MATHEMATICAL ASPECTS OF A MODEL FOR GRANULAR FLOW

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Abstract. The model for granular flow being studied by the authors was proposed by Hadeler and Kuttler in [20]. In one space dimension, by a change of variable, the system can be written as a 2×2 hyperbolic system of balance laws.

Various results are obtained for this system, under suitable assumptions on initial data which leads to a strictly hyperbolic system. For suitably small initial data, the solution remains smooth globally. Furthermore, the global existence of large BV solutions for Cauchy problem is established for initial data with small height of moving layer. Finally, at the slow erosion limit as the height of moving layer tends to zero, the slope of the mountain provides the unique entropy solution to a scalar integro-differential conservation law, implying that the profile of the standing layer depends only on the total mass of the avalanche flowing downhill.

Various open problems and further research topics related to this model are discussed at the end of the paper.

Key words. Granular matter, balance laws, weakly linearly degenerate systems, global large BV solutions, slow erosion.

AMS(MOS) subject classifications. Primary 35L45, 35L50, 35L60, 35L65; Secondary 35L40, 58J45

1. Introduction. In [20] the following model was proposed to describe granular flows

$$\begin{cases} h_t = \operatorname{div}(h\nabla u) - (1 - |\nabla u|)h\,, \\ u_t = \big(1 - |\nabla u|\big)h\,. \end{cases} \tag{1.1}$$

These equations describe conservation of masses. The material is divided in two parts: a moving layer with height h on top and a standing layer with height u at the bottom. The moving layer slides downhill, in the direction of steepest descent, with speed proportional to the slope of the standing layer. If the slope $|\nabla u| > 1$ then grains initially at rest are hit by rolling grains of the moving layer and start moving as well. Hence the moving layer gets bigger. On the other hand, if $|\nabla u| < 1$, grains which are rolling can be deposited on the bed. Hence the moving layer becomes smaller.

This model is studied in one space dimension by the authors [29, 2, 3]. Define $p \doteq u_x$, and assume $p \geq 0$, one can rewrite (1.1) into the following 2×2 system of balance laws

$$\begin{cases} h_t - (hp)_x = (p-1)h\,, \\ p_t + \big((p-1)h\big)_x = 0\,. \end{cases} \tag{1.2}$$

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Writing the system of balance laws (1.2) in quasilinear form, the corresponding Jacobian matrix is computed as

$$A(h,p) = \begin{pmatrix} -p & -h \\ p-1 & h \end{pmatrix} .$$

For $h \geq 0$ and $p > 0$, one finds two real distinct eigenvalues $\lambda_1 < 0 \leq \lambda_2$, with r_1, r_2 the corresponding eigenvectors. Denote "$\bullet$" as the directional derivative, a direct computation gives

$$r_1 \bullet \lambda_1 = -\frac{2(\lambda_1+1)}{\lambda_2-\lambda_1} \approx \frac{2(p-1)}{p}, \qquad r_2 \bullet \lambda_2 = -\frac{2\lambda_2}{\lambda_2-\lambda_1} \approx -2\frac{h}{p^2} .$$

This shows the fact that the first characteristic field is genuinely nonlinear away from the line $p=1$ and the second field is genuinely nonlinear away from the line $h=0$, therefore the system is weakly linearly degenerate at the point $(h,p)=(0,1)$. Also, the direction of increasing eigenvalues, for the first family, changes with the sign of $p-1$. The lines $p=1$, $h=0$ are characteristic curves of the first, second family respectively, along which the system becomes separated and linearly degenerate:

$$p=1, \quad h_t - h_x = 0 ; \qquad\qquad h=0, \quad p_t = 0 .$$

See Figure 1 for the characteristic curves.

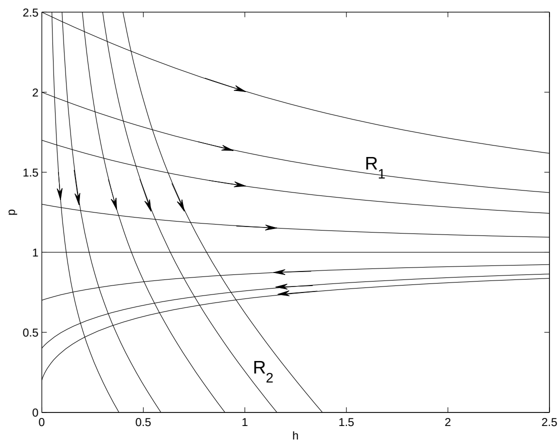

FIG. 1. *Characteristic curves of the two families in the h-p plane. The arrows point in the direction of increasing eigenvalues.*

In this paper we review some recent results about the existence of solutions for system (1.2), that are shown to exist globally in time for suitable classes of initial data. For systems of conservation laws with source term, some dissipation conditions are known in the literature that ensure the global in time existence of (smooth or weak) solutions; we refer to [22], to Kawashima–Shizuta condition (see [21]) for smooth solutions and to [15, 24] for the weak solutions. These conditions exploit a suitable balance between the differential terms and the source term that enable to control the nonlinearity of the system. It is interesting to remark that system (1.2) does not satisfy any of these conditions, nevertheless it admits global in time solutions.

For a derivation of the model (1.1) of granular flow we refer to [20]. Other models can be found in [7, 16, 27]. A mathematical analysis of steady state solutions for (1.1) was carried out in [10, 11]. We remark that, besides [29], the papers [2, 3] provide the first analytical study of time dependent solutions to this system.

2. Global smooth solutions. The global existence of smooth solutions is established in [29], under suitable assumptions on the initial data.

Let's first define the **decoupled initial data**

$$h(0,x) = \phi(x) \qquad\qquad p(0,x) = 1 + \psi(x) \tag{2.1}$$

with ϕ, ψ satisfying

$$\begin{cases} \phi(x) = 0 & \text{if } \ x \notin [a,b]\,, \\ \psi(x) = 0 & \text{if } \ x \notin [c,d]\,. \end{cases}$$

The intervals are disjoint, i.e., $a < b < c < d$. Moreover we assume $\psi(x) > -1$ for all x. For decoupled initial data, a global solution of the Cauchy problem can be explicitly given, namely

$$h(t,x) = \phi(x+t)\,, \qquad\qquad p(t,x) = 1 + \psi(x)\,, \qquad\qquad x \in I\!R\,, \quad t \geq 0\,.$$

Our first result provides the stability of these decoupled solutions. More precisely, every sufficiently small, compactly supported perturbation of a Lipschitz continuous decoupled solution eventually becomes decoupled. Moreover, no gradient catastrophe occurs, i.e., solutions remain smooth for all time.

THEOREM 2.1. *Let $a < b < c < d$ be given, together with Lipschitz continuous, decoupled initial data as in (2.1). Then there exists $\delta > 0$ such that the following holds. For every perturbations $\tilde\phi, \tilde\psi$, satisfying*

$$\tilde\phi(x) = \tilde\psi(x) = 0 \qquad \text{if } \ x \notin [a,d]\,, \qquad \left|\tilde\phi'(x)\right| \leq \delta\,, \left|\tilde\psi'(x)\right| \leq \delta\,, \tag{2.2}$$

the Cauchy problem for (1.2) with initial data

$$h(0,x) = \phi(x) + \tilde\phi(x)\,, \qquad\qquad p(0,x) = 1 + \psi(x) + \tilde\psi(x)\,, \tag{2.3}$$

has a unique solution, defined for all $t \geq 0$ and globally Lipschitz continuous. Moreover, this solution becomes decoupled in finite time.

The proof replies on the method of characteristics [22]. One must bound the $\mathbf{L}^\infty$ and $\mathbf{L}^1$ norms of h_x and p_x. For details, we refer to [29].

3. Global existence of large BV solutions. For more general initial data, due to the nonlinearity of the flux, the solutions will develop discontinuities (shocks) in finite time. Solutions should be defined in the space of BV functions. Assuming the height of the moving layer h sufficiently small, in [2] we prove the global existence of large BV solutions, for a class of initial data with bounded but possibly large total variation.

More precisely, consider initial data of the form

$$h(0,x) = \bar{h}(x) \geq 0\,, \qquad p(0,x) = \bar{p}(x) > 0\,. \tag{3.1}$$

which satisfy the following properties:

$$\text{Tot.Var.}\{\bar{p}\} \leq M\,, \qquad \text{Tot.Var.}\{\bar{h}\} \leq M\,, \tag{3.2}$$

$$\|\bar{h}\|_{\mathbf{L}^1} \leq M\,, \qquad \|\bar{p}-1\|_{\mathbf{L}^1} \leq M\,, \qquad \bar{p}(x) \geq p_0 > 0\,, \tag{3.3}$$

for some constants M (possibly large) and p_0. The following theorem is proved in [2].

Theorem 3.1. *For any constants M, $p_0 > 0$, there exists $\delta > 0$ small enough such that, if (3.2)–(3.3) hold together with*

$$\|\bar{h}\|_{\mathbf{L}^\infty} \leq \delta\,, \tag{3.4}$$

then the Cauchy problem (1.2)–(3.1) has an entropy weak solution, defined for all $t \geq 0$, with uniformly bounded total variation.

Compared with previous literature, the main novelty of this result stems from the fact that:

(i) We have arbitrarily large BV data;
(ii) We assume a small $\mathbf{L}^\infty$ bound on $\bar{h}$, but not on both initial data;
(iii) The system is strictly hyperbolic, but one of the characteristic fields is neither genuinely nonlinear nor linearly degenerate;
(iv) The system (1.2) contains source terms.

In the literature, for systems without source terms and small BV data, the global existence and uniqueness of entropy-weak solutions to the Cauchy problem are well known, using techniques such as the Glimm scheme [18, 23, 25], front tracking approximations [9, 5, 6], and vanishing viscosity approximations [8]. In some special cases, one has the existence and uniqueness of global solutions in the presence of a source term [15, 24, 12, 14, 1, 13].

However, global existence of solutions to hyperbolic systems with large BV data is a more difficult, still largely open problem. In addition to the special system [26], two main cases are known in the literature, where global existence of large BV solutions is achieved.

One is the case of Temple class systems [28], where one can measure the wave strengths in terms of Riemann invariants, so that the total strength of all wave fronts does not increase in time, across each interaction. A second major result [19] refers to general 2×2 systems, where again we can measure wave strengths in terms of Riemann coordinates; here, if the $\mathbf{L}^\infty$ norm of the solution is sufficiently small, the increase of total variation produced by the interaction is very small, and a global existence result of large BV solutions can then be established.

The validity of Theorem 3.1 relies heavily on some special properties of the hyperbolic system (1.2). First, the system is linearly degenerate along the straight line where $h = 0$. In the region where h is very small, the second field of the system is "almost-Temple class". Rarefaction curve and shock curve through the same point are very close to each other. This allows us to deduce refined interaction estimates, in which the effect of the nonlinearity is controlled by the quantity $\|h\|_{\mathbf{L}^\infty}$.

Second, the source term involves the quadratic form $h(p-1)$. Here the quantities h and $p-1$ have large, but bounded $\mathbf{L}^1$ norms. Moreover, they are transported with strictly different speeds. The total strength of the source term is thus expected to be $\mathcal{O}(1)\cdot\|h\|_{\mathbf{L}^1}\cdot\|p-1\|_{\mathbf{L}^1}$. In addition, since h itself is a factor in the source term, one can obtain a uniform bound on the norm $\|h\|_{\mathbf{L}^\infty}$, valid for all times $t \geq 0$.

For details of the proof of Theorem 3.1 we refer to [2].

4. Global large BV solutions of an initial boundary value problem. Next, we study how the mountain profile evolves when the thickness of the moving layer approaches zero, but the total mass of sliding material remains positive. This result is best formulated in connection with an initial-boundary value problem. On $\mathbb{R}_- \doteq \{x < 0\}$, consider the initial-boundary value problem for (1.2), with initial data (3.1) and the following boundary condition at $x = 0$

$$p(t,0)\,h(t,0) \;=\; F(t)\,. \tag{4.1}$$

Notice that here we prescribe the incoming flux $F(t)$ of the moving material, through the point $x = 0$. We assume

$$F(t) \geq 0\,, \qquad \text{Tot.Var}\{F\} \leq M\,, \qquad 0 < M' < \int_0^\infty F(\tau)\,d\tau \leq M\,. \tag{4.2}$$

As a partial step toward the slow erosion limit, we prove in [3] next theorem on the global existence of large BV solutions to this initial-boundary value problem, provided that $\|\bar h\|_{\mathbf{L}^\infty}$ and $\|F\|_{\mathbf{L}^\infty}$ are sufficiently small.

Theorem 4.1. *Given M, $p_0 > 0$, there exists $\delta > 0$ such that the assumptions (3.2)–(3.3) and (4.2), together with*

$$\|\bar h\|_{\mathbf{L}^\infty} \leq \delta\,, \qquad\qquad \|F\|_{\mathbf{L}^\infty} \leq \delta\,, \tag{4.3}$$

imply that the initial-boundary value problem (1.2)-(3.1), (4.1) has a global solution, with uniformly bounded total variation for all $t \geq 0$.

The proof for Theorem 4.1 follows a similar setting as for Theorem 3.1. The additional difficulty lies in the treatment of the boundary condition at

$x = 0$. Fortunately, the addition waves generated at the boundary, such as reflection waves, new entering wave, all contain a factor of $\|h\|_{\mathbf{L}^\infty}$ or the term $\|F\|_{\mathbf{L}^\infty}$, which are arbitrarily small. Therefore, same global a priori estimates as for Theorem 3.1 can be established, proving the global existence of large BV solution. For details, see [3].

5. Slow erosion limit. We now study the slow erosion limit. Numerical simulations in [29] show the following observation. When the height of the moving layer h is very small, the profile of the standing layer depends only on the total mass of the avalanche flowing downhill, not on the time-law describing at which rate the material slides down. This observation is proved rigorously in [3].

We define a new variable which measures the total mass of avalanche flowing down:

$$\mu(t) \doteq \int_0^t F(\tau)\,d\tau\,.$$

Recalling that $F(t) \geq 0$, the above function is monotone non-decreasing. Let $t(\mu)$ be its generalized inverse, and reparametrize the solution in terms of μ:

$$(\tilde{h}, \tilde{p})(\mu, x) = (h, p)\,(t(\mu), x)\,.$$

The last Theorem gives the slow erosion limit.

THEOREM 5.1. *Assume all the assumption in Theorem 4.1 hold. Then, as* $\|\bar{h}\|_{\mathbf{L}^\infty} \to 0$ *and* $\|F\|_{\mathbf{L}^\infty} \to 0$*, the rescaled* p *component of the solutions to the initial boundary value problem (1.2)–(3.1)–(4.1) converges to a limit function* $\hat{p}$*, which provides the unique entropy solution to the scalar integro-differential conservation law*

$$p_\mu + \left(\frac{p-1}{p} \cdot \exp \int_x^0 \frac{p(\mu, y) - 1}{p(\mu, y)}\,dy\right)_x = 0\,, \tag{5.1}$$

with initial data $\hat{p}(0, x) = \bar{p}(x)$ *for* $x < 0$.

A formal derivation is obtained as follows. For simplicity, assume $\bar{h}(x) = 0$ for $x < 0$. We stretch a given boundary data $\bar{F}(t) > 0$ by defining $F^\varepsilon(t) = \varepsilon\,\bar{F}(\varepsilon\,t)$. Introduce the new variable $\mu = \mu^\varepsilon(t)$,

$$\mu^\varepsilon(t) \doteq \int_0^t F^\varepsilon(s)\,ds = \int_0^{\varepsilon t} \bar{F}(s)\,ds\,, \qquad \text{so} \;\; \mu'(t) = F^\varepsilon(t) = \varepsilon\bar{F}(\varepsilon t)\,.$$

Using $\mu = \mu^\varepsilon$ as a rescaled time variable, the equations in (1.1) can now be rewritten as

$$\begin{cases} \mu' h^\varepsilon_\mu - (h^\varepsilon p^\varepsilon)_x &= (p^\varepsilon - 1)h^\varepsilon\,, \\ \mu' p^\varepsilon_\mu + \big((p^\varepsilon - 1)h^\varepsilon\big)_x &= 0\,. \end{cases} \tag{5.2}$$

Taking the limit as $\varepsilon \to 0$ in the first equation, the term $\mu' h^\varepsilon_\mu$ turns out to be of higher order w.r.t. ε, while $h = \mathcal{O}(\varepsilon)$ and $p = \mathcal{O}(1)$.

Introducing the new variable $m \doteq hp/\varepsilon$, from (5.2) we formally obtain

$$-m_x = \frac{p-1}{p}\, m\,, \tag{5.3}$$

$$\bar F(\mu) p_\mu + \left(\frac{p-1}{p}\, m\right)_x = 0\,. \tag{5.4}$$

Integrating the equation in (5.3) with proper boundary conditions, one obtains

$$m(\mu, x) = \exp\left(\int_x^\infty \frac{p(\mu,y)-1}{p(\mu,y)}\, dy\right) \bar F(\mu)\,. \tag{5.5}$$

Roughly speaking, this is the size of the avalanche at the time when it crosses a given point $x < 0$. By inserting (5.5) in (5.4), and dividing both terms by the common factor $\bar F(\mu)$, we obtain (5.1).

The key point in the proof of Theorem 5.1 is to show that, taking a converging sequence of p-component of solutions to the initial boundary value problem (1.2)-(3.1)-(4.1), the limit $\hat p$ is a weak solution to the conservation law (5.1). This is achieved by passing to the limit in the corresponding weak formulations. Here one needs the weak convergence of the flux hp and the strong convergence of the function $\frac{p-1}{p}$

$$\tilde p(\mu, x)\, \tilde h(\mu, x) \rightharpoonup \exp \int_x^0 \frac{\hat p(\mu,\zeta)-1}{\hat p(\mu,\zeta)}\, d\zeta\,, \qquad \frac{\tilde p(\mu,x)-1}{\tilde p(\mu,x)} \to \frac{\hat p(\mu,x)-1}{\hat p(\mu,x)}\,.$$

The above convergence is obtained in [3] using a compactness argument. By showing that the limiting integro-differential equation (5.1) is well posed, the convergence is extended to the whole sequence. The well-posedness of (5.1) is non-trivial because the flux is a global function. In the forthcoming work [4], we prove that the flow generated by the integro-differential equation (5.1) is Lipschitz continuous restricted to the domain of functions satisfying the bounds:

$$\inf_{x<0} p(x,t) \geq p_o > 0\,, \quad \text{Tot.Var.}\ p(\cdot, t) \leq M\,, \quad \|p(\cdot,t) - 1\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq M\,.$$

6. Further discussion. In this final section we discuss various interesting open problems related to this model.

(A). Uniqueness of entropy weak solutions for (1.2). After having established the global existence of large BV solutions to (1.2), it remains open the question of uniqueness of entropy-weak solutions. This does not immediately follow from the known results [14], because one of the characteristic fields is neither genuinely nonlinear, nor linearly degenerate. Uniqueness and continuous dependence of large BV solutions may be obtained by putting together the techniques in [5, 14, 12].

(B). The Cauchy problem with a source term $f = f(t,x)$. The original model in [20] takes into account also precipitation effects. This corresponds to supplement the first equation in (1.1) by an extra term $f = f(t,x) \geq 0$ in the right hand side, representing additional material that increments the size of the moving layer. In one dimension we are lead to

$$\begin{cases} h_t - (hp)_x &= (p-1)h + f\,, \\ p_t + \big((p-1)h\big)_x &= 0\,. \end{cases} \tag{6.1}$$

Notice that this system still decouples for $p \equiv 1$: in this case, h satisfies the scalar equation $h_t - h_x = f(t,x)$.

For system (6.1), can one still establish the global existence result? What will be the proper assumptions on $f(t,\cdot)$? To start, one can assume that the support of $f(t,\cdot)$ is uniformly bounded, that

$$\int_0^\infty [\text{Tot.Var.}\, f(t,\cdot)]\ dt \leq C\,,$$

and that the norm $\|f\|_{\mathbf{L}^\infty}$ is sufficiently small, depending on the above constant C.

(C). Mountain slope changes sign. It is interesting to check whether the equations (1.1) have meaningful solutions also when the slope $p = u_x$ changes sign. In this case, the one-dimensional model takes the form

$$\begin{cases} h_t - (hp)_x &= (|p|-1)h + f\,, \\ p_t + \big((|p|-1)h\big)_x &= 0\,. \end{cases} \tag{6.2}$$

Since all these models do not account for the conservation of momentum, one might wonder if the predictions are realistic, also in cases where $p \approx 0$ and the actual motion of the granular matter may be dominated by inertial forces. We remark that, when p is allowed to change sign, the flux in (6.2) is no longer smooth but only Lipschitz continuous. From the point of view of basic theory, this is a situation not covered by standard existence and uniqueness results, and should be examined specifically.

(D). Radially symmetric solutions. As an intermediate step toward the fully two-dimensional model, one can study radially symmetric solutions in $\mathbb{R}^2$. By writing the system (1.1) into polar coordinates (r,θ) and looking for solutions $(h,u)(t,r)$, one reduces to

$$\begin{cases} h_t - (hp)_r &= (|p|-1)h + \frac{p}{r}h\,, \\ p_t + ((|p|-1)h)_r &= 0 \end{cases} \tag{6.3}$$

where $p \doteq u_r$ and $r > 0$. One may reduce to consider a conic-shaped mountain, therefore assuming $p < 0$. The resulting system is quite similar to (1.2), but differs for the additional source term ph/r, whose dependence on the space variable r is not integrable on the half line.

Notice that, in the simple case $p \equiv -1$, (6.3) reduces to the equation $h_t + h_r = -h/r$. Assuming that

$$h(0,r) \;=\; h_0(r) = \frac{\Phi(r)}{r}$$

with $h_0(r) \to 0$ as $r \to 0+$, then the solution is given by

$$h(r,t) = \begin{cases} \dfrac{\Phi(r-t)}{r} & \text{if } r > t \\ 0 & \text{if } 0 < r < t\,. \end{cases}$$

Here, due to the spreading of the mass, h becomes smaller as r grows.

A very interesting problem is to reach a priori BV bounds for the system (6.3), for suitable boundary conditions at $r = r_0 > 0$. For this case, the BV bound estimates run into several difficulties: (i) the source term depends on the space variable r in a non-integrable way; and (ii) the characteristic speed is not strictly bounded away from 0, unless p is proved to remain strictly different from 0.

(E). The two dimensional case. To present date, the mathematical analysis of two-dimensional granular flow has been mainly concerned with steady state solutions [10, 11] and has been approached numerically in [17]. Furthermore, the general existence-uniqueness result for a two-dimensional hyperbolic system such as (1.1) is not yet available. However, the special structure of this system suggests a possible line of attack, based on a multi-dimensional extension of the integro-differential formula (5.1).

For example, consider the case where we initially have a steady sand-pile of height $u_0(x)$ on a table Ω and $h_0 \equiv 0$. Start to pour more sand on top of it at a very slow rate $f = \varepsilon \hat{f}(x)$; then the sand falls off as it reaches the edge of the table. As $\varepsilon \to 0$, a formal extension of the formula (5.1) leads to the following approximate evolution equation:

$$u_t(t,x) = (1 - |\nabla u|)\, H(t,x)\,,$$

where H solves the following linear equation for every fixed t

$$\nabla u \cdot \nabla H + \big(|\nabla u| - 1 + \Delta u\big) H + \hat{f} = 0\,,$$

with the initial and boundary data

$$u(0,x) = u_0(x)\,,\ h(0,x) \equiv 0 \quad \text{for} \quad x \in \Omega\,, \qquad u(x) \equiv 0\,, \quad \text{for} \quad x \in \partial\Omega\,.$$

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