The entropy dissipation method for spatially inhomogeneous reaction–diffusion type systems

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Motivation

Many physical, chemical and biological processes are driven by diffusion and reactions, representing instantaneous interactions between particles. Typical examples are chemical kinetics, population dynamics, flame propagation and combustion, movement of biological cells in plants and animals, and charge carrier transport in semiconductors.

A significant variant to many of these situations is given by the presence of a *heterogeneous enviroment*.

Basic modeling

Let $U = U(x,t) \in \mathbb{R}^N$ denote the concentration vector describing N interacting species, where $x \in \mathbb{R}^n$ denotes the position variable and $t > 0$ time. Then, the diffusion part of the motion is described by

$$U_t = \operatorname{div}(D \nabla U),$$

where $D = D(x,t,U)$ is a positive definite, symmetric diffusion matrix, in general depending on position x, time t, and on the concentration vector U itself.

Given a reaction process in terms of a 'local' dynamical system of the form $U_t = F(x,t,U)$, then the interaction of both reaction and diffusion leads to

$$U_t = \operatorname{div}(D(x,t,U) \nabla U) + F(x,t,U).$$

Classical literature

Reaction–diffusion systems are a classical topic, going (at least) back to the pivotal works of e.g. Fisher and Kolmogorov et al.

Vast literature: see the textbooks by Smoller, Rothe, Amann, and the references therein for existence of solutions and their global boundedness, stability and asymptotics, traveling waves, geometry and topology of attracting sets and singular limits.

Stability vs. instability: in 1952, A. M. Turing first pointed out diffusion–induced instability of stable homogeneous reaction systems in chemistry. The classical mathematical analysis of the long–time asymptotic behavior involves linearized stability techniques, spectral theory, perturbation and invariant regions arguments, and Lyapunov stability.

Entropy method

Our approach is different and motivated by the recent great progress in the understanding of long–time asymptotics of scalar linear and nonlinear diffusion and diffusion–convection equations due to the so called *entropy approach*. Literature:

- Arnold, Markowich, Toscani, Unterreiter 2000 – 2001
- Markowich, Villani 2001
- Carrillo, Toscani 2000
- Carrillo, Juengel, Markowich, Toscani, Unterreiter 2001

Basic ideas:

- A *functional inequality relation* between an *entropy functional* of a system and its monotone change in time, usually called the *entropy dissipation*.
- Such an entropy–entropy dissipation inequality entails convergence to an entropy minimizing equilibrium state, at first in entropy and further in L^1 using Cziszár-Kullback-Pinsker type inequalities (cf. Otto, Carrillo et al., Markowich et al.)

Advantages:

- It's a nonlinear method avoiding any kind of linearization and capable of providing explicitly computable convergence rates.
- Being based on functional inequalities rather than particular differential equations, it has the advantage of being quite robust with respect to model variations.
- It does not require any restriction on the space dimension.

Previous results

Related to the context of this paper, the entropy approach has already been applied to semiconductor drift–diffusion–Poisson systems (Arnold et al.), or to drift–diffusion–reaction–Poisson systems on bounded domains in $\mathbb{R}^2$ (Glitzki et al.) but with the drawback of a proof based on an indirect contradiction argument without control on the rates and constants. We also mention a paper by Degond – Genieys – Juengel dealing with general cross–diffusion systems. It was first in the works by Desvillettes and Fellner that explicit exponential convergence to equilibrium has been shown via entropy methods for nonlinear reaction–diffusion systems modeling reversible mass action kinetics of two, three, and four species. A general framework for reaction–diffusion systems however (long–time asymptotic convergence, convergence rates, etc.) is still lacking.

Simple examples

In order to demonstrate the entropy approach we consider the simple problem

$$\begin{cases} u_t = \Delta u, \quad u(t=0,x) = u_I(x), & x \in \Omega \\ n(x) \cdot \nabla u(x,t) = 0. & x \in \partial\Omega \end{cases}$$

All constants are stationary states and the equilibrium state is determined by the conservation of the initial mass:

$$\overline{u}(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t)\, dx = \frac{1}{|\Omega)|} \int_\Omega u_I(x)\, dx =: \overline{u_I} \qquad \text{for all } t > 0.$$

A simple computation gives

$$\frac{d}{dt}\int_\Omega (u-\overline{u})^2\,dx = -2\int_\Omega |\nabla u|^2\,dx \le -\frac{2}{D_\Omega^2}\int_\Omega (u-\overline{u})^2\,dx\,,$$

where we have used the $H^1(\Omega)$–Poincaré inequality with constant $1/D_\Omega^2$, which is the spectral gap of the homogeneous Neumann–Laplace operator. Hence, after integration in time we obtain the sharp decay estimate:

$$\int_\Omega (u(x,t)-\overline{u_I})^2\,dx \le \exp\left(-\frac{2}{D_\Omega^2}t\right)\int_\Omega (u_I-\overline{u_I})^2(x)\,dx.$$

Let us add now a linear absorption term with a constant rate:

$$\begin{cases} u_t = \Delta u - \lambda u, \quad u(t=0,x) = u_I(x), & x \in \Omega \\ n(x) \cdot \nabla u(x,t) = 0. & x \in \partial\Omega \end{cases}$$

Clearly, this shifts the spectrum and convergence to the unique equilibrium state $u_\infty = 0$ ($\overline{u}$ is no more conserved) follows from the sharp estimate:

$$\int_\Omega u(x,t)^2\,dx \leq \exp(-2\lambda t) \int_\Omega u_I(x)^2\,dx.$$

Thus, even in this most simplistic scalar example equation, diffusion and stable reaction do not 'cooperate' in the rate of decay to equilibrium (since constant states are not affected by diffusion). Nevertheless, full asymptotic cooperation between diffusion and stable reaction is observed for the intermediate asymptotic state $\overline{u}(t)$:

$$\int_\Omega \left(u(x,t) - \overline{u}(t)\right)^2 \, dx \leq \exp\left(-2\left(\lambda + \frac{1}{D_\Omega^2}\right)t\right) \int_\Omega \left(u_I(x) - \overline{u_I}\right)^2 \, dx$$

Remarks:

- Linear systems combining 'purely diffusive' modes obeying conservation laws with 'diffusive–reactive' modes are particular interesting as far as (non)cooperation of reaction and diffusion effects on the convergence rates are concerned.

- An explicitly computable 2×2 system with constant coefficients demonstrates the complicated system related interaction effects of diffusion and reaction. Such systems are typical in many physical and biological applications.

Our aim is to start developing a framework for the quantitative analysis of the large–time asymptotics of stable reaction–diffusion–convection systems based on the entropy approach. For systems on the whole space with confinement, the present paper provides the first attempt in this generality.

Outline

1. First we outline the presented approach in the clearest possible way, namely for systems posed on bounded domains with constant equilibrium states. We prove their exponential stability.

2. Then we turn to the analysis of whole–space systems with confinement in each component. In the general case of N species we prove exponential stability of constant equilibria under a quite elaborated set of assumptions.

3. A refined convergence result for a linearized 2×2 drift–diffusion–recombination system with confining potentials is also shown, under a lighter (and more physically reasonable) set of assumptions.

4. For a nonlinear 2×2 drift–diffusion–recombination system we show exponential convergence towards the inhomogeneous equilibria.

Linear systems on bounded domains with constant equilibria

We consider linear systems in the symmetrized form

$$\partial_t(SU) = \operatorname{div}(\widetilde{D}(x)\nabla U) + \widetilde{R}(x)U, \quad x \in \Omega \subset \mathbb{R}^n, \quad U = U(x,t) \in \mathbb{R}^N \tag{1}$$

We further prescribe the initial datum

$$U(x,0) = U_0(x), \tag{2}$$

and assume zero-flux boundary conditions

$$(\widetilde{D}(x)\nabla U(x))\cdot\nu(x) = 0 \qquad \text{on } \partial\Omega. \tag{3}$$

Structural assumptions:

(A1) The matrix $S \in \mathbb{R}^{N\times N}$ is constant, symmetric and strictly positive definite. More precisely, there exists a constant $\underline{s} > 0$ such that $S\xi \cdot \xi \geq \underline{s}|\xi|^2$ for all $\xi \in \mathbb{R}^N$.

(A2) The matrix $\widetilde{D}(x)$ is symmetric with eigenvalues $(0, \dots, 0, \mu_{d+1}(x), \dots, \mu_N(x))$ for an integer $d < N$ and $0 < \underline{\mu} \leq \mu_j(x)$ for all $j \in \{d+1, \dots, N\}$. Moreover, $\widetilde{D}(x)$ has a constant eigenvector othonormed matrix F.

(A3) The matrix $\widetilde{R}(x)$ is symmetric with eigenvalues $(0, \dots, 0, \lambda_{c+1}(x), \dots, \lambda_N(x))$ for an integer $1 \leq c < N$ and $\lambda_j(x) \leq -\underline{r} < 0$ for all $j \in \{c+1, \dots, N\}$. Moreover, $\widetilde{R}(x)$ admits a constant orthogonal eigenvector matrix E.

(A4) The following *Kawashima–type* condition holds

$$\mathrm{Ker}\widetilde{D}(x) \cap \mathrm{Ker}\widetilde{R}(x) = \{0\} \qquad \text{for all } \ x \in \Omega. \tag{4}$$

Example 1: Semiconductor devices with trapped state.

Let us consider the system

$$\begin{cases} \partial_t n - \Delta n = C_c n_{tr} - C_a n(N_{tr} - n_{tr}) =: R_n \\ \partial_t p - \Delta p = C_d(N_{tr} - n_{tr}) - C_b\, p\, n_{tr} =: R_p \\ \partial_t n_{tr} = R_p - R_n \end{cases} \tag{5}$$

modeling transport and diffusion of charged particles in a semiconductor device combined with a recombination–generation mechanism called *band–trap capture and emission* (due to the presence of impurities, see the book by Markowich–Ringhofer–Schmeiser). For simplicity, we have neglected the effect of the self–consistent potential, therefore coupling occurs only due to the recombination–generation terms.

The equilibrium vectors $(n^\infty, p^\infty, n_{tr}^\infty)$ satisfy

$$n^\infty p^\infty = \frac{C_c C_d}{C_a C_b}, \qquad n_{tr}^\infty = N_{tr} \frac{C_a n^\infty + C_d}{C_a n^\infty + C_c + C_b p^\infty + C_d} < N_{tr},$$

whereas the uniqueness of the equilibrium state is achieved by imposing

$$|\Omega|(n^\infty - p^\infty + n_{tr}^\infty) = M.$$

The linearization of system (5) around the unique equilibrium state gives the linear system

$$U_t - \Delta(DU) = R \tag{6}$$

where

$$D = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} C_a(n_{tr}^\infty - N_{tr}) & 0 & C_c + C_a n^\infty \\ 0 & -C_b n_{tr}^\infty & -C_d - C_b p^\infty \\ -C_a(n_{tr}^\infty - N_{tr}) & -C_b n_{tr}^\infty & -C_d - C_b p^\infty - C_c - C_a n^\infty \end{bmatrix}.$$

A symmetrizer for R is the diagonal matrix $S = \mathrm{diag}\left(\frac{1}{n^\infty},\ \frac{1}{p^\infty},\ \frac{1}{n^\infty}\left(1+\frac{C_d}{p^\infty C_b}\right)\right)$ and the symmetrized reaction matrix reads

$$\widetilde{R} = SR = \begin{bmatrix} \frac{C_a}{n^\infty}(n_{tr}^\infty - N_{tr}) & 0 & \frac{C_c}{n^\infty} + C_a \\ 0 & -\frac{C_b n_{tr}^\infty}{p^\infty} & -C_b - \frac{C_d}{p^\infty} \\ \frac{C_c}{n^\infty} + C_a & -C_b - \frac{C_d}{p^\infty} & -N_{tr}(C_a n^\infty + C_d)\frac{C_d + C_b p^\infty}{C_b p^\infty} \end{bmatrix}.$$

The matrix $\widetilde{R}$ has the only 0–eigenvector $E^1 := (n^\infty, p^\infty, \frac{C_c C_d}{C_a(p^\infty C_b + C_d)})$, which implies that the assumptions above are satisfied. An elementary but tedious calculations shows that the two remaining eigenvalues λ_2 and λ_3 of $\widetilde{R}$ are both negative. Therefore, the symmetrized form of the linear system (6) satisfies all assumptions (A1)–(A4).

Example 2: Reaction–diffusion system with four species

We consider a diffusive and reversible chemical reaction of the type $\mathcal{A}_1 + \mathcal{A}_3 \rightleftharpoons \mathcal{A}_2 + \mathcal{A}_4$ on a bounded domain $\Omega \subset \mathbb{R}^n$. Denote by $a_i(x,t)$, $i = 1, \ldots, 4$ the concentration of the four reacting species $\mathcal{A}_1, \ldots, \mathcal{A}_4$ and by $d_i > 0$, $i = 1, \ldots, 4$ their respective diffusivity constants. Assuming mass action kinetics for the reactions, we obtain the system

$$\partial_t a_i = d_i \Delta a_i + (-1)^i (a_1 a_3 - a_2 a_4),$$

where we have rescaled – without loss of generality – the reaction rates to one.

The stationary states $(a_1^\infty, \ldots, a_4^\infty)$ consists of the unique set of positive constants, which balance the reaction, i.e. $a_1^\infty a_3^\infty = a_2^\infty a_4^\infty$, and satisfy the conservation laws of the systems. Linearization around those states produces the linearized reaction matrix R and the associated diagonal symmetrizer matrix S

$$R := \begin{bmatrix} -a_3^\infty & a_4^\infty & -a_1^\infty & a_2^\infty \\ a_3^\infty & -a_4^\infty & a_1^\infty & -a_2^\infty \\ -a_3^\infty & a_4^\infty & -a_1^\infty & a_2^\infty \\ a_3^\infty & -a_4^\infty & a_1^\infty & -a_2^\infty \end{bmatrix}, \qquad S := \operatorname{diag}\left(\frac{1}{a_1^\infty}, \ldots, \frac{1}{a_4^\infty}\right).$$

It is easy to check that all assumptions (A1)–(A4) are satisfied in this case.

Stationary states and their exponential stability

Lemma 1. [Conserved quantities] *Let $U(x,t)$ be a classical solution to system (1). Then, for every $j \in \{1,\dots,c\}$, the quantity*

$$l_j := \int_\Omega E^j S U(x,t) dx \tag{7}$$

is conserved for all times $t \geq 0$.

Proposition 1. [Stationary solutions] *For every fixed set of quantities $l_1,\dots,l_c$, there exists a unique (constant) stationary solution U^∞ to (1) such that*

$$l_j := |\Omega| E^j S U^\infty \tag{8}$$

for all $j \in \{1,\dots,c\}$.

Two facts justify symmetrization as a suitable method:

- In case R and D and S be constant matrices, then, the functional defined in (10) is a Lyapunov functional for our linear system if and only if $SD + DS$ and $-(SR + R^T S)$ are nonnegative definite.

- In case of absence of symmetry of R, one can easily construct examples of D producing Turing instabilities.

We introduce the *normal modes* vector variable

$$V(x,t) := EU(x,t) \tag{9}$$

and the following *energy functional*

$$\mathcal{E}[U] := \frac{1}{2}\int_\Omega U(x)^T S\ U(x)dx. \tag{10}$$

Theorem 1. [Exponential convergence towards constant equilibrium states] *Let $U(x,t)$ be a classical solution to (1) with initial datum $U_0(x) \in L^2(\Omega)$ satisfying*

$$l_j := \int_\Omega E^j S U_0(x) dx \tag{11}$$

for some given $l_1, \ldots, l_c \in \mathbb{R}$. Then, for all times $t \geq 0$ the following estimate holds

$$\mathcal{E}[U(t) - U^\infty] \leq \mathcal{E}[U(0) - U^\infty] \exp\left(- \left(\frac{K \underline{r}}{\overline{s}(K+L)} - \delta \right) t \right), \tag{12}$$

where $K := \frac{\underline{\mu}}{\overline{=}} \frac{C_F}{C_P(\Omega)}$ and $C_P(\Omega)$ is the optimal constant in Poincaré inequality on the domain Ω, L and C_F are constants depending on the matrices E and F respectively, the constant $\overline{s}$, $\underline{\mu}$ and $\underline{r}$ are defined in the assumptions (A1)–(A3) and δ is an arbitrary positive small constant, $0 < \delta \ll 1$. In particular, $U(t) \to U^\infty$ exponentially fast in L^2 as $t \to +\infty$.

Confined linear systems on $\mathbb{R}^n$ with integrable equilibria

We consider now

$$\partial_t U = \operatorname{div}\left(D(x)\nabla(S(x)U)\right) + R(x)U, \tag{13}$$

with $U = U(x,t) \in \mathbb{R}^N$, $d \geq 1$, $t \geq 0$, $x \in \mathbb{R}^n$. We prescribe the initial datum $U(x,0) = U_0(x)$. System (13) generalizes a model of N reacting–diffusing species convected by N external potentials.

We develop:

1. a general theory for $N \times N$ systems including an exponential asymptotic stability result;

2. an improved result on the 2×2 case, obtained under less restrictive assumptions.

Structural assumptions:

(B1) *Essentially convex and diagonal confinement:* The confinement matrix is of the form $S(x) = \mathrm{diag}(s_1(x), \ldots, s_N(x))$ and such that there exist two functions $\underline{s}, \overline{s} > 0$ and a constant $K > 0$ such that $\underline{s}^{-1} \in L^1_+(\mathbb{R}^N)$ and, for all j's,

$$\underline{s}(x) \leq s_j(x) \leq \overline{s}(x), \qquad \overline{s}(x) \leq K\underline{s}(x). \tag{14}$$

Moreover, we shall assume that the functions $\mathbb{R}^n \ni x \mapsto \log s_j(x)$ are L^∞ perturbations of uniformly convex functions for all $j = 1, \ldots, N$.

(B2) *Positive definite diffusion matrix:* The diffusion matrix $D(x)$ is symmetric and positive definite. Moreover, the following inequality is satisfied for a certain positive constant d_0

$$\xi^T D(x)\xi \geq d_0 \xi^T S^{-1}(x)\xi, \quad \text{for all } \xi \in \mathbb{R}^N. \tag{15}$$

(B3) *Confinement compatible reaction matrix:* The confinement matrix $S(x)$ is a symmetrizer for the reaction matrix $R(x)$, i. e.

$$\widetilde{R}(x) := S(x)R(x) \qquad \text{is symmetric.}$$

Moreover, the symmetrized reaction matrix $\widetilde{R}(x)$ has eigenvalues

$$\lambda_1(x), \ldots, \lambda_N(x)$$

satisfying $\lambda_1(x) \equiv \ldots \equiv \lambda_d(x) \equiv 0$, $\lambda_j(x) \leq 0$ for all $j \in \{d+1, \ldots, N\}$ and there exists two positive constant C_1 and C_2 such that

$$C_1 s_j(x) \leq -\lambda_j(x) \leq C_2 s_j(x), \tag{16}$$

for all $x \in \mathbb{R}^N$ and for all $j \in \{d+1, \ldots, N\}$.

(B4) *Bounded eigenvector matrix E of the reaction matrix $\widetilde{R}$:* The orthogonal eigenvector matrix $E(x)$ of $\widetilde{R}(x)$ satisfies

$$E(x)\widetilde{R}(x)E(x)^T = \Lambda(x), \quad \Lambda(x) := \mathrm{diag}(\underbrace{0,\ldots,0}_{d}, \lambda_{d+1}(x),\ldots,\lambda_N(x)).$$

Moreover, $E(x)$ has uniformly bounded coefficients $e_{ij}(x)$, $i,j = 1,\ldots,N$, such that $|e_{ij}| \geq C_E > 0$ for all i,j.

(B5) *Conservation laws:* The reaction matrix $R(x)$ has d constant left zero–eigenvectors, i. e. there exist $F_1,\ldots,F_d \in \mathbb{R}^N$ such that

$$F_j^T R(x) = 0, \qquad j \in \{1,\ldots,d\}.$$

Example 3: $N \times N$ reaction drift–diffusion system on $\mathbb{R}^n$

$$\partial_t u_j = \operatorname{div}(\nabla u_j + u_j \nabla V_j) + R^j(x) \cdot U, \quad j = 1, \dots N, \tag{17}$$

where $U(x,t) = (u_1(x,t), \dots, u_N(x,t))$, $x \in \mathbb{R}^n$, $t \geq 0$, R^j denotes the j–th row of a reaction matrix R and V_j is a *confining potential* acting on the j–th species u_j. A simple computation shows that the system (17) can be written in the above form (13) with

$$S(x) = \operatorname{diag}(e^{V_1(x)}, \dots, e^{V_N(x)}), \qquad D(x) := S(x)^{-1}$$

and $R(x)$ any reaction matrix such that $S(x)R(x)$ is symmetric. The assumption (B1) can be fulfilled by assuming that all potentials $V_j(x)$ are all continuous and convex with equal asymptotic behaviour of the tails as $|x| \to +\infty$. However, this assumption on the tails is often too restrictive.

Lemma 2. [Conserved quantities] *Let $U(x,t)$ be a classical solution to system (13) such that U decays rapidly at $|x| \to +\infty$. Then, for every $j \in \{1,\dots,d\}$, the quantity $l_j := \int_{\mathbb{R}^N} (F^j)^T U(x,t)dx$ is conserved for all times $t \geq 0$.*

Proposition 2. [Stationary solutions] *For every fixed set of real numbers $l_1,\dots,l_d$, there exists a unique stationary solution $U^\infty(x)$ to (13) such that*

$$l_j = \int_{\mathbb{R}^N} F^j U^\infty(x)dx \tag{18}$$

for all $j \in \{1,\dots,d\}$. Moreover, $U^\infty(x)$ has the form

$$U^\infty(x) = S^{-1}(x)C^\infty \tag{19}$$

for a certain constant vector $C^\infty = (C_1^\infty,\dots,C_N^\infty)^T$.

We introduce the following variables

$$W(x,t) := E(x)U(x,t)$$
$$Z(x,t) := S(x)U(x,t) = S(x)E^T(x)W(x,t).$$

Similarly to the previous case, we shall consider the *energy functional*

$$\mathcal{E}[U] := \frac{1}{2}\int_{\mathbb{R}^N} U^T S(x) U dx = \frac{1}{2}\int_{\mathbb{R}^N} Z^T S^{-1}(x) Z dx.$$

Theorem 2. [Exponential convergence towards inhomogeneous equilibria] *Let $U(x,t)$ be a classical solution to system (13) with initial datum U_0, such that U decays rapidly at $|x| \to +\infty$. Suppose*

$$l_j = \int_{\mathbb{R}^N} (F^j)^T U_0(x) dx, \quad j \in \{1, \dots, d\}, \tag{20}$$

for fixed quantities $l_1, \dots, l_d \in \mathbb{R}$. Let the stationary state U^∞ be uniquely determined by the l_j's as in Proposition 2. Then, there exists a fixed constant $\varepsilon > 0$ (depending on the structural assumptions (B1)–(B5)) such that

$$\mathcal{E}[U(t) - U^\infty] \le \mathcal{E}[U_0 - U^\infty] e^{-\varepsilon t} \tag{21}$$

for all $t \ge 0$.

A typical 2×2 example case

We consider a 2×2 whole space drift–diffusion–recombination system (the linearization of the nonlinear semiconductor model to be studied later on)

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla V_1) - F(n_\infty, p_\infty, x)(p_\infty u + n_\infty v) \\ \partial_t v = \nabla \cdot (\nabla v + v \nabla V_2) - F(n_\infty, p_\infty, x)(p_\infty u + n_\infty v). \end{cases} \tag{22}$$

where

$$n_\infty(x) := e^{-V_1(x)}, \quad p_\infty(x) := e^{-V_2(x)}, \qquad V_1,\ V_2 \in C^2(\mathbb{R}^n),$$
$$V_1 \text{ and } V_2 \text{ are } \ L^\infty \text{ perturbation of uniformly convex functions} \tag{23}$$

which implies in particular that $n_\infty, p_\infty \in L^1_+(\mathbb{R}^n)$.

The reaction rate $F(\cdot,\cdot,\cdot)$ shall denote a continuous function, typically thought to be of *Shockley–Read–Hall* type

$$F(n,p,x) = (r_1 + r_2 n + r_3 p)^{-1}, \tag{24}$$

for some positive, bounded below functions $r_1(x), r_2(x), r_3(x) \geq r > 0$.

We use the entropy (or free energy) functional

$$E(u,v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{u^2}{n_\infty} + \frac{v^2}{p_\infty}\right) dx.$$

We introduce the variables $z_1 = \frac{u}{n_\infty}$ and $z_2 = \frac{v}{p_\infty}$ and we obtain

$$\frac{d}{dt}E(z_1,z_2) = \frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^N} z_1^2\, dn_\infty + \int_{\mathbb{R}^N} z_2^2\, dp_\infty\right) = -D(z_1,z_2)\,,$$

with

$$D(z_1, z_2) = \int_{\mathbb{R}^N} |\nabla z_1|^2 dn_\infty + \int_{\mathbb{R}^N} |\nabla z_2|^2 dp_\infty + \int_{\mathbb{R}^N} F\, n_\infty p_\infty (z_1 + z_2)^2\, dx\,,$$

and the measures $dn_\infty = n_\infty dx$, $dp_\infty = p_\infty dx$.

We denote

$$d\xi_1 = \frac{dn_\infty}{N_\infty}\,, \quad N_\infty = \int_{\mathbb{R}^N} dn_\infty\,, \qquad d\xi_2 = \frac{dp_\infty}{P_\infty}\,, \quad P_\infty = \int_{\mathbb{R}^N} dp_\infty\,,$$

and we estimate the Fisher–information terms using a weighted Poincaré inequality

$$\int_{\mathbb{R}^N} |\nabla z_i|^2 d\xi_i \geq P_i \int_{\mathbb{R}^N} (z_i - \overline{z_i})^2\, d\xi_i\,, \quad \text{with } \overline{z_i} := \int z_i\, d\xi_i\,.$$

The constants P_1 and P_2 are the whole space Poincaré constants with respect to ξ_1 and ξ_2 respectively, therefore they depend on V_1 and V_2. Then, for a suitable constant $C > 0$, we are looking for the following entropy–entropy dissipation estimate

$$
\begin{aligned}
D &\geq P_1 \int_{\mathbb{R}^N} (z_1 - \overline{z_1})^2 dn_\infty + P_2 \int_{\mathbb{R}^N} (z_2 - \overline{z_2})^2 dp_\infty + \int_{\mathbb{R}^N} F\, n_\infty p_\infty \, (z_1 + z_1)^2 \, dx \\
&\geq C\, E = \frac{C}{2} \left(\int_{\mathbb{R}^N} z_1^2 dn_\infty + \int_{\mathbb{R}^N} z_2^2 dp_\infty \right) ,
\end{aligned}
$$

under the constraint for the conservation of mass :

$$
\int_{\mathbb{R}^N} z_1 \, dn_\infty = \int_{\mathbb{R}^N} z_2 \, dp_\infty \, . \tag{25}
$$

Lemma 3.
Consider measurable functions z_i, $i = 1, 2$ *such that* (25) *holds. Let* $F(n_\infty, p_\infty)$ *be integrable with respect to the measure* $n_\infty p_\infty dx$ *and satisfy*

$$Fn_\infty \leq \frac{1}{\tau_n}, \qquad Fp_\infty \leq \frac{1}{\tau_p}. \tag{26}$$

Then

$$D \geq \frac{K(1-\varepsilon)}{K_1} E$$

holds provided that

$$K \leq \min\left\{ \frac{P_1}{\frac{2}{\tau_n}\frac{1-\varepsilon}{\varepsilon} + \frac{1}{2k_1}}, \frac{P_2}{\frac{2}{\tau_p}\frac{1-\varepsilon}{\varepsilon} + \frac{1}{2k_1}} \right\}, \tag{27}$$

with K_1 *defined in* (**??**) *and where* $0 < \varepsilon < 1$ *can be chosen in order to maximise the constant* K.

A reaction–diffusion model with nonlinear reaction

In this section we study the nonlinear model system arising in semiconductor and plasma physics

$$\begin{cases} n_t = \mathrm{div} J_n - R(n,p)\,, & J_n := \nabla n + n\nabla V_n \\ p_t = \mathrm{div} J_p - R(n,p)\,, & J_p := \nabla p + p\nabla V_p \end{cases} \tag{28}$$

where n and p model two species of charged particles subject to confinement and to a recombination–generation mechanism $R(n,p)$. We suppose non-negative initial data

$$n(x,0) = n_I(x) \geq 0\,, \qquad\qquad p(x,0) = p_I(x) \geq 0\,,$$

and the following assumptions:

(NL1) The confining potentials V_n and V_p satisfy

$$D^2 V_i(x) \geq \sigma_i \mathbb{I}_N, \qquad \text{for certain constants } \sigma_n, \sigma_p > 0, \quad i \in \{n, p\}.$$

Moreover, $\|\Delta V_i\|_{L^\infty(\mathbb{R}^n)}$ is finite for $i \in \{n, p\}$, and we define $\mu_i := e^{-V_i(x)}$ and introduce the related measures $d\mu_i := \mu_i(x)dx$.

(NL2) The recombination–generation term is of the form $R(n, p) = F(n, p, x)(np - \delta^2 \mu_n \mu_p)$ for a constant $\delta > 0$, which – without loss of generality – shall be rescaled as $\delta = 1$. The scalar function $F(n, p, x) \geq 0$ is assumed to be such that

$$R(n, p) \leq A_1 + A_2 n + A_3 p, \tag{29}$$

for constants $A_1 > 0$, $A_2, A_3 \geq 0$, which includes the typical *Shockley–Read–Hall* form (24).

We recall that the initial mass $M \in \mathbb{R}$ is conserved for all $t > 0$

$$\int_{\mathbb{R}^N} n(x,t)\,dx - \int_{\mathbb{R}^N} p(x,t)\,dx = M := \int_{\mathbb{R}^N} n_I(x)\,dx - \int_{\mathbb{R}^N} p_I(x)\,dx\,. \tag{30}$$

Given M as fixed, the equilibrium $n = n_\infty$, $p = p_\infty$ is uniquely determined by

$$\begin{cases} n_\infty(x) = C_n\, e^{-V_n(x)}, \quad p_\infty(x) = C_p\, e^{-V_p(x)}, \\ C_n, C_p > 0 : \quad C_n C_p = 1, \quad C_n \int e^{-V_n} dx + C_p \int e^{-V_p} dx = M. \end{cases} \tag{31}$$

The *relative entropy* $E = E(n,p)$ of the system (28) with respect to the equilibrium (31)

$$E = \int_{\mathbb{R}^N} \left[n \ln \frac{n}{n_\infty} - (n - n_\infty) + p \ln \frac{p}{p_\infty} - (p - p_\infty) \right] dx \tag{32}$$

dissipates (i.e. $\frac{d}{dt}E(n,p) = -D(n,p)$) with the entropy dissipation

$$D = \int_{\mathbb{R}^N} \frac{|J_n|^2}{n} dx + \int_{\mathbb{R}^N} \frac{|J_p|^2}{p} dx + \int_{\mathbb{R}^N} F\,\mu^2 \left(\frac{n}{n_\infty}\frac{p}{p_\infty} - 1\right) \ln\left(\frac{n}{n_\infty}\frac{p}{p_\infty}\right) dx\,. \tag{33}$$

Uniform L^∞ bound

Lemma 4. [Uniform $L^1 \cap L^\infty$ bounds] *Assume the initial data n_I, p_I are in $L^1 \cap L^\infty(\mathbb{R}^n)$ with finite entropy $E(n_I, p_I) < +\infty$. Then the solution (n, p) of (28) satisfies*

$$\sup_{t \geq 0} \left[\|n(t)\|_{L^r} + \|p(t)\|_{L^r} \right] < \infty$$

for all $r \in [1, +\infty]$.

Exponential convergence to equilibrium

In the following we show exponential convergence (with constants that can all be made explicit) towards the unique equilibrium states n_∞, p_∞ as defined in (31). In addition to the assumptions (NL1) and (NL2) we will suppose that :

(NL3) The confining potentials are equal $V := V_n = V_p$ and $\mu := e^{-V(x)}$ is – without loss of generality – normalized with $\int_{\mathbb{R}^N} d\mu = \int_{\mathbb{R}^N} \mu \, dx = 1$.

(NL4) There exists a constant lower bound $F(n,p)\mu(x) \geq C_F(\|n\|_\infty^{-1}, \|p\|_\infty^{-1})$ and moreover, due to the bounds of Lemma 4, $C_F(\|n\|_\infty^{-1}, \|p\|_\infty^{-1}) \geq C_F > 0$.

Lemma 5. [Entropy entropy-dissipation inequality] *Let n and p be nonnegative functions in $L^1 \cap L^\infty(\mathbb{R}^n)$ satisfying the conservation law $N - P = C_n - C_p$ as given in* (30)*. Suppose the assumptions (NL1), (NL3), and (NL4) hold. Then, the following inequality holds for a constant K depending only on the stated quantities*

$$E(n,p) \leq K(\|n\|_1, \|p\|_1, C_F)\, D(n,p)\,. \tag{34}$$

Thanks for your attention!