepsilon^2}\|\bar\rho(s)\|\|\bar u(s)\|ds \le\\
\le& O(1)\varepsilon\Delta^{1/2}\int_0^t \left[\frac{\|\bar\rho(s)\|^2}{\lambda\varepsilon}(s+1)^{\nu} + \lambda\frac{\|\bar u(s)\|^2}{\varepsilon^3}\right]ds+\\
&+O(\delta)\int_0^t \frac{\|\bar\rho(s)\|^2}{\varepsilon^2}(s+1)^{-1}ds + O(\delta)\int_0^t \frac{\|\bar u(s)\|^2}{\varepsilon^2}ds \le\\
\le& O(1)\frac{\varepsilon^2}{\lambda}(t+1)^{1+2\nu}\mathcal{E}^n(t) + \left(O(\lambda)+O(\delta)\right)\mathcal{E}^n(t)(t+1)^{\nu}. \qquad (2.2.19)
\end{aligned}$$

By combining all these estimates, dividing both sides of (2.2.12) by $(t+1)^{\nu}$, taking the $\sup_{0\le t\le T}$ and by suitably choosing Δ and λ such that

$$0<\Delta<1, \qquad \Delta > \tilde{C}\lambda,$$

for a fixed constant $\tilde{C}$, we obtain the following

Lemma 2.2.3 *Suppose $\delta+\varepsilon+\varepsilon T^{\frac{1+\nu}{2}} \le \lambda \ll 1$. Then, there exists a positive fixed constant Δ such that*

$$\sup_{0\le t\le T}\left\{\frac{1}{(1+t)^{\nu}}\left[\frac{1}{\varepsilon^2}\|\bar\rho(t)\|^2 + \|\bar u(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t \|\bar u(s)\|^2 ds\right]\right\} \le \frac{\Delta}{4}, \qquad (2.2.20)$$

In a similar fashion, we can derive L_2 estimates for the derivatives of $\bar\rho$ and $\bar u$. By differentiatiting (2.2.4)-(2.2.5) w.r.t. x, we obtain

$$\begin{aligned}
\bar\rho_{xt} + \widehat{u}\bar\rho_{xx} + \widehat{\rho}\bar u_{xx} =& -\widehat{u}_x\bar\rho_x - \widehat{\rho}_x\bar u_x - (\bar{\bar{u}}_x + w_x)\widetilde{\rho}_x - (\bar{\bar{u}}+w)\widetilde{\rho}_{xx}+\\
&-\bar{\bar{\rho}}_x\widetilde{u}_x - \bar{\bar{\rho}}\widetilde{u}_{xx} - \widehat{\rho}_x w_x - \widehat{\rho}w_{xx}, \qquad (2.2.21)\\
\bar u_{xt} + \widehat{u}\bar u_{xx} + \frac{1}{\varepsilon^2}\pi(\widehat{\rho})\bar\rho_{xx} =& -\widehat{u}_x\bar u_x - \frac{1}{\varepsilon^2}\pi'(\widehat{\rho})\widehat{\rho}_x\bar\rho_x - \widetilde{u}_{xt} - \widehat{u}_x(\widetilde{u}_x + w_x)+\\
&-\widehat{u}(\widetilde{u}_{xx}+w_{xx}) - \frac{1}{\varepsilon^2}\left(\pi(\widehat{\rho}) - \pi(\widetilde{\rho})\right)\widetilde{\rho}_{xx}+\\
&-\frac{1}{\varepsilon^2}\left(\pi'(\widehat{\rho})\widehat{\rho}_x - \pi'(\widetilde{\rho})\widetilde{\rho}_x\right)\widetilde{\rho}_x - \frac{1}{\varepsilon^2}\bar u_x. \qquad (2.2.22)
\end{aligned}$$

It is clear that the system (2.2.21)-(2.2.22) has the same stucture as system (2.2.4)-(2.2.5). Thus, by multiplying the first equation by $\frac{1}{\varepsilon^2}\pi(\widehat{\rho})\bar{\rho}_x$ and the second equation by $\widehat{\rho}\bar{u}_x$, we obtain an energy identity similar to (2.2.8). Then, by integrating w.r.t. time, and from the same considerations as those in the remark 2.2.2, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\|\bar{\rho}_x(t)\|^2 + \|\bar{u}_x(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t \|\bar{u}_x(s)\|^2 ds \leq \\
\leq &O(1)\int_0^t [|\widehat{\rho}_t|_\infty + |\widehat{\rho}_x\widehat{u}|_\infty + |\widehat{u}_x|_\infty]\left[\frac{\|\bar{\rho}_x(s)\|^2}{\varepsilon^2} + \|\bar{u}_x(s)\|^2\right] ds + \\
+ &\frac{O(1)}{\varepsilon^2}\int_0^t |\widehat{\rho}_x|_\infty \|\bar{\rho}_x(s)\|\|\bar{u}_x(s)\| ds + \int_0^t G(s)ds + \int_0^t F(s)ds,
\end{aligned}$$

where we denoted all the integrals involving $\widetilde{\rho}$, $\widetilde{u}$ and w by $\int_0^t F(s)ds$, and where

$$\begin{aligned}
\int_0^t G(s)ds = &\frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty} \widehat{u}_x\bar{\rho}_x^2 dxds + \frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty} \widehat{\rho}_x\bar{u}_x\bar{\rho}_x dxds + \\
&+ O(1)\int_0^t\int_{-\infty}^{+\infty} \widehat{u}_x\bar{u}_x^2 dxds. \qquad (2.2.23)
\end{aligned}$$

The terms (2.2.23) are to be treated as the terms $\sum_{h=1}^2 J_h$ of the previous lemma. The integrals denoted by $\int_0^t F(s)ds$ are made up by bilinear terms (in the variables marked by $^-$ and $^=$), where the terms estimated in L^∞ depends only on the corrector w and on the derivatives of asymptotic profile $\widetilde{\rho}$, as in the estimates of the integrals $\int_0^t I_k(s)$ of the previous lemma (we also have a faster decay for these terms, which involve second order derivatives). Hence, we easily obtain the following

Lemma 2.2.4 *Suppose $\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \leq \lambda \ll 1$. Then, there exists a positive fixed constant Δ such that*

$$\begin{aligned}
\sup_{0\leq t\leq T}\Bigg\{\frac{1}{(1+t)^\nu}&\left[\frac{1}{\varepsilon^2}\|\bar{\rho}_x(t)\|^2 + \|\bar{u}_x(t)\|^2\right. \\
&\left.+\frac{1}{\varepsilon^2}\int_0^t \|\bar{u}_x(s)\|^2 ds\right]\Bigg\} \leq \frac{\Delta}{4}, \qquad (2.2.24)
\end{aligned}$$

Remark 2.2.5 To complete the proof of the theorem 2.1.1, we differentiate w.r.t. x in order to get estimates for second and third derivatives of $(\bar{\rho}, \bar{u})$. The analogous of terms (2.2.23) behave the same as above (there is always a coefficient with order of derivation less then or equal to 2, to be estimated in

$L^\infty(\mathbb{R})$). Since these computations are very similar to those concerning the preceding L_2 estimates, we skip the details about them. Hence, the proof of the proposition is complete. $\square$

We now prove the contraction of the sequence $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ in the following

Proposition 2.2.6 *Let us denote, for any $\varepsilon > 0$, $n \in \mathbb{N}$, $0 < \nu < 1/2$,*

$$\mathcal{F}^n_\varepsilon(T) = \sup_{0\le t\le T}\left\{\frac{1}{(t+1)^\nu}\left[\frac{1}{\varepsilon^2}\|\rho^\varepsilon_{(n)}(t) - \rho^\varepsilon_{(n-1)}(t)\|^2_{L_2} + \|u^\varepsilon_{(n)}(t) - u^\varepsilon_{(n-1)}(t)\|^2_{L_2} + \right.\right.$$
$$\left.\left. + \frac{1}{\varepsilon^2}\int_0^t \|u^\varepsilon_{(n)}(s) - u^\varepsilon_{(n-1)}(s)\|^2_{L_2} ds\right]\right\}.$$

Then, under the condition $\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \le \lambda$, for $\lambda \ll 1$, there exists a positive constant $\mu < 1$ such that

$$\mathcal{F}^n_\varepsilon(T) \le \mu \mathcal{F}^{n-1}_\varepsilon(T). \tag{2.2.25}$$

Proof. We denote

$$\begin{array}{lll} \rho^\varepsilon_{(n-2)} = \widehat{\widehat{\rho}} & \rho^\varepsilon_{(n-1)} = \widehat{\rho} & \rho^\varepsilon_{(n)} = \rho \\ u^\varepsilon_{(n-2)} = \widehat{\widehat{u}} & u^\varepsilon_{(n-1)} = \widehat{u} & u^\varepsilon_{(n)} = u \\ \bar{\rho} = \rho - \widehat{\rho} & \bar{\bar{\rho}} = \widehat{\rho} - \widehat{\widehat{\rho}} & \bar{u} = u - \widehat{u} \quad \bar{\bar{u}} = \widehat{u} - \widehat{\widehat{u}}. \end{array}$$

With this notation, we can write system (2.2.1) as

$$\begin{cases} \bar{\rho}_t + \widehat{\rho}\bar{u}_x + \widehat{u}\bar{\rho}_x = -\bar{\bar{\rho}}\widehat{u}_x - \bar{\bar{u}}\widehat{\rho}_x \\ \bar{u}_t + \widehat{u}\bar{u}_x + \dfrac{1}{\varepsilon^2}\pi(\widehat{\rho})\bar{\rho}_x = -\bar{\bar{u}}\widehat{u}_x - \left(\dfrac{1}{\varepsilon^2}\pi(\widehat{\rho}) - \dfrac{1}{\varepsilon^2}\pi(\widehat{\widehat{\rho}})\right)\widehat{\rho}_x - \dfrac{\bar{u}}{\varepsilon^2}. \end{cases} \tag{2.2.26}$$

As in the preceding proposition, we symmetrize the system (2.2.26) by

$$\begin{bmatrix} \frac{1}{\varepsilon^2}\pi(\widehat{\rho}) & 0 \\ 0 & \widehat{\rho} \end{bmatrix}$$

and obtain the standard enegy identity

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\left[\frac{p'(\widehat{\rho})}{\varepsilon^2\widehat{\rho}}\frac{\bar{\rho}^2}{2}+\widehat{\rho}\frac{\bar{u}^2}{2}\right]dx=$$

$$=\int_{-\infty}^{+\infty}\left[(\pi'(\widehat{\rho})(\widehat{\rho}_t+\widehat{\rho}_x\widehat{u})+\pi(\widehat{\rho})\widehat{u}_x)\frac{\bar{\rho}^2}{2\varepsilon^2}\right]dx+$$

$$+\int_{-\infty}^{+\infty}\left[(\widehat{\rho}_t+\widehat{\rho}_x\widehat{u}+\widehat{\rho}\widehat{u}_x)\frac{\bar{u}^2}{2}+\frac{1}{\varepsilon^2}p''(\widehat{\rho})\widehat{\rho}_x\bar{\rho}\bar{u}\right]dx+$$

$$-\int_{-\infty}^{+\infty}\pi(\widehat{\rho})\bar{\rho}\bar{u}\bar{\bar{u}}\widehat{\rho}_x dx-\int_{-\infty}^{+\infty}\pi(\widehat{\rho})\bar{\rho}\bar{\bar{\rho}}\widehat{u}_x dx+$$

$$-\int_{-\infty}^{+\infty}\widehat{\rho}\bar{u}\bar{\bar{u}}\widehat{u}_x dx-\int_{-\infty}^{+\infty}\widehat{\rho}\bar{u}(\pi(\widehat{\rho})-\pi(\widehat{\widehat{\rho}}))\widehat{\rho}_x dx-\int_{-\infty}^{+\infty}\widehat{\rho}\frac{\bar{u}^2}{\varepsilon}dx. \qquad (2.2.27)$$

As a consequence of the proposition (2.2.1), after some considerations about the symmetrizing coefficients (same as those in the remark 2.2.2), we obtain the same estimates as (2.2.9), (2.2.10), (2.2.11), under the condition (2.2.7). After time integration in the interval $[0,t]$ for $0<t<T$, we obtain

$$\frac{1}{\varepsilon^2}\|\bar{\rho}(t)\|^2+\|\bar{u}(t)\|^2+\frac{1}{\varepsilon^2}\int_0^t\|\bar{u}(s)\|^2 ds\leq$$

$$\leq O(1)\int_0^t[|\widehat{\rho}_t|_\infty+|\widehat{\rho}_x\widehat{u}|_\infty+|\widehat{u}_x|_\infty]\left[\frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2}+\|\bar{u}(s)\|^2\right]ds+$$

$$+\frac{O(1)}{\varepsilon^2}\int_0^t|\widehat{\rho}_x|_\infty\|\bar{\rho}(s)\|\|\bar{u}(s)\|ds+\frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\bar{\rho}\bar{\bar{\rho}}\widehat{u}_x dxds+$$

$$+\frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\bar{\rho}\bar{\bar{u}}\widehat{\rho}_x+O(1)\int_0^t\int_{-\infty}^{+\infty}\bar{\bar{u}}\widehat{u}_x\bar{u}dxds+\frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\bar{\bar{\rho}}\bar{u}\widehat{\rho}_x dxds=$$

$$=\sum_{k=1}^{6}L_k(t).$$

We now consider each term separately, using the result of the proposition (2.2.1).

$$L_1(t)\leq O(1)\mathcal{F}^n(t)\int_0^t\Big[|\widehat{\rho}_x(s)|_\infty\Big(|\widehat{\widehat{u}}(s)|_\infty+|\widehat{u}(s)|_\infty\Big)+|\widehat{u}_x(s)|_\infty\Big](s+1)^\nu ds\leq$$

$$\leq O(1)\mathcal{F}^n(t)\int_0^t\Big[|(\widehat{u}-\widetilde{u}-w)_x(s)|_\infty+|\widetilde{u}_x(s)|_\infty+|w_x(s)|_\infty+$$

$$+\Big(|(\widehat{\rho}-\widetilde{\rho})_x(s)|_\infty+|\widetilde{\rho}_x(s)|_\infty\Big)\Big(|(\widehat{u}-\widetilde{u}-w)(s)|_\infty+$$

$$+|(\widehat{\widehat{u}}-\widetilde{u}-w)(s)|_\infty+|\widetilde{u}(s)|_\infty+|w(s)|_\infty\Big)\Big](s+1)^\nu ds.$$

We estimate this term as in (2.2.16), (2.2.17), (2.2.18) of the proposition (2.2.1) and obtain

$$L_1(t) \leq \big(O(\delta) + O(\lambda) + O(\varepsilon^2)\big)\,(t+1)^\nu \mathcal{F}^n(t).$$

Then, in a similar fashion, we estimate the term $L_2(t)$ as in (2.2.19). Let us compute the remaining terms;

$$\begin{aligned}
L_3(t) \leq& \frac{O(1)}{\varepsilon^2}\int_0^t |\widehat{u}_x(s)|_\infty \|\bar{\rho}(s)\|\|\bar{\bar{\rho}}(s)\| ds \leq \\
\leq& \frac{O(1)}{\varepsilon^2}\int_0^t \left[\frac{\|\bar{\rho}(s)\|^2}{(s+1)^\nu} + \frac{\|\bar{\bar{\rho}}(s)\|^2}{(s+1)^\nu}\right] |\widehat{u}_x(s)|_\infty (s+1)^\nu ds \leq \\
\leq& O(1)\big[\mathcal{F}^n(t) + \mathcal{F}^{n-1}(t)\big]\int_0^t \left[\frac{\lambda|(\widehat{u}-\widetilde{u}-w)_x|_\infty^2}{\varepsilon^2} + \frac{\varepsilon^2(s+1)^{2\nu}}{\lambda} + \right.\\
&\left. + O(\delta)(s+1)^{\nu-1}\right] ds \leq [O(\lambda) + O(\delta)]\,(t+1)^\nu\left(\mathcal{F}^n(t) + \mathcal{F}^{n-1}(t)\right),
\end{aligned}$$

where we have used the condition (2.2.7).

$$\begin{aligned}
L_4(t) \leq& \frac{O(1)}{\varepsilon^2}\int_0^t |\widehat{\rho}_x(s)|_\infty \|\bar{\rho}(s)\|\|\bar{\bar{u}}(s)\| ds \leq \\
\leq& O(1)\int_0^t \left[\frac{\lambda\|\bar{\bar{u}}(s)\|^2}{\varepsilon^2} + \frac{|\widehat{\rho}_x(s)|_\infty^2\|\bar{\rho}(s)\|^2}{\lambda\varepsilon^2(s+1)^\nu}(s+1)^\nu\right] ds \leq \\
\leq& O(\lambda)\mathcal{F}^{n-1}(t)(t+1)^\nu + O(1)\frac{1}{\lambda}\mathcal{F}^n(t)\int_0^t \big[|(\widehat{\rho}-\widetilde{\rho})_x(s)|_\infty^2 (s+1)^\nu + \\
&+ O(\delta)(s+1)^{\nu-1}\big]\, ds \leq \\
&\leq [O(\lambda) + O(\delta)]\,(t+1)^\nu\left(\mathcal{F}^n(t) + \mathcal{F}^{n-1}(t)\right).
\end{aligned}$$

The integrals $L_5(t)$ and $L_6(t)$ can be treated as above. Thus, by suitably choosing δ and λ small, the proof is complete. $\square$

We finally arrive to the convergence of the approximating sequence. Let ν, ε and T be fixed in the usual way. Since $(\rho_\varepsilon^n, u_\varepsilon^n)$ is a Cauchy sequence in the norm expressed by $\mathcal{F}^n$, by interpolation we have

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left\{\frac{1}{(t+1)^\nu}\left[\frac{1}{\varepsilon^2}\|(\rho^\varepsilon_{(n)} - \rho^\varepsilon_{(m)})(t)\|^2_{H^{3\theta}} + \|(u^\varepsilon_{(n)} - u^\varepsilon_{(m)})(t)\|^2_{H^{3\theta}} + \right.\right.\\
&\left.\left. + \frac{1}{\varepsilon^2}\int_0^t \|(u^\varepsilon_{(n)} - u^\varepsilon_{(m)})(t)\|^2_{H^{3\theta}} ds\right]\right\} \to 0 \qquad \text{as} \quad n,m\to\infty. \qquad (2.2.28)
\end{aligned}$$

for any $\theta \in (0,1)$. Thus, the approximating sequence $(\rho_\varepsilon^n, u_\varepsilon^n)$ converges in the norm expressed by (2.2.28) to (ρ^*, u^*). By choosing $\theta \in (0,1)$ big enough, we obtain

$$(\rho_\varepsilon^n, u_\varepsilon^n) \to (\rho^*, u^*) \quad \text{in} \quad L^\infty([0,T]; H^2(\mathbb{R})). \tag{2.2.29}$$

Hence, we can identify the limit as the solution $(\rho_\varepsilon, u_\varepsilon)$ to the system (2.1.2) and carry out the limit as $n \to \infty$ in the estimate (2.2.3), with $H^{3\theta}$ in place of H^3, and the proof of the theorem (2.1.1) is complete.

Chapter 3

A one dimensional model for viscoelasticity

The present chapter is devoted to the study of the diffusive relaxation model for viscoelasticity described in section 1.4. The next section 3.1 contains the first rigorous justification of the relaxation scheme in theorem 3.1.1, which is proved via standard symmetrization and compactness arguments. In section 3.2 we provide a rate of convergence in L^2 norm. Finally, in section 3.3 we prove a result concerning travelling waves for the model considered.

3.1 Global existence and singular convergence of H^1 solutions

In this section we study the global existence of H^1 solutions and the relaxation limit for the nonhomogeneous strictly hyperbolic system

$$\begin{cases} u_t - v_x = 0 \\ v_t - z_x = 0 \\ \varepsilon^2 z_t - \mu v_x = -z + \sigma(u), \end{cases} \tag{3.1.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ are the independent variables, u, v and z take values in $\mathbb{R}$, μ is a positive constant, $\varepsilon > 0$ is the relaxation parameter and $\sigma : \mathbb{R} \to \mathbb{R}$ is a smooth function. We rewrite the semilinear system (3.1.1) as follows

$$W_t + \mathcal{A}^\varepsilon W_x = \mathcal{Q}^\varepsilon(W), \tag{3.1.2}$$

where

$$W = \begin{pmatrix} u \\ v \\ z \end{pmatrix} \quad \mathcal{A}^\varepsilon(W) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -\frac{\mu}{\varepsilon^2} & 0 \end{pmatrix} \quad \mathcal{Q}^\varepsilon(W) = \frac{1}{\varepsilon^2} \begin{pmatrix} 0 \\ 0 \\ -z + \sigma(u) \end{pmatrix}.$$

We recall that, for system (3.1.2) with fixed ε and μ, a local existence result holds. More precisely, if the initial datum $(u(\cdot,0), v(\cdot,0), z(\cdot,0))$ belongs in $L^\infty(\mathbb{R})$, then there exists a positive time T such that the solution $(u(t), v(t), z(t))$ exists in $L^\infty(\mathbb{R})$ for $t \in [0,T]$. Since the matrix $\mathcal{A}^\varepsilon$ is constant, the solution to (3.1.1) is global in time once the following estimate is verified

$$\lim_{t \uparrow T^*} \left[\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} + \|v(\cdot,t)\|_{L^\infty(\mathbb{R})} + \|z(\cdot,t)\|_{L^\infty(\mathbb{R})} \right] < +\infty, \tag{3.1.3}$$

where $[0, T^*]$ is the maximal time interval of existence of the solution. The estimate (3.1.3) and the global existence of solutions to (3.1.1) for $\varepsilon > 0$ fixed are well known, provided that the following globally Lipschitz condition on the function σ holds

$$\sup_{u \in \mathbb{R}} |\sigma'(u)| < +\infty. \tag{3.1.4}$$

However, we shall obtain this a priori L^∞ bound, which guarantees the global existence of solutions, as a consequence of the following theorem, which actually provides a stronger estimate, namely an estimate for the H^1–norm of the solutions, with initial datum $(u(\cdot,0), v(\cdot,0), z(\cdot,0)) \in H^1(\mathbb{R})$, *uniform* with respect to the parameter ε.

Theorem 3.1.1 *Let $(u,v,z)(\cdot,t)$ be the solution to the system (3.1.1) with initial data $(u,v,z)(\cdot,0) \in H^1(\mathbb{R})$. Suppose that the function σ satisfies condition (3.1.4). Then, the following inequality holds for any $t > 0$*

$$\begin{aligned} &\varepsilon^2 \|z(t)\|^2_{H^1(\mathbb{R})} + \|u(t)\|^2_{H^1(\mathbb{R})} + \|v(t)\|^2_{H^1(\mathbb{R})} + \int_0^t \|z(s)\|^2_{H^1(\mathbb{R})} ds \\ &\leq \left[\varepsilon^2 \|z(0)\|^2_{H^1(\mathbb{R})} + \|u(0)\|^2_{H^1(\mathbb{R})} + \|v(0)\|^2_{H^1(\mathbb{R})} \right] e^{Ct}, \end{aligned} \tag{3.1.5}$$

where C is a positive constant depending only on $\sup_{u \in \mathbb{R}} |\sigma'(u)|$.

Proof. We prove estimate (3.1.5) by means of energy estimates. To this aim, we employ that system (3.1.1) admits a symmetrizer, which is positive definite for small values of ε, namely

$$\mathcal{B}^\varepsilon = \begin{pmatrix} \frac{\mu}{\varepsilon^2} & 0 & -1 \\ 0 & \frac{\mu}{\varepsilon^2} - 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Hence, we define the energy

$$\mathcal{E}^\varepsilon(W) = (\mathcal{B}^\varepsilon W, W)_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} \left[\frac{\mu}{\varepsilon^2} u^2 - 2uz + \left(\frac{\mu}{\varepsilon^2} - 1 \right) v^2 + z^2 \right] dx$$

and we observe that, for $\varepsilon^2 \leq \frac{\mu}{3}$,

$$\mathcal{E}^\varepsilon(W) \geq \frac{1}{2}\int_{-\infty}^{+\infty}\left[\frac{\mu}{\varepsilon^2}u^2 + \frac{\mu}{\varepsilon^2}v^2 + z^2\right]dx. \tag{3.1.6}$$

Thus, (3.1.2) and integration by parts yields

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^\varepsilon(W(t)) &= -2\int_{-\infty}^{+\infty}\mathcal{B}^\varepsilon\mathcal{A}^\varepsilon W_x W\,dx + 2\int_{-\infty}^{+\infty}\mathcal{B}^\varepsilon\mathcal{Q}^\varepsilon(W)\,W\,dx\\ &= 2\int_{-\infty}^{+\infty}\mathcal{B}^\varepsilon\mathcal{Q}^\varepsilon(W)\,W\,dx.\end{aligned}$$

After integration over the time interval $[0,t]$, we get

$$\mathcal{E}^\varepsilon(W(t)) - \mathcal{E}^\varepsilon(W(0)) = 2\int_0^t\int_{-\infty}^{+\infty}\mathcal{B}^\varepsilon\mathcal{Q}^\varepsilon(W(s))\,W(s)dxds =: I(W). \tag{3.1.7}$$

We now estimate the term $I(W)$ in (3.1.7) as follows

$$\begin{aligned}I(W) &= \frac{2}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\left[zu - u\sigma(u) - z^2 + z\sigma(u)\right]dxds\\ &\leq -\frac{1}{\varepsilon^2}\int_0^t\|z(s)\|^2_{L^2(\mathbb{R})}ds + \frac{3}{\varepsilon^2}\int_0^t\Bigg[\|u(s)\|^2_{L^2(\mathbb{R})}\\ &\quad + \int_{-\infty}^{+\infty}|\sigma(u(x,s))|^2dx\Bigg]ds\\ &\leq -\frac{1}{\varepsilon^2}\int_0^t\|z(s)\|^2_{L^2(\mathbb{R})}ds + \frac{3}{\varepsilon^2}\left(1 + \sup_u|\sigma'(u)|^2\right)\int_0^t\|u(s)\|^2_{L^2(\mathbb{R})}ds,\end{aligned}$$

if we assume, without loss of generality, $\sigma(0) = 0$. Thus, combining (3.1.7) and (3.1.6) one has

$$\begin{aligned}&\varepsilon^2\|z(t)\|^2_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})} + \|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t\|z(s)\|^2_{L^2(\mathbb{R})}ds\\ &\leq C_1\left(\varepsilon^2\|z(0)\|^2_{L^2(\mathbb{R})} + \|u(0)\|^2_{L^2(\mathbb{R})} + \|v(0)\|^2_{L^2(\mathbb{R})}\right) + C_2\int_0^t\|u(s)\|^2_{L^2(\mathbb{R})}ds,\end{aligned} \tag{3.1.8}$$

where C_1 and C_2 are fixed positive constant with C_2 depending on σ'. We now perform a similar estimate for the spatial derivative W_x. By differentiating (3.1.2) with respect to x, we get

$$W_{xt} + \mathcal{A}^\varepsilon W_{xx} = \mathcal{Q}^\varepsilon(W)_x,$$

Thus, proceeding as before, we get

$$\mathcal{E}^\varepsilon\left(W_x(t)\right)-\mathcal{E}^\varepsilon\left(W_x(0)\right)=2\int_0^t\int_{-\infty}^{+\infty}\mathcal{B}^\varepsilon\mathcal{Q}^\varepsilon\left(W(s)\right)_x W_x(s)dxds=:J\left(W_x\right). \tag{3.1.9}$$

We estimate the integral term $J\left(W_x\right)$ as follows

$$\begin{aligned}J\left(W_x\right)&=\frac{2}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\mathcal{B}^\varepsilon\mathcal{Q}^\varepsilon\left(W\right)_x W_xdxds\\&=\frac{2}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\left[-z_x^2-\sigma'(u)u_x^2+(\sigma'(u)+1)u_xz_x\right]dxds\\&\leq-\frac{1}{2\varepsilon^2}\int_0^t\|z_x(s)\|^2_{L^2(\mathbb{R})}ds+C_3\int_0^t\|u_x(s)\|^2_{L^2(\mathbb{R})}ds, \end{aligned}\tag{3.1.10}$$

with C_3 depending only on σ', as in the previous estimate. From (3.1.6), (3.1.9) and (3.1.10), we get

$$\begin{aligned}&\varepsilon^2\|z_x(t)\|^2_{L^2(\mathbb{R})}+\|u_x(t)\|^2_{L^2(\mathbb{R})}+\|v_x(t)\|^2_{L^2(\mathbb{R})}+\int_0^t\|z_x(s)\|^2_{L^2(\mathbb{R})}ds\\&\leq C_4\left(\varepsilon^2\|z_x(0)\|^2_{L^2(\mathbb{R})}+\|u_x(0)\|^2_{L^2(\mathbb{R})}+\|v_x(0)\|^2_{L^2(\mathbb{R})}\right)\\&+C_5\int_0^t\|u_x(s)\|^2_{L^2(\mathbb{R})}ds,\end{aligned}\tag{3.1.11}$$

with C_4 and C_5 positive constants, C_5 depending on σ'. Thus, from (3.1.8), (3.1.11) and from the Gronwall Lemma, we recover the estimate (3.1.5) and the proof is complete. □

The following estimates for the time derivatives of u, v and z are an immediate consequence of (3.1.5)

$$\begin{aligned}&\|u_t(t)\|^2_{L^2(\mathbb{R})}+\int_0^t\|v_t(s)\|^2_{L^2(\mathbb{R})}ds\leq Ke^{Ct}, \qquad (3.1.12)\\&\int_0^t\|\varepsilon z_t(s)\|^2_{L^2(\mathbb{R})}ds\leq Ke^{Ct},\end{aligned}$$

with K positive constant depending on the initial datum. In particular, (3.1.5) and (3.1.12) imply that the solution (u,v,z) verifies $u\in H^1([0,T]\times\mathbb{R})$, $v\in H^1([0,T]\times\mathbb{R})$ and $\varepsilon z\in H^1([0,T]\times\mathbb{R})$. Hence, as a first consequence of the above estimates, we have obtained the well known relation (3.1.3), and thus the global existence of solutions to (3.1.1) in L^∞, for $\varepsilon>0$ fixed. Moreover, since our estimates are uniform in ε, we can prove a first result of convergence for our semilinear relaxation approximation.

Corollary 3.1.2 *Let $(u^\varepsilon, v^\varepsilon, z^\varepsilon)(\cdot, t)$ be the solution to the system (3.1.1) with initial data $(u, v, z)(\cdot, 0) \in H^1(\mathbb{R})$. Suppose that the function σ satisfies condition (3.1.4). Then,*

$$\begin{aligned} u^\varepsilon &\to u, && \textit{strongly in } L^p_{loc}([0,T]\times\mathbb{R}),\ 2 \le p < +\infty, \\ v^\varepsilon &\to v, && \textit{strongly in } L^p_{loc}([0,T]\times\mathbb{R}),\ 2 \le p < +\infty, \end{aligned}$$

and (u, v) is the solution of

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx} \end{cases}$$

with (u_0, v_0) as initial condition.

Proof. Since estimates (3.1.5) and (3.1.12) are uniform in ε, the family of solutions $\{u^\varepsilon, v^\varepsilon, z^\varepsilon\}_{\varepsilon>0}$ to system (3.1.1) satisfies the following properties, for any $T > 0$,

$$\{u^\varepsilon\} \quad \text{uniformly bounded in} \quad H^1([0,T]\times\mathbb{R}), \tag{3.1.13}$$

$$\{v^\varepsilon\} \quad \text{uniformly bounded in} \quad H^1([0,T]\times\mathbb{R}), \tag{3.1.14}$$

$$\{z^\varepsilon\} \quad \text{uniformly bounded in} \quad L^2([0,T]; H^1(\mathbb{R})). \tag{3.1.15}$$

Hence, from (3.1.13) and (3.1.14) we deduce that the sequence $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ is compact in $L^p_{loc}([0,T]\times\mathbb{R})$ for any $2 \le p < +\infty$. Therefore, as $\varepsilon \downarrow 0$, we obtain, passing if necessary to subsequences,

$$u^\varepsilon \to u, \quad v^\varepsilon \to v, \quad \text{strongly in } L^p_{loc}([0,T]\times\mathbb{R}),\ 2 \le p < +\infty.$$

Finally, by taking advantage of (3.1.15), we pass in the limit as $\varepsilon \downarrow 0$ into system (3.1.1) with initial datum $(u_0, v_0, z_0) \in H^1(\mathbb{R})$, and we recover that the limit functions u and v satisfy

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx} \\ u(\cdot, 0) = u_0 \\ v(\cdot, 0) = v_0 \end{cases} \tag{3.1.16}$$

in distributional sense.

Now, for any $T > 0$, the initial value problem (3.1.16) admits a unique solution $(u, v) \in L^\infty([0,T], H^1(\mathbb{R}))$ (see Appendix A). Therefore, all the sequence $\{(u^\varepsilon, v^\varepsilon)\}$ converges to (u, v) strongly in L^p_{loc}, for any $2 \le p < +\infty$ and the proof is complete. □

3.2 Rate of convergence in L^2 norm

In the previous section we proved the strong convergence of H^1 solutions of the semilinear relaxation approximation (3.1.1) toward the solutions of the limit (3.1.16) via compactness arguments. Here, we shall prove such convergence by showing directly the L^2 norm of the difference between the approximating and the limit solutions tends to zero as $\varepsilon \downarrow 0$, with a polynomial rate of convergence w.r.t. ε. To this aim, let $\{(u^\varepsilon, v^\varepsilon)\}_{\varepsilon>0}$ be any solution of

$$\begin{cases} u_t^\varepsilon - v_x^\varepsilon = 0 \\ v_t^\varepsilon - z_x^\varepsilon = 0 \\ \varepsilon^2 z_t^\varepsilon - \mu v_x^\varepsilon = -z^\varepsilon + \sigma(u^\varepsilon), \end{cases} \tag{3.2.1}$$

and (u, v) be the solution of the hyperbolic–parabolic system (3.1.16) with initial data in $H^2(\mathbb{R})$. We assume the initial data for the hyperbolic system (3.2.1) to be "well–prepared" in the following sense: we require $(u^\varepsilon, v^\varepsilon, z^\varepsilon)\,|_{t=0}$ to converge in L^2 to $(u, v, z)\,|_{t=0}$, where z is determined by the relation

$$z = \sigma(u) + \mu v_x. \tag{3.2.2}$$

Let us first denote, for fixed $\varepsilon > 0$,

$$\overline{u} = u^\varepsilon - u, \quad \overline{v} = v^\varepsilon - v, \quad \overline{z} = z^\varepsilon - z,$$

where z is given by (3.2.2). The vector $(\overline{u}, \overline{v}, \overline{z})$ satisfies the system of equations

$$\begin{cases} \overline{u}_t - \overline{v}_x = 0 \\ \overline{v}_t - \overline{z}_x = 0 \\ \overline{z}_t - \frac{\mu}{\varepsilon^2}\overline{v}_x = -z_t - \frac{1}{\varepsilon^2}\left[\overline{z} - (\sigma(\overline{u}+u) - \sigma(u))\right]. \end{cases} \tag{3.2.3}$$

Again, we use the vectorial notation

$$\overline{W} = \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{z} \end{pmatrix} \qquad \mathcal{A}^\varepsilon(\overline{W}) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -\frac{\mu}{\varepsilon^2} & 0 \end{pmatrix}$$

$$\mathcal{P}^\varepsilon(\overline{W}, u, z) = \frac{1}{\varepsilon^2}\begin{pmatrix} 0 \\ 0 \\ -z_t - \frac{1}{\varepsilon^2}\left[\overline{z} - (\sigma(\overline{u}+u) - \sigma(u))\right] \end{pmatrix}.$$

Hence, system (3.2.3) becomes

$$\overline{W}_t - \mathcal{A}^\varepsilon \overline{W}_x = \mathcal{P}^\varepsilon(\overline{W}, u, z). \tag{3.2.4}$$

We state the convergence result in following theorem.

Theorem 3.2.1 *Let (u, v) be the solution of the parabolic system (3.1.16) with initial data $u_0, v_0 \in H^2(\mathbb{R})$. For fixed $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon, z^\varepsilon)$ be any solution of the hyperbolic system (3.2.1) with initial data $(u_0^\varepsilon, v_0^\varepsilon, z_0^\varepsilon)$ satisfying the condition*

$$\|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R})} + \|v_0^\varepsilon - v_0\|_{L^2(\mathbb{R})} + \varepsilon\|z_0^\varepsilon - z_0\|_{L^2(\mathbb{R})} \leq \omega(\varepsilon), \quad \textit{as } \varepsilon \to 0, \tag{3.2.5}$$

for some continuous function ω such that $\omega(0) = 0$, with z_0 given by

$$z_0 = \sigma(u_0) - \mu v_{0,x}.$$

Moreover, let the function σ satisfies condition (3.1.4). Then, for any fixed $t > 0$, the following estimate holds

$$\begin{aligned} &\varepsilon^2\|z^\varepsilon(t) - z(t)\|^2_{L^2(\mathbb{R})} + \|u^\varepsilon(t) - u(t)\|^2_{L^2(\mathbb{R})} + \|v^\varepsilon(t) - v(t)\|^2_{L^2(\mathbb{R})} \\ &\quad + \int_0^t \|z^\varepsilon(s) - z(s)\|^2_{L^2(\mathbb{R})} ds \leq Ce^{C_\sigma t}\left(\omega(\varepsilon)^2 + \varepsilon^4 C_0\right), \end{aligned} \tag{3.2.6}$$

for some positive constants C, C_0, C_σ, with C_0 depending on (u_0, v_0) and on t, and C_σ depending on σ.

Proof. We prove (3.2.6) by means of the same energy method as in the proof of Theorem 3.1.1. We employ once again the symmetrizer

$$\mathcal{B}^\varepsilon = \begin{pmatrix} \frac{\mu}{\varepsilon^2} & 0 & -1 \\ 0 & \frac{\mu}{\varepsilon^2} - 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We apply $\mathcal{B}^\varepsilon$ to system (3.2.3). Therefore, we end up with the same energy of the previous section

$$\mathcal{E}^\varepsilon\left(\overline{W}\right) = \left(\mathcal{B}^\varepsilon \overline{W}, \overline{W}\right)_{L^2(\mathbb{R})}.$$

We have, as in the proof of Theorem 3.1.1, for small ε,

$$\mathcal{E}^\varepsilon\left(\overline{W}\right) \geq \frac{1}{2}\int_{-\infty}^{+\infty} \left[\frac{\mu}{\varepsilon^2}\overline{u}^2 + \frac{\mu}{\varepsilon^2}\overline{v}^2 + \overline{z}^2\right] dx. \tag{3.2.7}$$

Then, integration by parts yields

$$\frac{d}{dt}\mathcal{E}^\varepsilon\left(\overline{W}(t)\right) = 2\int_{-\infty}^{+\infty} \mathcal{B}^\varepsilon \mathcal{P}^\varepsilon\left(\overline{W}(t)\right)\overline{W}(t) dx.$$

After integration over the time interval $[0,t]$, we get

$$\mathcal{E}^{\varepsilon}\left(\overline{W}(t)\right)-\mathcal{E}^{\varepsilon}\left(\overline{W}(0)\right)=2\int_0^t\int_{-\infty}^{+\infty}\mathcal{B}^{\varepsilon}\mathcal{P}^{\varepsilon}\left(\overline{W}(s)\right)\overline{W}(s)dxds=:H\left(\overline{W}\right). \tag{3.2.8}$$

We estimate the integral term $H\left(\overline{W}\right)$ as follows

$$\begin{aligned}
H\left(\overline{W}\right)&=2\int_0^t\int_{-\infty}^{+\infty}z_t\overline{u}dxds-2\int_0^t\int_{-\infty}^{+\infty}z_t\overline{z}dxds+\int_0^t\int_{-\infty}^{+\infty}\overline{z}\,\overline{u}dxds\\
&-\frac{2}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{z}^2dxds-\frac{2}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\left[\sigma(\overline{u}+u)-\sigma(u)\right]\overline{u}\;dxds\\
&+\frac{2}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\left[\sigma(\overline{u}+u)-\sigma(u)\right]\overline{z}dxds\\
&\le\varepsilon^2\int_0^t\int_{-\infty}^{+\infty}z_t^2dxds+\frac{1}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{u}^2dxds+4\varepsilon^2\int_0^t\int_{-\infty}^{+\infty}z_t^2dxds\\
&+\frac{1}{4\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{z}^2dxds+\frac{1}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{u}^2dxds+\frac{1}{4\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{z}^2dxds\\
&-\frac{2}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{z}^2dxds+\frac{2\sup|\sigma'|}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{u}^2dxds\\
&+\frac{1}{4\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{z}^2dxds+\frac{4\sup|\sigma'|}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\overline{u}^2dxds.
\end{aligned}$$

Hence, since $\sup|\sigma'|<\infty$ and in view of (3.2.7), we obtain

$$\begin{aligned}
&\varepsilon^2\|\overline{z}(t)\|^2_{L^2(\mathbb{R})}+\|\overline{u}(t)\|^2_{L^2(\mathbb{R})}+\|\overline{v}(t)\|^2_{L^2(\mathbb{R})}+\int_0^t\|\overline{z}(s)\|^2_{L^2(\mathbb{R})}ds\\
&\le C\Bigg[\varepsilon^2\|\overline{z}(0)\|^2_{L^2(\mathbb{R})}+\|\overline{u}(0)\|^2_{L^2(\mathbb{R})}+\|\overline{v}(0)\|^2_{L^2(\mathbb{R})}\\
&+C_\sigma\int_0^t\|\overline{u}(s)\|^2_{L^2(\mathbb{R})}ds+\varepsilon^4\int_0^t\|z_t(s)\|^2_{L^2(\mathbb{R})}ds\Bigg],
\end{aligned}$$

for fixed constants C, $C_\sigma>0$ with C_σ depending on σ'. Therefore, Gronwall inequality implies

$$\begin{aligned}
&\varepsilon^2\|\overline{z}(t)\|^2_{L^2(\mathbb{R})}+\|\overline{u}(t)\|^2_{L^2(\mathbb{R})}+\|\overline{v}(t)\|^2_{L^2(\mathbb{R})}+\int_0^t\|\overline{z}(s)\|^2_{L^2(\mathbb{R})}ds\\
&\le C\left[\varepsilon^2\|\overline{z}(0)\|^2_{L^2(\mathbb{R})}+\|\overline{u}(0)\|^2_{L^2(\mathbb{R})}+\|\overline{v}(0)\|^2_{L^2(\mathbb{R})}+\varepsilon^4\int_0^t\|z_t(s)\|^2_{L^2(\mathbb{R})}ds\right]e^{C_\sigma t}.
\end{aligned}$$

We then employ the estimate for the term z_t in Appendix A and thus the proof is complete. □

3.3 Travelling waves

In the previous sections we studied the convergence as $\varepsilon \downarrow 0$ in L^p norms of solutions to the hyperbolic system

$$\begin{cases} u_t - v_x = 0 \\ v_t - z_x = 0 \\ \varepsilon^2 z_t - \mu v_x = -z + \sigma(u). \end{cases} \tag{3.3.1}$$

In this section we analyze the behavior as $\varepsilon \downarrow 0$ of travelling–wave–type solutions to the same model and we obtain a pointwise convergence of such solutions. To this aim, we restrict ourselves to the case $\sigma' > 0$, that is when the system

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = 0 \end{cases} \tag{3.3.2}$$

is strictly hyperbolic, but we do not require the function σ to satisfy the globally Lipschitz condition (3.1.4).

Let us consider the viscous profiles

$$(U(\xi), V(\xi)), \quad \xi = x - st$$

for the model

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx}, \end{cases} \tag{3.3.3}$$

with given condition at $\pm\infty$. These limiting conditions and the speed s of our profile are chosen in agreement with the Rankine–Hugoniot conditions and the Lax conditions of the system (3.3.2). Then, we show that these solutions are limits, as $\varepsilon \downarrow 0$, of travelling–wave–type solutions to (3.3.1) with the same limiting conditions. More precisely, we consider solutions to (3.3.3) of the form

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = x - st, \quad s \in \mathbb{R},$$

with limiting conditions

$$U(\pm\infty) = u^{\pm}, \quad V(\pm\infty) = v^{\pm}. \tag{3.3.4}$$

Thus, U and V satisfy the following second–order system of ordinary differential equations

$$\begin{cases} -sU' - V' = 0 \\ -sV' - \sigma'(U)U' = \mu V''. \end{cases} \tag{3.3.5}$$

Let us now establish the conditions at infinity. The Rankine–Hugoniot conditions for the discontinuity $((u^-, v^-), (u^+, v^+), s)$ of system (3.3.2) are given by

$$\begin{aligned} s(u^+ - u^-) &= v^- - v^+ \\ s(v^+ - v^+) &= \sigma(u^-) - \sigma(u^+). \end{aligned} \tag{3.3.6}$$

The Lax conditions for the 1–shock for the same system are given by

$$-\sqrt{\sigma'(u^+)} < s < -\sqrt{\sigma'(u^-)}. \tag{3.3.7}$$

Thus, if the function σ is convex on the range of the profile of the variable u, then we must require $u^- < u^+$ to get the existence of the corresponding admissible shock profile. Conversely, if σ is concave, then $u^- > u^+$. Similarly, in the case of a 2–shock, Lax conditions read

$$\sqrt{\sigma'(u^+)} < s < \sqrt{\sigma'(u^-)} \tag{3.3.8}$$

and we have $u^- > u^+$ whenever σ is convex, $u^- < u^+$ whenever σ is concave. We observe that in the cases of physical interest, the function σ may change its convexity. A typical example is given by the function $u + \alpha u^3$. In this case, for positive (negative, respectively) values of u^- and u^+, we must choose $u^- < u^+$ ($u^- > u^+$, respectively) in case of 1–shock, $u^- > u^+$ ($u^- < u^+$, respectively) in case of 2–shock. In order to treat the general case, one has to consider the Oleinik chord condition, besides the Lax conditions quoted above. For the sake of clearness, we treat the case of convex σ with $u^- < u^+$ (1–shock), but the remaining cases can be handled similarly. Thus, (3.3.7) implies in particular $s < 0$, which combined with (3.3.6) gives $v^- < v^+$. The existence of profiles U and V satisfying (3.3.5), with infinity data (3.3.4) obeying the conditions established above, is equivalent to the global existence of solutions to the ordinary differential equation

$$V' = -sV + c_2 - \sigma\left(\frac{1}{s}(c_1 - V)\right), \tag{3.3.9}$$

with the profile of the variable u given by

$$U = \frac{1}{s}(c_1 - V),$$

where we have set

$$\begin{aligned} c_1 &= su^+ + v^+ = su^- + v^- \\ c_2 &= sv^+ + \sigma(u^+) = sv^- + \sigma(u^-). \end{aligned} \tag{3.3.10}$$

Thus, the global existence of profiles U and V is a consequence of the strict monotonicity of V, which comes directly from the Rankine–Hugoniot conditions (3.3.6) and the Lax conditions (3.3.7) (see, for instance, [Ser99]). In particular, both functions U and V are strictly increasing. Moreover, due to the monotonicity of the profiles, the following estimates hold

$$\begin{aligned} &\big(u^+ - U(\xi)\big) \le \big(u^+ - U(0)\big)\, e^{-C\xi} &&\text{for } \ \xi > 0 \\ &\big(U(\xi) - u^-\big) \le \big(U(0) - u^-\big)\, e^{C\xi} &&\text{for } \ \xi < 0, \end{aligned}$$

where C is a strictly positive constant depending only on $u^-, u^+, U(0), \mu, s$, for $\varepsilon \ll 1$. The above estimate implies also

$$\int_{-\infty}^{0} \big[U(\xi) - u^-\big]\, d\xi + \int_{0}^{+\infty} \big[u^+ - U(\xi)\big]\, d\xi < +\infty.$$

Similar estimates hold for the variable V as well. The profile (U, V) is not unique, since if $(U, V)(\xi)$ is a solution to (3.3.4)–(3.3.5), so is $(U, V)(\xi + k)$ for any constant $k \in \mathbb{R}$. We will prove that any of these solutions is the limit, as the relaxation parameter vanishes, of a family of travelling wave profiles which solve (3.3.1).

Let us then consider travelling wave type solutions to the hyperbolic system (3.3.1) where $\varepsilon > 0$ is fixed, namely

$$u^\varepsilon(x,t) = U^\varepsilon(x - st), \quad v^\varepsilon(x,t) = V^\varepsilon(x - st), \quad z^\varepsilon(x,t) = Z^\varepsilon(x - st), \quad s \in \mathbb{R}.$$

Hence, the profiles of U^ε, V^ε and Z^ε satisfy the following system

$$\begin{cases} -s(U^\varepsilon)' - (V^\varepsilon)' = 0 \\ -s(V^\varepsilon)' - (Z^\varepsilon)' = 0 \\ -s(Z^\varepsilon)' - \frac{\mu}{\varepsilon^2}(V^\varepsilon)' = -\frac{1}{\varepsilon^2}\left(Z^\varepsilon - \sigma(U^\varepsilon)\right). \end{cases} \tag{3.3.11}$$

The limiting conditions are given by

$$U^\varepsilon(\pm\infty) = u^\pm, \quad V^\varepsilon(\pm\infty) = v^\pm, \quad Z^\varepsilon(\pm\infty) = \sigma(u^\pm). \tag{3.3.12}$$

As in the parabolic case, we require the end states $(u^\pm, v^\pm)$ and the velocity of propagation s of the profile $(U^\varepsilon, V^\varepsilon, Z^\varepsilon)$ satisfies the Rankine–Hugoniot conditions (3.3.6) and the Lax condition for the 1–shock (3.3.7). The global well posedness of system (3.3.11) with limiting conditions (3.3.12) is equivalent to the global existence of solutions to the single first order differential equation

$$\left(\mu - \varepsilon^2 s^2\right)(Z^\varepsilon)' = -s\left(Z^\varepsilon - \sigma\left(U^\varepsilon\right)\right), \tag{3.3.13}$$

where U^ε and V^ε are given by

$$\begin{aligned} U^\varepsilon &= u^- + \frac{1}{s^2}\left(Z^\varepsilon - \sigma\left(u^-\right)\right) \\ V^\varepsilon &= v^- - \frac{1}{s}\left(Z^\varepsilon - \sigma\left(u^-\right)\right). \end{aligned}$$

Again, the Rankine–Hugoniot conditions (3.3.6) and the Lax condition (3.3.7), for $\varepsilon \ll 1$, imply that Z^ε is strictly increasing. This yields to the global existence of the profile Z^ε we are looking for. As in the parabolic case, if $\left(U^\varepsilon, V^\varepsilon, Z^\varepsilon\right)(\xi)$ is a solution to (3.3.11), so is $\left(U^\varepsilon, V^\varepsilon, Z^\varepsilon\right)(\xi + k)$, for any constant k. Again, the monotonicity of the solutions also provide the estimates

$$\begin{aligned} &\left(u^+ - U^\varepsilon(\xi)\right) \le \left(u^+ - U^\varepsilon(0)\right) e^{-C\xi} \qquad && \text{for } \ \xi > 0 \\ &\left(U^\varepsilon(\xi) - u^-\right) \le \left(U^\varepsilon(0) - u^-\right) e^{C\xi} \qquad && \text{for } \ \xi < 0, \end{aligned}$$

together with the integral estimate

$$\int_{-\infty}^{0} \left[U^\varepsilon(\xi) - u^-\right] d\xi + \int_{0}^{+\infty} \left[u^+ - U^\varepsilon(\xi)\right] d\xi < +\infty. \tag{3.3.14}$$

We refer to [YZ97] for a more general result of existence of relaxation profiles for a hyperbolic system of conservation laws.

Our next purpose is to investigate the behavior of the profiles U^ε and V^ε as $\varepsilon \downarrow 0$. Our result is contained in the following theorem, which concerns the pointwise convergence of the profiles $U^\varepsilon, V^\varepsilon$ as $\varepsilon \downarrow 0$.

Theorem 3.3.1 *Let $\varepsilon > 0$. Let $(U, V)(\xi)$ be a solution to (3.3.5) with the limiting conditions (3.3.4). Let $\left(U^\varepsilon, V^\varepsilon, Z^\varepsilon\right)(\xi)$ be the unique solution to (3.3.11) with limiting conditions (3.3.12) and such that*

$$U^\varepsilon(0) = U(0) \tag{3.3.15}$$

for any $\varepsilon > 0$. Then, as $\varepsilon \downarrow 0$,

$$U^\varepsilon(\xi) \longrightarrow U(\xi) \tag{3.3.16}$$

$$V^\varepsilon(\xi) \longrightarrow V(\xi), \tag{3.3.17}$$

uniformly on compact intervals.

Proof. Since (3.3.17) is a straightforward consequence of (3.3.16) and of the relation

$$U^\varepsilon - U = -\frac{1}{s}\left(V^\varepsilon - V\right),$$

it suffices to prove (3.3.16). Writing (3.3.13) in terms of U^ε gives

$$(U^\varepsilon)' = \frac{1}{s(\mu - \varepsilon^2 s^2)} \left[s^2 \left(u^+ - U^\varepsilon\right) - \left(\sigma\left(u^+\right) - \sigma\left(U^\varepsilon\right)\right)\right]. \tag{3.3.18}$$

The equation for the viscous profile U is given by

$$U' = \frac{1}{s\mu} \left[s^2 \left(u^+ - U\right) - \left(\sigma\left(u^+\right) - \sigma\left(U\right)\right)\right]. \tag{3.3.19}$$

We now integrate both (3.3.18) and (3.3.19) in the interval $(0, \xi)$, for $\xi > 0$. Then, by employing the condition (3.3.15), we obtain the following integral equation for the difference $U^\varepsilon - U$

$$\begin{aligned} U^\varepsilon(\xi) - U(\xi) = &\left[\frac{1}{s(\mu - \varepsilon^2 s^2)} - \frac{1}{s\mu}\right] \int_0^\xi \left[s^2 \left(u^+ - U^\varepsilon(\eta)\right)\right. \\ &\left. - \left(\sigma(u^+) - \sigma\left(U^\varepsilon(\eta)\right)\right)\right] d\eta + \frac{1}{s\mu} \int_0^\xi \left[s^2 \left(U^\varepsilon(\eta) - U(\eta)\right)\right. \\ &\left. - \left(\sigma\left(U^\varepsilon(\eta)\right) - \sigma\left(U(\eta)\right)\right)\right] d\eta =: I_1 + I_2. \end{aligned}$$

Now, since the quantity

$$\left[\frac{1}{s\mu} - \frac{1}{s(\mu - \varepsilon^2 s^2)}\right]$$

is an $O\left(\varepsilon^2\right)$, from the continuity of σ' and from the estimate

$$u^- < U^\varepsilon(\xi) < u^+,$$

we obtain

$$|I_1| \leq O\left(\varepsilon^2\right) \int_0^{+\infty} \left(u^+ - U^\varepsilon(\eta)\right) d\eta.$$

Thus, from the integral estimate (3.3.14) we get $I_1 = O\left(\varepsilon^2\right)$. Similarly, we obtain

$$|I_2| \leq O(1) \int_0^\xi \left|U^\varepsilon(\eta) - U(\eta)\right| d\eta.$$

Hence, we apply Gronwall Lemma to obtain the pointwise estimate

$$|U^\varepsilon(\xi) - U(\xi)| \leq C_1 \varepsilon^2 e^{C_2 \xi},$$

where C_1 is a positive constant depending on the limiting conditions, on the values of U^ε and U at $\xi = 0$ and on the function σ. By means of the relations (3.3.10) we can write the equations (3.3.18) and (3.3.19) in terms of u^-. Hence, by integrating on $(\xi, 0)$ for $\xi < 0$ and by performing the same calculations of the previous case, we obtain a similar estimate for negative ξ and the proof is complete. □

Remark 3.3.2 The condition

$$U^\varepsilon(0) = U(0), \tag{3.3.20}$$

is to be interpreted as follows. As it is known, in the case of hyperbolic–hyperbolic relaxation (namely in case of $\varepsilon^2\mu$ instead of μ in the third equation of system (3.3.1)), the relaxation profiles are of the form

$$U^\varepsilon(x - st) = \widetilde{U}\left(\frac{x - st}{\varepsilon^2}\right), \tag{3.3.21}$$

where $\widetilde{U}$ solves the same system with $\varepsilon = 1$. Hence, the mass condition

$$\int_{-\infty}^{0} \left[U^\varepsilon(\xi + k) - u^-\right] d\xi + \int_{0}^{+\infty} \left[u^+ - U^\varepsilon(\xi + k)\right] d\xi = 0,$$

which select a unique profile among those admissible (here U is the shock profile of the problem with $\varepsilon = 0$), is automatically satisfied for any $\varepsilon > 0$, once we require the same condition to be satisfied by $\widetilde{U}$ instead of U^ε. In the present case we cannot express the travelling wave profiles in the form (3.3.21) for a *fixed* $\widetilde{U}$. Hence, we must require (3.3.20) for any $\varepsilon > 0$.

Chapter 4

The Hamer model for radiating gases

This chapter describes the well–posedness theory and the relaxation limits for the radiating gas model presented in the section 1.5 of the introduction. In the following section we describe the global existence theory in $L^1 \cap L^\infty$. We also provide an alternative way of determining the well–posedness via the theory of nonlinear semigroups in the subsection 4.1.1. In section 4.2 we analyze both the hyperbolic to hyperbolic and the hyperbolic to parabolic relaxations for this model.

4.1 Global existence of solutions

In this section, we study the existence and uniqueness of weak, entropy solutions to the Cauchy problem for the hyperbolic-elliptic coupled system

$$\begin{cases} u_t + \operatorname{div} f(u) = -\operatorname{div} q \\ -\nabla \operatorname{div} q + q = -\nabla u \end{cases} \tag{4.1.1}$$

where $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$ are the independent variables, u and q are dependent variables with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively and f is a smooth mapping from $\mathbb{R}$ into $\mathbb{R}^d$, which we assume, without loss of generality, to satisfy $f(0) = f'(0) = 0$. We first suppose that the initial datum to the variable u is given by a function u_0 in the space $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The case of L^∞ data will be covered later on. We start by rewriting the system (4.1.1) as a scalar balance law of the form

$$u_t + \operatorname{div} f(u) = -u + K * u, \tag{4.1.2}$$

where the kernel K is given by the *Bessel potential*

$$K(x) = \frac{1}{(4\pi)^{d/2}} \int_0^{+\infty} \frac{e^{-s-\frac{|x|^2}{4s}}}{s^{d/2}} ds.$$

As it is well-known, a bounded measurable function u is a weak solution to (4.1.2) if it verifies the equation in distributional sense and the test functions are smooth functions with compact support, intersecting the line $\{t = 0\}$. A weak solution is said to be entropic if, in addition, it verifies the inequality

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} [\eta(u)\psi_t + q(u) \cdot \nabla\psi] dxdt + \int_{\mathbb{R}^d} \eta(u_0(x))\psi(x,0)dx \\ &\geq \int_0^T \int_{\mathbb{R}^d} \eta'(u)[u - K * u]\psi dxdt, \end{aligned} \tag{4.1.3}$$

for any convex entropy η with flux $q : \mathbb{R} \to \mathbb{R}^d$ given by

$$q_\alpha(u) = \int^u f'_\alpha(s)\eta'(s)ds \tag{4.1.4}$$

and for any nonnegative Lipschitz continuous test function ψ on $\mathbb{R}^d \times [0,T]$ with compact support, intersecting the line $\{t = 0\}$.

In order to show the existence of weak entropy solutions to the equation (4.1.2) with initial datum $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we apply the method of vanishing viscosity, namely, we study the behavior of the solutions to the parabolic approximation

$$u_t + \operatorname{div} f(u) = -u + K * u + \mu\Delta u \tag{4.1.5}$$

as $\mu \downarrow 0$. We state the following theorem of local existence and uniqueness of solutions to the equation (4.1.5) with initial datum in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The proof of this theorem is contained in the Appendix B.

Theorem 4.1.1 *Let the parameter $\mu > 0$ be fixed. Let $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then, there exist a positive time $T_0 > 0$ (depending on μ, u_0 and f) such that the equation (4.1.5) has an unique solution $u \in C([0,T_0]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ with u_0 as initial datum. Moreover, the spatial derivatives of any order of u are also in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.*

We start the analysis of the limit as $\mu \downarrow 0$ by proving the following theorem.

Theorem 4.1.2 *Let u and $\bar{u}$ be solutions to (4.1.5) with initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for any $t > 0$,*

$$\int_{\mathbb{R}^d} (u(x,t) - \bar{u}(x,t))^+ dx \leq \int_{\mathbb{R}^d} (u_0(x) - \bar{u}_0(x))^+ dx; \tag{4.1.6}$$

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R}^d)}. \tag{4.1.7}$$

Moreover, if $u_0(x) \leq \bar{u}_0(x)$ a.e. on $\mathbb{R}^d$, then $u(x,t) \leq \bar{u}(x,t)$ a.e. on $\mathbb{R}^d \times [0,T]$. In addition, the range of both u and $\bar{u}$ is contained in $[-a,a]$, where

$$a = \max\{\|u_0\|_{L^\infty(\mathbb{R}^d)}, \|\bar{u}_0\|_{L^\infty(\mathbb{R}^d)}\}.$$

Proof. Let $0 < T < T_0$, where T_0 is the time of existence of the solution given by the Theorem 4.1.1. We prove the relation (4.1.6) for any $t \in [0,T]$. For $\varepsilon > 0$, we define

$$\eta_\varepsilon(\xi) = \begin{cases} 0 & \text{if } -\infty < \xi \leq 0 \\ \frac{\xi^2}{4\varepsilon} & \text{if } 0 < \xi \leq 2\varepsilon \\ \xi - \varepsilon & \text{if } 2\varepsilon < \xi < +\infty. \end{cases}$$

We multiply the equation for $(u - \bar{u})$ by $\eta'_\varepsilon(u - \bar{u})$ and obtain

$$\begin{aligned} &\eta_\varepsilon(u-\bar{u})_t + \operatorname{div}[\eta'_\varepsilon(u-\bar{u})(f(u) - f(\bar{u}))] - \eta''_\varepsilon(u-\bar{u})(f(u)-f(\bar{u})) \cdot \nabla(u-\bar{u}) \\ &\quad = \mu\Delta\eta_\varepsilon(u-\bar{u}) - \mu\eta''_\varepsilon(u-\bar{u})\left|\nabla(u-\bar{u})\right|^2 - \eta'_\varepsilon(u-\bar{u})[u - \bar{u} - K*(u-\bar{u})]. \end{aligned} \tag{4.1.8}$$

Integrating (4.1.8) in x and t, and since $\eta''_\varepsilon > 0$, we have the following inequality

$$\begin{aligned} \int_{\mathbb{R}^d} \eta_\varepsilon(u(x,t) - \bar{u}(x,t))dx \leq &\int_{\mathbb{R}^d} \eta_\varepsilon(u_0(x) - \bar{u}_0(x))dx \\ &+ \int_0^t \int_{\mathbb{R}^d} \eta''_\varepsilon(u-\bar{u})(f(u)-f(\bar{u})) \cdot \nabla(u-\bar{u})dxds \\ &- \int_0^t \int_{\mathbb{R}^d} \eta'_\varepsilon[u - \bar{u} - K*(u-\bar{u})]dxds. \end{aligned} \tag{4.1.9}$$

Now, as $\varepsilon \downarrow 0$, we have, pointwise,

$$\begin{aligned} &\eta_\varepsilon(u(x,t) - \bar{u}(x,t)) \to (u(x,t) - \bar{u}(x,t))^+ \\ &\eta'_\varepsilon(u(x,t) - \bar{u}(x,t)) \to \operatorname{sgn}(u(x,t) - \bar{u}(x,t))^+ \\ &\eta''_\varepsilon(u(x,t) - \bar{u}(x,t))(f(u) - f(\bar{u})) \to 0. \end{aligned}$$

Now, since f is a smooth mapping (in particular, f is Lipschitz continuous), the quantity

$$\eta_\varepsilon''(u(x,t)-\bar{u}(x,t))|f(u)-f(\bar{u})|$$

is bounded uniformly in ε. Therefore, letting $\varepsilon \downarrow 0$ in (4.1.9), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}(u-\bar{u})^+dx &\leq \int_{\mathbb{R}^d}(u_0-\bar{u}_0)^+dx \\ &-\int_0^t\int_{\mathbb{R}^d}[(u-\bar{u})^+-\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]dxds. \qquad (4.1.10)\end{aligned}$$

We now estimate the convolution term in the above relation as follows

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})dxd\tau \\ &=\int_0^t\int_{\mathbb{R}^d}\operatorname{sgn}(u-\bar{u})^+\int_{\mathbb{R}^d}K(x-y)(u-\bar{u})(y,\tau)dydxd\tau \\ &\leq \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x-y)(u-\bar{u})^+(y,\tau)dydxd\tau \\ &=\int_0^t\int_{\mathbb{R}^d}(u-\bar{u})^+(y,s)dyd\tau,\end{aligned}$$

where we have used the property of the convolution kernel

$$\int_{\mathbb{R}^d}K(z)dz=1.$$

Thus, the integral inequality (4.1.10) reduces to

$$\int_{\mathbb{R}^d}(u-\bar{u})^+dx \leq \int_{\mathbb{R}^d}(u_0-\bar{u}_0)^+dx$$

which is exactly (4.1.6). Interchanging the roles of u and $\bar{u}$ we derive a similar inequality which added to (4.1.6) yields (4.1.7). The monotonicity property stated in the theorem is again a consequence of (4.1.6). To prove the last assertion of the theorem, namely the bound of the solutions in $L^\infty(\mathbb{R}^d)$, we perform an uniform control of the L^p norms of the solutions. For $\varepsilon > 0$, let $\theta_\varepsilon(\xi)$ be a convex regularization of the function $|\xi|^p$, $1 < p < \infty$. We multiply the equation (4.1.5) by $\theta_\varepsilon'(u)$ to obtain

$$\theta_\varepsilon(u)_t+\operatorname{div}(F(u))=-\theta_\varepsilon'(u)\,(u-K*u)+\mu\Delta\,(\theta_\varepsilon(u))-\mu\theta_\varepsilon''(u)\,|\nabla u|^2\,,$$

where

$$F_j(\xi)=\int^\xi\theta_\varepsilon'(s)f_j'(s)ds,$$

for $j = 1, ..., d$. Integrating in x and t and letting $\varepsilon \downarrow 0$ we obtain

$$\int_{\mathbb{R}^d} |u|^p dx \le \int_{\mathbb{R}^d} |u_0|^p dx \\ - \int_0^t \int_{\mathbb{R}^d} p|u|^p dx ds + \int_0^t \int_{\mathbb{R}^d} p|u|^{p-1} |K * u| dx ds. \tag{4.1.11}$$

We estimate the convolution integral as follows

$$\int_{\mathbb{R}^d} |u|^{p-1} |K * u| dx \le \|u\|_{L^p}^{p-1} \|K * u\|_{L^p} \le \|u\|_{L^p}^{p-1} \|u\|_{L^p} \|K\|_{L^1} = \|u\|_{L^p}^p,$$

where again we used the identity $\int_{\mathbb{R}^d} K(x)dx = 1$. Hence, (4.1.11) becomes

$$\|u\|_{L^p} \le \|u_0\|_{L^p} \le \|u_0\|_{L^\infty}^{\frac{p-1}{p}} \|u_0\|_{L^1}^{\frac{1}{p}}.$$

Thus we estimate the L^∞ norm of u as follows

$$\|u\|_{L^\infty} \le \limsup_{p \to +\infty} \|u\|_{L^p} \le \lim_{p \to +\infty} \|u_0\|_{L^\infty}^{\frac{p-1}{p}} \|u_0\|_{L^1}^{\frac{1}{p}} = \|u_0\|_{L^\infty}$$

and the last assertion of the theorem follows. Finally, both this last L^∞ estimate and the L^1 estimate coming from (4.1.7) imply in particular that the local-in-time solution provided by the Theorem 4.1.1 is indeed global and all the estimate we have proved are satisfied for any $t > 0$. $\square$

In the following lemma we provide the estimate (uniformly with respect to μ) of the L^1-modulus of continuity of the solution u^μ to (4.1.5) with $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Lemma 4.1.3 *Let u^μ be the solution to (4.1.5) with $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ as initial datum. In particular*

$$\int_{\mathbb{R}^d} |u_0(x+h) - u_0(x)| dx \le \omega(|h|), \qquad \textit{for any } h \in \mathbb{R}^d, \tag{4.1.12}$$

for some nondecreasing function ω on $[0, +\infty)$ with $\omega(r) \downarrow 0$ as $r \downarrow 0$. Then there exists a constant C, depending on $\|u_0\|_{L^\infty(\mathbb{R}^d)}$ and on the mapping f such that, for any $t > 0$,

$$\int_{\mathbb{R}^d} |u^\mu(x+h,t) - u^\mu(x,t)| dx \le \omega(|h|), \qquad \textit{for any } h \in \mathbb{R}^d \tag{4.1.13}$$

and

$$\int_{\mathbb{R}^d} |u^\mu(x,t+k) - u^\mu(x,t)| dx \le C(k + k^{\frac{2}{3}} + \mu k^{\frac{1}{3}}) \|u_0\|_{L^1(\mathbb{R}^d)} + 4\omega(k^{\frac{1}{3}}), \tag{4.1.14}$$

for any $k > 0$.

Proof. Fix $t > 0$. Applying (4.1.7) with $\bar{u}(x,t) = u(x+h,t)$, we obtain (4.1.13).

Now, let $k > 0$ be fixed and let ϕ be a smooth, bounded function on $\mathbb{R}^d$. Then, multiplying the equation (4.1.5) by ϕ and integrating by parts the resulting equation on the domain $\mathbb{R}^d \times (t, t+k)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^d} \phi(x)[u(x,t+k) - u(x)]dx \\ &= \int_t^{t+k} \int_{\mathbb{R}^d} [\nabla\phi(x) \cdot f(u(x,\tau)) + \mu\Delta\phi(x)u(x,\tau) \\ &+ \phi(x)(-u(x,\tau) + (K * u)(x,\tau))]dxd\tau. \end{aligned} \tag{4.1.15}$$

We now set ϕ to be a suitable regularization of the function $\operatorname{sgn}[u(x,t+k) - u(x,t)]$, in order to get the desired estimate (4.1.14). More precisely, we choose

$$\phi(x) = \int_{\mathbb{R}^d} k^{-\frac{d}{3}} \prod_{j=1}^{d} \rho\left(\frac{x_j - \xi_j}{k^{\frac{1}{3}}}\right) \operatorname{sgn}(u(\xi,t+k) - u(\xi,t))d\xi,$$

where ρ is a smooth, nonnegative function on $\mathbb{R}$ with support contained in $[-1,1]$ and total mass one. Since $|\phi| \le 1$, $|\nabla\phi| \le c_1 k^{-\frac{1}{3}}$ and $|\Delta\phi| \le c_2 k^{-\frac{2}{3}}$, from (4.1.15) we obtain

$$\int_{\mathbb{R}^d} \phi(x)(u(x,t+k) - u(x,t))dx \le C(k + k^{\frac{2}{3}} + \mu k^{\frac{1}{3}})\|u_0\|_{L^1(\mathbb{R}^d)}, \tag{4.1.16}$$

where the constant C depends on $\|u_0\|_{L^\infty}$ and on the Lipschitz constant of the mapping f. Then, we observe that

$$\begin{aligned}&|u(x,t+k) - u(x,t)| - \phi(x)(u(x,t+k) - u(x,t)) \\ &= \int_{\mathbb{R}^d} k^{-\frac{d}{3}} \prod_{j=1}^{d} \rho\left(\frac{x_j - \xi_j}{k^{\frac{1}{3}}}\right) [|u(x,t+k) - u(x,t)| \\ &- (u(x,t+k) - u(x,t)) \operatorname{sgn}(u(\xi,t+k) - u(\xi,t))]d\xi \\ &\le 2\int_{\mathbb{R}^d} k^{-\frac{d}{3}} \prod_{j=1}^{d} \rho\left(\frac{x_j - \xi_j}{k^{\frac{1}{3}}}\right) |u(x,t+k) - u(x,t) - (u(\xi,t+k) - u(\xi,t))|d\xi.\end{aligned}$$

Therefore, after integration on $\mathbb{R}^d$ and from (4.1.13), we obtain the inequality

$$\int_{\mathbb{R}^d} [|u(x,t+k) - u(x,t)| - \phi(x)(u(x,t+k) - u(x,t)]\,dx \le 4\omega(k^{\frac{1}{3}})$$

which, combined with (4.1.16), implies (4.1.14). $\square$

By virtue of Lemma 4.1.3, the family $\{u^\mu\}$ is compact in $L^1_{loc}(\mathbb{R}^d\times[0,\infty))$, and then, passing if necessary to a subsequence, it converges strongly (and boundedly almost everywhere in $\mathbb{R}^d\times[0,\infty)$, from Theorem 4.1.2) to a function $u\in L^1(\mathbb{R}^d\times[0,T])\cap L^\infty(\mathbb{R}^d\times[0,T])$. Moreover, due to the strong convergence of the sequence and due to its boundedness in L^∞, it is easy to verify that the limit function u is an entropy solution to (4.1.2) with u_0 as initial datum. Hence, the following theorem holds.

Theorem 4.1.4 *Let u^μ be the solution to (4.1.5) with $u_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ as initial datum. Then, as $\mu\downarrow 0$ (passing if necessary to a subsequence), for any $T>0$,*

$$u^\mu \longrightarrow u, \qquad\qquad \textit{strongly in } L^p_{loc}(\mathbb{R}^d\times[0,T]),\ p<+\infty,$$

and $u\in L^1(\mathbb{R}^d\times[0,T])\cap L^\infty(\mathbb{R}^d\times[0,T])$ is an entropy solution to (4.1.2) with u_0 as initial datum.

We now pass to the study of the uniqueness of the weak, entropy solutions to (4.1.2) with initial datum $u_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. As in the homogeneous case (see [Kru70], [Daf00]), we have the following theorem.

Theorem 4.1.5 *Let $u,\bar u\in L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$ be weak entropy solutions to (4.1.2) with initial data $u_0,\bar u_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then, for any $t\in[0,T]$,*

$$\int_{\mathbb{R}^d}(u(x,t)-\bar u(x,t))^+dx \le \int_{\mathbb{R}^d}(u_0(x)-\bar u_0(x))^+dx; \tag{4.1.17}$$

$$\|u(\cdot,t)-\bar u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \|u_0(\cdot)-\bar u_0(\cdot)\|_{L^1(\mathbb{R}^d)}. \tag{4.1.18}$$

Moreover, if $u_0(x)\le\bar u_0(x)$ a.e. on $\mathbb{R}^d$, then $u(x,t)\le\bar u(x,t)$ a.e. on $\mathbb{R}^d\times[0,T]$. In addition, the essential range of both u and $\bar u$ is contained in $[-a,a]$, where

$$a=\max\{\|u_0\|_{L^\infty(\mathbb{R}^d)},\|\bar u_0\|_{L^\infty(\mathbb{R}^d)}\}.$$

Proof. We proceed by using the following relation, as in the homogeneous case,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}[\psi_t(u-\bar u)^+ + \mathrm{sgn}(u-\bar u)^+\nabla\psi\cdot(f(u)-f(\bar u))]dxdt\\
&+\int_{\mathbb{R}^d}\psi(x,0)(u_0-\bar u_0)^+dx\\
&\ge\int_0^T\int_{\mathbb{R}^d}\psi[(u-\bar u)^+ - \mathrm{sgn}(u-\bar u)^+K*(u-\bar u)]dxdt,
\end{aligned}\tag{4.1.19}$$

for any nonnegative Lipschitz continuous test function ψ on $\mathbb{R}^d \times [0,T]$ with compact support, intersecting the line $\{t = 0\}$. The inequality (4.1.19) is obtained by considering the entropy $(u(x,t)-\bar{u}(y,s))^+$ either in the (x,t) and in the (y,s) variables, and by choosing a suitable test function, converging to δ functions centered at $\{y = x\}$ and at $\{t = s\}$ (see [Kru70] and [Daf00]). Now, we employ (4.1.19) as follows. Let $R > 0$, $t \in [0,T)$ and $\varepsilon > 0$ be fixed, and let us write (4.1.19) with the test function given by $\psi(x,\tau) = \chi(x,\tau)\theta(\tau)$, where

$$\chi(x,\tau) = \begin{cases} 1 & \text{if } 0 \le \tau < T,\ 0 \le |x| < R + s(t-\tau) \\ \frac{1}{\varepsilon}[R + s(t-\tau) - |x|] + 1 & \text{if } 0 \le \tau < T, \\ & R + s(t-\tau) \le |x| < R + s(t-\tau) + \varepsilon \\ 0 & \text{if } 0 \le \tau < T,\ |x| \ge R + s(t-\tau) + \varepsilon, \end{cases}$$

$$\theta(\tau) = \begin{cases} 1 & \text{if } 0 \le \tau < t \\ \frac{1}{\varepsilon}(t-\tau) + 1 & \text{if } t \le \tau < t + \varepsilon, \\ 0 & \text{if } t + \varepsilon \le \tau < T, \end{cases}$$

and

$$s = \max\left\{\frac{|f(u) - f(\bar{u})|}{|u - \bar{u}|}\right\}, \tag{4.1.20}$$

for u and $\bar{u}$ in the range of solutions. Computing explicitly the derivatives of ψ (see [Daf00]), we get

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<R+s(t-\tau)} (u-\bar{u})^+ dx d\tau \le \int_{|x|<R+st} (u_0 - \bar{u}_0)^+ dx \\
&-\frac{1}{\varepsilon}\int_0^T\int_{R+s(t-\tau)\le|x|<R+s(t-\tau)+\varepsilon}\left[s(u-\bar{u})^+ + \operatorname{sgn}(u-\bar{u})^+\frac{x}{|x|}\cdot(f(u)-f(\bar{u}))\right] dx d\tau \\
&-\int_0^T\int_{\mathbb{R}^d} \chi(x,\tau)\theta(\tau)[(u-\bar{u})^+ - \operatorname{sgn}(u-\bar{u})^+ K * (u-\bar{u})] dx d\tau + O(\varepsilon) \\
&\le \int_{|x|<R+st} (u_0 - \bar{u}_0)^+ dx - I(u-\bar{u}) + O(\varepsilon).
\end{aligned} \tag{4.1.21}$$

The last inequality is due to the special choice of the constant s, which implies

$$\left[s(u-\bar{u})^+ + \frac{x}{|x|}\cdot(f(u) - f(\bar{u}))\operatorname{sgn}(u-\bar{u})^+\right] > 0.$$

Moreover, we denoted the contribution due to the source term by $I(u - \bar{u})$. The special form of the function ψ yields to

$$I(u-\bar{u}) = \int_0^t\int_{|x|<R+s(t-\tau)} [(u-\bar{u})^+ - \operatorname{sgn}(u-\bar{u})^+ K * (u-\bar{u})] dx d\tau + O(\varepsilon).$$

Hence, letting $\varepsilon \downarrow 0$, we have

$$\int_{|x|<R}(u-\bar{u})^+dx \le \int_{|x|<R+st}(u_0-\bar{u}_0)^+dx - \int_0^t\int_{|x|<R+s(t-\tau)}(u-\bar{u})^+dxd\tau$$
$$+\int_0^t\int_{|x|<R+s(t-\tau)}\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]dxd\tau. \quad (4.1.22)$$

From the above inequality, with $R\to\infty$, we obtain

$$\int_{\mathbb{R}^d}(u-\bar{u})^+dx \le \int_{\mathbb{R}^d}(u_0-\bar{u}_0)^+dx - \int_0^t\int_{\mathbb{R}^d}(u-\bar{u})^+dxd\tau$$
$$+\int_0^t\int_{\mathbb{R}^d}\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]dxd\tau. \quad (4.1.23)$$

We estimate the convolution term as follows

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d}\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]dxd\tau\\
&=\int_0^t\int_{\mathbb{R}^d}\operatorname{sgn}(u-\bar{u})^+\int_{\mathbb{R}^d}K(x-y)(u-\bar{u})(y,\tau)dydxd\tau\\
&\le\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x-y)(u-\bar{u})^+(y,\tau)dydxd\tau\\
&=\int_0^t\int_{\mathbb{R}^d}(u-\bar{u})^+(y,\tau)dyd\tau.
\end{aligned}$$

Hence, (4.1.17) is proved. As usual, interchanging the roles of u and $\bar{u}$ we obtain the L^1-contraction property (4.1.18), while the monotonicity property stated in the theorem is again a direct consequence of (4.1.17). Now, as in the case of the viscous approximation, we perform the L^∞ estimate by means of a uniform bound for the L^p norms. As in the Theorem 4.1.2, we obtain

$$\int_{\mathbb{R}^d}|u|^pdx \le \int_{\mathbb{R}^d}|u_0|^pdx - \int_0^t\int_{\mathbb{R}^d}p|u|^pdxds + \int_0^t\int_{\mathbb{R}^d}p|u|^{p-1}|K*u|dxds,$$

which yields to the desired L^∞ estimate, and the proof is complete. □

As a consequence of the last theorem, we have the following

Corollary 4.1.6 *There exists at most one entropy solution of (4.1.2), belonging in the space $L^\infty([0,T];L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$, having initial datum in $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$.*

Remark 4.1.7 As a consequence of the uniqueness result we have established, we can conclude that any weak, entropy solution to (4.1.2) belonging in the space $L^\infty([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ verifies the estimate

$$\int_{\mathbb{R}^d} |u(x,t+k) - u(x,t)|dx \le C(k + k^{\frac{2}{3}})\|u_0\|_{L^1(\mathbb{R}^d)} + 4\omega(k^{\frac{1}{3}})$$

(where ω represents the L^1–modulus of continuity of the initial datum), which can be easily obtained by sending $\mu \to 0$ in (4.1.14). Thus, $u \in C([0,T]; L^1(\mathbb{R}^d))$ for any $T > 0$.

4.1.1 Well–posedness via nonlinear semigroups

In view of the previous results, the solutions to (4.1.2) may be viewed as the trajectories of a contraction semigroup $\{S(t),\ t > 0\}$ defined on the space $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ endowed with the L^1–norm. Our next aim is to show that the existence and uniqueness of weak, entropy solutions to the balance law (4.1.2), together with its contraction properties, can be obtained through the theory of nonlinear contraction semigroups in Banach spaces ([Cra72, CL71]).

To construct the semigroup, we should realize (4.1.2) as an abstract differential equation

$$\frac{du}{dt} + A(u) \ni 0, \tag{4.1.24}$$

for a suitably defined transformation A, with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ both contained in $L^1(\mathbb{R}^d)$. This operator may, in general, be multivalued, even though, for u smooth one should expect $A(u) = \operatorname{div} f(u) + u - K * u$. However, while performing the extension of A to those function that are not smooth, we must consider the admissibility condition encoded in the relation (4.1.3). This leads us to the following abstract definition of the transformation A. We first define the simpler extension $\widehat{A}$.

Definition 4.1.8 *The (possibly multivalued) transformation $(\widehat{A}, \mathcal{D}(\widehat{A}))$, with domain $\mathcal{D}(\widehat{A})$ contained in $L^1(\mathbb{R}^d)$, is determined by $u \in \mathcal{D}(\widehat{A})$ and $w \in \mathcal{R}(\widehat{A})$ if u, w and $f(u)$ are all in $L^1(\mathbb{R}^d)$ and if the inequality*

$$\begin{aligned}\int_{\mathbb{R}^d} &\{\nabla\psi(x) \cdot q(u(x)) + \eta'(u(x))\,[-u(x) + (K * u)(x)]\,\psi(x) \\ &+ \psi(x)\eta'(u(x))w(x)\}\,dx \ge 0,\end{aligned} \tag{4.1.25}$$

holds for any convex entropy function η, such that η' is bounded on $\mathbb{R}$, with associated entropy flux $q : \mathbb{R} \to \mathbb{R}^d$ determined as in (4.1.4) and for all non-negative Lipschitz continuous test functions ψ on $\mathbb{R}^d$ with compact support.

Applying (4.1.25) for the pairs $(\pm u, \pm f(u))$, we recover that

$$\widehat{A}(u) = \operatorname{div} f(u) + u - K * u$$

in the sense of distributions for any $u \in \mathcal{D}(\widehat{A})$. In particular, A is a single–valued transformation. Moreover, the relation (4.1.4) implies, for any $u \in C_0^1(\mathbb{R}^d)$ and for any nonnegative test function ψ,

$$\int_{\mathbb{R}^d} \{\nabla\psi \cdot q(u) + \psi\eta'(u) \operatorname{div} f(u)\}\, dx = 0,$$

which yields

$$\int_{\mathbb{R}^d} \left\{\nabla\psi \cdot q(u) + \psi\eta'(u)[\widehat{A}(u) - u + K * u]\right\} dx = 0.$$

Hence, $C_0^1(\mathbb{R}^d) \in \mathcal{D}(\widehat{A})$ and, in particular, $\mathcal{D}(\widehat{A})$ is dense in $L^1(\mathbb{R}^d)$. Thus, we finally define the transformation A as follows

Definition 4.1.9 *The (possibly multivalued) transformation $(A, \mathcal{D}(A))$, with domain $\mathcal{D}(A)$ contained in $L^1(\mathbb{R}^d)$, is the graph closure of $\widehat{A}$, i.e., $u \in \mathcal{D}(A)$ and $w \in A(u)$ iff (u, w) is the limit in $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ of a sequence $\{(u_k, w_k)\}$ such that $u_k \in \mathcal{D}(\widehat{A})$ and $w_k \in \widehat{A}(u_k)$.*

We now establish the properties of A which guarantee that it is the generator of a contraction semigroup.

Theorem 4.1.10 *The transformation A is accretive, that is, if u and $\bar{u}$ are in $\mathcal{D}(A)$, then, for any $\lambda > 0$, $w \in A(u)$, $\bar{w} \in A(\bar{u})$, the following inequality holds*

$$\|(u + \lambda w) - (\bar{u} + \lambda\bar{w})\|_{L^1(\mathbb{R}^d)} \geq \|u - \bar{u}\|_{L^1(\mathbb{R}^d)}. \tag{4.1.26}$$

Proof. It is sufficient to prove (4.1.26) for $u, \bar{u} \in \mathcal{D}(\widehat{A})$. Thus, let $w = A(u)$, $\bar{w} = A(\bar{w})$. We proceed by considering the following relation, as in the homogeneous case (see [Daf00]),

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \operatorname{sgn}(u(x) - \bar{u}(\bar{x}))\, \{(\nabla_x + \nabla_{\bar{x}})\, \phi(x, \bar{x}) \cdot (f(u(x)) - f(\bar{u}(\bar{x}))) \\ &\quad -2\,[u(x) - \bar{u}(\bar{x}) - (K * u)(x) + (K * \bar{u})(\bar{x})]\, \phi(x, \bar{x}) \\ &\quad +\phi(x, \bar{x})[w(x) - \bar{w}(\bar{x})]\}\, dx d\bar{x} \geq 0, \end{aligned} \tag{4.1.27}$$

which holds for any test function $\phi \geq 0$ on $\mathbb{R}^d \times \mathbb{R}^d$. The inequality (4.1.27) is obtained by writing the relation (4.1.25) for the entropy–entropy flux pairs

$$\begin{aligned}
&\eta(u; \bar{u}(\bar{x})) = |u - \bar{u}(\bar{x})|, \\
&q(u; \bar{u}(\bar{x})) = \operatorname{sgn}(u - \bar{u}(\bar{x}))(f(u) - f(\bar{u}(\bar{x})))
\end{aligned}$$

and by interchanging the roles of u and $\bar{u}$. We now choose the test function ϕ as follows, for fixed $\varepsilon > 0$,

$$\phi(x, \bar{x}) = \varepsilon^{-d} \psi\left(\frac{x + \bar{x}}{2}\right) \prod_{\alpha=1}^{d} \rho\left(\frac{x_\alpha - \bar{x}_\alpha}{2\varepsilon}\right),$$

where ρ is a standard mollifier as in the proof of Lemma 4.1.3 and ψ is a smooth function on $\mathbb{R}^d$ such that $\psi(x) = 1$ for $|x| < R$, and $\psi(x) = 0$ for $|x| > R + 1$. Since

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [u(x) - \bar{u}(\bar{x}) - (K * u)(x) + (K * \bar{u})(\bar{x})] \, dx d\bar{x} \geq 0,$$

by explicitly computing the derivatives of ϕ, by letting $\varepsilon \downarrow 0$ and then by sending $R \to \infty$, we obtain

$$\int_{\mathbb{R}^d} \operatorname{sgn}(u(x) - \bar{u}(x))[w(x) - \bar{w}(x)] dx \geq 0.$$

Now, let $\lambda > 0$. From the last inequality we recover

$$\begin{aligned}
&\|(u + \lambda w) - (\bar{u} + \lambda \bar{w})\|_{L^1(\mathbb{R}^d)} \\
&\quad \geq \int_{\mathbb{R}^d} \operatorname{sgn}(u(x) - \bar{u}(x))\{u(x) - \bar{u}(x) + \lambda[w(x) - \bar{w}(x)]\} dx \\
&\quad \geq \int_{Rd} \operatorname{sgn}(u(x) - \bar{u}(x))[u(x) - \bar{u}(x)] dx = \|u - \bar{u}\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

The proof is complete. □

Theorem 4.1.11 *The transformation A is maximal, that is*

$$\mathcal{R}(\mathbb{I} + \lambda A) = L^1(\mathbb{R}^d), \quad \textit{for any} \ \ \lambda > 0.$$

Proof. It will suffice to show that $\mathcal{R}(\mathbb{I} + \lambda A)$ is dense in $L^1(\mathbb{R}^d)$; indeed, we will prove $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset \mathcal{R}(\mathbb{I} + \lambda A)$. Let $g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We seek solutions $u \in \mathcal{D}(\widehat{A})$ of the equation

$$u + \lambda \widehat{A}(u) = g \tag{4.1.28}$$

We construct solutions to this equation as the $\mu \downarrow 0$ limit of the elliptic equations

$$u(x) + \lambda \{\operatorname{div} f(u(x)) + u(x) - (K * u)(x)\} - \mu \Delta u(x) = g(x), \quad x \in \mathbb{R}^d. \tag{4.1.29}$$

From the standard theory of elliptic equations, (4.1.29) admits a solution in $H^2(\mathbb{R}^d)$. The convergence of u^μ to some function u which is the solution to (4.1.28) comes from the following two lemmas.

Lemma 4.1.12 *Let u, $\bar{u}$ be solutions to (4.1.29) with respective source terms g and $\bar{g}$ that are in $L^1(\mathbb{R}^d)$ with range contained in the interval $[a,b]$. Then*

$$\int_{\mathbb{R}^d} [u(x) - \bar{u}(x)]^+ dx \leq \int_{\mathbb{R}^d} [g(x) - \bar{g}(x)]^+ dx,$$

$$\|u - \bar{u}\|_{L^1(\mathbb{R}^d)} \leq \|g - \bar{g}\|_{L^1(\mathbb{R}^d)}. \tag{4.1.30}$$

Moreover, if

$$g(x) \leq \bar{g}(x), \quad on \ \ \mathbb{R}^d$$

then

$$u(x) \leq \bar{u}(x), \quad on \ \ \mathbb{R}^d.$$

Moreover, the range of both u and $\bar{u}$ is contained in $[a,b]$.

We skip the detail of the proof of this lemma, which are very similar to those in the proof of Theorem 4.1.2. Once again, the contraction properties of the source term $-u + K * u$ are employed. Hence, in a similar way as in the case of the vanishing viscosity approximation, we obtain the compactness of the sequence $\{u^\mu\}$ needed to pass the limit as $\mu \downarrow 0$ in the equation (4.1.29).

Lemma 4.1.13 *Let u^μ denote the solution of (4.1.29), with source term given by $g \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then, as $\mu \downarrow 0$, $\{u^\mu\}$ converges boundedly a.e. to the solution u of (4.1.28).*

Proof. By applying (4.1.30) with $\bar{g}(x) = g(x+y)$, we obtain

$$\int_{\mathbb{R}^d} |u^\mu(x+y) - u^\mu(x)| dx \leq \int_{\mathbb{R}^d} |g(x+y) - g(x)| dx.$$

Thus the family $\{u^\mu\}$ is compact in L^1_{loc}. Hence, passing if necessary to subsequences, $u^\mu \to u$ boundedly a.e. on $\mathbb{R}^d$. To complete the proof, we have to show that u is the unique solution to (4.1.28). Consider any smooth convex entropy $\eta(u)$ with associated entropy flux q determined by (4.1.4). Then u^μ satisfies the identity

$$\eta'(u^\mu)u^\mu + \lambda \operatorname{div} q(u^\mu) + \lambda\eta'(u^\mu)(u - (K * u)) - \mu\Delta\eta(u^\mu) \\ + \mu\eta''(u^\mu)\,|\nabla u^\mu|^2 = \eta'(u^\mu)g.$$

Hence, multiplying the last identity by any nonnegative smooth test function ψ on $\mathbb{R}^d$, with compact support, and integrating over $\mathbb{R}^d$ yields

$$\int_{\mathbb{R}^d} \{\lambda\nabla\psi \cdot q(u^\mu) + \psi\eta'(u^\mu)(g - u^\mu - \lambda(u^\mu - (K * u^\mu)))\}\, dx \\ \geq -\mu \int_{\mathbb{R}^d} \Delta\psi\eta dx.$$

Finally, by sending $\mu \downarrow 0$ we get

$$\int_{\mathbb{R}^d} \left\{\nabla\psi \cdot q(u) + \psi\eta'(u)\left[\frac{1}{\lambda}(g - u) - (u - (K * u))\right]\right\} \geq 0,$$

which means that u is a solution to (4.1.28). Now, as a consequence of the previous theorem, the solution u to (4.1.28) is unique. Therefore, the entire sequence u^μ converges to u as $\mu \downarrow 0$. This completes the proof of the lemma and the proof of Theorem 4.1.11. □

Finally, once we have established accretiveness and maximality for the transformation A, we can employ the Crandall–Liggett theory of semigroups in nonreflexive Banach spaces to obtain that A generates a contraction semigroup $S(\cdot)$ on $\overline{\mathcal{D}(A)} = L^1(\mathbb{R}^d)$. We summarize the properties of the $S(\cdot)$ in the following theorem, whose proof can be found in [Cra72, CL71].

Theorem 4.1.14 *The transformation A generates a contraction semigroup*

$$S(\cdot) : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d),\ t \in [0, +\infty)$$

which is continuous with respect to t. Moreover, for any $u_0, \bar{u}_0 \in L^1(\mathbb{R}^d)$ such that

$$u_0 \leq \bar{u}_0 \quad a.e.\ on\ \ \mathbb{R}^d,$$

we have

$$S(t)u_0 \leq S(t)\bar{u}_0, \qquad a.e.\ on\ \ \mathbb{R}^d.$$

Moreover, for $1 \leq p \leq \infty$, the sets $L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ are positively invariant under $S(t)$ and, for any $t \geq 0$,

$$\|S(t)u_0\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)}, \quad for\ all\ \ u_0 \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d).$$

If $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then $S(\cdot)u_0$ is the admissible weak, entropy solution of (4.1.2) with initial datum u_0.

4.1.2 Extension to L^∞

We now conclude this section by discussing the general case of initial data in $L^\infty(\mathbb{R}^d)$. More precisely, we extend the semigroup $S(t)$ to the space $L^\infty(\mathbb{R}^d)$ endowed with the following norm

$$|||u||| = \int_{\mathbb{R}^d} \phi(x)|u(x)|dx, \tag{4.1.31}$$

where ϕ is a positive, smooth function on $\mathbb{R}^d$ satisfying

$$\begin{aligned} \phi(x) &= e^{-|x|} && \text{for any } |x| \geq 2, \\ \phi(x) &= 1 && \text{for any } |x| \leq 1. \end{aligned}$$

We remark that the uniqueness of solutions with initial data in L^∞ has been proved only in the one–dimensional case (see [Ser03]). This problem is still open in the multi–dimensional case. The extension of the semigroup to L^∞ leaves entirely open the question of uniqueness in L^∞. We show the continuity of the semigroup S_t on the space $(L^\infty \cap L^1, |||\cdot|||)$, which is a dense subspace of $\big(L^\infty(\mathbb{R}^d), |||\cdot|||\big)$. We also observe that the following procedure extends the semigroup S_t to the Banach space $L^1(\mathbb{R}^d, \phi dx)$ which is the closure of our space $\big(L^\infty(\mathbb{R}^d), |||\cdot|||\big)$ with respect to the norm $|||\cdot|||$.

Theorem 4.1.15 *Let u, $\bar{u} \in L^\infty([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ be weak entropy solutions of (4.1.2) with initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for any $t \in [0,T]$,*

$$|||u(\cdot,t) - \bar{u}(\cdot,t)||| \leq e^{Ct}|||u_0 - \bar{u}_0|||, \tag{4.1.32}$$

where the constant C depends on ϕ, on the mapping f and on $\|u_0\|_{L^\infty}$, $\|\bar{u}_0\|_{L^\infty}$.

Proof. Due to the uniqueness result, we can view the solutions $u, \bar{u}$ as the limit of their viscous approximations $u^\mu, \bar{u}^\mu$ respectively. We then multiply the equation for $u^\mu - \bar{u}^\mu$ by the weight function ϕ to obtain

$$\begin{aligned} &[\phi(u^\mu - \bar{u}^\mu)]_t + \operatorname{div}[\phi(f(u^\mu) - f(\bar{u}^\mu))] \\ &= -\phi\,[u^\mu - \bar{u}^\mu - K * (u^\mu - \bar{u}^\mu)] \\ &+ \nabla\phi \cdot (f(u^\mu) - f(\bar{u}^\mu)) + \mu\phi\Delta(u^\mu - \bar{u}^\mu). \end{aligned} \tag{4.1.33}$$

Let $\eta_\varepsilon(\xi)$ be a convex regularization of the function $|\xi|$ (in the spirit of the function used in the proof of Theorem 4.1.2. Multiplying (4.1.33) by $\eta'_\varepsilon(u^\mu -$

$\bar{u}^{\mu}$) we obtain

$$\begin{aligned}
&\phi\eta_{\varepsilon,t} + \operatorname{div}\left[\phi\eta_\varepsilon'(f(u^\mu) - f(\bar{u}^\mu))\right] \\
&\quad = -\phi\eta_\varepsilon'\left[u^\mu - \bar{u}^\mu - K * (u^\mu - \bar{u}^\mu)\right] \\
&\quad + \phi\eta_\varepsilon''(f(u^\mu) - f(\bar{u}^\mu)) \cdot \nabla(u^\mu - \bar{u}^\mu) \\
&\quad + \eta_\varepsilon'\nabla\phi \cdot (f(u^\mu) - f(\bar{u}^\mu)) \\
&\quad + \mu \operatorname{div}\left[\phi\eta_\varepsilon'\nabla(u^\mu - \bar{u}^\mu)\right] - \mu\eta_\varepsilon'\nabla\phi \cdot \nabla(u^\mu - \bar{u}^\mu) \\
&\quad - \mu\eta_\varepsilon''\phi\left|\nabla(u^\mu - \bar{u}^\mu)\right|^2 . \qquad (4.1.34)
\end{aligned}$$

Thus, proceeding as in the proof of Theorem 4.1.2 we integrate (4.1.34) in dx and dt and we let $\varepsilon \downarrow 0$ to obtain (after integration by parts)

$$\begin{aligned}
\int_{\mathbb{R}^d} \phi(x)|u^\mu - \bar{u}^\mu|dx &\leq \int_{\mathbb{R}^d} \phi(x)|u_0 - \bar{u}_0|dx \\
&+ C\int_0^t \int_{\mathbb{R}^d} \left(|\nabla\phi| + |\Delta\phi|\right)|u^\mu - \bar{u}^\mu|dsdx, \qquad (4.1.35)
\end{aligned}$$

where the constant C depends on $\|u\|_\infty, \|u_0\|_\infty$ and on the mapping f, for $\mu < 1$. Since $(|\nabla\phi| + |\Delta\phi|) \leq C_1\phi$, from (4.1.35) and from the Gronwall lemma, we obtain (4.1.32) as $\mu \downarrow 0$. □

4.2 Relaxation Limits

In this section we analyze the convergence of the relaxation limits for our model

$$\begin{cases} U_s + \operatorname{div} f(U) = -\operatorname{div} Q \\ -\nabla \operatorname{div} Q + Q = -\nabla U, \end{cases} \qquad (4.2.1)$$

where $(y, s) \in \mathbb{R}^d \times \mathbb{R}_+$ are the independent variables.

4.2.1 Hyperbolic-hyperbolic relaxation limit

We start the analysis with the hyperbolic-hyperbolic relaxation. Therefore, in order to obtain an hyperbolic-type limit, we perform the following scaling

$$\begin{aligned}
u^\varepsilon(x,t) &= U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \\
q^\varepsilon(x,t) &= Q\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right).
\end{aligned}$$

Thus, the system (4.2.1) becomes

$$\begin{cases} u_t^\varepsilon + \operatorname{div} f(u^\varepsilon) = -\operatorname{div} q^\varepsilon \\ -\varepsilon^2 \nabla \operatorname{div} q^\varepsilon + q^\varepsilon = -\varepsilon \nabla u^\varepsilon. \end{cases} \tag{4.2.2}$$

We give an initial datum $u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We postpone the discussion of L^∞ initial data at the end of the subsection.

Remark 4.2.1 In the construction of the Cauchy problem for (4.2.2), we scale only the terms for $t > 0$, without scaling the initial datum, which is given *a posteriori* as a fixed function in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Indeed, the scaled initial datum is given by the sequence $u_0^\varepsilon(x) = U_0\left(\frac{x}{\varepsilon}\right)$, which converges to zero strongly in $L^1(\mathbb{R}^d)$ because $U_0 \in L^1(\mathbb{R}^d)$. Thus, in this way we can recover at the limit only the null solution. In other words, we do not investigate the relaxation limit of the scaled solution, but we use the scaling only to detect the terms which are physically negligible in the equations and we study the singular limit of the new Cauchy problem, with fixed datum.

Letting $\varepsilon \downarrow 0$ in the system (4.2.2), we see that formally we obtain $q = 0$ and the limit equation is given by

$$u_t + \operatorname{div} f(u) = 0. \tag{4.2.3}$$

To justify rigorously this limit, it is convenient once again to write the system (4.2.2) as the balance law

$$u_t^\varepsilon + \operatorname{div} f(u^\varepsilon) = -\frac{1}{\varepsilon}\left(u^\varepsilon - K^\varepsilon * u^\varepsilon\right), \tag{4.2.4}$$

where the convolution kernel is given by

$$K^\varepsilon(x) = \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}\right),$$

and K is again

$$K(x) = \frac{1}{(4\pi)^{d/2}} \int_0^{+\infty} \frac{e^{-s-\frac{|x|^2}{4s}}}{s^{d/2}} ds.$$

We observe that the new convolution kernel K^ε is scaled such that $\|K^\varepsilon\|_{L^1} = \|K\|_{L^1} = 1$. Thus, the contraction properties of the source term we employed in the previous section are still valid. Therefore, the same properties established in the Theorem 4.1.2 for the viscous approximation are satisfied by the family $\{u^\varepsilon\}$, as we state in the following result. We skip the details of the proof, which follows the same ideas of Theorem 4.1.2.

Theorem 4.2.2 *Let $\varepsilon > 0$. Let u^ε, $\bar{u}^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ be weak entropy solutions of (4.2.4) with initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for any $t \in [0,T]$,*

$$\int_{\mathbb{R}^d} (u^\varepsilon(x,t) - \bar{u}^\varepsilon(x,t))^+ dx \leq \int_{\mathbb{R}^d} (u_0(x) - \bar{u}_0(x))^+ dx;$$
$$\|u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R}^d)}.$$

Moreover, if $u_0(x) \leq \bar{u}_0(x)$ a.e. on $\mathbb{R}^d$, then $u^\varepsilon(x,t) \leq \bar{u}^\varepsilon(x,t)$ a.e. on $\mathbb{R}^d \times [0,T]$. In addition, the essential range of both u^ε and $\bar{u}^\varepsilon$ is contained in $[-a,a]$, where

$$a = \max\{\|u_0\|_{L^\infty(\mathbb{R}^d)}, \|\bar{u}_0\|_{L^\infty(\mathbb{R}^d)}\}.$$

Again, as in the study of the vanishing viscosity approximation, we carry out the following estimates for the L^1- modulus of continuity of the solutions.

Lemma 4.2.3 *Let $u^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ be the weak, entropy solution to (4.2.4) with $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ as initial datum. In particular*

$$\int_{\mathbb{R}^d} |u_0(x+h) - u_0(x)| dx \leq \omega(|h|), \qquad \text{for any } h \in \mathbb{R}^d,$$

for some nondecreasing function ω on $[0,+\infty)$ with $\omega(r) \downarrow 0$ as $r \downarrow 0$. Then there exists a constant C, depending only on $\|u_0\|_{L^\infty(\mathbb{R}^d)}$ and on the mapping f such that, for any $t > 0$,

$$\int_{\mathbb{R}^d} |u^\varepsilon(x+h,t) - u^\varepsilon(x,t)| dx \leq \omega(|h|), \qquad \text{for any } h \in \mathbb{R}^d \tag{4.2.5}$$

and

$$\int_{\mathbb{R}^d} |u^\varepsilon(x,t+k) - u^\varepsilon(x,t)| dx \leq C k^{\frac{2}{3}} \|u_0\|_{L^1(\mathbb{R}^d)} + 4\omega(k^{\frac{1}{3}}), \tag{4.2.6}$$

for any $k > 0$.

Proof. As in the proof of Lemma 4.1.3, the results of Theorem 4.2.2 implies (4.2.5). Now, due to the time regularity of the weak, entropy solution u^ε (see the Remark 4.1.7), we have

$$\int_{\mathbb{R}^d} \phi(x)[u^\varepsilon(x,t+k) - u^\varepsilon(x)] dx = \int_t^{t+k} \int_{\mathbb{R}^d} \Bigg[\nabla\phi(x) \cdot f(u^\varepsilon(x,\tau)) + \frac{\phi(x)}{\varepsilon}(-u^\varepsilon(x,\tau) + (K^\varepsilon * u^\varepsilon)(x,\tau))\Bigg] dx d\tau, \tag{4.2.7}$$

where ϕ is the same regularization of the sign function considered in the proof of Lemma 4.1.3. The only difference with the case of the vanishing viscosity approximation stands in the source terms, which, however, can be controlled in a similar way, due to its good contraction properties. Indeed, we have

$$
\begin{aligned}
&-\frac{1}{\varepsilon}\int_t^{t+k}\int_{\mathbb{R}^d}\phi(x)\,(u^\varepsilon - K^\varepsilon * u^\varepsilon)\,dxd\tau \\
&= -\frac{1}{\varepsilon}\int_t^{t+k}\int_{\mathbb{R}^d}\phi(x)\int_{\mathbb{R}^d}K(\xi)\,(u^\varepsilon(x,\tau) - u^\varepsilon(x-\varepsilon\xi,\tau))\,d\xi dxd\tau \\
&= -\frac{1}{\varepsilon}\int_t^{t+k}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\phi(x+\varepsilon\xi)-\phi(x))K(\xi)u^\varepsilon(x,\tau)dxd\tau d\xi \\
&= \varepsilon\frac{1}{\varepsilon}\int_t^{t+k}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(\xi)u^\varepsilon(x,\tau)\nabla\phi(\zeta)\cdot\xi dxdtd\xi \\
&\leq Ck^{\frac{2}{3}}\|u_0\|_{L^1(\mathbb{R}^d)},
\end{aligned}
$$

where the constant C is independent from ε. Finally, the relation (4.2.6) can be proved as before starting from (4.2.7) and the proof is complete. □

Now we can prove our first relaxation result.

Theorem 4.2.4 *Let $u^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$ be the weak, entropy solution of (4.2.4) with initial datum $u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then, as $\varepsilon \downarrow 0$,*

$$
u^\varepsilon \longrightarrow u, \qquad \textit{strongly in } L^p_{loc}(\mathbb{R}^d\times[0,T]),\ p<+\infty.
$$

Moreover, $u \in L^\infty([0,T]; L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$ is the unique entropy solution to (4.2.3) with u_0 as initial datum.

Proof. Applying the results of Lemma 4.2.3 and of the Theorem 4.2.2, we obtain the compactness of the sequence u^ε in L^1_{loc} and its boundedness in L^∞. Thus, passing to subsequence, we have

$$
u^\varepsilon \longrightarrow u, \qquad \text{strongly in } L^p_{loc}(\mathbb{R}^d\times[0,T]),\ p<+\infty,
$$

and boundedly almost everywhere on $\mathbb{R}^d\times[0,\infty)$. Now, as usual in this framework, this conditions guarantee that the first-order terms in the weak, entropy formulation of (4.2.4) converge, as $\varepsilon\downarrow 0$, to the distribution $u_t + \operatorname{div} f(u)$. Hence, we can conclude that the limit function u is the unique entropy solution of (4.2.3) if, as $\varepsilon \downarrow 0$,

$$
-\frac{1}{\varepsilon}\,(u^\varepsilon - K^\varepsilon * u^\varepsilon) \longrightarrow 0 \tag{4.2.8}
$$

in the sense of distributions. We remark that once we prove that u is the unique entropy solution to (4.2.3), the whole sequence u^ε will converge to u. To analyze the convergence of the source term, let $\psi(x,t)$ be a smooth, compactly supported test function. Then, we have

$$
\begin{aligned}
&-\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^d}\psi(x,t)\left(u^\varepsilon - K^\varepsilon * u^\varepsilon\right)dxd\tau \\
&= -\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^d}\psi(x,t)\int_{\mathbb{R}^d}K(\xi)\left(u^\varepsilon(x,\tau)-u^\varepsilon(x-\varepsilon\xi,\tau)\right)d\xi dxd\tau \\
&= -\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\psi(x+\varepsilon\xi)-\psi(x,t))K(\xi)u^\varepsilon(x,\tau)dxd\tau d\xi.
\end{aligned}
$$

Now, we observe that

$$
|\psi(x+\varepsilon\xi,t)-\psi(x,t)-\varepsilon\nabla\psi(x,t)\cdot\xi| \le \frac{1}{2}\varepsilon^2\left|\xi\right|^2\|\psi\|_{C_0^2}
$$

and hence we have

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^d}\psi(x,t)\left(u^\varepsilon - K^\varepsilon * u^\varepsilon\right)dxd\tau\right| \\
&\le \left|\varepsilon\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(\xi)u^\varepsilon(x,t)\xi\cdot\nabla\psi(x,t)dxd\psi dt\right| \\
&\quad+\frac{\varepsilon}{2}\|\psi\|_{C^2}\int_0^T\int_{\mathbb{R}^d}|u^\varepsilon(x,t)|dxdt\int_{\mathbb{R}^d}|\xi|^2K(\xi)d\xi := I_1+I_2.
\end{aligned}
$$

Now, since

$$
\int_{\mathbb{R}^d}K(\xi)\xi_j d\xi = 0, \quad j=1,...,d,
$$

then, $I_1=0$. Moreover, since the function $K(\xi)|\xi|^2$ has finite mass, the term I_2 is bounded by $O(\varepsilon)T\|\psi\|_{C^2}\|u_0\|_{L^1(\mathbb{R}^d)}$. This proves (4.2.8). $\square$

Remark 4.2.5 In the proof of the hyperbolic–hyperbolic relaxation limit, we take advantage of the L^1 contraction of the source of (4.2.4), which are due solely to the property of the convolution kernel K^ε, namely

- $K^\varepsilon(x) = K^\varepsilon(|x|)$;
- $K^\varepsilon \ge 0$;
- $\|K^\varepsilon\|_{L^1} = 1$.

Moreover, the above conditions imply also that the first momenta of K^ε are zero and this feature allows to control the singular limit of the source in the sense of distribution (see the last line in the proof of Theorem 4.2.4). Thus, the results we obtained are still valid for any equation of the form (4.2.4), with a convolution kernel satisfying the above conditions.

We conclude this subsection with the case of L^∞ initial data. Although we cannot state a relaxation result for *any* solution in L^∞ (because of the lack in the well–posedness in the multi–dimensional case), we can restrict ourselves to the case of L^∞ solutions defined via the extended semigroup determined by the result in Theorem 4.1.15. In other words, given $u_0 \in L^\infty$, we consider the L^∞ solution with initial datum u_0 defined via the density argument performed at the end of the previous section. As we pointed out before, the only difference with the non-scaled equation lies in the source term, due to the singular coefficient $\frac{1}{\varepsilon}$ and the scaled kernel K^ε. However, these changements do not affect the monotonicity properties of the source term and therefore we can repeat the argument of the previous section to prove the following theorem.

Theorem 4.2.6 *Let u^ε, $\bar{u}^\varepsilon \in L^\infty([0,T] \times \mathbb{R}^d)$ be weak entropy solutions of (4.2.4) with initial data u_0, $\bar{u}_0 \in L^\infty(\mathbb{R}^d)$, u^ε, $\bar{u}^\varepsilon$ determined by the extended semigroup defined by Theorem 4.1.15. Then, for any $t \in [0,T]$,*

$$|||u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)||| \le e^{Ct}|||u_0 - \bar{u}_0|||, \tag{4.2.9}$$

where the constant C depends only on ϕ and on $\|u_0\|_{L^\infty}$, $\|\bar{u}_0\|_{L^\infty}$ and $|||\cdot|||$ is the norm defined (4.1.31).

Finally, once we have the property (4.2.9), we can easily obtain the results of Lemma 4.2.3 even for the norm (4.1.31) and therefore we get the convergence of the relaxation limit for the L^∞ solutions of (4.2.4) determined by the extended semigroup defined above.

Remark 4.2.7 As we pointed out in Remark 4.2.1, we consider only relaxation limits with fixed initial data, without scaling the function at $t = 0$. However, if we choose the initial data U_0 of (4.2.1) only in $L^\infty(\mathbb{R}^d)$, then we can consider the genuine relaxation limit of the corresponding weak solution, by scaling also the initial datum $u_0^\varepsilon(x) = U_0\left(\frac{x}{\varepsilon}\right)$. Indeed, this time the sequence u_0^ε is only bounded in L^∞ and therefore the solutions u^ε of (4.2.2) will converge to the solution of its formal limit (4.2.3) with the weak-* limit in L^∞ of u_0^ε as initial condition, provided u_0 is also in $\mathcal{BV}(\mathbb{R}^d)$. Indeed, in this case, the estimate (4.2.9) gives the necessary uniform control of the modulus of continuity of the sequence u^ε in the L^1–weighted norm even for the scaled initial data $u_0^\varepsilon(x) = U_0\left(\frac{x}{\varepsilon}\right)$.

4.2.2 Hyperbolic-parabolic relaxation limit

In this section we analyze the hyperbolic-parabolic relaxation limit for the system (4.2.1). We perform the transformation

$$
\begin{aligned}
u^\varepsilon(x,t) &= \frac{1}{\varepsilon} U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\\
q^\varepsilon(x,t) &= \frac{1}{\varepsilon^2} Q\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right).
\end{aligned}
$$

Hence, the system (4.2.1) becomes

$$
\begin{cases}
u_t^\varepsilon + \frac{1}{\varepsilon^2}\operatorname{div} f(\varepsilon u^\varepsilon) = -\operatorname{div} q^\varepsilon\\
-\varepsilon^2 \nabla \operatorname{div} q^\varepsilon + q^\varepsilon = -\nabla u^\varepsilon.
\end{cases}
\tag{4.2.10}
$$

As in the previous case, we start by taking initial datum $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and postpone the L^∞ case to the end of this subsection.

Remark 4.2.8 As we pointed out in Remark 4.2.1, we perform the above scaling only to show which terms are negligible in the equations and we do not consider also the scaled initial datum. With this scaling, namely, in the parabolic regime, it turns out that the negligible term is $\nabla \operatorname{div} q$, as proposed in [KNN99] in the one dimensional case. Once again, we do not scale the initial datum, because, in this way, the sequence we end up is given by $\frac{1}{\varepsilon} U_0\left(\frac{x}{\varepsilon}\right)$, which is uniformly bounded in $L^1(\mathbb{R}^d)$, but not in $L^\infty(\mathbb{R}^d)$, if $U_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, without further restrictions.

Letting $\varepsilon \downarrow 0$, we recover that the formal limit of the second equation in (4.2.10) is given by $q = -\nabla u$. Moreover, since we supposed the mapping f to be smooth and such that $f(0) = f'(0) = 0$, we obtain, at a formal level, the following limit equation

$$
u_t + \frac{1}{2} f''(0) \cdot \nabla(u^2) = \Delta u. \tag{4.2.11}
$$

Once again, we rewrite the scaled system (4.2.10) as a nonhomogeneous conservation law, namely

$$
u_t^\varepsilon + \frac{1}{\varepsilon^2}\operatorname{div} f(\varepsilon u^\varepsilon) = -\frac{1}{\varepsilon^2}\left(u^\varepsilon - K^\varepsilon * u^\varepsilon\right), \tag{4.2.12}
$$

with the same convolution kernel of the previous case. To simplify the aspect of equation (4.2.12) for small ε, we set

$$
f(\varepsilon u^\varepsilon) = \varepsilon^2 \frac{f''(0)}{2}(u^\varepsilon)^2 + \varepsilon^3 g(u^\varepsilon),
$$

where g is a smooth function such that $g(0) = g'(0) = g''(0) = 0$. Hence, (4.2.12) becomes

$$u_t^\varepsilon + \frac{f''(0)}{2} \cdot \nabla(u^\varepsilon)^2 + \varepsilon \operatorname{div} g(u^\varepsilon) = -\frac{1}{\varepsilon^2} \left(u^\varepsilon - K^\varepsilon * u^\varepsilon\right), \qquad (4.2.13)$$

Hence, with the exception of the extra term $\varepsilon \operatorname{div} g(u^\varepsilon)$, which doesn't affect the asymptotic analysis, we recover the same structure as in the hyperbolic-hyperbolic relaxation limit. Thus, the following theorem holds.

Theorem 4.2.9 *Let u^ε, $\bar{u}^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ be weak entropy solutions of (4.2.13) with initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for any $t \in [0,T]$,*

$$\int_{\mathbb{R}^d} (u^\varepsilon(x,t) - \bar{u}^\varepsilon(x,t))^+ dx \le \int_{\mathbb{R}^d} (u_0(x) - \bar{u}_0(x))^+ dx;$$
$$\|u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R}^d)}.$$

Moreover, if $u_0(x) \le \bar{u}_0(x)$ a.e. on $\mathbb{R}^d$, then $u^\varepsilon(x,t) \le \bar{u}^\varepsilon(x,t)$ a.e. on $\mathbb{R}^d \times [0,T]$. In addition, the essential range of both u^ε and $\bar{u}^\varepsilon$ is contained in $[-a,a]$, where

$$a = \max\{\|u_0\|_{L^\infty(\mathbb{R}^d)}, \|\bar{u}_0\|_{L^\infty(\mathbb{R}^d)}\}.$$

Once again, the above results yield the compactness of the relaxation approximations.

Lemma 4.2.10 *Let $u^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ be the weak, entropy solution to (4.2.13) with $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ as initial datum. In particular*

$$\int_{\mathbb{R}^d} |u_0(x+h) - u_0(x)| dx \le \omega(|h|), \qquad \text{for any } h \in \mathbb{R}^d,$$

for some nondecreasing function ω on $[0,+\infty)$ with $\omega(r) \downarrow 0$ as $r \downarrow 0$. Then there exists a constant C, depending on $\|u_0\|_{L^\infty(\mathbb{R}^d)}$ and on the mapping f such that, for any $t > 0$,

$$\int_{\mathbb{R}^d} |u^\varepsilon(x+h,t) - u^\varepsilon(x,t)| dx \le \omega(|h|), \qquad \text{for any } h \in \mathbb{R}^d \qquad (4.2.14)$$

and

$$\int_{\mathbb{R}^d} |u^\varepsilon(x,t+k) - u^\varepsilon(x,t)| dx \le C(k^{\frac{2}{3}} + k^{\frac{1}{3}}) \|u_0\|_{L^1(\mathbb{R}^d)} + 4\omega(k^{\frac{1}{3}}), \quad (4.2.15)$$

for any $k > 0$.

Proof. As in the proof of Lemma 4.2.3, (4.2.14) follows from Theorem 4.2.9. Moreover, our weak, entropy solution verifies

$$\int_{\mathbb{R}^d} \phi(x)[u^\varepsilon(x,t+k) - u^\varepsilon(x)]dx = \int_t^{t+k} \int_{\mathbb{R}^d} \Bigg[\nabla\phi(x) \cdot \left(\frac{f''(0)}{2}(u^\varepsilon)^2 + \varepsilon g(u^\varepsilon)\right)$$
$$+\frac{\phi(x)}{\varepsilon^2}(-u^\varepsilon(x,\tau) + (K^\varepsilon * u^\varepsilon)(x,\tau))\Bigg]\, dxd\tau, \tag{4.2.16}$$

where $\phi(x)$ stands for the smooth function considered in the proofs of Lemma 4.1.3 and Lemma 4.2.3. Once again, we must control the singularity of the source term. This can be done using the second derivatives of the smooth function ϕ. Indeed,

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_t^{t+k}\int_{\mathbb{R}^d} \phi(x)\,(u^\varepsilon - K^\varepsilon * u^\varepsilon)\,dxd\tau \\
&= -\frac{1}{\varepsilon^2}\int_t^{t+k}\int_{\mathbb{R}^d} \phi(x)\int_{\mathbb{R}^d} K(\xi)\,(u^\varepsilon(x,\tau) - u^\varepsilon(x-\varepsilon\xi,\tau))\,d\xi dxd\tau \\
&= -\frac{1}{\varepsilon^2}\int_t^{t+k}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\phi(x+\varepsilon\xi) - \phi(x))K(\xi)u^\varepsilon(x,\tau)dxd\tau d\xi \\
&= -\varepsilon\frac{1}{\varepsilon}\int_t^{t+k}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(\xi)u^\varepsilon(x,\tau)\nabla\phi(\zeta)\cdot\xi dxd\tau d\xi \\
&-\frac{\varepsilon^2}{2\varepsilon^2}\int_t^{t+k}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(\xi)u^\varepsilon(x,\tau)\left(D^2\phi(\zeta)\xi\right)\cdot\xi dxd\xi d\tau \\
&\le Ck^{\frac{1}{3}}\|u_0\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

since, as we pointed out before,

$$\int_{\mathbb{R}^d} K(\xi)\xi_j d\xi = 0, \quad j = 1, ..., d,$$

and the function $K(\xi)|\xi|^2$ has finite mass. As before, the constant C is independent from ε and therefore the relation (4.2.15) is an easy consequence of (4.2.16). The proof is complete. □

Finally, the convergence result is contained in the next theorem.

Theorem 4.2.11 *Let $u^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$ be the weak, entropy solution of (4.2.13) with initial datum $u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then, as $\varepsilon \downarrow 0$,*

$$u^\varepsilon \longrightarrow u, \qquad \textit{strongly in } L^p_{loc}(\mathbb{R}^d \times [0,T]),\ p < +\infty,$$

and u is the unique solution to (4.2.11) with u_0 as initial datum.

Proof. As for the proof of Theorem 4.2.4, we obtain, up to subsequences,

$$u^\varepsilon \longrightarrow u, \qquad \text{strongly in } L^p_{loc}(\mathbb{R}^d \times [0,T]),\ p < +\infty,$$

and boundedly almost everywhere in $\mathbb{R}^d \times [0,+\infty)$, where $u \in L^1(\mathbb{R}^d \times [0,T]) \cap L^\infty(\mathbb{R}^d \times [0,T])$, thanks to the compactness coming from the results of Lemma 4.2.10. Once again, these conditions allow us to pass to the limit into the weak, entropy formulation of (4.2.13) as $\varepsilon \downarrow 0$, as in the previous case. In particular, the additional term $\varepsilon \operatorname{div} g(u^\varepsilon)$ tends to zero in distributional sense. Hence, to conclude that u is the unique solution to (4.2.11) with u_0 as initial datum, we need to prove

$$-\frac{1}{\varepsilon^2}\left(u^\varepsilon - K^\varepsilon * u^\varepsilon\right) \longrightarrow \Delta u \tag{4.2.17}$$

in the sense of distributions, as $\varepsilon \downarrow 0$. We proceed as in the proof oh Lemma 4.2.10. Also in this case, the uniqueness of solutions to this Cauchy problem implies that the whole sequence u^ε will converge. Let $\psi(x,t)$ be a smooth, compactly supported test function. Since

$$\begin{aligned}&\psi(x+\varepsilon\xi,t) - \psi(x,t)\\ &\quad = \varepsilon\nabla\psi(x,t)\cdot\xi + \frac{1}{2}\varepsilon^2\left(D^2\psi(x,t)\xi\right)\cdot\xi + \frac{1}{6}\varepsilon^3\xi\cdot\left((D^3\psi(\zeta^\varepsilon,t)\xi)\cdot\xi\right)\end{aligned}$$

we have

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^d}\psi(x,\tau)\left(u^\varepsilon - K^\varepsilon * u^\varepsilon\right)dxd\tau\\
&= -\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^d}\psi(x,\tau)\int_{\mathbb{R}^d}K(\xi)\left(u^\varepsilon(x,\tau) - u^\varepsilon(x-\varepsilon\xi,\tau)\right)d\xi dxd\tau\\
&= -\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\psi(x+\varepsilon\xi,\tau) - \psi(x,\tau))K(\xi)u^\varepsilon(x,\tau)dxd\tau d\xi\\
&= -\varepsilon\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(\xi)u^\varepsilon(x,\tau)\nabla\psi(x,\tau)\cdot\xi dxd\tau d\xi\\
&-\frac{\varepsilon^2}{2\varepsilon^2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(\xi)u^\varepsilon(x,\tau)\left(D^2\psi(x,\tau)\xi\right)\cdot\xi dxd\xi d\tau\\
&-\frac{\varepsilon^3}{6\varepsilon^2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(\xi)u^\varepsilon(x,\tau)\ \xi\cdot\left((D^3\psi(\zeta^\varepsilon,\tau)\xi)\cdot\xi\right)d\xi dxd\tau := J_1 + J_2 + J_3.
\end{aligned}$$

Once again, since

$$\int_{\mathbb{R}^d}K(\xi)\xi_j d\xi = 0, \quad j = 1,...,d,$$

we have $J_1 = 0$. Moreover, since the function $K(\xi)|\xi|^3$ has finite mass, the integral J_3 is controlled by $\varepsilon CT\|\psi\|_{C^2}\|u_0\|_{L^1}$. We write the term J_2 as follows

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(\xi)u^\varepsilon(x,\tau)\ {}^T\xi D^2\psi(x,t)\xi dx d\xi d\tau \\
&= \frac{1}{2}\sum_{i,j=1}^d \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u^\varepsilon(x,\tau)\xi_i\xi_j K(\xi)\frac{\partial^2\psi}{\partial x_i \partial x_j}(x,\tau) d\xi dx d\tau \\
&= \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u^\varepsilon(x,\tau)\Delta\psi(x,\tau) d\xi dx d\tau,
\end{aligned}$$

because of the properties of the kernel K

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d} \xi_i^2 K(\xi) d\xi = 1 \quad i = 1, ..., d \\
&\int_{\mathbb{R}^d} \xi_i\xi_j K(\xi) d\xi = 0 \ \text{ for } i,j = 1, ..., d, \quad i \neq j.
\end{aligned}$$

Hence, passing into the limit in the above relation, we recover (4.2.17) and the proof is complete. □

Remark 4.2.12 As we pointed out in Remark 4.2.5, the results established above are valid for any nonnegative convolution kernel K such that $K(x) = \widehat{K}(|x|)$ and such that $\|K^\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$. In this case, the fact that the first momenta of K^ε are zero guarantees the control of the L^1 modulus of continuity in the t variable (see the proof of Lemma 4.2.10). The matrix of the second momenta of K^ε, necessarily diagonal and positive, gives the double of the viscosity tensor in the limit equation (see the relations above).

We treat now the case of L^∞ initial data. The same remarks given at the and of the previous subsection about the well–posedness in L^∞ are valid here. Once again, our parabolic scaling introduces a singular coefficient in front of the source term, namely $\frac{1}{\varepsilon^2}$, and it scales the kernel K, but, as in the previous case, it preserves the monotonicity of the source itself. Therefore, repeating step by step the procedure in the proof of the Theorem 4.1.15, we can prove Lipschitz continuity in the weighted norm considered in the previous section. Hence, we have the following theorem.

Theorem 4.2.13 *Let u^ε, $\bar{u}^\varepsilon \in L^\infty([0,T]\times\mathbb{R}^d)$ be weak entropy solutions of (4.2.13) with initial data u_0, $\bar{u}_0 \in L^\infty(\mathbb{R}^d)$, u^ε, $\bar{u}^\varepsilon$ determined by the extended semigroup defined by Theorem 4.1.15. Then, for any $t \in [0,T]$,*

$$|||u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)||| \leq e^{Ct}|||u_0 - \bar{u}_0|||, \tag{4.2.18}$$

where $|||\cdot|||$ *is the norm defined (4.1.31) and the constant* C *depends only on the weight* ϕ *and on* $\|u_0\|_{L^\infty}$, $\|\bar{u}_0\|_{L^\infty}$.

Once we have proved (4.2.18), we can recover the result of Lemma 4.2.10 in terms of the norm (4.1.31) and hence we obtain the convergence of the relaxation limit even in the case of the L^∞ solutions of (4.2.4) determined by the extended semigroup defined in the previous section.

Part II

Long time asymptotics

Chapter 5

Nonlinear diffusion equations

In this chapter we use the entropy dissipation method described in section 1.7 of the introduction in order to detect asymptotic self similar profiles as the typical intermediate asymptotic states for a class of general nonlinear diffusion equations. In section 5.1 we recall some preliminary results and perform the basic time dependent scaling which allows to view intermediate asymptotic profile as stationary states. In section 5.2 we prove the result concerning convergence in relative entropy. In the last section we use the p–Wasserstein distances in order to find an optimal estimate for the speed of propagation of the support of the solutions.

5.1 Preliminaries

We consider the Cauchy problem for a general nonlinear diffusion equation, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\phi(u) \\ u(\cdot, 0) = u_0. \end{cases} \tag{5.1.1}$$

Here, $(x, t) \in \mathbb{R}^N \times \mathbb{R}_+$, while the initial datum u_0 is taken to be nonnegative and belonging in $L^1(\mathbb{R}^N)$. We shall assume throughout this chapter that the nonlinearity function ϕ satisfies the following assumptions

(NL1) $\phi'(u) > 0$ for all $u > 0$

(NL2) $\phi(u) = u^m\psi(u)$ for some $m > \frac{N-2}{N}$, where the 'perturbation' function ψ satisfies the following properties

(P1) $\exists \lim_{u\to 0^+} \psi(u) = l \in (0, +\infty)$ (for simplicity we assume $l = 1$)

(P2) $\psi \in C[0,+\infty) \cap C^1(0,+\infty)$

(P3) $\psi'(u) = O(u^k)$ as $u \to 0^+$, for some $k > -1$.

Under the above assumptions, it is well–known that the Cauchy problem (5.1.1) is well–posed for any initial datum in $L^1_+(\mathbb{R}^N)$. Moreover, the following conservation law holds

$$\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx.$$

The unique solution to (5.1.1) may not be classical in general. This fact leads to a definition of generalized solution (see, for instance [Váz90] and the references therein, for the porous medium case $\phi(u) = u^m$). We refer to [Bén76, Vér] for the existence and regularity theory for equation (5.1.1) with initial data in L^1, obtained by means of the Crandall–Ligget formula for nonlinear semigroups. In case of *slow* diffusion, i.e. in case $\phi'(0) = 0$, the support of the solution travels with finite speed. This is due to the degeneracy of the parabolic operator as u approaches zero (see [Kal87, Kne77] and the references therein for the general diffusion equation 5.1.1; see also the extremely complete survey paper by Vazquez [Váz03] and the references therein for the porous medium equation). Concerning the evolution of the integral norms, the equation (5.1.1) induces a contraction with respect to all the L^p norms. More precisely, if $u_0 \in L^p(\mathbb{R}^N)$, so is the solution $u(\cdot,t)$ at any time $t > 0$, and we have

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \leq \|u_0\|_{L^p(\mathbb{R}^N)}. \tag{5.1.2}$$

Moreover, under an extra assumption on the nonlinearity ϕ, namely

(NL3) $\exists C > 0$ such that $\phi'(u) \geq Cu^{m-1}$ for all $u > 0$,

the equation (5.1.1) enjoys an L^1–L^∞ regularizing property. Indeed, it was proved that the solution to (5.1.1) with intial datum in L^1 satisfies the following temporal decay estimate (see [AB79, Váz90] for the power law case, [Vér] for the general nonlinear case).

Theorem 5.1.1 ($\mathbf{L^1}$–$\mathbf{L^\infty}$ regularizing effect) *Let $u(x,t)$ be the solution to the Ca– uchy problem (5.1.1), with $u_0 \in L^1_+(\mathbb{R}^N)$. Let the nonlinearity ϕ satisfy the assumption **(NL3)** above. Then, at any $t > 0$, $u(x,t) \in L^\infty(\mathbb{R}^N)$ and the following estimate hold*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{N}{N(m-1)+2}}\|u_0\|_{L^1(\mathbb{R}^N)}. \tag{5.1.3}$$

We are interested in the study of the time–asymptotic behaviour of the solution to (5.1.1). It is already known that such behaviour is well described by the Barenblatt–Prattle self–similar solutions to the porous medium equation $u_t = \Delta u^m$ (where m is the same exponent as in condition **(NL2)** above, see [Kam76, BDE02]). We now perform the following time–dependent scaling, in order to put in evidence the role of the term u^m in the nonlinearity function ϕ.

$$\begin{aligned} &u(x,t) = R(t)^{-N\lambda} v(y,s) \qquad y = xR(t)^{-\lambda} \qquad s = \lambda \log R(t) \\ &R(t) = \left(1 + \frac{t}{\lambda}\right) \qquad \lambda = \frac{1}{N(m-1)+2} \end{aligned} \tag{5.1.4}$$

As usual in this framework, equation (5.1.1) turns into the following non linear (time–dependent) Fokker–Planck type equation

$$\frac{\partial v}{\partial s} = \nabla \cdot (yv) + e^{Nms} \Delta \phi(e^{-Ns} v), \tag{5.1.5}$$

with initial datum $v(y, 0) = u_0(y)$. In the sequel it will be useful to write equation (5.1.5) as follows

$$\frac{\partial v}{\partial s} = \nabla \cdot \left[v \nabla \left(\frac{|y|^2}{2} + e^{N(m-1)s} h(e^{-Ns} v) \right) \right], \tag{5.1.6}$$

where $h(u)$ is the generalized *enthalpy*

$$h(u) = \int_1^u \frac{\phi'(\eta)}{\eta} d\eta. \tag{5.1.7}$$

The function h is well defined on $(0, +\infty)$, and it may eventually have a singularity at $u = 0$ (e.g. in case of *fast* diffusion, that is $\phi'(0) > 0$). In order to define a generalized entropy functional we will need afterwards, we require the natural condition

(NL4) $h(\cdot) \in L^1_{loc}([0, +\infty))$.

We close this section by re–stating the contraction property (5.1.2) and temporal decay estimate (5.1.3) in terms of the new unknown function v and the new independent variables (y, s). The estimate for the L^p–norm, for $p \in [1, +\infty)$, follows by interpolation.

Proposition 5.1.2

(a) Let $v(y,s)$ be the solution to (5.1.5) with initial datum $u_0 \in L^1_+(\mathbb{R}^N)$. Then, for all $p \in [1,+\infty)$ the following estimate holds for some fixed constant $C > 0$

$$\|v(\cdot,s)\|_{L^p(\mathbb{R}^N)} \leq Ce^{N\left(1-\frac{1}{p}\right)s}\left[k\left(e^{\frac{s}{k}}-1\right)\right]^{-kN\left(1-\frac{1}{p}\right)}\|u_0\|_{L^1(\mathbb{R}^N)}, \quad (5.1.8)$$

while, for $p = +\infty$ we have

$$\|v(\cdot,s)\|_{L^\infty(\mathbb{R}^N)} \leq Ce^{Ns}\left[k\left(e^{\frac{s}{k}}-1\right)\right]^{-kN}\|u_0\|_{L^1(\mathbb{R}^N)}. \quad (5.1.9)$$

Hence, for any fixed $s_0 > 0$ and for all $p \in [0,+\infty]$, we have

$$\sup_{s\geq s_0}\|v(\cdot,s)\|_{L^p(\mathbb{R}^N)} \leq C(s_0)\|u_0\|_{L^1(\mathbb{R}^N)}, \quad (5.1.10)$$

where $C(s_0)$ doesn't depend on p.

(b) Let $v(y,s)$ be the solution to (5.1.5) with initial datum $u_0 \in L^p(\mathbb{R}^N)$. Then, the following local stability property holds at any $s \geq 0$

$$\|v(\cdot,s)\|_{L^p(\mathbb{R}^N)} \leq e^{N\left(1-\frac{1}{p}\right)s}\|u_0\|_{L^p(\mathbb{R}^N)}. \quad (5.1.11)$$

A note about the notation. In the sequel we denote by C a generic positive constant. Sometimes we shall indicate its dependence on some parameters by means of the expressions $C(\ldots)$ or $C_{\ldots}$.

5.2 The relative entropy method

5.2.1 Statement of the problem and result

Let v be the solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial v}{\partial s} = \nabla\cdot(yv) + e^{Nms}\Delta\phi(e^{-Ns}v) \\ v(y,0) = u_0(y). \end{cases} \quad (5.2.1)$$

Due to the conservation of the mass, we can set

$$\int_{\mathbb{R}^N} u_0(y)dy = \int_{\mathbb{R}^N} u(y,s)dy = M.$$

We expect the solution $v(y,s)$ to behave like the rescaled Barenblatt similarity function

$$v^{\infty}(y) = \begin{cases} \left(C_M - \lambda\frac{|y|^2}{2}\right)_+^{\frac{1}{m-1}} & \text{if } \ m \neq 1 \\ C_M e^{-\frac{|y|^2}{2}} & \text{if } \ m = 1, \end{cases} \tag{5.2.2}$$

as $s \to +\infty$, where the constant C_M is chosen in such a way that

$$\int_{\mathbb{R}^N} v^{\infty}(y)dy = M,$$

and λ depends only on m and on the space dimension N. We emphasize that v^{∞} is *not* a solution to equation (5.2.1). Let us then define our entropy functional

$$H(v) = \begin{cases} \dfrac{1}{m-1}\displaystyle\int_{\mathbb{R}^N} v(y)^m dy + \frac{1}{2}\int_{\mathbb{R}^N} |y|^2 v(y)dy & \text{if } \ m \neq 1 \\ \displaystyle\int_{\mathbb{R}^N} v(y)\log v(y)dy + \frac{1}{2}\int_{\mathbb{R}^N} |y|^2 v(y)dy & \text{if } \ m = 1. \end{cases} \tag{5.2.3}$$

Here m is the exponent describing the behaviour of the nonlinearity in zero, given by condition **(NL2)**. We recall that the convex functional $H(v)$ attains its minimum over $L^1_+(\mathbb{R}^N)$, under the constraint $\int_{\mathbb{R}^N} v =$ constant, at the state v^{∞} with mass M (see [CT00]). The *relative* entropy is defined, as usual, by

$$H\left(v|v^{\infty}\right) = H(v) - H(v^{\infty}).$$

The above functional $H\left(v|v^{\infty}\right)$ is related to a Dirichlet–type integral, the so–called *entropy production* or *generalized Fisher information*, by means of the following Sobolev–type inequality (see [CT00, AMTU00, Gro75]).

Theorem 5.2.1 *Let $v \in L^1_+(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} v(y)dy = M$, let v^{∞} be the ground state defined in (5.2.2) with mass M. Then, the following inequality holds,*

$$H(v) - H(v^{\infty}) \leq \frac{1}{2} I(v|v^{\infty}), \tag{5.2.4}$$

where

$$I(v|v^{\infty}) = \int_{\mathbb{R}^N} v \left|\nabla\left(\frac{m}{m-1}v^{m-1} + \frac{|y|^2}{2}\right)\right|^2 dy,$$

where m *is the same exponent as in the definition (5.2.3) of the entropy functional .*

We also recall the generalized *Csiszár–Kullback* inequality, which provides an upper bound of the L^1 norm of the difference between any positive density v and the ground state v^∞ having the same mass as v, in terms of their relative entropy. More precisely, we have

Theorem 5.2.2 *Let $v \in L^1_+(\mathbb{R}^N)$, with $\int_{\mathbb{R}^N} v(y)dy = \int_{\mathbb{R}^N} v^\infty(y)dy$. Then, the following holds*

$$\|v - v^\infty\|^\alpha_{L^1(\mathbb{R}^N)} \leq C[H(v) - H(v^\infty)], \tag{5.2.5}$$

where

$$\alpha = \begin{cases} 2 & \text{if } m \leq 2 \\ m & \text{if } m \geq 2, \end{cases} \tag{5.2.6}$$

where m *is again the exponent in the definition (5.2.3) of the entropy functional.*

In the sequel we shall also need the *modified* entropy functional (see [BDE02])

$$E(v,s) = e^{mNs} \int_{\mathbb{R}^N} F(e^{-Ns}v)dy + \frac{1}{2}\int_{\mathbb{R}^N} |y|^2 v dy, \tag{5.2.7}$$

where

$$F(u) = \int_0^u h(\theta)d\theta, \tag{5.2.8}$$

and h is the enthalpy defined in (5.1.7). The function F is well–defined thanks to condition **(NL4)** in the previous section; moreover, F is everywhere nonnegative. We observe that the following identity holds

$$F(u) = uh(u) - \phi(u).$$

In what follows, we shall assume the solution to enjoy enough regularity in order to be treated as a classical solution; this can be justified, for instance, by supposing that the initial datum u_0 is strictly positive. In that case, the solution $u(y,s)$ is C^∞ at any $s > 0$. The rigorous justification of our result for general data then follows by a standard density argument (see [Váz90]). Let us then state the main result of this section.

Theorem 5.2.3 *Let $v(y,s)$ be the solution to the Cauchy problem (5.2.1). Let the initial datum $v_0 \in L^1_+(\mathbb{R}^N)$ be such that*

$$\int_{\mathbb{R}^N} F(v_0(y))dy + \frac{1}{2}\int_{\mathbb{R}^N} |y|^2 v_0(y)dy + \int_{\mathbb{R}^N} v_0(y)^m dy < +\infty, \tag{5.2.9}$$

where F is defined in (5.2.8). Then, the entropy functional $H(v(s)|v^\infty)$ satisfies

$$H(v(s)|v^\infty) \leq Ce^{-\delta s}, \quad \textit{for all} \;\; s \geq 0, \tag{5.2.10}$$

*where $\delta = \min\{2, N(k+1)\}$, k is the exponent in the structural condition **(NL2)**, and $C > 0$ is a constant depending on the mass of v_0 and on the bounded quantity in (5.2.9).*

The Csiszár Kullback inequality (5.2.5) above and the time dependent scaling (5.1.4), then, provides the rate of convergence in L^1 for the solution to the original problem (5.1.1) towards the Barenblatt self–similar functions

$$u^\infty(x,t) = \left(C - \lambda \frac{|x|^2}{t^{\frac{2}{N(m-1)+2}}} \right)_+^{\frac{1}{m-1}}. \tag{5.2.11}$$

Corollary 5.2.4 *Let $u(x,t)$ be the solution to the Cauchy problem (5.1.1), with initial datum $u_0 \geq 0$ such that*

$$\int_{\mathbb{R}^N} \left[u_0^m(x) + |x|^2 u_0(x) + F(u_0(x)) \right] dx < +\infty.$$

Let $u^\infty(x,t)$ be the Barenblatt self–similar function given by (5.2.11) with the constant C such that $\int u_\infty = \int u_0$. Then, the following estimate holds for all $t \geq 0$

$$\|u(t) - u^\infty(t)\|_{L^1(\mathbb{R}^N)} \leq C(t+1)^{-\frac{\lambda\delta}{\alpha}}, \tag{5.2.12}$$

where λ is defined in (5.1.4) and α is defined by (5.2.6).

Remark 5.2.5 The hypotesis on the initial data 5.2.9 can be relaxed in case a further assumption on the nonlinearity ϕ holds, namely $\phi'(u) \leq \widetilde{C} u^{m-1}$ for some $\widetilde{C} > 0$. This comes by direct computation of the evolution of the second moment.

Remark 5.2.6 The rate of convergence in (5.2.12) has been already obtained in [BDE02]. However, our result includes more general nonlinearity functions. In particular, we don't need to require the technical condition $(m-1)uh(u) - mf(u) \leq 0$. Moreover, our approach seems to be much simpler than that in [BDE02], since we make use the usual nonlinear version of the Log–Sobolev inequality, and we use directly the Csiszar–Kullback inequality (5.2.5) in order to get a rate of convergence in L^1.

5.2.2 Proof of Theorem 5.2.3

In order to prove theorem 5.2.3, we first perform an estimate for large times s, which is obtained basically by means of the regularizing effect (5.1.9) and of the Sobolev–type inequality (5.2.4). Then we use the modified entropy functional E defined above in (5.2.7) in order to control the evolution of the entropy in a finite time interval.

Proposition 5.2.7 *Under the assumptions* ***(N1)*****–*****(N2)*****–*****(N3)*** *on the nonlinearity ϕ, there exist an $s_0 > 0$ and a positive constant $C(s_0)$ depending on s_0 such that, if $H(v(s_0)) < +\infty$, then we have*

$$H(v(s)|v^\infty) \leq C(s_0)e^{-\min\{2,N(k+1)\}s}. \tag{5.2.13}$$

Proof. We recall that the equation (5.2.1) can be written in the alternative way (5.1.6). Then, integration by parts yields

$$\begin{aligned}\frac{d}{ds}H(v(s)) =& \int_{\mathbb{R}^N}\left[\frac{m}{m-1}v^{m-1}+\frac{|y|^2}{2}\right]\nabla\cdot\left[v\nabla\left(\frac{|y|^2}{2}+e^{N(m-1)s}h(e^{-Ns}v)\right)\right]dy\\ =& -\int_{\mathbb{R}^N} v\nabla\left(\frac{m}{m-1}v^{m-1}+\frac{|y|^2}{2}\right)\cdot\nabla\left(\frac{|y|^2}{2}+e^{N(m-1)s}h(e^{-Ns}v)\right)dy.\end{aligned}$$

Hence, we employ the structural condition **(NL2)** to obtain

$$\begin{aligned}\frac{d}{ds}H(v(s)) =& -\int_{\mathbb{R}^N}\psi(e^{-Ns}v)\left|\nabla\left(\frac{m}{m-1}v^{m-1}-\frac{|y|^2}{2}\right)\right|^2dy+\\ & - me^{-Ns}\int_{\mathbb{R}^N}v^{2m-2}\psi'(e^{-Ns}v)\left|\nabla v\right|^2dy+\\ & - e^{-Ns}\int_{\mathbb{R}^N}v^m\psi'(e^{-Ns}v)y\cdot\nabla\left(\frac{m}{m-1}v^{m-1}\right)dy\\ =& -\int_{\mathbb{R}^N}v\left(\psi(e^{-Ns}v)+\frac{1}{m}\psi'(e^{-Ns}v)e^{-Ns}v\right)\left|\nabla\left(\frac{m}{m-1}v^{m-1}-\frac{|y|^2}{2}\right)\right|^2dy+\\ & + e^{Nms}\int_{\mathbb{R}^N}\left(e^{-Ns}v\right)^m\psi'(e^{-Ns}v)y\cdot\nabla(e^{-Ns}v)dy+\\ & + \frac{1}{m}\,e^{-Ns}\int_{\mathbb{R}^N}v^2\psi'(e^{-Ns}v)|y|^2dy = \sum_{j=1}^3 I_j.\end{aligned} \tag{5.2.14}$$

We first compute the term I_2. We observe that, from the hypothesis **(P3)** on the nonlinearity, the function $g(u) = u^m\psi'(u)$ is is summable over any interval $[0, L)$, $L > 0$. Hence, the primitive

$$G(u) = \int_0^u \theta^m\psi'(\theta)d\theta$$

is well defined on $[0,+\infty)$. As a consequence of that, and after integration by parts, I_2 may be written as follows

$$I_2 = -Ne^{Nms}\int_{\mathbb{R}^N} G(e^{-Ns}v)dy.$$

Again from the structural hypothesis **(P3)**, it follows easily that $G(u)$ is a $(m+k+1)$–Hölder function on a neighborhood of $u=0$. This fact, together with estimate (5.1.9), yields

$$\begin{aligned} I_2 &\le C(M,s_0)e^{-N(k+1)s}\int_{\mathbb{R}^N} v^{m+k+1}dy \\ &\le C(M,s_0)\|v\|^{k+1}_{L^\infty(\mathbb{R}^N)}e^{-N(k+1)s}\int_{\mathbb{R}^N} v^m dy \\ &\le C(M,s_0)e^{-N(k+1)s}H(v(s)), \end{aligned}$$

for some $s_0>0$ (chosen in order to have $e^{-Ns}v$ small) and for any $s\ge s_0$. In a very similar way, we estimate I_3

$$I_3 \le C(M,s_0)e^{-N(k+1)}\int_{\mathbb{R}^N} v^{2+k}|y|^2 \le C(M,s_0)e^{-N(k+1)}H(v(s)),$$

for s_0 large and $s\ge s_0$.

The integral term I_1 may be written as follows.

$$I_1 = -\int_{\mathbb{R}^N}\alpha(s)v\left|\nabla\left(\frac{m}{m-1}v^{m-1}+\frac{|y|^2}{2}\right)\right|^2 dy,$$

where $\alpha(s) = \psi(e^{-Ns}v) + \frac{1}{m}\psi'(e^{-Ns}v)e^{-Ns}v$. By means of the structural condition **(NL2)** and the first order Taylor expansion, we have

$$\alpha(s) = 1 + e^{-Ns}v\psi'(e^{-Ns}\eta) + \frac{1}{m}e^{-Ns}v\psi'(e^{-Ns}v), \quad \eta\in[0,v(s)].$$

We employ once again the regularizing effect (5.1.9) to get

$$\alpha(s) \ge 1 - C(M,s_0)e^{-N(k+1)s}, \quad s\ge s_0,$$

and we can choose s_0 large enough in order to have $\alpha(s)\ge 0$ for any $s\ge s_0$. Hence, we put the above estimates into (5.2.14) and we use the Sobolev–type inequality (5.2.4) to recover

$$\begin{aligned} \frac{d}{ds}H(v(s)|v^\infty) &\le -2\left(1-C(M,s_0)e^{-N(k+1)}\right)H(v(s)|v^\infty)+ \\ &\quad + C(M,s_0)e^{-N(k+1)s}H(v(s)) \\ &\le -2\left(1-C(M,s_0)e^{-N(k+1)}\right)H(v(s)|v^\infty)+ \\ &\quad + C(M,s_0)e^{-N(k+1)s}H(v^\infty). \end{aligned}$$

Finally, we use the variation of constants formula and obtain the desired estimate (5.2.13). □

We now perform a local–in–time estimate of the entropy, in order to control the constant $C(s_0)$ in the inequality (5.2.13).

Lemma 5.2.8 *Suppose that the initial datum u_0 satisfies (5.2.9). Then, the following inequality holds at any $s \geq 0$,*

$$E(v(s)) \leq e^{(m-1)s} E(v_0), \tag{5.2.15}$$

where $E(u)$ is defined in (5.2.7). In particular, the entropy $H(v(s))$ is uniformly bounded on any finite time interval $[0, s_0]$.

Proof. We calculate the evolution in time of the functional $E(v(s), s)$ defined in (5.2.7) (see [BDE02]). After integration by parts, we get

$$\begin{aligned}\frac{d}{ds}E(v(s)) = &- \int_{\mathbb{R}^N} v \left| y + e^{(m-1)Ns} \nabla h(e^{-Ns}v) \right|^2 dy + \\ &+ e^{mNs} \int_{\mathbb{R}^N} \left[(m-1)ve^{-Ns}h(e^{-Ns}v) - m\phi(e^{-Ns}v) \right] dy \\ &\leq e^{mNs}(m-1) \int F(e^{-Ns}v)dy \leq (m-1)E(v(s)),\end{aligned}$$

which proves (5.2.15). The last assertion comes directly from (5.1.11). □

Remark 5.2.9 The entropy dissipation method has been successfully used in [CF03] in order to prove convergence towards Barenblatt solutions for diffusion dominated convection–diffusion equations. By means of the same approach in the present chapter, one can generalized the results in [CF03] to a generalized convection–diffusion model where the power law in the diffusion term is replaced by a more general ϕ satisfying the hypothesis above.

5.3 Evolution of the 1-d Wasserstein distances

5.3.1 Preliminaries and results

In this section we analyze the Cauchy problem (5.1.1) in one space dimension, i.e.

$$\begin{cases} u_t = \phi(u)_{xx} \\ u(x,0) = u_0(x), \end{cases} \tag{5.3.1}$$

where the initial datum u_0 is taken in $L^1_+(\mathbb{R})$.

We require the nonlinearity function ϕ to satisfy the conditions **(NL1)** and **(NL2)** stated at the beginning of section 5.1. To simplify the calculations below, we can express condition **(NL2)** in the following alternative way

(NL2) $\phi(u) = u^m + \psi(u)$, where $\psi(u) = O(u^n)$ as $u \to 0$, for some $n > m$, $\psi \in C^1((0, +\infty))$.

Moreover, we restrict ourselves to the *slow* diffusion cases by requiring

(SD) $\phi'(0) = 0$ iff $m > 1$.

As in the previous section, we perform the time dependent scaling (5.1.4), which turns equation (5.3.1) into

$$\begin{cases} v_s = (yv + e^{ms}\phi(e^{-s}v)_y)_y \\ v(y,0) = u_0(y). \end{cases} \tag{5.3.2}$$

In the sequel we shall assume for simplicity that $\int_{-\infty}^{+\infty} u_0(x)dx = 1$. Our aim is to study, for any $p \in [1, +\infty]$, the dynamic induced by the above equation (5.3.2) on the metric space

$$\mathcal{M}_{2p} = \{U(\cdot) \in L^1_+(\mathbb{R}), \int_{-\infty}^{+\infty} |x|^{2p} u(x)dx < \infty\}$$

endowed with the p–*Wasserstein distance*

$$W_{2p}(U,V) = \inf \left[\int_{-\infty}^{+\infty} |x - Tx|^{2p} U(x)dx \right]^{\frac{1}{2p}}, \tag{5.3.3}$$

where the infimum is taken over the *admissible* maps $T : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} \psi(x)V(x)dx = \int_{-\infty}^{+\infty} \psi(Tx)U(x)dx, \quad \text{for all } \psi \in C^0(\mathbb{R}). \tag{5.3.4}$$

The condition (5.3.4) is often referred as T being a *push–forward* of the measure Udx onto the measure Vdx (notation: $V = T_\sharp U$).

The precise definition of the Wasserstein distances comes from a *relaxed* variational problem, the so–called Monge–Kantorowich mass transportation problem. More precisely, the set of admissible maps defined in (5.3.4) is embedded into the set of all probability measures π on $\mathbb{R}^2$ with marginals given by Udx and Vdx. The cost defined in the definition (5.3.3) is then converted into

$$\left[\int\int_{\mathbb{R}^2} (x-y)^{2p} \mu(dx,dy) \right]^{\frac{1}{2p}}.$$

It has been proved (in a much more general context! See [Vil03] and the references therein), that the optimal measure π^*, which minimizes the relaxed variational problem, is supported on the graph of a map $T^* : \mathbb{R} \to \mathbb{R}$. Hence, the infimum in (5.3.3) is actually a minimum. We refer to [Vil03] for a detailed explanation of these topics. For further reference, we only recall the property

$$p \leq q \quad \Rightarrow \quad W_{2p}(U,V) \leq W_{2q}(U,V). \tag{5.3.5}$$

In one space dimension, the optimal mat T^* can be expressed in a very simple way (see also [CT03, CGT03]). Given two probability measures $Udx, Vdx \in \mathcal{M}_{2p}$, we define the distribution functions

$$F(x) = \int_{-\infty}^{x} U(y)dy \quad G(x) = \int_{-\infty}^{x} V(y)dy,$$

and their pseudo–inverses $F^{-1}, G^{-1} : [0,1] \to \mathbb{R}$

$$F^{-1}(\rho) = \inf\{\omega : \ F(\omega) > \rho\} \quad G^{-1}(\rho) = \inf\{\omega : \ G(\omega) > \rho\}$$

(eventually F^{-1} and G^{-1} may attain the values $\pm\infty$ at $\rho = 0$ or at $\rho = 1$). Then, it can be easily proved that the optimal map T^* between Udx and Vdx is

$$T^* = G^{-1} \circ F.$$

Hence, by writing down the definition (5.3.3) of Wasserstein distance in terms of the optimal T^*, and after a change of variable, we get

$$W_{2p}(U,V) = \left[\int_0^1 \left|F^{-1}(\rho) - G^{-1}(\rho)\right|^{2p} d\rho\right]^{\frac{1}{2p}}. \tag{5.3.6}$$

Thanks to the monotonicity property (5.3.5), one can eventually send $p \to \infty$ to obtain

$$W_\infty(U,V) = \sup_{\rho \in (0,1)} \left|F^{-1}(\rho) - G^{-1}(\rho)\right|$$

One can easily see that, whenever U and V have compact support, the above quantity $W_\infty(U,V)$ controls the 'distance' between the supports of U and V respectively. More precisely, we have (see [CT03])

$$\begin{aligned} &|\inf\{supp\ U\} - \inf\{supp\ V\}| \leq W_\infty(U,V) \\ &|\sup\{supp\ U\} - \sup\{supp\ V\}| \leq W_\infty(U,V) \end{aligned} \tag{5.3.7}$$

Let us now turn back to the rescaled equation (5.3.2). From the results in the previous section, we already know (in some sense), that the evolution of the solution v for large s is dominated by the power nonlinearity u^m. Thus, we also expect that the Wasserstein distance between the solution $v(\cdot, s)$ of equation (5.3.2) and the corresponding Barenblatt profile v^∞ with mass 1 (defined in (5.2.11)) tends to zero as s goes to $+\infty$. In order to study the evolution of such a quantity, because of all the above observations about the one–dimensional case, we set

$$F(y,s) = \int_{-\infty}^{y} v(z,s)dz$$
$$G(y,s) = \int_{-\infty}^{y} v^\infty(z)dz.$$

Let $F^{-1}, G^{-1} : (0,1) \to \mathbb{R}$ be the pseudo–inverses of F and G respectively. Then, F^{-1} satisfies the following equation (similar computations can be found in [CT03, CGT03]),

$$\frac{\partial F^{-1}}{\partial s} = -F^{-1} - \frac{\partial}{\partial \rho}\left\{\left[\frac{\partial F^{-1}}{\partial \rho}\right]^{-m} + e^{ms}\psi\left(e^{-s}\left(\frac{\partial F^{-1}}{\partial \rho}\right)^{-1}\right)\right\}, \quad (5.3.8)$$

while G^{-1} satisfies

$$G^{-1} + \frac{\partial}{\partial \rho}\left[\left(\frac{\partial G^{-1}}{\partial \rho}\right)^{-m}\right] = 0. \quad (5.3.9)$$

We next state our result concerning Wasserstein distances.

Theorem 5.3.1 *Let $\phi(u)$ satisfy conditions **(NL1)**, **(NL2)**, and **(SD)** above.*

(a) Let $v(y,s)$ be the solution to (5.3.2) with $u_0 \in L^1_+(\mathbb{R})$ having mass 1 and finite second moment. Let v^∞ be the rescaled Barenblatt profile with mass 1 defined by (5.2.2). Then, for any $p \geq 1$, the following holds

$$W_{2p}(v(s), v^\infty) = \left[\int_0^1 \left|F^{-1}(\rho) - G^{-1}(\rho)\right|^{2p} d\rho\right]^{\frac{1}{2p}} \leq Ce^{-s}, \quad (5.3.10)$$

where $C = C_0 + W_{2p}(u_0, v^\infty)$ and C_0 depends only on ϕ.

(b) Let $v(y,s)$ be the solution to (5.3.2) with $u_0 \in L^1_+(\mathbb{R})$ having mass 1 and compact support. Let v^∞ be the rescaled Barenblatt profile with mass 1 defined by (5.2.2). Then

$$W_\infty(v(s), v^\infty) \leq Ce^{-s}, \quad (5.3.11)$$

where $C = C_0 + W_\infty(u_0, v^\infty)$ and C_0 depends only on ϕ.

In the original variables (5.3.1), the part (b) of the previous theorem provides our result concerning with the speed of propagation of the support of *any* solution $u(x,t)$ having compactly supported initial datum u_0. Indeed, since the support of the Barenblatt profile is a ball of radius $C(t+1)^{\frac{1}{m+1}}$ for some fixed constant $C>0$, we easily obtain the following result.

Corollary 5.3.2 *Let $u(x,t)$ be the solution to (5.3.1) with $u_0 \in L^1_+(\mathbb{R})$ having compact support. Then, there exist two positive constants $C_1 < C_2$ such that*

$$\begin{aligned} &|\inf\{supp\ u(t)\} - \inf\{supp\ u^\infty(t)\}| \leq C_1 \\ &|\sup\{supp\ u(t)\} - \sup\{supp\ u^\infty(t)\}| \leq C_2 \end{aligned} \tag{5.3.12}$$

Remark 5.3.3 We found a lot of references in the literature concerning the finite speed of propagation property in slow diffusion equations (see [Kne77, Kal87] for the general nonlinear case). Most of them are based on heavy analytic tools. Our result is more complete in general nonlinear case, and covers a wide class of nonlinearities. Moreover, our technique seems to be applied to this problem in a very natural way.

5.3.2 Proof of Theorem 5.3.1

To perform the proof of theorem 5.3.1, we compute the evolution of the Wasserstein distance $W_{2p}(v(s), v^\infty)$ by means of the one–dimensional representation formula (5.3.6). The calculations below are formal, in the sense that we should need the pseudo–inverse function F^{-1} to be smooth enough. We observe that this occurs when the initial datum u_0 is supported on a interval. We could make this argument rigorous by means of standard approximation tools (see [CGT03]). We skip these details and suppose that F^{-1} is smooth. Moreover, we need to know a priori that the speed of propagation of the support of the solution is finite. This property, which actually characterizes slow diffusion equations, was proved by Kalashnikov, Oleinik and Yiu–Lin (see [Kal87] and the references therein). Using the notations of the previous subsection, thanks to (5.3.9) and after integration by parts, we have

$$
\begin{aligned}
&\frac{d}{ds}\int_0^1 \left[F^{-1}-G^{-1}\right]^{2p} d\rho = 2p\int_0^1 \left[F^{-1}-G^{-1}\right]^{2p-1}\frac{\partial}{\partial s}F^{-1}(\rho,s)d\rho\\
&= 2p\int_0^1\left[F^{-1}-G^{-1}\right]^{2p-1}\left[-F^{-1}-\frac{\partial}{\partial\rho}\left(\left(\frac{\partial F^{-1}}{\partial\rho}\right)^{-m}+e^{ms}\psi\left(e^{-s}\left(\frac{\partial F^{-1}}{\partial\rho}\right)^{-1}\right)\right)+\right.\\
&\quad\left.+G^{-1}+\frac{\partial}{\partial\rho}\left(\left(\frac{\partial G^{-1}}{\partial\rho}\right)^{-m}\right)\right]\\
&= -2p\int_0^1\left[F^{-1}-G^{-1}\right]^{2p}d\rho-2p(2p-1)\int_0^1\left[F^{-1}-G^{-1}\right]^{2p-2}\left(\frac{\partial F^{-1}}{\partial\rho}-\frac{\partial G^{-1}}{\partial\rho}\right)\times\\
&\quad\times e^{ms}\left[\phi\left(e^{-s}\left(\frac{\partial F^{-1}}{\partial\rho}\right)^{-1}\right)-\phi\left(e^{-s}\left(\frac{\partial G^{-1}}{\partial\rho}\right)^{-1}\right)\right]d\rho\\
&\quad-2p\int_0^1\left[F^{-1}-G^{-1}\right]^{2p-1}\frac{\partial}{\partial\rho}\left(e^{ms}\psi\left(e^{-s}\left(\frac{\partial G^{-1}}{\partial\rho},s\right)^{-1}\right)\right). \qquad (5.3.13)
\end{aligned}
$$

We observe that, due to the compact support of the solutions, the boundary term coming from integration by parts disappears (see [CT03, CGT03]). In fact, this boundary term is given by

$$
\begin{aligned}
&\sum_{i=0,1}(-1)^i 2p\left[F^{-1}(i,s)-G^{-1}(i)\right]^{2p-1}\left[\left(\frac{\partial F^{-1}}{\partial\rho}\right)^{-m}(i,s)+\right.\\
&\left.e^{ms}\psi\left(e^{-s}\left(\frac{\partial F^{-1}}{\partial\rho}\right)^{-1}\right)(i,s)-\left(\frac{\partial G^{-1}}{\partial\rho}\right)^{-m}(i,s)\right].
\end{aligned}
$$

The first bracket is bounded at any s because of the finite speed of propagation property of the solutions. The second bracket is a sum of positive powers of the solution v and of the Barenblatt function v^∞ evaluated at the boundary of their support respectively. Hence, this second bracket equals zero.

Now, since the function ϕ is increasing, the second integral at the end of (5.3.13) is nonnegative. This observation is the key point in this computation (see again [CGT03]). In fact, thanks to this we can get rid of the nonlinearity term, and we have only to estimate the term depending on the Barenblatt profile, which is known. Indeed, after some calculations in the very last term of (5.3.13), due to (5.3.9), we obtain the following inequality

$$\frac{d}{ds}\int_0^1 \left[F^{-1}-G^{-1}\right]^{2p} d\rho \le -2p\int_0^1 \left[F^{-1}-G^{-1}\right]^{2p} d\rho$$
$$-\frac{2p}{m}e^{(m-1)s}\int_0^1 \left[F^{-1}-G^{-1}\right]^{2p-1} G^{-1}\psi'\left(e^{-s}\left(\frac{\partial G^{-1}}{\partial\rho}\right)^{-1}\right)\left(\frac{\partial G^{-1}}{\partial\rho}\right)^{m-1} d\rho.$$

Because of the condition **(NL2)** on ψ, we can assume that $\psi(u) = u^n g(u)$, with $g'(u) = O(u^k)$, $k > -1$, as $u \to 0$. Then, it follows that $\psi'(u) = O(u^{n-1})$, as $u \to 0$. Hence, thanks to Hölder inequality, the last integral above can be estimated from above by the term

$$C(v^\infty)pe^{-(n-m)s}\left(\int_0^1 \left|F^{-1}-G^{-1}\right|^{2p-1} d\rho\right)^{\frac{2p-1}{2p}},$$

where the constants $C(v^\infty)$ is given by

$$C(v^\infty) = \|v^\infty\|_{L^\infty(\mathbb{R})}\max\{|\inf\{supp\ v^\infty\}|, |\sup\{supp\ v^\infty\}|\}$$

(this quantity depends only on the mass and on the exponent m). We now apply the variation of constants formula in order to get the rate of convergence to zero of $W_p(v(s), v^\infty)$. In order to perform this task, we set for simplicity

$$\mathcal{X}_p(s) = \int_0^1 \left[F^{-1}(\rho,s)-G^{-1}(\rho)\right]^{2p} d\rho.$$

Hereafter, C denotes a fixed positive constant independent on p and s. So far we have proved that

$$\frac{d}{ds}\mathcal{X}_p(s) \le -2p\ \mathcal{X}_p(s) + 2p\ Ce^{-(n-m)s}\mathcal{X}_p(s)^{\frac{2p-1}{2p}}. \tag{5.3.14}$$

By Young inequality we get

$$\frac{d}{ds}\mathcal{X}_p(s) \le -2p\ \left(1 - Ce^{-(n-m)s}\right)\mathcal{X}_p(s) + Ce^{-(n-m)s}.$$

In a similar fashion to the computation of the entropy in the previous section, by means of the variation of constants formula we easily obtain

$$\mathcal{X}_p(s) \le \left(\mathcal{X}_p(0) + C\right)e^{-\min\{2p,(n-m)\}s}. \tag{5.3.15}$$

In case that $n - m < 2p$, the exponential rate of convergence in (5.3.15) can be improved iteratively by substituting the above inequality in the last addend of (5.3.14), until it reaches the value e^{-2ps}. Obviously, the number of steps depends on p. We have thus proved (5.3.10). The inequality (5.3.11), then, easily follows by sending $p \to \infty$.

Remark 5.3.4 It is worth to remark that in the above computation we don't need the hypothesis **(NL3)** on the nonlinearity, since we don't need the apriori estimates (5.1.8)–(5.1.9) on the solution v (which are consequences of the L^1–L^∞ regularizing effect), which was one the tools in the proof of the entropy dissipation result in the previous section.

Chapter 6

A small perturbation result for nonlinear diffusion far from vacuum

This chapter contains a stability under small perturbation result for the porous medium equation far from vacuum.

6.1 Statement of the problem and result

In this chapter we prove the asymptotic stability of certain *caloric* self–similar solutions to the generalized porous medium equation

$$\rho_t = p(\rho)_{xx}, \tag{6.1.1}$$

where the pressure p is smooth and strictly increasing. By caloric self–similar solutions we mean a class of similarity solutions of the form

$$\widetilde{\rho}(x,t) = f\left(\frac{x^2}{t+1}\right)$$

satisfying the limiting conditions

$$\widetilde{\rho}(\pm\infty, t) = \rho^{\pm},$$

for strictly positive ρ^+, ρ^-. For the existence of such solutions we refer to [AP71, AP74, vDP77, vDP77]. We recall here that for the caloric profile $\widetilde{\rho}$

the following estimates hold

$$\begin{aligned}\left|\frac{\partial^{\alpha+\beta}\widetilde{\rho}(t)}{\partial x^\alpha \partial t^\beta}\right|_\infty =& O(\delta)\frac{1}{(t+1)^{\frac{\alpha}{2}+\beta}} \quad \alpha,\beta>0\\ \int_{-\infty}^{+\infty}\left|\frac{\partial^{\alpha+\beta}\widetilde{\rho}(x,t)}{\partial x^\alpha \partial t^\beta}\right|^2 dx =& O(\delta^2)\frac{1}{(t+1)^{\alpha+2\beta-\frac{1}{2}}} \quad \alpha,\beta>0,\end{aligned} \tag{6.1.2}$$

where $\delta = |\rho^+ - \rho^-|$. Let us denote by $\check{\rho}$ the solution to the same equation (6.1.1) with the same limiting conditions at infinity and with the initial datum given by a small perturbation of $\widetilde{\rho}(x,0)$. Let us denote

$$r(x,t) = \check{\rho}(x,t) - \widetilde{\rho}(x+x_0,t),$$

where x_0 will be determined later on. By integrating w.r.t. x the equation satisfied by r, we get

$$\frac{d}{dt}\int_{-\infty}^{+\infty} r(x,t)dx = \Big[p(\check{\rho}(x,t)) - p(\widetilde{\rho}(x+x_0,t))\Big]_x\Big|_{-\infty}^{+\infty} = 0\ ,$$

Thus, if one has

$$\int_{-\infty}^{+\infty}\Big[\check{\rho}_0(x) - \widetilde{\rho}_0(x+x_0)\Big]\,dx = 0\ ,$$

it follows both

$$x_0 = \frac{1}{\rho^+ - \rho^-}\int_{-\infty}^{+\infty}\left[\check{\rho}_0(x) - \widetilde{\rho}_0(x)\right]dx \tag{6.1.3}$$

and

$$\int_{-\infty}^{+\infty} r(x,t)dx = 0.$$

Let us define the primitive variable

$$R(x,t) = \int_{-\infty}^{x} r(\xi,t)d\xi\ , \tag{6.1.4}$$

which satisfies the following problem

$$\begin{cases} R_t = p(\widetilde{\rho}+R_x)_x - p(\widetilde{\rho})_x \\ R(x,0) = \displaystyle\int_{-\infty}^{x}[\check{\rho}_0(\xi) - \widetilde{\rho}_0(\xi+x_0)]d\xi \\ R(\pm\infty,t) = 0. \end{cases} \tag{6.1.5}$$

Then, the small perturbation analysis with respect to the caloric self-similar solution is given by the following result.

Theorem 6.1.1 *Let us suppose that $\|R(0)\|_5^2$ is sufficiently small. Then, for any $t \geq 0$, we have*

$$\sum_{k=0}^{5}(t+1)^k\|R^{(k)}(t)\|^2 + \int_0^t (\tau+1)^k\|R^{(k+1)}(\tau)\|^2 d\tau \leq C\|R(0)\|_5^2. \quad (6.1.6)$$

The proof of the Theorem (6.1.1) will be given in the next section.

6.2 The Proof of the Theorem 6.1.1

In this section we proof the asymptotic stability result (6.1.6) by means of a continuation principle. We start with the a priori condition

$$\sup_{0\leq t\leq T}\sum_{k=0}^{5}(1+t)^k\|R^{(k)}(t)\|^2 \leq \sigma. \quad (6.2.1)$$

Lemma 6.2.1 *Suppose $\sigma \ll 1$. Then*

$$\|R(t)\|^2 + \int_0^t \|R_x(s)\|^2 ds \leq O(1)\|R_0\|^2. \quad (6.2.2)$$

Proof. By multiplying the first equation in (6.1.5) by R and after integration over $\mathbb{R}$, we get

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|R(t)\|^2 &= \int_{-\infty}^{+\infty} [p(\widetilde{\rho}+R_x)_x - p(\widetilde{\rho})_x]\, R dx = \\
&= -\int_{-\infty}^{+\infty} [p(\widetilde{\rho}+R_x) - p(\widetilde{\rho})]\, R_x dx = \\
&= -\int_{-\infty}^{+\infty} \left[p'(\widetilde{\rho})R_x^2 + \mathcal{R}_1(p,\widetilde{\rho},R_x)R_x\right] dx \leq \\
&\leq -O(1)\|R_x(t)\|^2 + O(\sigma)\|R_x(t)\|^2.
\end{aligned}$$

We denoted by $\mathcal{R}_1(p,\widetilde{\rho},R_x)$ the remainder in the first order Taylor expansion of p' around $\widetilde{\rho}$. The last inequality is due to the uniform boundedness of the coefficient $p''(\zeta)$ when $\zeta \in (\widetilde{\rho},\widetilde{\rho}+R_x)$ (as a consequence of the maximum principle). Then, we integrate over $[0,t]$ and get the desired estimate (6.2.2). □

Lemma 6.2.2 *Suppose $\sigma \ll 1$. Then we have*

$$(1+t)\|r(t)\|^2 + \int_0^t (1+s)\|r_x(s)\|^2 ds \leq O(1)\|R_0\|_1^2. \quad (6.2.3)$$

Proof. By differentiating the equation (6.1.5) w.r.t. x, we get

$$\begin{aligned} r_t =& p(\check{\rho})_{xx} - p(\widetilde{\rho})_{xx} = (p'(\check{\rho})\check{\rho}_x - p'(\widetilde{\rho})\widetilde{\rho}_x)_x = \\ =& \left[p'(\check{\rho})r_x + (p'(\check{\rho}) - p'(\widetilde{\rho}))\,\widetilde{\rho}_x\right]_x . \end{aligned} \tag{6.2.4}$$

We multiply (6.2.4) by $(1+t)r$ and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned} &\frac{d}{dt}\left[(1+t)\frac{\|r(t)\|^2}{2}\right] - \|r(t)\|^2 = \\ =&(1+t)\int_{-\infty}^{+\infty} (p'(\check{\rho})r_x)_x\, r dx + (1+t)\int_{-\infty}^{+\infty} \left((p'(\check{\rho}) - p'(\widetilde{\rho}))\,\widetilde{\rho}_x\right)_x r dx \\ =&-(1+t)\int_{-\infty}^{+\infty} p'(\widetilde{\rho}) r_x^2 dx - (1+t)\int_{-\infty}^{+\infty} (p'(\check{\rho}) - p'(\widetilde{\rho}))\,\widetilde{\rho}_x r_x dx. \end{aligned}$$

Finally, by integrating w.r.t. to time, we get

$$\begin{aligned} &(1+t)\frac{1}{2}\|r(t)\|^2 + \int_0^t (1+s)\|r_x(s)\|^2 ds \le O(1)\|r(0)\|^2 + \\ &+O(1)\int_0^t \|r(s)\|^2 ds + O(\delta)\int_0^t (1+s)^{1/2}\|r(s)\|\|r_x(s)\| ds \le \\ &\le O(1)\|R(0)\|_1^2 + O(1)\int_0^t \left[\|r(s)\|^2 + O(\delta)(1+s)\|r_x(s)\|^2\right] ds \le \\ &\le O(1)\|R(0)\|_1^2 + O(\delta)\int_0^t (1+s)\|r_x(s)\|^2 ds, \end{aligned}$$

where we have used (6.1.2) and Lemma 6.2.1. Thus, for $\delta \ll 1$, we have the desired estimate (6.2.3).□

Let us write the equation satisfied by r_x:

$$r_{xt} = (p'(\check{\rho})\check{\rho}_x - p'(\widetilde{\rho})\widetilde{\rho}_x)_{xx} . \tag{6.2.5}$$

Hence, we obtain the following Lemma.

Lemma 6.2.3 *Let $\sigma \ll 1$. Then*

$$(1+t)^2\|r_x(t)\|^2 + \int_0^t (1+s)^2\|r_{xx}(s)\|^2 ds \le O(1)\|R(0)\|_2^2. \tag{6.2.6}$$

Proof. From (6.2.5) we have

$$r_{xt} - (p'(\check{\rho})r_x)_{xx} = \left[(p'(\check{\rho}) - p'(\widetilde{\rho}))\,\widetilde{\rho}_x\right]_{xx} .$$

We multiply by $(1+t)^2 r_x$ and integrate over $\mathbb{R}$ to get

$$\frac{d}{dt}\left[(1+t)^2\frac{1}{2}\|r_x(t)\|^2\right]-(1+t)\|r_x(t)\|^2$$

$$+(1+t)^2\int_{-\infty}^{+\infty} p'(\check{\rho})r_{xx}^2dx=-(1+t)^2\int_{-\infty}^{+\infty}[p''(\check{\rho})\check{\rho}_x r_x r_{xx}$$

$$+\left(p'(\check{\rho})-p'(\widetilde{\rho})\right)\widetilde{\rho}_{xx}r_{xx}+\left(p'(\check{\rho})-p'(\widetilde{\rho})\right)_x\widetilde{\rho}_x r_{xx}]\,dx.$$

Then, by integrating w.r.t. time, we obtain

$$(1+t)^2\|r_x(t)\|^2+\int_0^t(1+s)^2\|r_{xx}(s)\|^2ds\leq$$

$$\leq O(1)\|r_x(0)\|^2+O(1)\int_0^t(1+s)\|r_x(s)\|^2ds$$

$$+O(1)\int_0^t(1+s)^2\int_{-\infty}^{+\infty}\left[r_x^2r_{xx}+\widetilde{\rho}_x r_x r_{xx}+r r_{xx}\widetilde{\rho}_{xx}\right.$$

$$\left.+\,r_x\widetilde{\rho}_x r_{xx}+r\widetilde{\rho}_x^2 r_{xx}\right]dxds\leq O(1)\|R(0)\|_2^2$$

$$+O(1)\int_0^t(1+s)^{1/2}|r_x(s)|_\infty\left[(1+s)\|r_x(s)\|^2+(1+s)^2\|r_{xx}(s)\|^2\right]ds$$

$$+O(\delta)\int_0^t\left[(1+s)\|r_x(s)\|^2+(1+s)^2\|r_{xx}(s)\|^2\right.$$

$$\left.+\,\|r(s)\|^2+(1+s)^2\|r_{xx}(s)\|^2+O(\delta)(1+s)\|r_{xx}(s)\|^2\right]ds,$$

where we have used Young inequality. Thus, using (6.2.1), (6.2.2) and (6.2.3), together with $\delta \ll 1$, we get

$$(1+t)^2\|r_x(t)\|^2+\int_0^t(1+s)^2\|r_{xx}(s)\|^2ds\leq O(1)\|R(0)\|_2^2+$$

$$+O(\sigma)\int_0^t\left[(1+s)^2\|r_{xx}(s)\|^2+(1+s)\|r_x(s)\|^2\right]ds.$$

Finally, by means of (6.2.3) and since $\sigma \ll 1$, we get the desired result (6.2.6).□

In order to complete the energy estimate (6.1.6) we have to carry out the time dacay estimates for the higher order derivatives r_{xx}, r_{xxx}, r_{xxxx}, , which can be done by following the same technique as above. We omit the details about this calculations.

Chapter 7

The viscous Burgers Equation

This chapter has to do with the classical viscous Burgers' equation described in section 1.8 of the introduction. We use the entropy dissipation approach to recover optimal rates of convergence towards diffusive waves. In the next section 7.1 we show how this equation inherits the gradient flow structure of the heat equation (cfr. section 1.7) by means of the famous Hopf–Cole transformation. Section 7.2 is devoted to the statement and the proof of the results concerning with the relative entropy approach. In the last section we prove a simple stability result for the 2–Wasserstein distance.

7.1 The Hopf–Cole Transformation

The time–asymptotic analysis for the viscous Burgers' equation

$$u_t + uu_x = u_{xx}, \tag{7.1.1}$$

is based on the classical Hopf–Cole transformation, which reduces (7.1.1) to the linear heat equation. In this section, we first explain the classical process used to obtain the typical intermediate–asymptotic states for this equation (see [Hop50]). Afterwards, we set up a different framework of notations, in order to convert (7.1.1) into the linear Fokker–Planck equation.

7.1.1 The classical setting

We consider the initial value problem

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = u_{xx} \\ u(x,0) = u_0(x) \end{cases} \tag{7.1.2}$$

where $x \in \mathbb{R}$, $t > 0$, $u \in \mathbb{R}$ and u_0 is a given function in $L^1(\mathbb{R})$. It is well known (see [Hop50]) that, if $u(\cdot, t)$ is the solution of (7.1.2) at a positive time t, then the following conservation property holds

$$\int_{-\infty}^{+\infty} u(x,t)dx = \int_{-\infty}^{+\infty} u_0(x)dx.$$

We denote in the following

$$\int_{-\infty}^{+\infty} u_0(x)dx = M. \tag{7.1.3}$$

The unique solution of (7.1.2) can found explicitely by means of the Hopf–Cole tranformation

$$\begin{aligned} \phi(x,t) &= \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u(y,t)dy\right) \\ u(x,t) &= -2\frac{\phi_x(x,t)}{\phi(x,t)}, \end{aligned} \tag{7.1.4}$$

which reduces (7.1.2) to the linear heat equation $\phi_t = \phi_{xx}$. Indeed, via the convolution formula for the heat equation, one obtains

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} e^{-\frac{|x-y|^2}{2t} - \int_{-\infty}^{y} u_0(z)dx} dy}{\int_{-\infty}^{+\infty} e^{-\frac{|x-y|^2}{2t} - \int_{-\infty}^{y} u_0(z)dx} dy}. \tag{7.1.5}$$

Moreover, one can construct a diffusion wave type solution U_M for (7.1.2) with mass M, corresponding to the solution

$$\Phi_M(x,t) = 1 - C_M \int_{-\infty}^{x(2t+1)^{-1/2}} e^{-\frac{\zeta^2}{2}} d\zeta \tag{7.1.6}$$

of the heat equation. The costant C_M in the above formula is determined in order to match condition (7.1.3).

To construct Φ_M, we observe that the spatial derivative $z(x,t) = -\phi_x(x,t)$ (ϕ given by (7.1.4)) satisfies again the heat equation $z_t = z_{xx}$ and it has an initial datum $z_0 \in L^1(\mathbb{R})$. Hence, one can consider the gaussian solution of the heat equation with the same mass as z_0, namely

$$Z_M(x,t) = C_M(2t+1)^{-1/2} e^{-\frac{x^2}{2(2t+1)}}, \tag{7.1.7}$$

and write the corresponding Φ_M by taking the spatial primitive of Z_M (the limiting conditions for Φ are determined by the conservation of mass). Finally, by replacing ϕ with Φ_M into (7.1.4), one obtains the diffusion wave

$$U_M(x,t) = 2C_M(2t+1)^{-\frac{1}{2}} \frac{\exp\left(-\frac{x^2}{2(2t+1)}\right)}{1 - C_M \int_{-\infty}^{x(2t+1)^{-1/2}} e^{-\frac{\zeta^2}{2}} d\zeta}. \tag{7.1.8}$$

7.1.2 Intermediate asymptotics and zero–viscosity regime

Let us consider equation (7.1.1) with a small viscosity parameter $\mu > 0$, i. e.

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = \mu u_{xx} \\ u(x,0) = u_0(x), \end{cases} \tag{7.1.9}$$

with initial datum $u_0 \in L^1(\mathbb{R})$ eventually sign–changing. In the limit as $\mu \to 0$, one recovers the unique entropy solution of the Cauchy problem for the inviscid Burgers' equation

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,0) = u_0(x). \end{cases} \tag{7.1.10}$$

It is well–known that the quantities

$$p = -\inf_{x\in\mathbb{R}} \int_{-\infty}^{x} u_0(y)dy \quad q = \sup_{x\in\mathbb{R}} \int_{x}^{+\infty} u_0(y)dy$$

are invariant for equation (7.1.10). Hence one can construct the N–wave type solution

$$N_{p,q}(x,t) = \begin{cases} \frac{x}{t} & \text{if } -\sqrt{2pt} \le x \le \sqrt{2qt} \\ 0 & \text{otherwise,} \end{cases} \tag{7.1.11}$$

which turns out to be the an attractor in the L^1 norm for any solution of (7.1.10) with initial datum having compact support and negative and positive masses given by p and q respectively (see [Lax57, DP75]). In case of nonnegative data with mass M, (7.1.11) becomes

$$N_M(x,t) = \begin{cases} \frac{x}{t} & \text{if } 0 \le x \le \sqrt{2Mt} \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.12}$$

As we pointed out before, in the case of a nonnegative initial datum u_0 for the viscous Burgers' equation (7.1.2), the mass $M = \int u_0$ is the only information needed to construct the asymptotic diffusive wave U_M given in (7.1.8). With the small viscosity parameter μ as in (7.1.9), this solution becomes

$$U_{M,\mu}(x,t) = 2\mu C_{M,\mu}(2t+1)^{-\frac{1}{2}} \frac{\exp\left(-\frac{x^2}{2\mu(2t+1)}\right)}{1 - C_{M,\mu}\int_{-\infty}^{x(2t+1)^{-1/2}} e^{-\frac{\zeta^2}{2\mu}} d\zeta}, \tag{7.1.13}$$

with $C_{M,\mu} = 1 - e^{-\frac{1}{2\mu}}$. It can be easily checked that, for any fixed $(x,t) \in \mathbb{R} \times \mathbb{R}_+$,

$$U_{M,\mu}(x,t) \longrightarrow N_M(x,t+1/2) \quad \text{as } \mu \to 0,$$

where N_M is given by (7.1.12), that is, the diffusive wave $U_{M,\mu}$ approximates a positive N–wave in the zero–viscosity limit. As pointed out by T.P. Liu in his introduction to [Liu85], this situation is an example of how the behavior of a nonlinear conservation law, at the level of diffusion waves, changes considerably with the presence of the viscosity (even when this is small). In the case the of Burgers' equation, an intermediate state for the viscous case (for general possibly sign-changing initial data) is provided by

$$\widetilde{U}(x,t) = -2\mu \frac{\widetilde{\phi}(x,t)}{1 - \int_{-\infty}^{x} \widetilde{\phi}(y,t)dy}, \tag{7.1.14}$$

where

$$\widetilde{\phi}(x,t) = \frac{a}{\sqrt{2\pi\mu t}} e^{-\frac{x^2}{2\mu t}} - \frac{b}{\sqrt{2\pi\mu t}} e^{-\frac{x^2}{2\mu t}}$$

and

$$a = \frac{1}{2\mu} \int_{-\infty}^{+\infty} u_0^-(y) e^{-\frac{1}{2\mu}\int_{-\infty}^{y} u_0(z)dz} dy \quad b = \frac{1}{2\mu} \int_{-\infty}^{+\infty} u_0^+(y) e^{-\frac{1}{2\mu}\int_{-\infty}^{y} u_0(z)dz} dy.$$

For fixed (x,t), the function $\widetilde{U}$ tends to the N–wave defined in (7.1.11) as μ tends to zero. The very interesting paper by Kim and Tzavaras ([KT01]) provides a quantitative understanding of the long–time–small–viscosity interplay for the Burgers' equation. It turns out that that, when the viscosity is fixed small, at a first asymptotic stage the solution tends to take the shape of an approximate N–wave of the type (7.1.14) (thus, its behavior is mainly governed by convection). Then, at a very long time stage, the diffusion produces an interaction between the positive and the negative masses, and the smallest between them is consumed, while the profile of the solution tends to that of the diffusive wave U_M defined in (7.1.13).

In the present chapter we do not deal with the zero viscosity limit, even though an interesting problem could be the understanding of the limiting behavior of the estimates carried out in our work as μ approaches to zero.

We also mention that the entropy approach for the inviscid Burgers' equation has been recently used by Dolbeault and Escobedo in [DE], where the authors use a time–dependent scaling in order to view the N–wave type solution as a stationary state. The convergence towards equilibrium is then proven by means of an entropy functional which provides a decay in a weighted L^1–norm.

7.1.3 The time–dependent scaling

We are now interested in the study of the asymptotic convergence of the solution u of (7.1.2) towards U_M given by (7.1.8) as $t \to \infty$. To perform this task, we consider a time-dependent scaling which transforms this problem of the study of the asymptotic stability of a stationary state. This idea has been frequently employed in the study of the time–asymptotics for nonlinear diffusion equations (see, for instance, the pioneering paper by Barenblatt [Bar52]). More precisely, we set

$$\begin{aligned} y &= y(x,t) = xR(t)^{-1} \\ s &= s(t) = \log R(t) \\ u(x,t) &= R(t)^{-1}\rho(y(x,t), s(t)), \end{aligned} \tag{7.1.15}$$

where

$$R(t) = (2t+1)^{1/2}.$$

With this notation, (7.1.2) turns into the following Cauchy problem for the *Burgers–Fokker–Plank* equation

$$\begin{cases} \dfrac{\partial \rho}{\partial s} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \rho}{\partial y} + y\rho - \dfrac{\rho^2}{2}\right) \\ \rho(y,0) = \rho_0(y) = u_0(y). \end{cases} \tag{7.1.16}$$

The equation (7.1.16) is often referred to as the *viscous Burgers' equation in similarity variables.* Obviously, we have again the conservation of the mass

$$\int_{-\infty}^{+\infty} \rho(y,s)dy = \int_{-\infty}^{+\infty} \rho_0(y)dy = M.$$

To recover the suitable stationary solution with mass M of (7.1.16), we employ once again the Hopf–Cole transformation

$$\tau(y,s) = \exp\left(-\frac{1}{2}\int_{-\infty}^{y} \rho(\zeta,s)d\zeta\right), \tag{7.1.17}$$

which converts (7.1.16) into

$$\frac{\partial \tau}{\partial s} = \frac{\partial^2 \tau}{\partial y^2} + y\frac{\partial \tau}{\partial y}.$$

Hence, the spatial derivative

$$\psi(y,s) = -\frac{\partial \tau}{\partial y}(y,s) = \frac{1}{2}\rho(y,s)\exp\left(-\frac{1}{2}\int_{-\infty}^{y} \rho(\zeta,s)d\zeta\right) \tag{7.1.18}$$

satisfies the Cauchy problem for the linear *Fokker–Planck* equation

$$\begin{cases} \dfrac{\partial \psi}{\partial s} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi}{\partial y} + y\psi\right) \\ \psi(y,0) = \psi_0(y) = \frac{1}{2}\rho_0(y)\exp\left(-\frac{1}{2}\int_{-\infty}^{y}\rho_0(\zeta)d\zeta\right). \end{cases} \tag{7.1.19}$$

The inverse transformation of (7.1.18) is given by

$$\rho(y,s) = -2\frac{\psi(y,s)}{1-\int_{-\infty}^{y}\psi(\zeta,s)d\zeta}. \tag{7.1.20}$$

An easy computation gives

$$\int_{-\infty}^{+\infty}\psi_0(y)dy = -\int_{-\infty}^{+\infty}\frac{\partial}{\partial y}\left(\exp\left(-\frac{1}{2}\int_{-\infty}^{y}\rho_0(\zeta)d\zeta\right)\right)dy = 1-e^{-\frac{M}{2}} =: m.$$

Hence, conservation of the total mass for the Fokker–Planck equation (7.1.19), implies

$$\int_{-\infty}^{+\infty}\psi(y,s)dy = m$$

for any $s > 0$. It is well known that (7.1.19) has the unique Gaussian equilibrium Ψ_m with mass m, namely

$$\Psi_m(y) = \frac{m}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}. \tag{7.1.21}$$

By putting Ψ_m into (7.1.20), we recover our steady state for the Burgers–Fokker–Planck equation (7.1.16)

$$\rho_M^\infty(y) = \frac{\frac{2m}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}}{1-\frac{2m}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-\frac{\zeta^2}{2}}d\zeta}. \tag{7.1.22}$$

By returning back to the original variables x, t, u, it turns out that the stationary state ρ_M^∞ corresponds precisely to the diffusion wave U_M defined in (7.1.8).

We close this section by recalling an easy estimate for the function $\tau(y,t)$, defined by the Hopf–Cole transformation (7.1.17), which will be useful in the sequel. Since the initial datum for the original Burgers' equation u_0 belongs in $L^1(\mathbb{R})$, the quantities

$$p = -\inf_{x\in\mathbb{R}}\int_{-\infty}^{x}u_0(y)dy \tag{7.1.23}$$

$$q = \sup_{x\in\mathbb{R}}\int_{x}^{+\infty}u_0(y)dy = M + p$$

are then finite. In terms of τ, relations (7.1.23) provide the following property for $\tau(\cdot,0)$

$$e^{-\frac{(M+p)}{2}} \leq \tau(y,0) \leq e^{\frac{p}{2}},$$

for any $y \in \mathbb{R}$. Hence, by simple maximum principle, we obtain the estimate

$$e^{-\frac{(M+p)}{2}} \leq \tau(y,t) \leq e^{\frac{p}{2}}, \tag{7.1.24}$$

for all $(y,t) \in \mathbb{R} \times \mathbb{R}_+$.

As it was first proved by Hopf ([Hop50]), the large time behavior of solutions of the original problem (7.1.2) is described by the diffusion wave U_M defined by (7.1.8). Our purpose here is to investigate the large time behavior of $u(x,t)$ in terms of solutions of equation (7.1.16). Hence, hereafter we shall discuss the asymptotic stability of the stationary solution ρ_M^∞ of equation (7.1.16).

7.2 Trend to equilibrium in relative entropy

In this section we analyze the convergence towards the stationary profile ρ_M^∞ defined by (7.1.22) for the solution ρ of equation (7.1.16) with initial datum $\rho_0 \in L^1(\mathbb{R})$. We start by treating the case of non–negative initial data u_0. The general case will be covered later on. Our choice of the functionals used to control the distance between u and $\widetilde{u}$ are the relative entropy functionals

$$H_e(\rho(s)|\rho_M^\infty) = \int_{-\infty}^{+\infty} e\left(\frac{\rho(y,s)}{\rho_M^\infty(y)}\right)\rho_M^\infty(y)dy, \tag{7.2.1}$$

where $e: \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function satisfying the following conditions

$$\begin{aligned} & e(1) = 0 \\ & e''(h) \geq 0 \quad \text{for any } h \in \mathbb{R}_+, \quad e'' \text{ not identically } 0 \\ & (e''')^2 \leq \frac{1}{2} e'' e^{(IV)}. \end{aligned} \tag{7.2.2}$$

Such functions are called *entropy generating functions*, and the corresponding functionals H_e are called *admissible relative entropies* (see [AMTU01]). In particular, one can prove that functions e satisfying the above conditions are strictly convex. Moreover, since ρ and ρ_m^∞ have the same mass, we can obtain the same functional H_e by "normalizing" the generating function e in such a way that $e'(1) = 0$. As a consequence of the above conditions, each generating function satisfies (see [AMTU01], Lemma 2.6)

$$\chi(h) \leq e(h) \leq \varphi(h), \tag{7.2.3}$$

where we have denoted

$$\chi(h) = \alpha(h+\beta)\log\frac{h+\beta}{1+\beta} - \alpha(h-1), \qquad \varphi(h) = \mu_2(h-1)^2,$$

with μ_2, α, β nonnegative constants depending on $e''(1)$ and $e'''(1)$, namely

$$\mu_2 = e''(1), \qquad \alpha = -\frac{e''(1)^2}{e'''(1)}, \qquad \beta = -\frac{e''(1)+e'''(1)}{e'''(1)}. \tag{7.2.4}$$

The constants (7.2.4) are well defined if $e'''(1) \neq 0$. In the case $e'''(1) = 0$, one has to set $\chi(h) = \frac{\mu_2}{2}(h-1)^2$. Hence, the admissible relative entropy approach allows us to cover a large range of functionals, including the physical relative entropy $\int \rho \log \frac{\rho}{\rho_m^\infty} dx$. We refer to [AMTU01] for a detailed explanation of the mathematical properties of the relative entropies and their generating functions.

By means of so–called *Csiszár–Kullback* inequalities, one can control the L^1 norm of the difference $\rho - \rho_M^\infty$ in terms of the relative entropy functionals defined in (7.2.1).

Theorem 7.2.1 (Csiszár–Kullback) *There exists a positive constant C such that, for all functions $\rho_1, \rho_2 \in L^1_+(\mathbb{R})$, with*

$$\int_{\mathbb{R}} \rho_1(x)dx = \int_{\mathbb{R}} \rho_2(x)dx \tag{7.2.5}$$

and for all admissible generating functions e, we have

$$\|\rho_1 - \rho_2\|^2_{L^1(\mathbb{R})} \leq C H_e(\rho_1|\rho_2). \tag{7.2.6}$$

Moreover, in case of the quadratic generating function

$$e(h) = \varphi(h) = (h-1)^2,$$

the relation (7.2.6) holds for all $\rho_1 \in L^1(\mathbb{R})$ (eventually sign–changing and without requiring the integral condition (7.2.5)) and $\rho_2 \in L^1_+(\mathbb{R})$.

We refer to [AMTU00] for a detailed analysis of Csiszár–Kullback type inequalities.

Let us state our first result of convergence in relative entropy. In what follows we denote the primitives

$$F(y,s) := \int_{-\infty}^{y} \rho(\zeta,s)d\zeta, \quad F^\infty(y,s) := \int_{-\infty}^{s} \rho_M^\infty(\zeta,s)d\zeta. \tag{7.2.7}$$

Theorem 7.2.2 *Let ρ be the solution of (7.1.16) with $\rho_0 \in L^1_+(\mathbb{R})$. Let ρ^∞_M be given by (7.1.22). Let H_e be the admissible relative entropy functional generated by the function e. Then, there exists a positive constant C depending on the mass M such that the following estimate holds:*

$$\begin{aligned} H_e(\rho(s)|\rho^\infty_M) \leq C e^{-2s} \big[& H_e(\rho_0|\rho^\infty_M) + \|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^\infty(\mathbb{R})} \\ & + \|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^1(\mathbb{R})} \big] . \end{aligned} \tag{7.2.8}$$

Remark 7.2.3 An easy computation shows that the condition

$$F(\cdot,0) - F^\infty(\cdot,0) \in L^1(\mathbb{R}),$$

needed to control the bracket in the r.h.s. of (7.2.8), is equivalent to the conditions

$$\int_{-\infty}^0 \int_{-\infty}^x \rho_0(y) dy dx < +\infty \tag{7.2.9}$$

and

$$\int_0^{+\infty} \int_x^{+\infty} \rho_0(y) dy dx < +\infty. \tag{7.2.10}$$

The conditions (7.2.9)–(7.2.10) are satisfied, for instance, in the case of initial data with finite second moment, i. e.

$$\int_{-\infty}^{+\infty} |y|^2 \rho_0(y) dy < +\infty. \tag{7.2.11}$$

Now, it turns out that whenever ρ_0 has finite relative logarithmic entropy, then (7.2.11) holds (see [ABM96]). Hence, for all generating functions e we have

$$H_e\left(\rho_0|\rho^\infty_M\right) < +\infty \ \Rightarrow \ \int_{-\infty}^{+\infty} |y|^2 \rho_0(y) dy < +\infty.$$

Therefore, the decay rate statement of Theorem 7.2.2 is valid for initial data in $L^1_+(\mathbb{R})$ with finite relative entropy, without further assumptions.

To prove theorem 7.2.2, we recall more concepts concerning relative entropies. Let $\rho, \gamma \in L^1_+(\mathbb{R})$ such that $\int_{-\infty}^{+\infty} \rho(x) dx = \int_{-\infty}^{+\infty} \gamma(x) dx$ and let us denote $h(x) = \rho(x)/\gamma(x)$. The *entropy dissipation* generated by e is defined as

$$I_e(\rho|\gamma) = \int_{-\infty}^{+\infty} e''(h(y,s))(h_y(y,s))^2 \rho^\infty_M(y) dy.$$

Then, we recall the following *generalized logarithmic Sobolev inequality* (see [AMTU01, MV00]).

Theorem 7.2.4 *Let H_e be an admissible relative entropy with generating function e. Let $\rho, \gamma \in L^1_+(\mathbb{R}^d)$ be such that*

$$\int \rho(x)dx = \int \gamma(x)ds = M.$$

Then, the following inequality holds

$$H_e(\rho|\gamma) \leq \frac{1}{2} I_e(\rho|\gamma). \tag{7.2.12}$$

In the quadratic case, inequality (7.2.12) becomes a generalized Poincaré–type inequality. More precisely, if we set $h = \rho/\gamma$, we obtain

$$\int_{-\infty}^{+\infty} (h-1)^2 \gamma dx \leq C \int_{-\infty}^{+\infty} |h_x|^2 \gamma dx. \tag{7.2.13}$$

We remark that positivity of ρ is not needed for (7.2.13), i.e. it holds for $h = \rho/\gamma$ with $\rho \in L^1(\mathbb{R})$, $\gamma \in L^1_+(\mathbb{R})$ and $\int \rho dx = \int \gamma dx$. Inequalities (7.2.12) and (7.2.13) are crucial in the proof of the following theorem, which is the main ingredient for our results. Again, we refer to [AMTU01] for the proof.

Theorem 7.2.5 *Let ψ be the solution to the Cauchy problem for the Fokker Planck equation (7.1.19) with $\psi_0 \in L^1_+(\mathbb{R})$. Let Ψ_m be the stationary solution given in (7.1.21). Then, for any generating function e, the corresponding relative entropy functional $H_e(\psi(s)|\Psi_m)$ satisfies the following estimate*

$$H_e(\psi(s)|\Psi_m) \leq H_e(\psi_0|\Psi_m) e^{-2s}. \tag{7.2.14}$$

Moreover, in the quadratic case $\psi(h) = (h-1)^2$, (7.2.14) is also valid for an eventually sign–changing initial datum $v_0 \in L^1(\mathbb{R})$.

The following lemma is also proved in [AMTU01] (Lemma 2.9) and will be used in the sequel.

Lemma 7.2.6 *The generator e of any relative entropy functional H_e satisfies*

a) $\quad e(\sigma) \leq e(\sigma_0)\left(\frac{\sigma}{\sigma_0}\right)^2 + \mu\left(\frac{\sigma}{\sigma_0} - 1\right)(\sigma - 1), \quad \sigma \geq \sigma_0 > 0$

b) $\quad e(\sigma) \leq e(\sigma_0)\left(\frac{\sigma}{\sigma_0}\right) + \mu\left(\frac{\sigma}{\sigma_0} - 1\right)(\sigma - 1), \quad \sigma_0 \geq \sigma > 0,$

where $\mu = e''(1)$.

Now we can provide the proof of Theorem 7.2.2.

Proof of theorem 7.2.2. We write inequality (7.2.14) in terms of ρ and ρ_M^∞ by means of identity (7.1.18). We observe that

$$\rho(y,s) = \frac{2\psi(y,s)}{\tau(y,s)}$$
$$\rho_M^\infty(y) = \frac{2\Psi_m(y)}{\tau_m(y)},$$

where τ is given by (7.1.17) and $\tau_m(y) = 1 - \int_{-\infty}^{y} \Psi_m(\zeta)d\zeta$. Thus, we have

$$\int_{-\infty}^{+\infty} e\left(\frac{\rho(y,s)\tau(y,s)}{\rho_M^\infty(y)\tau_m(y)}\right)\rho_M^\infty(y)\tau_m(y)dy$$
$$\leq e^{-2s}\int_{-\infty}^{+\infty} e\left(\frac{\rho_0(y)\tau(y,0)}{\rho_M^\infty(y)\tau_m(y)}\right)\rho_M^\infty(y)\tau_m(y)dy. \tag{7.2.15}$$

We employ lemma 7.2.6 with

$$\sigma(y,s) = \frac{\rho(y,s)}{\rho_M^\infty(y)}, \qquad \sigma_0(y,s) = \frac{\rho(y,s)\tau(y,s)}{\rho_M^\infty(y)\tau_m(y)} .$$

We observe that estimate (7.1.24) ($p = 0$ in case of positive solutions) implies

$$\sup_{y,s}\max\left\{\left(\frac{\sigma(x,t)}{\sigma_0(x,t)}\right), \left(\frac{\sigma(x,t)}{\sigma_0(x,t)}\right)^2\right\} \leq e^M.$$

Hence, we have

$$H_e(\rho(s)|\rho_M^\infty(s)) = \int_{-\infty}^{+\infty} e\left(\frac{\rho(y,s)}{\rho_M^\infty(y)}\right)\rho_M^\infty(y)dy$$
$$\leq e^{\frac{3M}{2}}\int_{-\infty}^{+\infty} \psi\left(\frac{\rho(y,s)\tau(y,s)}{\rho_M^\infty(y)\tau_m(y)}\right)\rho_M^\infty(y)\tau_m(y)dy$$
$$+\mu\int_{-\infty}^{+\infty}(\tau_m(y)-\tau(y,s))\left(\rho(y,s)-\rho_M^\infty(y)\right)\tau(y,s)^{-1}dy := I_1 + I_2.$$

We estimate the term I_1 by means of (7.2.15),

$$I_1 \leq e^{\frac{3M}{2}}e^{-2s}\int_{-\infty}^{+\infty} e\left(\frac{\rho_0(y)\tau(y,0)}{\rho_M^\infty(y)\tau_m(y)}\right)\rho_M^\infty(y)\tau_m(y)dy.$$

By using again Lemma 7.2.6 with $\sigma(y) = \dfrac{\rho_0(y)\tau(y,0)}{\rho_M^\infty(y)\tau_m(y)}$, $\sigma_0(y) = \dfrac{\rho_0(y)}{\rho_M^\infty(y)}$, we obtain

$$
\begin{aligned}
I_1 &\leq e^{2M}e^{-2s}\left[\int_{-\infty}^{+\infty} e\left(\frac{\rho_0(y)}{\rho_M^\infty(y)}\right)\rho_M^\infty(y)dy\right.\\
&+\mu\int_{-\infty}^{+\infty}\frac{\tau(y,0)}{\tau_m(y)}\left(\tau(y,0)-\tau_m(y)\right)\left(\rho_0(y)-\rho_M^\infty(y)\right)dy\\
&\left.+\mu\int_{-\infty}^{+\infty}\frac{\rho_M^\infty(y)}{\tau_m(y)}\left(\tau(y,0)-\tau_m(y)\right)^2 dy\right]\\
&\leq e^{-2s}C(M,\mu)\left[H_e(\rho_0|\rho_M^\infty)\right.\\
&\left.+\|\rho_0-\rho_M^\infty\|_{L^1}\|\tau(\cdot,0)-\tau_m\|_{L^\infty(\mathbb{R})}+\|\tau(\cdot,0)-\tau_m\|^2_{L^\infty(\mathbb{R})}\right]\\
&\leq C(M,\mu)e^{-2s}\left[H_e(\rho_0|\rho_M^\infty)+\|\tau(\cdot,0)-\tau_m\|^2_{L^\infty(\mathbb{R})}\right], \qquad (7.2.16)
\end{aligned}
$$

where we have used the estimate (7.1.24) and the inequality (7.2.6). The constant $C(M,\mu)$ depends on M and μ. Let us estimate the integral term I_2 as follows,

$$
\begin{aligned}
I_2 &\leq \mu e^M\int_{-\infty}^{+\infty}|\tau_m(y)-\tau(y,s)|\,|\rho(y,s)-\rho_M^\infty(y)|\,dx\\
&\leq \frac{\mu e^M}{2}\left[\frac{1}{\varepsilon}\|\tau_m-\tau(s)\|^2_{L^\infty(\mathbb{R})}+\varepsilon\|\rho(s)-\rho_M^\infty\|^2_{L^1(\mathbb{R})}\right]\\
&\leq \frac{\mu e^M}{2}\left[\frac{1}{\varepsilon}\|\tau_m-\tau(s)\|^2_{L^\infty(\mathbb{R})}+C\varepsilon H_\psi(\rho(s)|\rho_M^\infty)\right], \qquad (7.2.17)
\end{aligned}
$$

where we have used once again inequality (7.2.6) and $\varepsilon > 0$ shall be fixed later on. The first term in the bracket of (7.2.17) is to be treated as follows. Using the notation in (7.1.15), we observe that

$$
\begin{aligned}
\tau(y,s) &= e^{-\frac{1}{2}\int_{-\infty}^{y}\rho(\zeta,s)d\zeta} = e^{-R(t(s))\frac{1}{2}\int_{-\infty}^{y}u(R(t(s))\zeta,t(s))d\zeta}\\
&= e^{-\frac{1}{2}\int_{-\infty}^{yR(t)}u(\zeta,t(s))d\zeta} =: \phi(x(y,s),t(s)).
\end{aligned}
$$

As we have seen in the previous section, the function $\phi(x,t)$ above satisfies the heat equation $\phi_t = \phi_{xx}$. We denote

$$
\widetilde{\phi}(x,t) = e^{-\frac{1}{2}\int_{-\infty}^{x}U_M(\zeta,t)d\zeta} = \tau_m(y(x,t),s(t)),
$$

$$
\overline{\phi} = \phi - \widetilde{\phi}.
$$

Hence, $\overline{\phi}$ satisfies $\overline{\phi}_t = \overline{\phi}_{xx}$ with initial datum $\overline{\phi}_0(x) = \tau(x,0) - \tau_m(x)$. The representation formula for the solution of the one–dimensional heat equation yields

$$\left|\overline{\phi}(x,t)\right| = \frac{C}{\sqrt{t}}\left|\int_{-\infty}^{+\infty}\overline{\phi}_0(y)e^{-\frac{(x-y)^2}{4t}}dy\right| \leq \frac{C}{\sqrt{t}}\|\overline{\phi}_0\|_{L^1(\mathbb{R})}. \tag{7.2.18}$$

Now, by (7.1.4) and since $0 \leq \int_{-\infty}^{x} u_0(y)dy \leq M$, there exists a constant K_M depending only on M such that

$$|\overline{\phi}_0(x)| \leq K_M|F(x,0) - F^\infty(x,0)|, \tag{7.2.19}$$

where F and F^∞ are defined by (7.2.7). Hence, there exists a fixed constant K_M' such that

$$\|\phi(t) - \widetilde{\phi}(t)\|^2_{L^\infty(\mathbb{R})} \leq \frac{K_M'}{t}\|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^1(\mathbb{R})}.$$

Moreover, since

$$\|\phi(t) - \widetilde{\phi}(t)\|^2_{L^\infty(\mathbb{R})} \leq \|\overline{\phi}_0\|^2_{L^\infty(\mathbb{R})},$$

we have, from (7.2.19),

$$\begin{aligned}\|\phi(t) - \widetilde{\phi}(t)\|^2_{L^\infty(\mathbb{R})} &\leq \frac{\widetilde{C}_M}{2t+1}\left[\|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^1(\mathbb{R})} + \|\overline{\phi}_0\|^2_{L^\infty(\mathbb{R})}\right]\\ &\leq \frac{\widetilde{K}_M}{2t+1}\left[\|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^1(\mathbb{R})} + \|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^\infty(\mathbb{R})}\right],\end{aligned} \tag{7.2.20}$$

for constants $\widetilde{C}_M, \widetilde{K}_M$ depending on M. In terms of the rescaled time s, (7.2.20) reads

$$\begin{aligned}\|\tau_m - \tau(s)\|^2_{L^\infty(\mathbb{R})} \leq e^{-2s}\widetilde{K}_M &\Big[\|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^1(\mathbb{R})}\\ &+\|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^\infty(\mathbb{R})}\Big].\end{aligned} \tag{7.2.21}$$

Hence, by choosing a sufficiently small $\varepsilon = \varepsilon(\mu, M) > 0$ in (7.2.17) and in view of (7.2.16), we obtain

$$\begin{aligned}H_e(\rho(s)|\rho_M^\infty) \leq Ce^{-2s}&\Big[H_e(\rho_0|\rho_M^\infty) + \|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^\infty(\mathbb{R})}\\ &+\|F(\cdot,0) - F^\infty(\cdot,0)\|^2_{L^1(\mathbb{R})}\Big]\end{aligned}$$

and the proof is complete. $\square$

The result in Theorem 7.2.2 is restricted to the case of positive initial data ρ_0 because the relative entropy functionals $H_e(\rho|\rho_M^\infty)$ are not defined for negative values of ρ, except for the one generated by the quadratic function $\varphi(h) = (h-1)^2$. It is possible to obtain a similar result for general sign–changing solutions and for the only case of quadratic entropy $H_\varphi(\rho|\rho_M^\infty) = \int_{-\infty}^{+\infty} \rho_M^\infty \left(\frac{\rho}{\rho_M^\infty} - 1\right)^2 dx$. We observe that we cannot employ lemma 7.2.6 in the quadratic case, since it is valid only for positive values of ρ.

Theorem 7.2.7 *Let $\rho(y,s)$ be the solution of the IVP (7.1.16) with*

$$\int \rho_0^2(y) e^{\frac{y^2}{2}} dy < \infty, \tag{7.2.22}$$

(ρ_0 eventually sign–changing). Let ρ_M^∞ be given by (7.1.22). Then, the following estimate holds

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left(\rho(y,s) - \rho_M^\infty(y)\right)^2 e^{\frac{y^2}{2}} dy \\ &\leq Ce^{-2s} \Big[\|F(\cdot,0) - F^\infty(\cdot,0)\|_{L^1(\mathbb{R})}^2 + \|F(\cdot,0) - F^\infty(\cdot,0)\|_{L^\infty(\mathbb{R})}^2 \\ &+ \int_{-\infty}^{+\infty} |\rho_0(y) - \rho_M^\infty(y)|^2 e^{\frac{y^2}{2}} dy \Big], \end{aligned} \tag{7.2.23}$$

with a constant C depending on the initial datum.

Proof.

From the estimate (7.1.24) (we recall that τ is expressed by (7.1.17)), we obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left(\rho(y,s) - \rho_M^\infty(y)\right)^2 e^{\frac{y^2}{2}} dy \\ &\leq e^{M+p} \int_{-\infty}^{+\infty} \left(\rho(y,s)\tau_m(y) - \rho_M^\infty(y)\tau_m(y)\right)^2 e^{\frac{y^2}{2}} dy \\ &\leq e^{M+p} \int_{-\infty}^{+\infty} \rho(y,s)^2 (\tau(y,s) - \tau_m(y))^2 e^{\frac{y^2}{2}} dy \\ &+ e^{M+p} \int_{-\infty}^{+\infty} \left(\rho(y,s)\tau(y,s) - \rho_M^\infty(y)\tau_m(y)\right)^2 e^{\frac{y^2}{2}} dy \\ &:= J_1 + J_2. \end{aligned} \tag{7.2.24}$$

We now employ the same argument as in the proof of the previous theorem to recover estimate (7.2.21). Hence, we estimate the term J_1 as follows

$$\begin{aligned} J_1 \leq C_0 e^{-2s} &\Big[\|F(\cdot,0) - F^\infty(\cdot,0)\|_{L^1(\mathbb{R})}^2 \\ &+ \|F(\cdot,0) - F^\infty(\cdot,0)\|_{L^\infty(\mathbb{R})}^2 \Big] \int_{-\infty}^{+\infty} \rho(y,s)^2 e^{\frac{y^2}{2}} dy. \end{aligned}$$

Then, inequality (7.2.14) in the case of quadratic entropy gives

$$J_1 \leq C_1 e^{-2s}\left[\|F(\cdot,0)-F^\infty(\cdot,0)\|^2_{L^1(\mathbb{R})}+\|F(\cdot,0)-F^\infty(\cdot,0)\|^2_{L^\infty(\mathbb{R})}\right],$$

where C_1 depends on the initial datum. By writing again inequality (7.2.14) in terms of ρ, ρ_M^∞ in the case of quadratic entropy, we obtain the estimate for the term J_2,

$$J_2 \leq C H_\varphi(\psi(s)|\Psi_m) \leq C e^{-2s} H_\varphi(\psi_0|\Psi_m),$$

for a constant C depending on the mass M. Now, we have

$$\begin{aligned}
H_\varphi(\psi_0|\Psi_m) &= \int_{-\infty}^{+\infty}(\psi_0(y)-\Psi_m(y))^2\Psi_m(y)^{-1}dy \\
&\leq \frac{1}{4}\int_{-\infty}^{+\infty}\left|\rho_0(y)\tau(y,0)-\rho_M^\infty(y)\tau_m(y)\right|^2 C_M e^{\frac{y^2}{2}}dy \\
&\leq C_M^1\int_{-\infty}^{+\infty}\left|\rho_0(y)-\rho_M^\infty(y)\right|^2 e^{\frac{y^2}{2}}dy \\
&+ C_M^2\int_{-\infty}^{+\infty}\left|\tau(y,0)-\tau_m(y)\right|^2\rho_M^\infty(y)^2 e^{\frac{y^2}{2}}dy \\
&\leq C_M^0\left[\int_{-\infty}^{+\infty}\left|\rho_0(y)-\rho_M^\infty(y)\right|^2 e^{\frac{y^2}{2}}dy+\|\tau(\cdot,0)-\tau_m(\cdot,0)\|^2_{L^\infty(\mathbb{R})}\right],
\end{aligned}$$

where all the constants above depend on the mass M. Hence, because of (7.2.19), we have

$$J_2 \leq C e^{-2s}\left[\int_{-\infty}^{+\infty}\left|\rho_0(y)-\rho_M^\infty(y)\right|^2 e^{\frac{y^2}{2}}dy+\|F(\cdot,0)-F^\infty(\cdot,0)\|^2_{L^\infty(\mathbb{R})}\right].$$

Thus, by substituting the estimates for the terms J_1 and J_2 into (7.2.24), we obtain (7.2.23). □

We remark that in the previous theorem condition (7.2.22) is sufficient to control the terms involving the primitive F in (7.2.23). This assertion follows from the observations in remark 7.2.3 and from the obvious inequality

$$\int_{-\infty}^{+\infty}\rho_0(y)y^2dy \leq C\left(\int_{-\infty}^{+\infty}\rho_0^2(y)e^{\frac{y^2}{2}}dy\right)^{\frac{1}{2}}.$$

The results in Theorems 7.2.2 and 7.2.7 can be easily converted in terms of L^1 decay for the solution ρ to equation (7.1.16) towards the stationary solution ρ_M^∞, by means of the Csiszár–Kullback inequality. Moreover, as usual in this framework, by returning to the original variable u one gets a polynomial rate of convergence towards diffusion waves for the viscous Burgers' equation. We collect all these results in the following two corollaries.

Corollary 7.2.8 *Let $\rho(y,s)$ be the solution to the Cauchy problem (7.1.16) with $\rho_0 \in L^1(\mathbb{R})$. Let ρ_M^∞ be given by (7.1.22). Suppose that one of the following two conditions is satisfied:*

i) $\rho_0 \geq 0, \quad \int_{\mathbb{R}} y^2 \rho_0(y)dy < +\infty, \text{ and } \int_{\mathbb{R}} \rho_0(y) \log \rho_0(y) dy < +\infty$

ii) $\int_{\mathbb{R}} \rho_0(y)^2 e^{\frac{y^2}{2}} dy < +\infty.$

Then, the following estimate holds

$$\|\rho(s) - \rho_M^\infty\|_{L^1(\mathbb{R})} \leq C_M e^{-s},$$

where C_M depends on the mass M of the initial datum.

Corollary 7.2.9 *Let u be the solution to (7.1.2) with initial datum $u_0 \in L^1(\mathbb{R})$. Let U_M given by (7.1.8). Suppose that one of the following two conditions is satisfied*

i) $u_0 \geq 0, \quad \int_{\mathbb{R}} x^2 u_0(x)dx < +\infty, \text{ and } \int_{\mathbb{R}} u_0(x) \log u_0(x) dx < +\infty$

ii) $\int_{\mathbb{R}} u_0(x)^2 e^{\frac{x^2}{2}} dx < +\infty$

Then, for all $t \geq 0$, the following inequality holds

$$\|u(t) - U_M(t)\|_{L^1(\mathbb{R})} \leq C(t+1)^{-\frac{1}{2}},$$

where C depends on the initial datum.

Remark 7.2.10 We mention here an alternative entropy dissipation approach to the Burgers'–Fokker–Planck equation (see [Cav00])

$$\frac{\partial \rho}{\partial s} = \frac{\partial}{\partial y}\left(\frac{\partial \rho}{\partial y} + y\rho - \frac{\rho^2}{2}\right).$$

We denote

$$\rho(y,s) = \widetilde{\rho}(y,s) e^{-\frac{y^2}{2}}$$
$$\rho_M^\infty(y) = \widetilde{\rho}_M^\infty(y) e^{-\frac{y^2}{2}},$$

where ρ_M^∞ is given in (7.1.22), as usual. Hence, the above equation becomes

$$e^{-\frac{y^2}{2}} \widetilde{\rho}_s = \left(e^{-\frac{y^2}{2}} \widetilde{\rho}_y - \frac{1}{2} e^{-y^2} \widetilde{\rho}^2\right)_y.$$

Since

$$-\frac{e^{-\frac{y^2}{2}}}{2} = \left(\frac{1}{\rho_M^\infty}\right)_y,$$

we have

$$e^{-\frac{y^2}{2}}\widetilde{\rho}_s = \left(e^{-\frac{y^2}{2}}\widetilde{\rho}^2\left(\frac{1}{\widetilde{\rho}_M^\infty} - \frac{1}{\widetilde{\rho}}\right)_y\right)_y$$

and finally we can rewrite the Burgers–Fokker–Planck equation in the following way

$$\rho_s = \left(e^{\frac{y^2}{2}}\rho^2\left(e^{-\frac{y^2}{2}}\left(\frac{1}{\rho_M^\infty} - \frac{1}{\rho}\right)\right)_y\right)_y.$$

This suggests the use of an alternative entropy, namely

$$H(\rho|\rho_M^\infty) = \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}}\left[\frac{\rho(y)}{\rho_M^\infty(y)} - 1 - \log\frac{\rho(y)}{\rho_M^\infty(y)}\right] dy.$$

Indeed, the entropy production $I = -\frac{d}{ds}H$ is given by

$$I(\rho|\rho_M^\infty) = \int_{-\infty}^{+\infty} \rho(y)^2 e^{\frac{y^2}{2}}\left[\left(e^{-\frac{y^2}{2}}\left(\frac{1}{\rho_M^\infty(y)} - \frac{1}{\rho(y)}\right)\right)_y\right]^2 dy > 0.$$

The use of the above entropy allows us to require less restrictive conditions on the initial data in order to obtain convergence to equilibrium. However, in this case no exponential decay is proven, and both functional $H(\rho)$ and $I(\rho)$ blow up if evaluated at a density $\rho(y)$ with a much 'faster' behavior than ρ_M^∞ at $|y| \to \infty$. Nevertheless, this approach can be generalized to convection diffusion equations with general nonlinear convection, since it does not require the use of the Hopf–Cole transformation.

7.3 Evolution of the Wasserstein metric

This section is devoted to the study of the Wasserstein metric of solutions of the Burgers–Fokker–Planck equation

$$\begin{cases} \dfrac{\partial \rho}{\partial s} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \rho}{\partial y} + y\rho - \dfrac{\rho^2}{2}\right) \\ \rho(y,0) = \rho_0(y) = u_0(y), \end{cases} \tag{7.3.1}$$

with initial datum $\rho_0 \in L^1_+(\mathbb{R})$ with finite second moment. Let us recall briefly some concepts concerning the Wasserstein metric. We denote by $\mathcal{M}_2$ the space of all probability densities on $\mathbb{R}$ with finite second moment, i. e.

$$\mathcal{M}_2 = \left\{ \rho \in L^1_+(\mathbb{R}), \ \int_{-\infty}^{+\infty} \rho(y)dy = 1, \ \int_{-\infty}^{+\infty} y^2 \rho(y)dy < \infty \right\}.$$

The Wasserstein metric $d_2(\cdot,\cdot)$ on the space $\mathcal{M}_2$ is defined as follows,

$$d_2^2(\rho_1,\rho_2) = \inf_{\rho_2 = T_\sharp \rho_1} \left\{ \int_{-\infty}^{+\infty} (y - T(y))^2 \rho_1(y)dy \right\}, \tag{7.3.2}$$

where the notation $\rho_2 = T_\sharp \rho_1$ means that the admissible maps T are the *push–forwards* between the two densities ρ_1 and ρ_2, i. e. the T's satisfy

$$\int_{-\infty}^{+\infty} \varphi(y)\rho_2(y)dy = \int_{-\infty}^{+\infty} \varphi(T(y))\rho_1(y)dy,$$

for any $\varphi \in C^0_c(\mathbb{R})$. The precise definition of the Wasserstein metric comes from a *relaxed* variational problem. More precisely, the set of admissible maps is embedded into the set of all probability measures μ on $\mathbb{R}^2$ with marginals given by ρ_1 and ρ_2. The quadratic cost defined above is converted into

$$\int\int_{\mathbb{R}^2} (y_0 - y_1)^2 \mu(dy_0, dy_1).$$

It turns out that the optimal measure μ^*, which minimizes the relaxed variational problem, is supported on the graph of a map $T^* : \mathbb{R} \to \mathbb{R}$, which is exactly the optimal map of the original variational problem (7.3.2). The relaxed problem above is a version of the Monge–Kantorovich mass transfer problem.

In the one dimensional case, the optimal map T^* can be expressed in the following simple way. Let us define the distribution functions

$$F_i(y) = \int_{-\infty}^{y} \rho_i(y)dy \quad i = 1,2$$

and their pseudo–inverses $F_i^{-1} : (0,1) \to \mathbb{R}$

$$F_i^{-1}(\eta) = \inf\{\omega : \ F_i(\omega) > \eta\}.$$

Then, it can be easily proven that the optimal map T^* between ρ_1 and ρ_2 is

$$T = F_2^{-1} \circ F_1.$$

Hence, by definition of Wasserstein metric, we have

$$d_2^2(\rho_1,\rho_2) = \int_{-\infty}^{+\infty} \left(y - \left(F_2^{-1} \circ F_1\right)(y)\right)^2 \rho_1(y)dy = \int_0^1 \left(F_1^{-1}(\eta) - F_2^{-1}(\eta)\right)^2 d\eta. \tag{7.3.3}$$

Let us now discuss the asymptotic behavior of the Wasserstein metric d_2 between a solution $\rho(y,s)$ of equation (7.3.1) with initial datum $\rho_0 \in \mathcal{M}_2$ and the stationary solution ρ_1^∞ defined in (7.1.22). To simplify the notation, we denote by ρ^∞ the stationary state with unit mass. We observe that all the computations below can be generalized to the case of $\int \rho_0 = M$ for any positive M. As in the previous section, we employ the Hopf–Cole transformation

$$\psi(y,s) = \frac{1}{2}\rho(y,s)\exp\left(-\frac{1}{2}\int_{-\infty}^{y} \rho(\zeta,s)d\zeta\right), \tag{7.3.4}$$

which reduces (7.3.1) to the linear Fokker–Planck equation

$$\begin{cases} \dfrac{\partial \psi}{\partial s} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi}{\partial y} + y\psi\right) \\ \psi(y,0) = \psi_0(y) = \frac{1}{2}\rho_0(y)\exp\left(-\frac{1}{2}\int_{-\infty}^{y}\rho_0(\zeta)d\zeta\right). \end{cases} \tag{7.3.5}$$

We recall that the initial datum ψ_0 has total mass equal to $m = 1 - e^{-\frac{1}{2}}$. Also, we recall the following theorem by Otto (see [Ott01]), giving a result of exponential decay of the Wasserstein metric for the linear Fokker–Planck equation. This result is a special case of a more general one for nonlinear diffusion equations contained in [Ott01].

Theorem 7.3.1 *Let $\psi(y,s)$ be the solution to (7.3.5) with initial datum $\psi_0 \in L^1_+(\mathbb{R})$, $\int \psi_0(y)dy = m$, $m = 1 - e^{-1/2}$. Let Ψ_m be the corresponding gaussian state defined in (7.1.21). Then the Wasserstein distance $d_2(\psi, \Psi_m)$ satisfies*

$$d_2^2(\psi(s), \Psi_m) \le e^{-2s} d_2^2(\psi_0, \Psi_m). \tag{7.3.6}$$

We now state our result for equation (7.3.1).

Theorem 7.3.2 *Let $\rho(y,s)$ be the solution to (7.3.1), with initial datum $\rho_0 \in L^1_+(\mathbb{R})$, $\int \rho_0(y)dy = 1$. Let ρ^∞ be the stationary solution*

$$\rho^\infty(y) = \frac{\frac{2m}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}}{1 - \frac{2m}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{\zeta^2}{2}}d\zeta}, \tag{7.3.7}$$

with $m = 1 - e^{-1/2}$. Then the Wasserstein distance $d_2(\rho(s), \rho^\infty)$ satisfies the exponential decay estimate

$$d_2^2(\rho(s), \rho^\infty) \le Ce^{-2s} d_2^2(\rho_0, \rho^\infty), \tag{7.3.8}$$

where C is a fixed constant.

Proof.

Let ρ be the solution to (7.3.1), we define the corresponding ψ by means of the transformation (7.3.4), i.e.

$$\psi(y, s) = \frac{1}{2}\rho(y, s) \exp\left(-\frac{1}{2}\int_{-\infty}^{y} \rho(\zeta, s) d\zeta\right). \tag{7.3.9}$$

We also recall that

$$\Psi_m(y) = \frac{1}{2}\rho^\infty(y) \exp\left(-\frac{1}{2}\int_{-\infty}^{y} \rho^\infty(\zeta) d\zeta\right).$$

Let us now define the distribution functions

$$\begin{aligned} F(y, s) &= \int_{-\infty}^{y} \rho(\zeta, s) d\zeta \\ F_\infty(y) &= \int_{-\infty}^{y} \rho^\infty(\zeta) d\zeta \\ G(y, s) &= \int_{-\infty}^{y} \psi(\zeta, s) d\zeta \\ G_\infty(y) &= \int_{-\infty}^{y} \Psi_m(\zeta) d\zeta. \end{aligned}$$

Hence, since ψ and Ψ_m satisfy the Fokker–Planck equation, Theorem 7.3.1 and the representation (7.3.3) of the Wasserstein metric in one space dimension imply

$$\int_0^m \left(G^{-1}(\eta, s) - G_\infty^{-1}(\eta)\right)^2 d\eta \le e^{-2s} \int_0^m \left(G^{-1}(\eta, 0) - G_\infty^{-1}(\eta)\right)^2 d\eta, \tag{7.3.10}$$

where the symbol $^{-1}$ stands for pseudo–inversion. Now, because of (7.3.9), it is possible to write F in terms of G (and similarly F_∞ in terms of G_∞). We have

$$\begin{aligned} G(\eta, s) &= \int_{-\infty}^{y} \psi(\zeta, s) d\zeta = \int_{-\infty}^{y} \frac{1}{2}\rho(\zeta, s) \exp\left(-\frac{1}{2}\int_{-\infty}^{\zeta} \rho(\xi, s) d\xi\right) d\zeta \\ &= -\int_{-\infty}^{y} \frac{\partial}{\partial y}\left(\exp\left(-\frac{1}{2}\int_{-\infty}^{\zeta} \rho(\xi, s) d\xi\right)\right) d\zeta \\ &= 1 - \exp\left(-\frac{1}{2}\int_{-\infty}^{y} \rho(\zeta, s) d\zeta\right) = 1 - \exp\left(-\frac{1}{2}F(y, s)\right). \end{aligned}$$

We denote by $\alpha : (0,1) \to (0,m)$ the bijective function

$$\alpha(t) = 1 - e^{-t/2},$$

which has inverse $\alpha^{-1}(\tau) = -2\log(1-\tau)$. Hence, we can write

$$G = \alpha \circ F \qquad G^{-1} \circ \alpha = F^{-1}.$$

Therefore, by simple change of variable, we obtain

$$\begin{aligned}
&\int_0^1 \left(F^{-1}(\eta,s) - F_\infty^{-1}(\eta)\right)^2 d\eta = \int_0^1 \left(G^{-1}(\alpha(\eta),s) - G_\infty^{-1}(\alpha(\eta))\right)^2 d\eta \\
&= \int_0^m \left(G^{-1}(\xi,s) - G_\infty^{-1}(\xi)\right)^2 \frac{2}{1-\xi} d\xi \\
&\leq 2\sqrt{e} \int_0^m \left(G^{-1}(\xi,s) - G_\infty^{-1}(\xi)\right)^2 d\xi \\
&\leq 2\sqrt{e}\, e^{-2s} \int_0^m \left(G^{-1}(\xi,0) - G_\infty^{-1}(\xi)\right)^2 d\xi \\
&= 2\sqrt{e}\, e^{-2s} \int_0^m \left(F^{-1}(\alpha^{-1}(\xi),0) - F_\infty^{-1}(\alpha^{-1}(\xi))\right)^2 d\xi \\
&= \sqrt{e}\, e^{-2s} \int_0^1 \left(F^{-1}(\eta,0) - F_\infty^{-1}(\eta)\right)^2 e^{-\frac{\eta}{2}} d\eta \\
&\leq \sqrt{e}\, e^{-2s} \int_0^1 \left(F^{-1}(\eta,0) - F_\infty^{-1}(\eta)\right)^2 d\eta,
\end{aligned}$$

which concludes the proof. □

In the original variables (7.1.2), the above theorem is converted into a stability result for the Wasserstein metric, as stated in the following corollary.

Corollary 7.3.3 *Let u be the solution of (7.1.2) with nonnegative initial datum $u_0 \in L^1(\mathbb{R})$ with finite second moment. Let U_M given by (7.1.8). Hence, there exists a fixed positive constant C (depending on the mass) such that*

$$d_2(u(t), U_M) \leq C d_2(u_0, U_M). \qquad (7.3.11)$$

We remark that such a result cannot be improved in order to obtain a decay at the level of the original variables, because of the translation–invariance and the representation (7.3.3) of the Wasserstein metric.

Chapter 8

A stability result for radiating gases

This last chapter deals with the asymptotic stability of diffusive waves for the Hamer model for radiating gases widely discussed in chapter 4. We use here the relative entropy method described in section 1.7 and developed in chapters 5 and 7, in order to detect the optimal rate of convergence in L^1 norm. The importance of such result lays on the fact that it is one of the first results involving entropy methods applied outside of the context of diffusion equations (see also [DE]). The next section is devoted to the precise statement of the problem and of the main result in theorem 8.1.4. In section 8.2 we prove such theorem.

8.1 Statement of the problem and result

In this section we analyze the time–asymptotic behaviour of the solutions to the Cauchy problem

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = -u + K * u \\ u(x,0) = u_0(x), \end{cases} \tag{8.1.1}$$

with $u_0 \in L^1(\mathbb{R})$ and the convolution kernel K is given by $K(x) = \dfrac{1}{2}e^{-|x|}$. We recall that the kernel K satisfies

(i) $\displaystyle\int_{-\infty}^{+\infty} K(x)dx = 1$

(ii) $K(-x) = K(x)$.

It can be easily checked that, if $u(\cdot,t)$ is the solution to (8.1.1) at a positive time t, then the following *mass* conservation property holds

$$\int_{-\infty}^{+\infty} u(x,t)dx = \int_{-\infty}^{+\infty} u_0(x)dx.$$

As pointed out in the introduction, it is well know that the behavior of the solutions to (8.1.1) for large times is described by the viscous Burgers' equation

$$u_t + \frac{1}{2}(u^2)_x = u_{xx}. \tag{8.1.2}$$

The typical asymptotic states for (8.1.2) are given by the self–similar waves

$$U_M(x,t) = 2(1-e^{-M/2})(2t+1)^{-\frac{1}{2}} \frac{\exp\left(-\frac{x^2}{2(2t+1)}\right)}{1 - C_M \int_{-\infty}^{x(2t+1)^{-1/2}} e^{-\frac{\zeta^2}{2}} d\zeta}, \tag{8.1.3}$$

where the parameter M is the integral of $U_M(\cdot,t)$ over $\mathbb{R}$ for all $t>0$. The expression above for the unique self–similar wave with mass M is obtained via the Hopf–Cole transformation

$$\phi(x,t) = \frac{1}{2}u(x,t)\exp\left(-\frac{1}{2}\int_{-\infty}^{x} u(z,t)dz\right), \tag{8.1.4}$$

which reduces (8.1.2) to the linear heat equation. Indeed, this transformation provides an explicit formula for the solution to (8.1.2). The diffusive wave (8.1.3) corresponds (in (8.1.4)) to a gaussian solution to the linear heat equation.

In order to view the asymptotic self–similar profile (8.1.3) as a stationary profile (in a similar fashion to many papers about self–similar asymptotics, see [Bar52, CT00, Ott01] e.g.), we consider the time–dependent scaling

$$\begin{aligned} y &= y(x,t) = xR(t)^{-1/2} \\ s &= s(t) = \frac{1}{2}\log R(t) \\ u(x,t) &= R(t)^{-1/2}\rho(y(x,t),s(t)), \end{aligned} \tag{8.1.5}$$

where

$$R(t) = (2t+1).$$

Then, equation (8.1.1) turns into

$$\rho_s + \left(\frac{\rho^2}{2} - y\rho\right)_y = -e^{2s}\left(\rho - K_s * \rho\right), \tag{8.1.6}$$

where the rescaled convolution kernel K_s is given by $K_s(\xi) = e^s K(e^s \xi)$. Now, it is known that the nonhomogeneous term $-e^{2s}(\rho - K_s * \rho)$ in (8.1.6) behaves like ρ_{yy} as $s \to \infty$ for sufficiently smooth solutions ρ (see [LM03]). Hence, one can expect the solution to (8.1.6) to be described asymptotically by the *Burgers'–Fokker–Planck* equation

$$\rho_s = \left(\rho_y + y\rho - \frac{\rho^2}{2}\right)_y, \tag{8.1.7}$$

also called *Burgers' equation in similarity variables.* The solutions to (8.1.7) converge as $s \to \infty$ (e.g. in L^1) to the unique stationary profile ρ_M^∞ (with the same mass as the initial datum) satisfying the relation

$$\rho_y + y\rho - \frac{\rho^2}{2} = 0, \tag{8.1.8}$$

i.e.

$$\rho_M^\infty(y) = \frac{\frac{2m}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}{1 - \frac{2m}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{\zeta^2}{2}} d\zeta}, \tag{8.1.9}$$

where $m = 1 - e^{-M/2}$ and $M = \int_{-\infty}^{+\infty} u_0(x)dx$ (We recall that the time–dependent scaling (8.1.5) is mass preserving). Such profile is again obtained via the Hopf–Cole transformation

$$\psi(y,s) = \frac{1}{2}\rho(y,s)\exp\left(-\frac{1}{2}\int_{-\infty}^{y} \rho(\zeta,s)d\zeta\right), \tag{8.1.10}$$

which turns (8.1.7) into the Fokker–Planck equation

$$\psi_s = (\psi_y + y\psi)_y \tag{8.1.11}$$

and the stationary profile satisfying (8.1.8) corresponds to a gaussian equilibrium of equation (8.1.11) via the relation (8.1.10).

Let us now return back to the equation (8.1.1) in the original variables. Our asymptotic analysis will include only solutions to that equations which are obtained as small perturbation in Sobolev norm of the zero state. However, the motivation of this restriction will come from the need of providing an a priori estimate of some error terms we will encounter in the computations. Therefore, our result can be extended to all those solution satisfying these a priori estimates, which are true in particular in a small perturbation setting (see [IK02]). For the sake of completeness we recall here the existence and regularity properties for the class of solutions we will consider. The proof of the following theorem can be found in the paper by Schochet and Tadmor [ST92], where equation (8.1.1) is referred to as the *Rosenau–Chapman–Enskog equation.*

Theorem 8.1.1 (Existence and regularity) *The unique entropy solution to the equation (8.1.1) remains as smooth as the initial datum* $u(x,0) = u_0(x)$ *(in* H^s *sense) provided the initial datum* u_0 *is sufficiently small so that*

$$2\|u_0\|_{L^\infty}^{1/2} + \|u_0'\|_{L^\infty} < 1. \tag{8.1.12}$$

We then state the following theorem by Iguchi and Kawashima [IK02], which holds in the more general framework of elliptic–hyperbolic coupled systems, as pointed out in the introduction.

Theorem 8.1.2 (Solutions with small initial datum) *Let* $s \geq 3$ *be an integer,* $\beta \geq 0$*. There exists a positive constant* $C_1 = C_1(s,\beta)$ *such that, if* $\|u_0\|_{H^3} + \|u_0\|_{L^1} \leq C_1$*, then for all* $0 \leq l \leq s-2$ *the solution* u *to (8.1.1) satisfies the following pointwise estimates*

$$\left|\partial_x^l u(x,t)\right| \leq C_{l,s}(1+t)^{-(1/2)(l+1)}\phi_\beta(x,t), \tag{8.1.13}$$

where

$$\phi_\beta(x,t) = \left(1 + \frac{x^2}{1+t}\right)^{-\beta/2} \tag{8.1.14}$$

and

$$C_{l,s} = C\left(1 + \|u_0\|_{H^s} + \|u_0\|_{L^1}\right)\sum_{k=1}^{l}\left\|(1+|x|)^\beta|\partial_x^k u_0(x)\right\|_{L_x^\infty}. \tag{8.1.15}$$

Next, we reformulate the above results in terms of the new variables ρ, y and s.

Corollary 8.1.3 *The solution* $\rho(y,s)$ *to equation (8.1.6) with initial datum* ρ_0 *remains as smooth as* ρ_0 *in* H^s *norm provided* rho_0 *satisfies*

$$2\|\rho_0\|_{L^\infty}^{1/2} + \|\rho_0'\|_{L^\infty} < 1. \tag{8.1.16}$$

Moreover, given $t \geq 3$ *integer,* $\beta \geq 0$*, there exists a positive constant* $C_1 = C_1(t,\beta)$ *such that, if*

$$\|\rho_0\|_{H^3} + \|\rho_0\|_{L^1} \leq C_1, \tag{8.1.17}$$

then for all $0 \leq l \leq t-2$ *the solution* ρ *to (8.1.6) satisfies the following pointwise estimates*

$$\left|\partial_y^l \rho(y,s)\right| \leq C_{l,t}\psi_\beta(y,s), \tag{8.1.18}$$

where

$$\psi_\beta(y,s) = \left(1+y^2\right)^{-\beta/2} \tag{8.1.19}$$

and

$$C_{l,t} = C\left(1 + \|\rho_0\|_{H^t} + \|\rho_0\|_{L^1}\right)\sum_{k=1}^{l}\left\|(1+|y|)^\beta|\partial_y^k \rho_0(y)\right\|_{L_y^\infty}. \tag{8.1.20}$$

We shall prove that the solution to the rescaled equation (8.1.6) with initial datum ρ_0 in $L^1_+(\mathbb{R})$ with mass M satisfying the hypothesis (8.1.16) and (8.1.17) of corollary 8.1.3 and such that the constant $C_{4,6}$ in the above (8.1.20) is finite, converge exponentially fast to the profile ρ^∞_M. To perform this task, we employ the entropy dissipation method developed in some of the previous chapters (see [CT00, Ott01, DFM]) in order to obtain the sharp rate of convergence in L^1 norm via Csiszár–Kullback inequality. This will improve the rate of convergence towards diffusive wave proven in [IK02] in our simpler scalar model.

We collect the above statements in the following

Theorem 8.1.4 *Let $\rho(y,s)$ be the solution to the equation (8.1.6) with initial datum $\rho_0 \in L^1_+(\mathbb{R}) \cap H^6(\mathbb{R})$ having mass M. Let us suppose that ρ_0 satisfies (8.1.16) and (8.1.17) of corollary 8.1.3 and such that the constant $C_{4,6}$ in (8.1.20) is finite. Then, there exists a positive fixed constant C such that the inequality*

$$\|\rho(\cdot,s) - \rho^\infty_M(\cdot)\|_{L^1(\mathbb{R})} \leq Ce^{-s} \tag{8.1.21}$$

is satisfied for any $s \geq 0$. As a consequence of (8.1.21), the solution u to the original problem (8.1.1) with $u_0 = \rho_0$ satisfies the inequality

$$\|u(\cdot,t) - U_M(\cdot)\|_{L^1(\mathbb{R})} \leq C(t+1)^{-\frac{1}{2}} \tag{8.1.22}$$

for all times $t \geq 0$.

8.2 Proof of the main theorem

We prove Theorem 8.1.4 via the entropy dissipation method already developed above. We recall the *admissible relative entropy* functionals of chapter 7, namely

$$H_e(\rho(s)|\rho^\infty_M) = \int_{-\infty}^{+\infty} e\left(\frac{\rho(y,s)}{\rho^\infty_M(y)}\right)\rho^\infty_M(y)dy, \tag{8.2.1}$$

where $e : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function satisfying the following conditions

$$\begin{aligned}
&e(1) = 0\\
&e''(h) \geq 0 \quad \text{for any } h \in \mathbb{R}_+, \quad e'' \text{ not identically } 0\\
&(e''')^2 \leq \frac{1}{2}e''e^{(IV)}.
\end{aligned} \tag{8.2.2}$$

We perform here a special choice of the generating function e, this choice being justified by technical reasons. We set

$$e(h) = (h+1)\log\frac{h+1}{2} - (h-1).$$

As a simple consequence of Lemma 7.2.6 in the previous chapter, we state the following lemma, the proof of which employs the uniform bound for u for bounded data.

Lemma 8.2.1 *Let ψ and ρ as in (8.1.10). Then we have*

$$H_e(\rho) \leq C_1 H_e(\psi) \leq C_2 H_e(\rho), \tag{8.2.3}$$

where the constants C_1 and C_2 depend on the masses M and m defined above.

Hereafter, we will work with the functional $H_e(\psi(s))$ where $\psi(s)$ is related to the solution $\rho(s)$ of equation (8.1.6) via (8.1.10). We recall that the functional $H_e(\psi)$ above attains its minimum at the gaussian state

$$\psi_m^\infty(y) = \frac{m}{\sqrt{2\pi}} e^{-\frac{y^2}{2}},$$

with $m = 1 - e^{-M/2}$ (see [AMTU01]). To simplify the notation, we will denote in the following

$$\Sigma(y,s) = \int_{-\infty}^{y} \rho(\xi, s) d\xi.$$

The evolution of Σ and ψ are governed by the relations

$$\Sigma_s + \frac{1}{2}(\Sigma_y)^2 - y\Sigma_y = -e^{2s}(\Sigma - K_s * \Sigma) \tag{8.2.4}$$

$$\psi_s - (y\psi)_y = e^{-\frac{1}{2}\Sigma}\left[-\rho\rho_y - \frac{1}{4}\rho^3 - e^{2s}(\rho - K_s * \rho) + e^{2s}\frac{\rho}{2}(\Sigma - K_s * \Sigma)\right] \tag{8.2.5}$$

Since we are dealing with nonnegative initial data, both ρ and ψ remain nonnegative by comparison principle (see Chapter 4). Hence, the functional $H_e(\psi(s))$ is well defined. Moreover, the term $e^{-\frac{1}{2}\Sigma(y,s)}$, which will appear very frequently in the calculations below, satisfies the estimate

$$0 < c \leq e^{-\frac{1}{2}\Sigma(y,s)} \leq C$$

uniformly w.r.t. s. We then compute the evolution of $H_e(\psi(s))$. We have

$$\frac{d}{ds}H_e(\psi(s)) = \int_{-\infty}^{+\infty} e'\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)\psi_s dy = \int_{-\infty}^{+\infty} e'\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)\Bigg\{(y\psi)_y$$
$$+e^{-\frac{1}{2}\Sigma}\left[-\rho\rho_y - \frac{1}{4}\rho^3 - e^{2s}(\rho - K_s * \rho) + e^{2s}\frac{\rho}{2}(\Sigma - K_s * \Sigma)\right]\Bigg\} dy \tag{8.2.6}$$

We then employ the following general relation (see Chapter 4 for the proof),

Lemma 8.2.2 *For any smooth function $f(y)$, the following expansion formula holds*

$$
\begin{aligned}
&-f(y)+K_s*f(y)=\int_{-\infty}^{+\infty}K(\xi)\left[-e^{-s}\xi f_y(y)+\frac{e^{-2s}}{2}\xi^2 f_{yy}(y)\right.\\
&\left.-\frac{e^{-3s}}{6}\xi^3 f_{yyy}(y)+\frac{1}{24}\int_0^{\xi e^{-s}}\theta^4 f_{yyyy}(y-\xi e^{-s}-\theta)d\theta\right]d\xi. \qquad (8.2.7)
\end{aligned}
$$

We then observe that the term e' in (8.2.6) can be estimated as follows.

$$
\left|e'\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)\right|\leq 1+\left|\log\left(\frac{\frac{\psi(y,s)}{\psi_m^\infty(y)}+1}{2}\right)\right|\leq C(1+|y|^2), \qquad (8.2.8)
$$

where we have used the nonnegativity and uniform bound from above for ψ and the explicit expression for $\psi_m^\infty(y)$. The above constant C is independent on y and depends on the initial datum. We then apply (8.2.7) both to ρ and to Σ into (8.2.6) to obtain

$$
\begin{aligned}
&\frac{d}{ds}H_e(\psi(s))=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K(\xi)e'\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)\left\{(y\psi)_y+e^{-\frac{1}{2}\Sigma}\left[-\rho\rho_y-\frac{1}{4}\rho^3+\right.\right.\\
&\left(-e^s\xi\rho_y(y)+\frac{1}{2}\xi^2\rho_{yy}(y)-\frac{e^{-s}}{6}\xi^3\rho_{yyy}(y)+\frac{e^{2s}}{24}\int_0^{\xi e^{-s}}\theta^4\rho_{yyyy}(y-\xi e^{-s}+\theta)d\theta\right)\\
&-e^{2s}\frac{\rho}{2}\left(-e^{-s}\xi\rho(y)+\frac{e^{-2s}}{2}\xi^2\rho_y(y)-\frac{e^{-3s}}{6}\xi^3\rho_{yy}(y)+\right.\\
&\left.\left.\left.\frac{1}{24}\int_0^{\xi e^{-s}}\theta^4\rho_{yyy}(y-\xi e^{-s}+\theta)d\theta\right)\right]\right\}d\xi dy=-\int_{-\infty}^{+\infty}e''\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)^2\psi_m^\infty(y)dy\\
&+O(1)e^{-2s}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K(\xi)(1+|y|^2)\xi^4\times\\
&\times\sup_{\theta\in[0,\xi e^{-s}]}\left[\left|\rho_{yyyy}(y-\xi e^{-s}+\theta)\right|+\left|\rho_{yyy}(y-\xi e^{-s}+\theta)\right|\right]d\xi dy
\end{aligned}
$$

where we have used integration by parts, the estimate (8.2.8), the properties

$$
K(-\xi)=K(\xi),\quad \int K(\xi)d\xi=1
$$

and the relation

$$
\psi_{yy}=\frac{1}{2}e^{-\frac{1}{2}\Sigma}\left(-\frac{3}{2}\rho\rho_y+\frac{\rho^3}{4}+\rho_{yy}\right).
$$

We then use estimate (8.1.18) to control the terms ρ_{yyy} and ρ_{yyyy} above. After simple calculations, by means of the properties of K, one can prove that the last integral above is finite and it can be controlled uniformly w.r.t. to s. We then obtain

$$\frac{d}{ds}H_e(\psi(s)) \leq -\int_{-\infty}^{+\infty} e''\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)^2 \psi_m^\infty(y)dy + O(1)e^{-2s}.$$

Hence, we apply the generalized Sobolev inequality

$$H_e(\psi(s)) \leq \frac{1}{2}I_e(\psi(s)),$$

where

$$I_e(\psi(s)) = \int_{-\infty}^{+\infty} e''\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)\left(\frac{\psi(y,s)}{\psi_m^\infty(y)}\right)^2 \psi_m^\infty(y)dy$$

is called *generalized Fisher information* or *generalized entropy production*. The– refore, we get

$$\frac{d}{ds}H_e(\psi(s)) \leq -2H_e(\psi(s)) + O(1)e^{-2s},$$

and, by variation of constants formula, the exponential decay

$$H_e(\psi(s)) \leq O(1)e^{-2s}.$$

By (8.2.3) we then recover

$$H_e(\rho(s)) \leq O(1)e^{-2s}.$$

Finally, we employ the following generalized *Csiszár–Kullback* inequality (see e.g. [AMTU00])

$$\|\rho(s) - \rho_M^\infty\|_{L^1}^2 \leq CH_e(\rho(s))$$

and return back to the original variables u, x, t. Hence, the proof of the theorem is complete.

Appendix A

Uniqueness and regularity of H^s solutions for a hyperbolic–parabolic system

In this section we prove some results concerning uniqueness and regularity of solutions in Sobolev spaces to the hyperbolic–parabolic system

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx} \\ u(x,0) = u_0(x) \\ v(x,0) = v_0(x), \end{cases} \tag{A.0.1}$$

under the assumption

$$\sup_{u \in \mathbb{R}} |\sigma'(u)| < +\infty. \tag{A.0.2}$$

Theorem A.0.3 *Let $(u^1, v^1), (u^2, v^2)$ be two solutions to the Cauchy problem (A.0.1) belonging in the space $L^\infty([0,T], H^1(\mathbb{R}))$, with $u_0, v_0 \in H^1(\mathbb{R})$ and let the function σ satisfies condition (A.0.2). Then, $u_1 = u_2, v_1 = v_2$ almost everywhere on $[0,T] \times \mathbb{R}$.*

Proof. Let $\overline{u} = u^1 - u^2$, $\overline{v} = v^1 - v^2$. Then, $\overline{u}$ and $\overline{v}$ satisfy

$$\begin{cases} \overline{u}_t - \overline{v}_x = 0 \\ \overline{v}_t - \left[\sigma(u^1) - \sigma(u^2)\right]_x = \mu \overline{v}_{xx}. \end{cases} \tag{A.0.3}$$

Multiplying the first equation in (A.0.3) by $\overline{u}$ and the second one by $\overline{v}$ and integrating in dx, after integration by parts, we get the energy identity

$$\frac{d}{dt}\frac{1}{2}\left[\|\overline{u}(t)\|^2_{L^2(\mathbb{R})}+\|\overline{v}(t)\|^2_{L^2(\mathbb{R})}\right]=\int_{-\infty}^{+\infty}\overline{v}_x\overline{u}dx$$
$$-\int_{-\infty}^{+\infty}\left[\sigma(u^1)-\sigma(u^2)\right]\overline{v}_x-\mu\|\overline{v}_x(t)\|^2_{L^2(\mathbb{R})}.$$

Hence, after time integration, from conditions (A.0.2) and from weighted Holder inequality, we obtain

$$\|\overline{u}(t)\|^2_{L^2(\mathbb{R})}+\|\overline{v}(t)\|^2_{L^2(\mathbb{R})}\le C\int_0^T\|\overline{u}(s)\|^2_{L^2(\mathbb{R})}ds,$$

where the constant C depends on σ'. Finally, by applying Gronwall Lemma, we recover

$$\|\overline{u}(t)\|^2_{L^2(\mathbb{R})}+\|\overline{v}(t)\|^2_{L^2(\mathbb{R})}=0,$$

which proves the theorem. □

We now perform an estimate in Sobolev norm for the hyperbolic–parabolic system (A.0.1), again under the assumption (A.0.2). Moreover, we require, without loss of generality, $\sigma(0)=0$.

Theorem A.0.4 *Let (u,v) be the solution to the hyperbolic–parabolic system (A.0.1) with initial data u_0,v_0 belonging in the space $H^2(\mathbb{R})$ and let the function σ satisfies condition (A.0.2). Then, for any $0<t<T$ with $T>0$ fixed, we have*

$$\|u(t)\|^2_{H^2(\mathbb{R})}+\|v(t)\|^2_{H^2(\mathbb{R})}+\int_0^t\|v_x(s)\|^2_{H^2(\mathbb{R})}ds\le C_0,\qquad\text{(A.0.4)}$$

for a constant C_0 depending on T, $\sup|\sigma'|$ and on the initial data.

Proof. Hereafter we denote by C a general positive constant depending on σ, T and on the H^2–norm of the initial data u_0,v_0. We compute, for $0<t<T$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|u(t)\|^2_{L^2(\mathbb{R})}+\|v(t)\|^2_{L^2(\mathbb{R})}\right]&=\int_{-\infty}^{+\infty}u_tudx+\int_{-\infty}^{+\infty}v_tvdx\\
&=\int_{-\infty}^{+\infty}v_xudx+\int_{-\infty}^{+\infty}\sigma(u)_xvdx+\mu\int_{-\infty}^{+\infty}v_{xx}vdx\\
&=\int_{-\infty}^{+\infty}v_xudx-\int_{-\infty}^{+\infty}v_x\sigma(u)-\mu\int_{-\infty}^{+\infty}v_x^2dx\\
&\le -c\int_{-\infty}^{+\infty}v_x^2dx+C\int_{-\infty}^{+\infty}u^2dx,\qquad\text{(A.0.5)}\end{aligned}$$

where $c > 0$ and we have used $\sigma(0) = 0$ and the condition (A.0.2). Hence, integration over the time interval $[0, t]$ and Gronwall inequality yields

$$\begin{aligned} &\|u(t)\|^2_{L^2(\mathbb{R})} + \|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v_x(s)\|^2_{L^2(\mathbb{R})} ds \\ &\leq C \left[\|u(0)\|^2_{L^2(\mathbb{R})} + \|v(0)\|^2_{L^2(\mathbb{R})}\right] e^{Ct}. \end{aligned} \tag{A.0.6}$$

We now write the system for the first spatial derivatives (u_x, v_x)

$$\begin{cases} u_{xt} - v_{xx} = 0 \\ v_{xt} - \sigma(u)_{xx} = \mu v_{xxx}. \end{cases} \tag{A.0.7}$$

As above, we compute

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|u_x(t)\|^2_{L^2(\mathbb{R})} + \|v_x(t)\|^2_{L^2(\mathbb{R})}\right] = \int_{-\infty}^{+\infty} u_{xt}u_x dx + \int_{-\infty}^{+\infty} v_{xt}v_x dx \\ &= \int_{-\infty}^{+\infty} v_{xx}u_x dx + \int_{-\infty}^{+\infty} \sigma(u)_{xx}v_x dx + \mu \int_{-\infty}^{+\infty} v_{xxx}v_x dx \\ &= \int_{-\infty}^{+\infty} v_{xx}u_x dx - \int_{-\infty}^{+\infty} v_{xx}\sigma'(u)u_x - \mu \int_{-\infty}^{+\infty} v_{xx}^2 dx \\ &\leq -c\int_{-\infty}^{+\infty} v_{xx}^2 dx + C\int_{-\infty}^{+\infty} u_x^2 dx, \end{aligned} \tag{A.0.8}$$

where, as before, $c > 0$ and thanks to the condition (A.0.2). Hence, Gronwall inequality gives

$$\begin{aligned} &\|u_x(t)\|^2_{L^2(\mathbb{R})} + \|v_x(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v_{xx}(s)\|^2_{L^2(\mathbb{R})} ds \\ &\leq C \left[\|u_x(0)\|^2_{L^2(\mathbb{R})} + \|v_x(0)\|^2_{L^2(\mathbb{R})}\right] e^{Ct}. \end{aligned} \tag{A.0.9}$$

As final step, we consider the system for the second spatial derivatives

$$\begin{cases} u_{xxt} - v_{xxx} = 0 \\ v_{xxt} - \sigma(u)_{xxx} = \mu v_{xxxx}. \end{cases} \tag{A.0.10}$$

Then we have, as above

$$\frac{1}{2}\frac{d}{dt}\left[\|u_{xx}(t)\|^2_{L^2(\mathbb{R})}+\|v_{xx}(t)\|^2_{L^2(\mathbb{R})}\right]=\int_{-\infty}^{+\infty}u_{xxt}u_{xx}dx+\int_{-\infty}^{+\infty}v_{xxt}v_{xx}dx$$
$$=\int_{-\infty}^{+\infty}v_{xxx}u_{xx}dx+\int_{-\infty}^{+\infty}\sigma(u)_{xxx}v_{xx}dx+\mu\int_{-\infty}^{+\infty}v_{xxxx}v_{xx}dx$$
$$=\int_{-\infty}^{+\infty}v_{xxx}u_{xx}dx-\int_{-\infty}^{+\infty}v_{xxx}\sigma''(u)u_x^2dx$$
$$-\int_{-\infty}^{+\infty}v_{xxx}\sigma'(u)u_{xx}dx-\mu\int_{-\infty}^{+\infty}v_{xxx}^2dx.$$

We now employ estimates (A.0.6), (A.0.9), the smoothness of function σ and the Sobolev inequality

$$\|f\|_{L^\infty(\mathbb{R})}\le\|f\|_{H^1(\mathbb{R})} \tag{A.0.11}$$

to obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|u_{xx}(t)\|^2_{L^2(\mathbb{R})}+\|v_{xx}(t)\|^2_{L^2(\mathbb{R})}\right]$$
$$\le-\frac{\mu}{2}\int_{-\infty}^{+\infty}v_{xxx}^2dx+C\int_{-\infty}^{+\infty}u_{xx}v_{xxx}dx+C\int_{-\infty}^{+\infty}u_x^4dx$$
$$\le-c\int_{-\infty}^{+\infty}v_{xxx}^2dx+C\int_{-\infty}^{+\infty}\left[u_{xx}^2+u_x^2\right]dx, \tag{A.0.12}$$

with $c>0$. Hence, by taking the sum of estimates (A.0.5), (A.0.8) and (A.0.12) and by Gronwall inequality, we obtain

$$\|u(t)\|^2_{H^2(\mathbb{R})}+\|v(t)\|^2_{H^2(\mathbb{R})}+\int_0^t\|v_x(s)\|^2_{H^2(\mathbb{R})}ds$$
$$\le C\left[\|u(0)\|^2_{H^2(\mathbb{R})}+\|v(0)\|^2_{H^2(\mathbb{R})}\right]e^{Ct} \tag{A.0.13}$$

and the proof is complete. □

As a straightforward consequence of the previous theorem, we have the following

Corollary A.0.5 *Let $z=\sigma(u)+\mu v_x$, where (u,v) is the solution of the system (A.0.1) with initial data $u_0,v_0\in H^2(\mathbb{R})$ and let the function σ satisfies condition (A.0.2). Then, for any $T>0$, the function $z_t(\cdot,\cdot)$ belongs in the space $L^2([0,T]\times\mathbb{R})$. More precisely, there exists a constant $C_0>0$, depending on T and on the initial data u_0 and v_0, such that*

$$\int_0^t\int_{-\infty}^{+\infty}z_t(x,s)^2dxds\le C_0. \tag{A.0.14}$$

Proof. We have

$$\begin{aligned} z_t &= \mu v_{xt} + \sigma'(u)u_t = \sigma'(u)v_x + \mu\sigma(u)_{xx} + \mu^2 v_{xxx} \\ &= \sigma'(u)v_x + \mu\sigma''(u)u_x^2 + \mu\sigma'(u)u_{xx} + \mu^2 v_{xxx}. \end{aligned}$$

Hence, the assertion follows from the previous theorem and from the Sobolev inequality (A.0.11). □

Appendix B

Local existence for the vanishing viscosity approximation of the Hamer model for radiating gases

We prove here the local-in-time existence of solutions to the equation

$$u_t + \operatorname{div} f(u) = -u + K * u + \mu \Delta u \tag{B.0.1}$$

when the initial data u_0 is chosen in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We show in particular that the time of existence depends on the L^1 and L^∞ norms of the initial data, on the Lipschitz norm (Sup norm plus Lipschitz seminorm) of the mapping f over the range of the solution and on the constant μ. As usual in this kind of problems (see, for instance, [Smo94]), we shall use a fixed point argument to prove the existence of such solutions. Therefore, let us consider the Banach space

$$\left\{ C([0,T_0]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \right\}$$

with the norm

$$|||u||| = \sup_{0 \le t \le T_0} \max \left\{ \|u(\cdot,t)\|_{L^1}, \|u(\cdot,t)\|_{L^\infty} \right\}$$

and let us consider its closed subset

$$\mathcal{B} = \left\{ u \in C([0,T_0]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \text{ such that } |||u - G_\mu * u_0||| \le |||u_0||| \right\},$$

where

$$G_\mu = \frac{1}{(4\pi\mu t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\mu t}}$$

and hence $G_\mu * u_0$ is the solution of the linear heat equation $u_t = \mu\Delta u$ with u_0 as initial datum. We will obtain the solution of (B.0.1) as fixed point of the following operator, defined on $\mathcal{B}$

$$\begin{aligned}\mathcal{T}u &= \int_{\mathbb{R}^d} G_\mu(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_\mu(x-y,t-s)\,\mathrm{div}\, f(u(y,s))dyds \\ &\quad - \int_0^t \int_{\mathbb{R}^d} G_\mu(x-y,t-s)\,[u(y,s)-(K*u)(y,s)]\,dyds \\ &= I_1 + I_2(u) + I_3(u). \qquad \text{(B.0.2)}\end{aligned}$$

Remark B.0.6 Since $|||G_\mu * u_0||| \le |||u_0|||$, $0 \in \mathcal{B}$ and $|||u||| \le 2|||u_0|||$ for any $u \in \mathcal{B}$.

The local existence for the solutions to (B.0.1) comes from the next theorem.

Theorem B.0.7 *There exists a positive time $T_0 > 0$ such that the operator $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is a contraction. In particular, there exists an unique solution $u \in C([0,T_0]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ to (B.0.1) with $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ as initial datum.*

Proof. In order to prove the theorem, we must find constants $T_0 > 0$ and $c_0 < 1$ such that

(i) $\mathcal{T}u \in \mathcal{B}$ for any $u \in \mathcal{B}$;

(ii) $|||\mathcal{T}u - \mathcal{T}v||| \le c_0|||u-v|||$ for any u, $v \in \mathcal{B}$.

Let us consider $u \in \mathcal{B}$. In order to control $|||\mathcal{T}u - G_\mu * u_0|||$, we have to estimate the L^1 and the L^∞ norms of I_2 and I_3 in the definition of $\mathcal{T}$. We have

$$\|I_2(u)\|_{L^\infty} \le \|f(u)\|_{L^\infty}\frac{C}{\sqrt{\mu}}\sqrt{t} \le C_\mu\sqrt{t}\|f'\|_{L^\infty(I)}\|u\|_{L^\infty},$$

where we denoted $I = [-\|u\|_{L^\infty}, \|u\|_{L^\infty}]$. Moreover

$$\|I_2(u)\|_{L^1} \le \|f(u)\|_{L^1}\frac{C}{\sqrt{\mu}}\sqrt{t} \le C_\mu\sqrt{t}\|f'\|_{L^\infty(I)}\|u\|_{L^1},$$

where C_μ depends only on μ and the L^1 norms of $K(x)$ and $K(x)x$. Therefore

$$|||I_2(u)||| \le C_\mu\sqrt{T_0}\|f'\|_{L^\infty(I)}|||u||| \le 2C_\mu\sqrt{T_0}\|f'\|_{L^\infty(J)}|||u_0|||.$$

where $J = [-2|||u_0|||, 2|||u_0|||]$. Moreover,

$$\|I_3(u)\|_{L^\infty} \le 2t\|u\|_{L^\infty}$$

and

$$\|I_3(u)\|_{L^1} \leq 2t\|u\|_{L^1},$$

that is

$$|||I_3(u)||| \leq 2T_0|||u||| \leq 4T_0|||u_0|||.$$

Thus, if

$$2C_\mu\sqrt{T_0}\|f'\|_{L^\infty(J)}|||u_0||| + T_0 4|||u_0||| \leq ||u_0|||, \tag{B.0.3}$$

(i) is satisfied. In order to fulfill (B.0.3), it is sufficient to choose

$$T_0 = \min\left\{\frac{1}{8}, \frac{1}{16C_\mu^2}\frac{1}{\|f'\|^2_{L^\infty(J)}}\right\}.$$

To show (ii), we have to estimate

$$|||\mathcal{T}u - \mathcal{T}v||| \leq |||I_2(u) - I_2(v)||| + |||I_3(u) - I_3(v)|||,$$

for any u, $v \in \mathcal{B}$. Since $I_3(u)$ is linear in u, we have

$$|||I_3(u) - I_3(v)||| = |||I_3(u-v)||| \leq 2T_0|||u-v|||.$$

Moreover,

$$\|I_2(u) - I_2(v)\|_{L^\infty} \leq \|f(u) - f(v)\|_{L^\infty}\frac{C}{\sqrt{\mu}}\sqrt{t} \leq C_\mu\sqrt{t}\|u-v\|_{L^\infty}\|f'\|_{L^\infty(J)},$$

where again $J = [-2|||u_0|||, 2|||u_0|||]$. Moreover

$$\|I_2(u) - I_2(v)\|_{L^1} \leq \|f(u) - f(v)\|_{L^1}\frac{C}{\sqrt{\mu}}\sqrt{t} \leq C_\mu\sqrt{t}\|u-v\|_{L^1}\|f'\|_{L^\infty(J)},$$

with C_μ as before. Hence,

$$|||I_2(u) - I_2(v)||| \leq C_\mu\sqrt{T_0}\|f'\|_{L^\infty(J)}|||u-v|||.$$

Therefore,

$$|||\mathcal{T}u - \mathcal{T}v||| \leq \left(C_\mu\sqrt{T_0}\|f'\|_{L^\infty(J)} + 2T_0\right)|||u-v||| \leq \frac{1}{2}|||u-v|||,$$

provided

$$T_0 = \min\left\{\frac{1}{8}, \frac{1}{16C_\mu^2}\frac{1}{\|f'\|^2_{L^\infty(J)}}\right\}.$$

Finally, the theorem is proved with the choice

$$T_0 = \min\left\{\frac{1}{8}, \frac{1}{16C_\mu^2}\frac{1}{\|f'\|^2_{L^\infty(J)}}\right\}.$$

□

Remark B.0.8 The smoothness of the solution u provided by the previous theorem comes easily from the integral representation of the equation given by (B.0.2) with $\mathcal{T}u$ replaced by u, thanks to the smoothing properties of the kernel G_μ.

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