University of L'Aquila

Master Degree in Computer Science

Course on Formal Methods

E-unification: some exercises

Monica Nesi

Let us consider the following E-unification problems taken from the collection of exercises in `http://www.di.univaq.it/monica/MFI/EserciziR.pdf`.

Exercise E2. Let R be the following canonical TRS that describes an equational theory E on the signature $\Sigma = \{a, f, g, h, k\}$:

$$\begin{array}{rcl} g(a,x) & \rightarrow & a \\ g(h(x),y) & \rightarrow & f(k(y), g(x,y)) \\ k(a) & \rightarrow & a \end{array}$$

Solve modulo E the equation $g(x, y) = k(y)$ by applying the E-unification algorithm based on normal and basic narrowing. Give the complete tree of the narrowing derivations.

The given TRS R is canonical, thus we can use the E-unification algorithm based on narrowing. Moreover, we will apply the basic and normal version of such an algorithm using the definition of basic positions and by reducing the goal to normal form in R before applying narrowing steps. With the expression "complete tree of the narrowing derivations" we mean the complete development *breadth-first* of the first levels of the tree, depending on the complexity of the E-unification problem. In this example it is sufficient to develop the first two complete levels of the tree.

We start with the initial goal $||(g(x, y), k(y))$, that labels the root of the tree of the narrowing derivations. The two terms of the goal are in normal form in R and do not unify syntactically. The initial set of positions is $Pos_0 = \{\epsilon, 1, 2\}$.

First level
It is possible to apply three narrowing steps, thus we have three branches exiting from the root:

I.1. Narrowing on $p = 1$ with the first rule (with variables suitably renamed[1]) $g(a, x_1) \to a$ and mgu $\sigma_1 = \{a/x,\ y/x_1\}$. The new goal is $||(a, k(y))$, is in normal form in R and the new set of basic positions is $Pos_1 = \{\epsilon, 1, 2\}$.

I.2. Narrowing on $p = 1$ with the second rule (with variables suitably renamed[2]) $g(h(x_1), y_1) \to f(k(y_1), g(x_1, y_1))$ and mgu $\sigma_1 = \{h(x_1)/x,\ y/y_1\}$. The new goal is $||(f(k(y), g(x_1, y)), k(y))$, is in normal form in R and the new set of basic positions is $Pos_1 = \{\epsilon, 1, 1.1, 1.2, 2\}$.

I.3. Narrowing on $p = 2$ with the third rule $k(a) \to a$ and mgu $\sigma_1 = \{a/y\}$. The new goal is $||(g(x, a), a)$, is in normal form in R and the new set of basic positions is $Pos_1 = \{\epsilon, 1, 2\}$.

Second level
II.1. Starting from the goal $||(a, k(y))$ of I.1 with $Pos_1 = \{\epsilon, 1, 2\}$, it is possible to apply only one narrowing step on $p = 2$ with the third rule $k(a) \to a$ and mgu $\sigma_2 = \{a/y\}$. The new goal is $||(a, a)$, whose terms unify syntactically with mgu $\mu = id$, thus we have termination with success along this path with the E-unifying substitution σ resulting from the composition $\sigma = \mu \circ \sigma_2 \circ \sigma_1 = \{a/y,\ a/x,\ a/x_1\}$. If we consider only the bindings for the variables in the initial equation, we have the solution $x = a,\ y = a$.
To verify that this is indeed a solution for the given equation, it is enough to substitute the computed values for the variables x and y in the initial equation and then use the decision procedure for the word problem in E. Therefore, the equivalence to be checked is $\sigma(g(x, y)) =_E \sigma(k(y))$, that is $g(a, a) =_E k(a)$. The two terms rewrite to the same normal form in R by applying the first and third rule, which are the rules applied in the narrowing tree along the derivation leading to the solution.

II.2. Starting from the goal $||(f(k(y), g(x_1, y)), k(y))$ of I.2 with $Pos_1 =$

[1]Variable renaming is necessary to avoid conflicts between the variables in the rule and those in the goal and with the variables of the other rules applied along the same derivation path. We recall that an E-unifier is obtained by composing the narrowing substitutions applied along a path that terminates with success and we are only interested in idempotent unifiers.

[2]Note that along a path it is possible to use renamings equal to those occurring along other paths.

$\{\epsilon, 1, 1.1, 1.2, 2\}$, it is possible to apply four narrowing steps:

II.2.1. Narrowing on $p = 1.1$ with the third rule $k(a) \to a$ and mgu $\sigma_2 = \{a/y\}$. The new goal is $||(f(a, g(x_1, a)), k(a))$ and the new set of basic positions is $\{\epsilon, 1, 1.1, 1.2, 2\}$. The new goal is not in normal form in R. By rewriting the goal with the third rule in position $p = 2$, we obtain the normalized goal $||(f(a, g(x_1, a)), a)$ and the new set of basic positions $Pos_2 = \{\epsilon, 1, 1.1, 1.2, 2\} = Pos_1$.

II.2.2. Narrowing on $p = 1.2$ with the first rule $g(a, x_2) \to a$ and mgu $\sigma_2 = \{a/x_1,\ y/x_2\}$. The new goal is $||(f(k(y), a), k(y))$, is in normal form in R and the new set of basic positions is $Pos_2 = \{\epsilon, 1, 1.1, 1.2, 2\} = Pos_1$.

II.2.3. Narrowing on $p = 1.2$ with rule $g(h(x_2), y_2) \to f(k(y_2), g(x_2, y_2))$ and mgu $\sigma_2 = \{h(x_2)/x_1,\ y/y_2\}$. We get the new goal $||(f(k(y), f(k(y), g(x_2, y))),$ $k(y))$, that is in normal form in R, and the new set of basic positions is $Pos_2 = \{\epsilon, 1, 1.1, 1.2, 1.2.1, 1.2.2, 2\}$.

II.2.4. Narrowing on $p = 2$ with the third rule $k(a) \to a$ and mgu $\sigma_2 = \{a/y\}$. The new goal is $||(f(k(a), g(x_1, a)), a)$ and the new set of basic positions is $\{\epsilon, 1, 1.1, 1.2, 2\}$. This goal is not normalized in R. By rewriting it with the third rule in position $p = 1.1$, we get the normalized goal $||(f(a, g(x_1, a)), a)$ and the new set of basic positions $Pos_2 = \{\epsilon, 1, 1.1, 1.2, 2\}$. Goal and positions coincide with those of node II.2.1.

II.3. Starting from the goal $||(g(x, a), a)$ of I.3 with $Pos_1 = \{\epsilon, 1, 2\}$, it is possible to apply two narrowing steps:

II.3.1. Narrowing on $p = 1$ with the first rule $g(a, x_2) \to a$ and mgu $\sigma_2 = \{a/x,\ a/x_2\}$. The new goal is $||(a, a)$, whose terms unify syntactically with mgu $\mu = id$, thus we have termination with success along this path with the E-unifying substitution $\sigma = \mu \circ \sigma_2 \circ \sigma_1 = \{a/x,\ a/x_2,\ a/y\}$. If we consider only the bindings for the variables in the initial equation, the solution is $x = a,\ y = a$, that means that we have a different path deriving the same solution of node II.1.

II.3.2 Narrowing on $p = 1$ with rule $g(h(x_2), y_2) \to f(k(y_2), g(x_2, y_2))$ and mgu $\sigma_2 = \{h(x_2)/x,\ a/y_2\}$. The new goal is $||(f(k(a), g(x_2, a)), a)$ and the new set of basic positions is $\{\epsilon, 1, 1.1, 1.2, 2\}$. This goal is not normalized in R. By rewriting it with the third rule in position $p = 1.1$, we get the normalized goal $||(f(a, g(x_2, a)), a)$ and the new set of basic positions $Pos_2 = \{\epsilon, 1, 1.1, 1.2, 2\} = Pos_1$.

This completes the development of the second level of the (infinite) tree of the narrowing derivations.

Exercise E4. Let R be the following canonical TRS that describes an equational theory E on the signature $\Sigma = \{a, f, g, h\}$:

$$\begin{array}{rcl} h(h(x)) & \to & x \\ g(f(x), y) & \to & f(g(x, y)) \\ g(a, x) & \to & x \end{array}$$

Solve modulo E the equation $g(x, f(h(y))) = f(x)$ by applying the E-unification algorithm based on normal and basic narrowing. Give the complete tree of the narrowing derivations.

Also in this case it is sufficient to develop the first two complete levels of the tree. Given the initial goal $||(g(x, f(h(y))), f(x))$, the two terms of the goal are in normal form in R and do not unify syntactically. The initial set of positions is $Pos_0 = \{\epsilon, 1, 1.2, 1.2.1, 2\}$.

First level
It is possible to apply three narrowing steps:

I.1. Narrowing on $p = 1$ with the second rule $g(f(x_1), y_1) \to f(g(x_1, y_1))$ and mgu $\sigma_1 = \{f(x_1)/x,\ f(h(y))/y_1\}$.
We get the goal $||(f(g(x_1, f(h(y)))), f(f(x_1)))$ (in normal form in R) and the set of basic positions $Pos_1 = \{\epsilon, 1, 1.1, 2\}$.

I.2. Narrowing on $p = 1$ with the third rule $g(a, x_1) \to x_1$ and mgu $\sigma_1 = \{a/x,\ f(h(y))/x_1\}$. We get the goal $||(f(h(y)), f(a))$ (in normal form in R) and the set of basic positions $Pos_1 = \{\epsilon, 2\}$. The terms of the goal do not unify syntactically and it is not possible to apply narrowing on the positions in Pos_1, therefore such a derivation path terminates with failure.

I.3. Narrowing on $p = 1.2.1$ with the first rule $h(h(x_1)) \to x_1$ and mgu $\sigma_1 = \{h(x_1)/y\}$. The new goal (in normal form in R) is $||(g(x, f(x_1)), f(x))$ and the set of basic positions is $Pos_1 = \{\epsilon, 1, 1.2, 2\}$.

Second level
II.1. Given the goal $||(f(g(x_1, f(h(y)))), f(f(x_1)))$ of I.1 with $Pos_1 = \{\epsilon, 1,$

$1.1, 2\}$, two narrowing steps can be applied:

II.1.1. Narrowing on $p = 1.1$ with the second rule $g(f(x_2), y_2) \to f(g(x_2, y_2))$ and mgu $\sigma_2 = \{f(x_2)/x_1,\ f(h(y))/y_2\}$.
The new goal is $||(f(f(g(x_2, f(h(y))))), f(f(f(x_2))))$ (in normal form in R) and $Pos_2 = \{\epsilon, 1, 1.1, 1.1.1, 2\}$.

II.1.2 Narrowing on $p = 1.1$ with the third rule $g(a, x_2) \to x_2$ and mgu $\sigma_2 = \{a/x_1,\ f(h(y))/x_2\}$. The new goal is $||(f(f(h(y))), f(f(a)))$ (in normal form in R) and $Pos_2 = \{\epsilon, 1, 2\}$. The terms of the goal do not unify syntactically and it is not possible to apply narrowing on the positions in Pos_2, therefore such a derivation path terminates with failure.

II.3. Given the goal $||(g(x, f(x_1)), f(x))$ of I.3 with $Pos_1 = \{\epsilon, 1, 1.2, 2\}$, two narrowing steps can be applied:

II.3.1. Narrowing on $p = 1$ with the second rule $g(f(x_2), y_2) \to f(g(x_2, y_2))$ and mgu $\sigma_2 = \{f(x_2)/x,\ f(x_1)/y_2\}$.
The new goal is $||(f(g(x_2, f(x_1))), f(f(x_2)))$ (in normal form in R) and $Pos_2 = \{\epsilon, 1, 1.1, 2\}$.

II.3.2 Narrowing on $p = 1$ with the third rule $g(a, x_2) \to x_2$ and mgu $\sigma_2 = \{a/x,\ f(x_1)/x_2\}$. The new goal is $||(f(x_1), f(a))$ (in normal form in R) and the new set of basic positions is $\{\epsilon, 2\}$. The terms of the goal unify syntactically with mgu $\mu = \{a/x_1\}$, thus we have termination with success along this path with the E-unifying substitution $\sigma = \{a/x_1,\ a/x,\ f(a)/x_2,\ h(a)/y\}$. If we consider only the bindings for the variables in the initial equation, the solution is $x = a,\ y = h(a)$.
Check: $\sigma(g(x, f(h(y)))) =_E \sigma(f(x))$, that is $g(a, f(h(h(a)))) =_E f(a)$. The term $f(a)$ is in normal form in R. The first term rewrites to $f(a)$ by applying the third and the first rule in R (or vice versa), that are the two rules applied along the derivation of the narrowing tree leading to the solution.

This completes the development of the second level of the (infinite) tree of the narrowing derivations. Note that in this exercise the use of basic positions allows us to “cut” some paths of the tree, that are possible derivations whenever the optimization based on basic positions is not applied.