Global solutions for a hyperbolic model of multiphase flow

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ABSTRACT. We study a strictly hyperbolic system of three balance laws arising in the modelling of fluid flows, in one space dimension. The fluid is a mixture of liquid and vapor, and pure phases may exist as well. The flow is driven by a reaction term depending either on the deviation of the pressure p from an equilibrium value p_e and on the mass density fraction of the vapor in the fluid; this makes possible for metastable regions to exist. A relaxation parameter is also involved in the model.

First, for the homogeneous system, we review a result about the global existence of weak solutions to the initial-value problem, for initial data with large variation. Then we focus on the inhomogeneous case. For initial data sufficiently close to the stable liquid phase we prove, through a fractional step algorithm, that weak global solutions still exist. At last, we study the relaxation limit under such assumptions, and prove that the solutions previously constructed converge to weak solutions of the homogeneous system for the pure liquid phase.

1. Introduction

In the last years the theory of hyperbolic balance laws in one space dimension has developed to a fairly satisfactory level. That is widely recognized and is proved by the many monographs and textbooks on this subject, [**Bre00, Daf05, HR02, LeF02, Ser00**]; in particular, the global existence and wellposedness of weak solutions to the initial-value problem, in the homogeneous case and for small BV initial data, has been intensively studied by using different methods of proofs, [**Gli65, Bre00, BB05**]. There are nevertheless some points of the general theory that leave in the shadow and would deserve further considerations. In the homogeneous case but for large initial data, for instance, solutions may blow up in finite time, [**BJ01**]; in other cases, on the contrary, solutions may exist globally in time with or without specific bounds on the variation of the data, [**Nis68, NS73, Bia00**]. Both previous issues are challenging and demand a deep

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understanding of how the nonlinearity affects the global existence versus finite time blow-up.

As far as the inhomogeneous case is concerned, the results of global existence of BV entropy solutions shrink considerably even for small initial data, [**DH82, Liu79, CP97**], and some issues as the relaxation approximation are still unclear for weak solutions, [**Nat99**].

In this paper we consider some of these open issues by focusing on a simple system arising in the modelling of multiphase flows, in one space dimension. The flow is assumed isothermal and inviscid; the model writes as

$$\begin{cases} v_t - u_x &= 0 \\ u_t + p(v,\lambda)_x &= 0 \\ \lambda_t &= \frac{1}{\tau}\left(p(v,\lambda) - p_e\right)\lambda(\lambda-1)\,. \end{cases} \tag{1.1}$$

Here above $t > 0$, $x \in \mathbb{R}$ and, as usual, $v > 0$ denotes the specific volume, u the velocity, p the pressure, τ a relaxation time. The state variable $\lambda \in [0,1]$ is the mass density fraction of the vapor in the fluid: the value $\lambda = 0$ identifies the pure liquid phase while $\lambda = 1$ the pure vapor phase. Moreover, $p_e > 0$ represents a constant equilibrium pressure; in particular the model allows for metastable regions. At last, the system is closed by the pressure law

$$p(v,\lambda) = \frac{a^2(\lambda)}{v} \tag{1.2}$$

where a is a smooth, positive and increasing function. This simple model is deduced from [**Fan00**] by dropping the viscosity and species diffusion terms but keeping the reaction source term which drives the dynamics of the phase changes. The pressure law (1.2) is a good approximation for fluids with large heat capacity, where the flows may be assumed to be nearly isothermal.

A detailed analysis of several hydrodynamical models related to (1.1) was given in [**BG91**], together with some numerical simulations; see also [**Pen94**] for a system close to (1.1). We refer to [**EK08**] for a related system, named drift-flux mixture model, where a viscosity term in the second equation is included; in that case the function $a(\lambda)$ above is assigned a specific expression. See also [**LMB**] for another model in geothermal energy recovery. Remark that the equilibrium points for the reaction term in (1.1) are $\lambda = 0$, $\lambda = 1$ and $p = p_e$; the Riemann problem for a p-system with states belonging to these sets has been studied in [**CF05**].

Here, we are concerned with the global existence of solutions to the initial-value problem both for the homogeneous and the complete system (1.1), with data

$$(v(0,x), u(0,x), \lambda(0,x)) = (v_o(x), u_o(x), \lambda_o(x)) \tag{1.3}$$

having bounded variation, and to the relaxation limit $\tau \to 0$ of such solutions.

We first briefly review the main results related to the homogeneous system

$$\begin{cases} v_t - u_x &= 0 \\ u_t + p(v,\lambda)_x &= 0 \\ \lambda_t &= 0\,. \end{cases} \tag{1.4}$$

Notice that in this case system (1.4) may also be written as

$$\begin{cases} v_t - u_x &= 0 \\ u_t + p(v,\lambda_o(x))_x &= 0 \end{cases} \tag{1.5}$$

with a inhomogeneous, possibly discontinuous, flux.

The case when λ_o is constant, and then $p = a^2/v$ for a constant a, was first considered in [**Nis68**], where it was proved by a Glimm scheme that the Cauchy problem has a global solution for every initial data (v_o, u_o) provided that $\mathrm{TV}(v_o, u_o) < \infty$. We also refer to [**CR98**] for a generalization and continuous dependence of solutions on data, [**AG01**] for a different proof and [**Asa05**]. In the case $p(v) = a^2/v^\gamma$, with $\gamma > 1$, the global existence of solutions was proved under the condition that $(\gamma - 1)\mathrm{TV}(u_o, v_o)$ is sufficiently small, [**NS73, DiP73**].

If λ_o is not constant, some results can be obtained by compensated compactness but for pressure laws differing from (1.2), [**BBL97, Gos01, Lu03**]. A model strictly related to (1.1) was recently studied in [**HRS**]; there, the adiabatic index $\gamma > 1$ entering in the pressure law $p = a^2/v^\gamma$ is assumed to vary in space and the last equation in (1.1) is replaced by $\gamma_t = 0$. In that interesting paper the authors prove by the Glimm scheme the existence of globally defined solutions under conditions similar to those in [**NS73**]. For completeness we mention that results for the close full Euler system have been given in [**Liu77, Tem81**].

Concerning the inhomogeneous case, we quote [**LT82**] for the p-system with $\gamma = 1$ with suitable source terms and [**PRV95**] for the isothermal Euler-Poisson system with large data. Remark that the diagonal dominance assumption [**DH82**] does not hold for (1.1).

At last we recall that the global existence of **BV** solutions for

$$\begin{cases} v_t - u_x & = 0 \\ u_t + (1/v)_x & = \frac{1}{\tau} r(v,u)\,, \end{cases}$$

for suitable reaction terms r, has been proved in [**AG01, LNY00**]; a typical example is $r(v,u) = A(v) - u$. Both papers consider as well the relaxation limit $\tau \to 0$ and, up to the knowledge of the authors, they are the only papers proving the convergence of relaxation approximation for weak solutions of quasi-linear systems; see also [**BS00, Bia01**].

In this paper we first recast a recent result concerning the Cauchy problem for the homogeneous system (1.4), [**AC08b**]: the existence of globally defined weak solutions is proved by means of a front-tracking algorithm, for initial data whose total variation is bounded by a quantity that may be explicitly specified. In particular, we emphasize that the variation of the initial data may be suitably large.

New results about the full system (1.1) are then presented. For $\tau > 0$ fixed we assume that the initial data are close to the pure stable liquid region, that is $p_o\left(v_o(x), \lambda_o(x)\right) > p_e$, and both $\|\lambda_o\|_\infty$ and $\mathrm{TV}\lambda_o$ are sufficiently small. Then global solutions to (1.1), (1.3) exist if v_o and u_o satisfy suitable bounds, which again may be given explicitly. We employ here a fractional step method and exploit the results in [**AC08b**]. At last we consider the relaxation limit $\tau \to 0$ of these weak solutions and prove that they converge to solutions of (1.5) with $\lambda_o = 0$. Complete proofs and more details are given in [**AC08c**].

2. The homogeneous system

In this section we review some results of [**AC08b**] concerning (1.4); at the same time we introduce some notation needed in the following sections. We assume that a is a $\mathbf{C^2}$ function defined on $[0,1]$ satisfying for every $\lambda \in [0,1]$

$$a(\lambda) > 0, \quad a'(\lambda) > 0\,. \tag{2.1}$$

We denote $U = (v, u, \lambda)^t \in \Omega = (0, +\infty) \times \mathbb{R} \times [0, 1]$ and

$$g(U) \quad = \quad (p(v, \lambda) - p_e)\lambda(1 - \lambda)\,. \tag{2.2}$$

The system (1.4) is strictly hyperbolic in Ω with eigenvalues $e_1 = -c$, $e_2 = 0$, $e_3 = c$, where $c = c(v, \lambda) = \sqrt{-p_v} = \frac{a(\lambda)}{v}$; the corresponding eigenvectors are $r_1 = (1, c, 0)$, $r_2 = (p_\lambda, 0, -p_v)$, $r_3 = (-1, c, 0)$. The eigenvalues e_1, e_3 are genuinely nonlinear with $\nabla e_i \cdot r_i = p_{vv}/(2c) > 0$, $i = 1, 3$, while e_2 is linearly degenerate.

We denote by $\mathrm{TV}(f)$ the total variation of a function f. For $f : \mathbb{R} \to (0, +\infty)$ we introduce the *weighted total variation* of f by

$$\mathrm{WTV}(f) = 2 \sup \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|}{f(x_j) + f(x_{j-1})},$$

where the sup is computed over all $n \geq 1$ and $(n + 1)$-tuples of points x_j, with $x_o < x_1 < \cdots < x_n$. The introduction of the weighted variation is suggested by the very definition of strength for the waves of the second family and allows for estimates which are slightly more precise than $\mathrm{TV}(\log f)$. Indeed, it can be proved, [**AC08b**], that

$$\frac{\inf f}{\sup f} \mathrm{TV}\left(\log(f)\right) \leq \mathrm{WTV}(f) \leq \mathrm{TV}\left(\log(f)\right)\,, \tag{2.3}$$

and $\mathrm{WTV}(f) = \mathrm{TV}\left(\log(f)\right)$ if $f \in C(\mathbb{R})$.

We next define $a_o(x) = a\left(\lambda_o(x)\right)$, $p_o(x) = p\left(v_o(x), \lambda_o(x)\right)$ and denote $A_o = \mathrm{WTV}(a_o)$. We assume:

$$v_o(x) \geq \underline{v} > 0\,, \qquad \lambda_o(x) \in [0, 1]\,. \tag{2.4}$$

Theorem 2.1 ([**AC08b**]). *Consider the system* (1.4) *with initial data* (1.3) *and assume* (2.4)*. For a suitable function $H : (0, 1/2] \to [0, \infty)$, satisfying*

$$H(1/2) = 0 \quad \text{and} \quad \lim_{A \to 0+} H(A) = +\infty\,,$$

the following is true: if

$$A_o \quad < \quad \frac{1}{2}\,, \tag{2.5}$$

$$\mathrm{TV}\left(\log(p_o)\right) + \frac{1}{\inf a_o}\mathrm{TV}(u_o) \quad < \quad H(A_o)\,, \tag{2.6}$$

then the Cauchy problem (1.4)-(1.3) *has a weak entropic solution $(v, u, \lambda)(\cdot, t)$ defined for $t \geq 0$, with uniformly bounded total variation.*

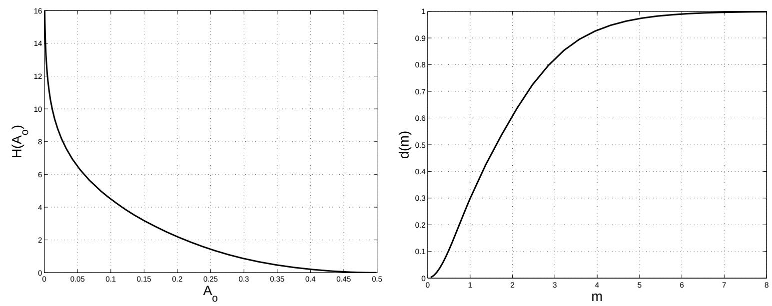
One of the interesting features of this result is that the function H above can be explicitly computed. Indeed, under the notations of [**AC08b**], we have

$$H(A_o) \doteq 2\left(1 - 2A_o\right) \cdot k^{-1}(A_o)\,,$$

see Figure 1*(a)*, where k^{-1} is the inverse function of $k(m) = (1 - \sqrt{d(m)})/(2 - \sqrt{d(m)})$. Here $d(m)$ is a crucial term measuring the damping of the reflected wave $\varepsilon_{\mathrm{refl}}$ in the interaction of two incoming waves δ', δ'' of the same family 1 or 3, both having strengths less than m:

$$|\varepsilon_{\mathrm{refl}}| \leq d(m) \cdot \min\{|\delta'|, |\delta''|\}\,, \tag{2.7}$$

see Figure 1*(b)*. We have that $d(m) < 1$ and $d(m) \to 1$ as $m \to \infty$; moreover, the function k^{-1} has the same monotonicity properties as H. Remark that, in the same spirit of [**NS73, Liu77, HRS**], if A_o is small, then $H(A_o)$ is large; on the contrary, if A_o is close to $1/2$, then $H(A_o)$ must be small.



(a) (b)

FIGURE 1. (a): the function $H(A_o)$, (b): the damping coefficient $d(m)$.

We now sketch the proof of Theorem 2.1 and introduce some quantities to be used in the next sections. First, the Riemann problem for (1.4), i.e., the Cauchy problem with initial data

$$(v,u,\lambda)(0,x) = \begin{cases} (v_\ell, u_\ell, \lambda_\ell) = U_\ell & x < 0 \\ (v_r, u_r, \lambda_r) = U_r & x > 0\,, \end{cases}$$

is solvable for any pair of initial data satisfying $v_\ell\,,\ v_r > 0$ and $\lambda_\ell,\ \lambda_r \in [0,1]$, [**AC08a**]. Remark that phase waves (i.e., waves of the second family) are stationary: both p and u are conserved across them. We call for short sonic waves the waves of the families 1 or 3.

The strength ε_i of an i-wave, $i = 1,2,3$, that connects a state U_l to a state $U_r = (v_r, u_r, \lambda_r)$ is defined by

$$\begin{aligned} \varepsilon_1 &= \frac{1}{2}\log\left(\frac{v_r}{v_l}\right) = \frac{1}{2}\log\left(\frac{p_l}{p_r}\right), \\ \varepsilon_2 &= 2\,\frac{a(\lambda_r) - a(\lambda_l)}{a(\lambda_r) + a(\lambda_l)}, \\ \varepsilon_3 &= \frac{1}{2}\log\left(\frac{v_l}{v_r}\right) = \frac{1}{2}\log\left(\frac{p_r}{p_l}\right). \end{aligned}$$

Since p is constant across phase waves, then $\sum_{i=1,3} |\varepsilon_i| = \frac{1}{2}\mathrm{TV}\log p$.

A notable feature of system (1.4) and of the above choice of the strengths is the possible amplification of reflected waves after an interaction of a sonic with a phase wave. More precisely, let $\delta_{\rm inc}$ be (the strength of) a sonic wave interacting with a 2-wave δ_2. As a consequence of the interaction three waves are emitted: a wave ε_2 of the second family, another wave $\varepsilon_{\rm trans}$ of the same family of $\delta_{\rm inc}$, a third wave $\varepsilon_{\rm refl}$ of the remaining family. One can then prove the estimates

$$|\varepsilon_{\rm trans}| \le \frac{1}{2}\,|\delta_2|\cdot|\delta_{\rm inc}|\,, \quad |\varepsilon_{\rm trans}| + |\varepsilon_{\rm refl}| - |\delta_{\rm inc}| \le |\delta_{\rm inc}| \cdot [\![\, \delta_2 \,]\!]_{\pm}\,,$$

where $[\![\, \delta_2 \,]\!]_{\pm}$ equals $[\![\, \delta_2 \,]\!]_{+}$ if the incoming wave belong to the first family and $[\![\, \delta_2 \,]\!]_{-}$ otherwise.

These estimates have two consequences: first, the reflected wave may be much larger than the incoming sonic wave and, second, the variation $|\varepsilon_{\rm trans}|+|\varepsilon_{\rm refl}|-|\delta_{\rm inc}|$ of the sonic strengths may be positive.

The interactions of sonic waves are much simpler: sonic waves of different families cross each other without changing their strength while waves of the same family satisfy $|\varepsilon_1| + |\varepsilon_3| \le |\delta'| + |\delta''|$, where ε_1, ε_3 (δ', δ'') are the outgoing (resp., incoming) waves. Therefore, in absence of 2-waves we recover that $L(t) = \sum_{i=1,3} |\varepsilon_i|$ is not increasing, see [**Nis68**]. In general, however, large reflected waves may interact with phase waves, producing even larger waves and so on; the improved estimate (2.7) on the reflected waves is used to measure as precisely as possible the amount of this increase.

The proof then goes on by using a suitable version of a wave-front tracking algorithm [**Bre00, AG01**]. More precisely we solve the interactions of sonic waves always with the accurate Riemann solver, while the interactions with the phase waves are solved according to the cases either with the accurate or the simplified Riemann solver by introducing the non-physical waves.

The following functionals are used to control the variations of the waves in the approximate solutions: for $\xi \ge 1$ and K, $K_{np} \ge 0$ we define

$$\begin{aligned} L_\xi &= \sum_{i=1,3,\,\gamma_i>0} |\gamma_i| \;+\; \xi \sum_{i=1,3,\,\gamma_i<0} |\gamma_i| \;+\; K_{np} \sum_{\gamma \in NP} |\gamma| \\ Q &= \sum_{\gamma_i,\,\delta_2 \text{approaching},\ i=1,3} |\gamma_i\,\delta_2| \\ F &= L_\xi + KQ \end{aligned} \tag{2.8}$$

and $L_{cd} = \sum |\gamma_2|$. The main result on approximate solutions to (1.4) is the following, from which the statement of Theorem 2.1 may be deduced from [**AC08b**]. Denote

$$\underline{m} = \frac{1}{2(1-2A_o)} \left(\mathrm{TV}\,(\log p_o) + \frac{1}{\inf a_o} \mathrm{TV} u_o \right)$$

and $m_o = k^{-1}(A_o)$. Note that, by (2.6), one has $\underline{m} < m_o$.

LEMMA 2.2. *Under the assumptions of Theorem 2.1, consider any approximate solution constructed as above. Then, for every* $m \in (\underline{m}, m_o)$ *there exist* $\xi \ge 1$, $K > 0$ *and* $K_{np} > 0$ *such that* $L_\xi(t) < m$, $L_{cd}(t) < k(m)$ *and* $F(t)$ *is non-increasing for all* $t \ge 0$. *Moreover,* $F(0+) \le m$.

Next, in order to define the algorithm for all times, one needs to prove that interaction points do not accumulate and then that wave-fronts do not focus; indeed, this could happen, see Figure 2.

To prevent this situation, we introduce non-physical waves, which are small artificial waves along which a small error term is propagated, and prove that their total size is small. Here, due to the special properties of this system (namely, a property of commutation between curves, [**AC08a**, Lemma 2]) the non-physical waves are discontinuous only in the u-component.

The proof is concluded by a compactness argument, which allows us the passage to the limit.

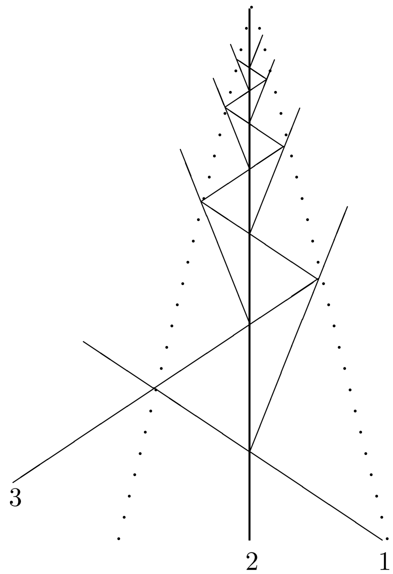


FIGURE 2. A case of possible interactions with a 2-wave. Numbers denote the three waves giving rise to the whole interaction pattern; the dotted lines pass through the interactions points of the waves in either phases, and show the focusing.

3. The reactive system

We consider now the full system (1.1) for $\tau > 0$ fixed. The reactive source term writes $-g(v,\lambda)/\tau$ and its set $E = \{(v,u,\lambda) : g = 0\}$ of equilibrium points consists of three subsets:

(i) E_o, where $\lambda = 0$, i.e. the liquid phase: the system reduces to

$$v_t - u_x = 0, \qquad u_t + p(v,0)_x = 0\,;$$

(ii) E_1, where $\lambda = 1$, the vapor phase: the system reduces to $v_t - u_x = 0$ and $u_t + p(v,1)_x = 0$;

(iii) E_p, where $p = p_e$: the pressure is at equilibrium; in this case we have

$$p = p_e\,, \qquad u = \text{const.}\,, \qquad \lambda = \lambda(x)\,.$$

Here we focus on the case $\lambda \sim 0$ (the other cases will be considered in a forthcoming paper). Remark that the sign of $p - p_e$ determines the behavior of the equation

$$\lambda_t = \frac{1}{\tau}(p - p_e)\lambda(\lambda - 1)\,, \tag{3.1}$$

for λ; from a physical point of view that sign, for $\lambda = 0$, characterizes the stable and metastable liquid regions. Here follows a result of global existence of solutions to (1.1).

Theorem 3.1. *Consider the system* (1.1) *with* $\tau > 0$ *fixed. Assume that the initial data* (1.3) *satisfy* (2.4)*, and*

$$\inf_{x\in\mathbb{R}} (p_o(x) - p_e) \quad \geq \quad \underline{c} \; > \; 0\,, \tag{3.2}$$

$$\mathrm{TV}\,(\log(p_o)) + \frac{1}{\inf a_o}\mathrm{TV}(u_o) \quad < \quad \log p_o(-\infty) - \log p_e\,, \tag{3.3}$$

for some constants $\underline{c}$, $\underline{v}$. *Then there exists* $\mu > 0$ *such that, if*

$$\|\lambda_o\|_\infty \leq \mu\,, \qquad \mathrm{TV}(\lambda_o) \leq \mu\,, \tag{3.4}$$

then the Cauchy problem (1.1), (1.3) *has a weak entropic solution* $(v, u, \lambda)(\cdot, t)$*. The solution exists for all* $t \geq 0$*, it has uniformly bounded total variation and satisfies, for a suitable constant* $c > 0$*,*

$$p(x,t) - p_e > c\,, \qquad 0 \leq \lambda(x,t) \leq \mu\,. \tag{3.5}$$

To avoid ambiguity in (3.5) we set $p(x,t) = \lim_{y\to x+} p(y,t)$, and similar for λ. As a consequence of (3.5), by (3.1) we have

$$\lambda_t \leq -\frac{c}{\tau}(1-\mu)\lambda\,,$$

which shows that the contribution of the source term in (3.1) decays exponentially in time, see [**DH82**].

The proof of Theorem 3.1 is performed by a fractional step scheme, [**DH82**], combined with the wave-front tracking algorithm exploited in [**AC08b**]; we now give a short sketch of the proof.

We first approximate the initial data with suitable piecewise constant functions $\big(v_o^\nu(x), u_o^\nu(x), \lambda_o^\nu(x)\big)$; in particular $\|\lambda_o^\nu\|_\infty \leq \mu$. For simplicity we drop the index ν from now on.

Then we fix a time mesh $\Delta t > 0$ and solve at time $t = 0+$ each Riemann problem arising at the points of jump of the approximate initial data. By slightly changing the speed of the waves, we may assume that at any time only two wave-fronts can interact; when this occurs, we solve a Riemann problem at the interaction point and extend the solution consequently. Assume that an approximate solution $U(x,t)$ is defined in this way until the time $t_n = n\cdot\Delta t$, for $n = 1, 2, \ldots$; as above, we assume that no wave interaction occurs at time t_n. Then, we update the solution by considering the source term:

$$\lambda(x, t_n) \quad \doteq \quad \lambda(x, t_n-) \; - \; \frac{\Delta t}{\tau}\cdot g(x, t_n-)\,. \tag{3.6}$$

At time t_n we solve each Riemann problem that arises for the updated U in correspondence to each discontinuity in U at time t_n-, see Figure 3. The whole procedure is then iterated.

We require that $\mu < 1/2$ and fix any m as in Lemma 2.2. Moreover, assume for the moment that $p - p_e$ is uniformly bounded from below. Then straightforward computations lead to the following estimates on the update states; we denote $g_n^\pm(x) = g(x, t_n\pm)$.

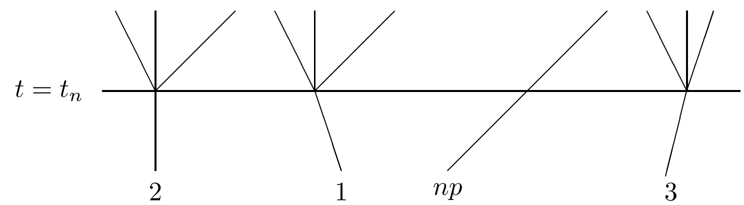


FIGURE 3. The fractional step scheme; the numbers 1, 2, 3 and *np* (non-physical) denote the wave family.

LEMMA 3.2. *For Δt sufficiently small there exists a small constant $c_1 > 0$ such that*

$$\begin{aligned} \lambda(x, t_n+) &\le \lambda(x, t_n-) \left(1 - c_1 \frac{\Delta t}{\tau}\right), \\ g_n^+(x) &\le g_n^-(x) \left(1 - c_1 \frac{\Delta t}{\tau}\right). \end{aligned}$$

Next, we consider the *variations* of the wave strengths at time steps. Non-physical waves do not change at t_n, since p and λ are continuous across them; then, only two cases need to be examined, according to the incoming wave is (a) a 2-wave or (b): either a 1- or a 3-wave, see Figure 4.

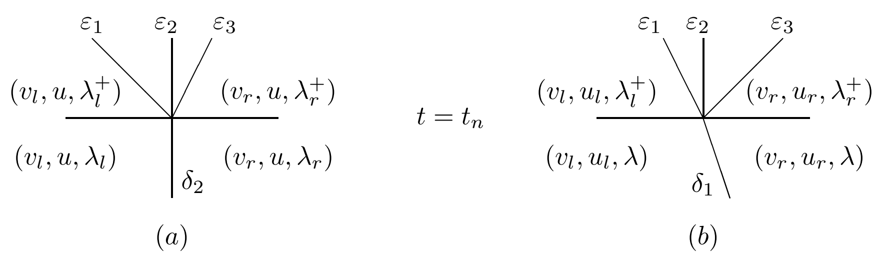


FIGURE 4. Incoming waves at time t_n: (a): a 2-wave; (b): a 1-wave. In case (a) we have $p_l = p_r$.

LEMMA 3.3. *In the notation in Figure 4, there exist positive constants c_o, C_o depending on m, such that the following estimates hold true.*

In case (a):

$$\left|\lambda_r^+ - \lambda_l^+\right| \le |\lambda_r - \lambda_l| \left(1 - c_o \frac{\Delta t}{\tau}\right),$$

$$|\varepsilon_1| + |\varepsilon_3| \le C_o \frac{\Delta t}{\tau} |\delta_2|, \qquad |\varepsilon_2| \le |\delta_2| \left(1 - c_o \frac{\Delta t}{\tau}\right);$$

in case (b), denoting with δ_i the incoming i-wave, $i = 1, 3$ and $j = 1, 3$, $j \neq i$:

$$\left|\lambda_r^+ - \lambda_l^+\right| \le C_o \frac{\Delta t}{\tau} |\delta_i| \lambda,$$

$$|\varepsilon_i - \delta_i| + |\varepsilon_j| \le C_o \frac{\Delta t}{\tau} |\delta_i| \lambda, \qquad |\varepsilon_2| \le C_o \frac{\Delta t}{\tau} |\delta_i| \lambda.$$

In order to control globally in time the variations at time steps we introduce the functionals

$$\Lambda(t) \;=\; \|\lambda(\cdot,t)\|_\infty\,, \qquad T(t) \;=\; \mathrm{TV}\lambda(\cdot,t)\,.$$

Then, we have that the functional $L_\xi + K_2\Lambda + K_3 T$ is non-increasing at times t_n for suitable values of the parameters K_2 and K_3, depending on ξ and on m. A further functional is also needed to balance the possible increase of the potential Q at time steps, namely,

$$Q_2(t) \;=\; \sum_{x_i<x_j} |\gamma_2^i \gamma_2^j|\,.$$

where γ_2^i (resp., γ_2^j) denotes the strength of the 2-wave located at point x_i (x_j) at time t.

Note that the functionals Λ, T, Q_2 may change only at time steps. Then, for a suitable choice of all the parameters, the functional

$$\mathcal{F} = F + K_2\Lambda + K_3 T + K_4 Q_2 \tag{3.7}$$

is proved to be non-increasing, where F was defined in (2.8).

A crucial point in the proof consists in checking that the term $p - p_e$ remains positive and uniformly bounded away from zero for any approximate solution; in turn, this is connected with the decaying of $\|\lambda(t)\|_\infty$. More precisely, under the assumptions (3.3), one can show that it is possible to choose μ small enough such that (3.4) implies $\mathcal{F}(0) \le m$. In turn, this inequality and the decreasing of (3.7) imply

$$V(t) \le m < \frac{1}{2}\left(\log p_o(-\infty) - \log p_e\right)\,.$$

Thanks to the identity

$$\mathrm{TV}\log p(\cdot,t) = \sum_{i=1,3} |\varepsilon_i| = 2V(t)\,,$$

one can then deduce that $p - p_e$ indeed remains uniformly bounded from below.

Moreover, for every times $t > s \ge 0$ and real numbers $a < b$, one has the $\mathbf{L^1}$-Lipschitz estimate

$$\int_a^b |\lambda(x,t) - \lambda(x,s)|\,dx \quad\le\quad L_\tau\left(t - s + \Delta t\right)\,,$$

where

$$L_\tau = C_1 + \frac{C_2}{\tau}\,\mathrm{e}^{-\frac{C_3 s}{\tau}}\,,$$

for suitable positive constants C_i, $i = 1, 2, 3$. The quantity L_τ is uniformly bounded as $\tau \to 0$, if $s \ge 1/n$ for some $n \in \mathbb{N}$; in the next section this makes possible a diagonalization argument that allows us to pass to the limit in λ as $\tau \to 0$ in $\mathbf{L^1_{\mathrm{loc}}}(\mathbb{R}\times(0,\infty))$.

The proof of Theorem 3.1 is then completed along the lines of [**Bre00**, §7.4].

4. The relaxation limit

We study in this section the relaxation limit $\tau \to 0$ of the solutions constructed in the previous section. The main result is the following.

THEOREM 4.1. *For $\tau > 0$, consider the system* (1.1) *and the initial data*

$$(v, u, \lambda)(0, x) = \left(v_o^\tau(x), u_o^\tau(x), \lambda_o^\tau(x)\right).$$

Assume that (v_o^τ, u_o^τ) is uniformly bounded and satisfies the bounds (2.4), (3.2), (3.3)*, for some constants $\underline{v} > 0$, $\underline{c} > 0$ independent of τ. Let $\mu > 0$ be given by Theorem 3.1 and assume that $\lambda_o^\tau \in [0, 1]$ satisfies* (3.4). *Finally, assume that for $\tau \to 0$*

$$(v_o^\tau, u_o^\tau) \to (v_o, u_o) \qquad in \ \ \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$$

and let $(v^\tau, u^\tau, \lambda^\tau)(x, t)$ be a solution provided by Theorem 3.1.

Then there exists a sequence $\tau_n \to 0$ such that

$$(v^{\tau_n}, u^{\tau_n}) \to (\tilde{v}, \tilde{u}) \quad in \ \ \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R} \times [0, \infty))\,, \tag{4.1}$$

$$\lambda^{\tau_n}(\cdot, t) \to 0 \quad in \ \ \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})\,, \quad \forall\, t > 0 \tag{4.2}$$

where $(\tilde{v}, \tilde{u})$ is a weak solution defined for $t \geq 0$ to the Cauchy problem for

$$\begin{cases} v_t - u_x & = 0 \\ u_t + p(v, 0)_x & = 0\,, \end{cases} \tag{4.3}$$

with initial data (v_o, u_o). Moreover, $\mathrm{TV}\,(\tilde{v}(\cdot, t), \tilde{u}(\cdot, t)) \leq C$ *for some constant C.*

At last, consider any smooth and convex entropy η to system (1.1) *with entropy flux q, and assume that η is dissipative, i.e., $\eta_\lambda \geq 0$. Then $(\tilde{v}, \tilde{u})$ is entropic with respect to the entropy pair $(\tilde{\eta}(v, u), \tilde{q}(v, u)) = (\eta(v, u, 0), q(v, u, 0))$, in the sense that*

$$\int_{-\infty}^{\infty} \int_0^{\infty} \eta(\tilde{v}, \tilde{u}, 0)\phi_t \ + \ q(\tilde{v}, \tilde{u}, 0)\phi_x \, dxdt \ + \ \int_{-\infty}^{\infty} \eta(v_o, u_o, 0)\phi(x, 0)\, dx \geq 0$$

for any $\phi \in C_o^\infty\left(\mathbb{R} \times [0, +\infty)\right),\ \phi \geq 0$.

The proof is along the lines of [**AG01**]. We emphasize that the sequence $\lambda^{\tau_n}(\cdot, t)$ converges only for $t > 0$, since no assumption of convergence on λ_o^τ was done.

A difficult and interesting issue would be to prove that the solution $(\tilde{v}, \tilde{u})$ constructed by passing to the relaxation limit in Theorem 4.1 is entropic with respect to *all* smooth and convex entropies of system (4.3). As an example, consider the standard entropy pair for system (4.3), i.e.,

$$\tilde{\eta}(v, u) = \frac{u^2}{2} - A(0)\log v, \qquad \tilde{q}(v, u) = \frac{A(0)u}{v}\,, \tag{4.4}$$

for $A = a^2$; the entropy $\tilde{\eta}$ is convex. Then,

$$\eta(v, u, \lambda) = \frac{u^2}{2} - A(\lambda)\log v + \phi(\lambda), \qquad q(v, u, \lambda) = \frac{A(\lambda)u}{v} \tag{4.5}$$

is an entropy-entropy flux pair for the complete system (1.1), for any smooth function ϕ. In particular, if $\phi(\lambda) = (C/2)\lambda^2 + D\lambda$, with C, $D > 0$, then η is (strictly) convex if

$$C - A''(\lambda)\log v - \frac{\left(A'(\lambda)\right)^2}{A(\lambda)} > 0 \tag{4.6}$$

and is dissipative if

$$\eta_\lambda = \phi' - A' \log v \quad = \quad C\lambda + D - A'(\lambda) \log v \geq 0\,. \tag{4.7}$$

Since v ranges over a bounded set $[v_{\min}, v_{\max}]$ with $v_{\min} > 0$, we can choose C, D so large that the previous conditions (4.6) and (4.7) are satisfied.

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