DECAY IN L^∞ FOR THE DAMPED SEMILINEAR WAVE EQUATION ON A BOUNDED 1D DOMAIN

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Abstract. In this paper we study the long time behavior for a semilinear wave equation with space-dependent and nonlinear damping term, rewritten as a first order system. Under appropriate assumptions on the nonlinearity, we prove the exponential convergence in L^∞, as $t \to +\infty$, of the solution towards a stationary solution.

1. **Introduction.** In this paper we study the initial–boundary value problem for the 2×2 system in one space dimension

$$\begin{cases} \partial_t \rho + \partial_x J = 0, \\ \partial_t J + \partial_x \rho = -2k(x) g(J) \end{cases} \tag{1}$$

where $x \in I = [0,1]$ and $t \geq 0$, and

$$(\rho, J)(x,0) = (\rho_0, J_0)(x)\,, \qquad J(0,t) = J(1,t) = J_b \tag{2}$$

for $(\rho_0, J_0) \in BV(I)$ and for a constant $J_b \in \mathbb{R}$. Assume that

$$0 < k_1 \leq k(x) \leq k_2 \quad \forall\, x\,, \qquad k_1,\ k_2 > 0 \tag{3}$$

and that

$$g \in C^1(\mathbb{R})\,, \qquad g(0) = 0\,, \qquad g'(J) > 0 \quad \forall\, J\,. \tag{4}$$

The long time behavior of the solutions to (1), (2) is addressed by means of the stationary equation

$$\partial_x J = 0\,, \qquad \partial_x \rho = -2k(x) g(J)\,.$$

The initial and boundary conditions (2) lead to a stationary solution $(\widetilde{\rho}, \widetilde{J})$:

$$\widetilde{\rho}(x) = -2g(J_b)\int_0^x k(y)\,dy + C\,, \qquad \widetilde{J}(x) = J_b\,, \tag{5}$$

the constant C being uniquely identified by

$$\int_0^1 \widetilde{\rho}(x)\,dx = \int_0^1 \rho_0(x)\,dx\,. \tag{6}$$

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By the change of variable $(\rho, J) \mapsto (\rho - \widetilde{\rho}, J - J_b)$ and $g(J) \mapsto g(J + J_b) - g(J)$, we can reduce to the case

$$J_b = 0\,, \qquad \int_0^1 \rho_0(x)\,dx = 0\,. \tag{7}$$

Problem (1), (2), (7) is related to the one-dimensional damped semilinear wave equation on a bounded interval: indeed the function

$$u(x,t) = -\int_0^x \rho(y,t)\,dy$$

satisfies $u_x = -\rho$, $u_t = J$ and

$$\partial_{tt} u - \partial_{xx} u + 2k(x) g(\partial_t u) = 0\,. \tag{8}$$

The equation (8) has been considered in several papers, see [9, 6, 7, 11], the recent monograph [8] and references therein. It is well known that the initial-boundary value problem for (8) is well-posed for initial data $(u_0, \partial_t u_0) \in H_0^1(I) \times L^2(I)$, for $k(x) \in L^\infty(I)$ with $k(x) \geq 0$, and decay estimates for the energy are obtained, either exponential or polynomial.

Moreover, in [7], L^p decay estimates with $2 \leq p \leq \infty$ are studied for the 1-dimensional problem. These estimates are obtained under the assumption that g' vanishes at 0, and using the hypotheses of sufficiently regular data, $(u_0, \partial_t u_0) \in W^{2,\infty}(I) \times W^{1,\infty}(I)$.

In this paper we study the decay in L^∞ for a very similar problem, assuming that the damping is space-dependent and that $g' > 0$, 4. Our main contribution is to develop an alternative approach that originates from the point of view of the hyperbolic systems of balance laws. In particular, we construct approximate solutions that allow us to get an accurate description of the solution, whose evolution is recast as a discrete time system. Then we provide a strategy for the analysis of this system, that makes use of a discrete representation formula. This eventually leads to the decay in L^∞ of the solution in terms of (u_x, u_t).

Here $u_x(\cdot,t)$, $u_t(\cdot,t))$ belong to $BV(I) \subset L^\infty(I)$ so that $(u(\cdot,t), u_t(\cdot,t))$ are in $W^{1,\infty}(I) \times L^\infty(I)$.

The main result of this paper here follows.

Theorem 1.1. *Let k satisfy (3) and g satisfy (4). Define*

$$d_1 = k_1 \min_{J \in D_J} g'(J) > 0\,, \qquad d_2 = k_2 \max_{J \in D_J} g'(J) \tag{9}$$

where D_J is a closed bounded interval depending on the data, which is invariant for J. Finally assume that

$$\mathrm{e}^{d_2} - d_2 < \mathrm{e}^{d_1}\,. \tag{10}$$

Let $(\rho, J)(x,t)$ be the solution of the problem (1), (2), (7) with $(\rho_0, J_0) \in BV(I)$.

Then there exist constant values $C_1 > 0$ and $C_2 > 0$, that depend only on the coefficients of the equation and on the initial and boundary data, such that

$$\begin{aligned} \|J(\cdot,t)\|_\infty &\leq C_1 \mathrm{e}^{-C_3 t}\,, \\ \|\rho(\cdot,t)\|_\infty &\leq C_2 \mathrm{e}^{-C_3 t}\,. \end{aligned} \tag{11}$$

where

$$C_3 = |\log C(d_1, d_2)|\,, \qquad C(d_1,d_2) = \mathrm{e}^{-d_1}(\mathrm{e}^{d_2} - d_2) < 1\,.$$

2. **Approximate solutions.** In this section we present our approach for the definition of approximate solutions. It consists of an adaptation of the scheme for the Cauchy problem developed in [3]. Our approach is based on the formulation of system (1) that is obtained by adding an equation for the antiderivative of $k(x)$

$$a(x) = \int_0^x k(s)\,ds\,. \tag{12}$$

More precisely, we introduce the non-conservative 3×3 system

$$\begin{cases} \partial_t\rho + \partial_x J & = 0\,,\\ \partial_t J + \partial_x\rho + 2g(J)\partial_x a & = 0\,,\\ \partial_t a & = 0\,, \end{cases} \tag{13}$$

and the piecewise constant initial data

$$\begin{aligned} \left(\rho_0^{\Delta x}, J_0^{\Delta x}, a^{\Delta x}\right)(x) &= \left(\rho_0(x_j+), J_0(x_j+), a(x_j)\right) \qquad x\in(x_j,x_{j+1})\\ x_j = j\Delta x \qquad & j=0,\dots,N, \qquad \Delta x = \frac{1}{N}, \end{aligned} \tag{14}$$

where $N \in 2\mathbb{N}$ is a fixed positive even number determining the size of the space mesh. In this way, we can set up a so-called Well-Balanced algorithm to construct approximate *wave-front tracking* solutions [5], with discontinuities uniformly distributed on a grid in the (x,t)-plane. We define an approximate solution as follows.

> An approximate solution $(\rho^{\Delta x}, J^{\Delta x}, a^{\Delta x})(x,t)$ is an **exact** solution to the initial-boundary value problem (13)–(14) with boundary condition $J^{\Delta x}(0,t) = J^{\Delta x}(1,t) = 0$. In particular, $a^{\Delta x}(x)$ is piecewise constant with discontinuities located at each x_j and $(\rho^{\Delta x}, J^{\Delta x})$ is a piecewise constant function, w.r.t. (x,t), with discontinuities traveling along segments in the (x,t)-plane with slopes $\in\{\pm1,0\}$.

As $\Delta x\to 0$, the approximate solutions converge in L^1_{loc} (up to a subsequence) to a weak solution of (13).

The characterization of such approximate solution is based on the Riemann problem for (13), that is the initial-value problem for (13) with unknown $U = (\rho, J, a)$ and data

$$U(x,0) = \begin{cases} U_\ell & x<0,\\ U_r & x>0, \end{cases} \tag{15}$$

for a given *left state* $U_\ell = (\rho_\ell, J_\ell, a_\ell)$ and *right state* $U_r = (\rho_r, J_r, a_r)$. By assuming (4) and that $a_\ell \le a_r$, this problem is uniquely solved by

$$U(x,t) = \begin{cases} U_\ell & x/t < -1,\\ U_* = (\rho_{*,\ell}, J_*, a_\ell) & -1 < x/t < 0,\\ U_{**} = (\rho_{*,r}, J_*, a_r) & 0 < x/t < 1,\\ U_r & x/t > 1, \end{cases} \tag{16}$$

where $\rho_{*,\ell}, \rho_{*,r}, J_*$ satisfy suitable conditions. See Figure (1) for a diagram of (16) in the (x,t)-plane, where the discontinuities travel along lines separating the couples (U_ℓ, U_*), (U_*, U_{**}) and (U_{**}, U_r), which stand for a -1-*wave*, a 0-*wave* and a $+1$-*wave*, respectively. In general, we call *i-wave* a couple of states (U_ℓ, U_r) separated

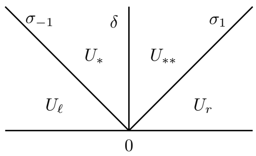


FIGURE 1. The solution to the Riemann problem (15).

by a discontinuity with *speed* (i.e. slope) $i \in \{0, \pm 1\}$ and we denote its size by

$$\begin{aligned} \sigma_{\pm 1} &= J_r - J_\ell = \pm(\rho_r - \rho_\ell) && \text{if } i = \pm 1, \\ \delta &= a_r - a_\ell && \text{if } i = 0. \end{aligned} \tag{17}$$

In the following we describe the approximate solutions in more detail; the procedure can be also regarded as a Well-Balanced scheme. See Figure (3) for a picture of the scheme in the case $N = 4$.

Step 1. The initial data is approximated as in (14); the 0-waves are located at each $0 < x_j < 1$, with size given by

$$\delta_j = a(x_j) - a(x_{j-1}) = \int_{x_{j-1}}^{x_j} k(x)dx \tag{18}$$

for $j = 1, \ldots, N-1$. Since $k \in L^\infty(I)$, we assume $\Delta x = 1/N$ to be sufficiently small so that

$$(\sup g') \cdot \delta_j < \frac{1}{2}. \tag{19}$$

Step 2. At time $t = 0+$ the solution is constructed by piecing together the solutions to the local Riemann problems at each $0 < x_j < 1$ (see (16)) and at the boundaries $x = 0$ and $x = 1$. Remark that at the boundaries the solution consists of a single $+1$-wave at $x = 0$ and of a single -1-wave at $x = 1$, respectively.

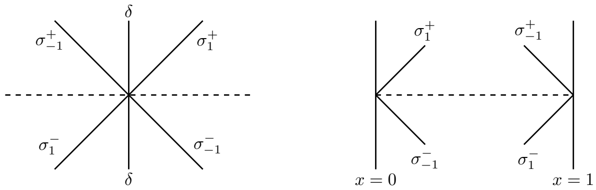
Step 3. At time $t = t^n = n\Delta t$ with $n \geq 1$ and $\Delta t = \Delta x$, multiple interactions of waves occur at $0 < x_j < 1$ (i.e. multiple segments intersect at each (x_j, t)) and the newly generated Riemann problems are solved according to

$$\begin{pmatrix} \sigma_{-1}^+ \\ \sigma_1^+ \end{pmatrix} = \begin{pmatrix} 1 - c_j & c_j \\ c_j & 1 - c_j \end{pmatrix} \begin{pmatrix} \sigma_{-1}^- \\ \sigma_1^- \end{pmatrix}, \qquad c_j := \frac{g'(s_j^n)\delta_j}{g'(s_j^n)\delta_j + 1}, \tag{20}$$

where $s_j^n \in D_J$, σ_{-1}^-, σ_1^- are the sizes of the incoming waves, σ_{-1}^+, σ_1^+ are the sizes of the outgoing ones and c is *transition coefficient*. The size of the 0-wave involved in the interaction remains constantly equal to δ_j (see (18)) across time t. Moreover, the waves hitting the boundaries $x = 0$ and $x = 1$ are both reflected and bounce back with the same size they had before the interaction. See Figure (2) for a picture of these two situations. We remark that a key property is that approximating $a(x)$ by a piecewise constant function implies that the source term is concentrated at the points x_j and results in the discontinuities with 0-slope in the solutions to the Riemann problems.

3. **The iteration matrix.** The semilinear character of system (1) and the presence of the (reflecting) boundary conditions allow us to view the problem as the time evolution of the solutions to a finite dimensional linear system of the form

$$\boldsymbol{\sigma}(t^n+) = B(t^n)\, \boldsymbol{\sigma}(t^{n-1}+) = \cdots = B(t^n)B(t^{n-1}) \cdots B(0+)\boldsymbol{\sigma}(0+). \tag{21}$$



(a) (b)

FIGURE 2. Interactions at $t = t_n > 0$: an example of multiple interaction at $0 < x_j < 1$ in (a); an example of interaction at the boundaries in (b).

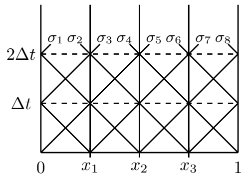


FIGURE 3. Well-balanced scheme for $N = 4$.

The components of the vector

$$\boldsymbol{\sigma}(t) = (\sigma_1, \ldots, \sigma_{2N}) \in \mathbb{R}^{2N}, \qquad N \in 2\mathbb{N},$$

are the wave sizes, see (17), that occur in the approximate solution to (13)–(14) at time t^n, ordered according to increasing space position; while the matrix $B \in \mathbb{R}^{2N\times 2N}$ is a doubly stochastic matrix (i.e. a nonnegative matrix for which the sum of all the elements by row is 1, as well as by column) given by

$$B(\boldsymbol{c}) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 1-c_1 & \cdots & 0 & 0 & 0 & 0 \\ 1-c_1 & 0 & 0 & c_1 & & \vdots & \vdots & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & c_{N-1} & 0 & 0 & 1-c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 1-c_{N-1} & 0 & 0 & c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\boldsymbol{c} = (c_1, \cdots, c_{N-1}) \in \mathbb{R}^{N-1}$ and by the smallness of δ_j (see (18), (19)) we have that

$$\frac{\inf g'}{2}\delta_j \le c_j \le (\sup g')\delta_j, \qquad j = 1, \ldots, N-1. \tag{22}$$

In general the vector $\boldsymbol{c}$ depends on n, which is the index for the time: $t = t^n = n\Delta t$. The eigenvalues λ_i of B satisfy $|\lambda_i| \le 1$ for all $i = 1, \ldots, 2N$. In particular, $\lambda = \pm 1$ are eigenvalues with corresponding (left and right) eigenvectors

$$\begin{aligned} \lambda_- &= -1, \qquad v_- = (1, -1, -1, 1, \ldots, 1, -1, -1, 1), \\ \lambda_+ &= 1, \qquad e = (1, 1, \ldots, 1, 1). \end{aligned} \tag{23}$$

Denote by E_- the $(2N-2)$–dim eigenspace related to λ_i with $|\lambda_i| < 1$.

It is well known (Birkhoff Theorem, [10, Theorem 8.7.2]) that doubly stochastic matrices can be written as a convex combination of permutations.

In case of $\boldsymbol{c} = c(1, \cdots, 1) \in \mathbb{R}^{N-1}$ for $c \in [0, 1/2)$, the decomposition is obtained with two terms:

$$B(\boldsymbol{c}) = (1-c)B(\mathbf{0}) + cB_1 = (1-c)\left[B(\mathbf{0}) + \gamma B_1\right], \tag{24}$$

where

$$\gamma = \frac{c}{1-c} = \frac{(\sup g')\bar{k}}{N} := \frac{d}{N},$$

$B(\mathbf{0})$ is the matrix $B(\boldsymbol{c})$ with $\boldsymbol{c} = \mathbf{0}$ and B_1 is a permutation matrix that switches two consecutive rows $(2k-1)$ and $2k$. We can rewrite (24) as

$$B(\boldsymbol{c}) = \left(1 + \frac{d}{N}\right)^{-1} \left[B(\mathbf{0}) + \frac{d}{N} B_1\right].$$

On the other hand, if $\boldsymbol{c}$ is not constant (that is the case for nonlinear damping), we can bound each matrix $B = B(\boldsymbol{c}^n)$, $n \in \mathbb{N}$ with a term-by-term inequality by

$$B(\boldsymbol{c}^n) \leq \left(1 + \frac{d_1}{N}\right)^{-1} \left[B(\mathbf{0}) + \frac{d_2}{N} B_1\right] \tag{25}$$

where d_1, d_2 are defined in (9).

3.1. **Total variation estimates.** Here we give a proof of the fact that

$$L_\pm(t) = \sum_{(\pm 1)-waves} |\Delta f^\pm| = \mathrm{TV}\, J^{\Delta x}(\cdot, t)$$

is not increasing in time, by means of the properties of doubly stochastic matrices. We recall here some results from [4, pp.149–153].

Definition 3.1 (Majorization of vectors)**.** Let $v, u \in \mathbb{R}^n$ and denote

$$v_{[1]} \geq v_{[2]} \geq \cdots \geq v_{[n]}, \qquad u_{[1]} \geq u_{[2]} \geq \cdots \geq u_{[n]},$$

the components rearranged in non-increasing order. We say that v is *majorized* by u if the following conditions hold:

$$\sum_{i=1}^n v_i = \sum_{i=1}^n u_i,$$
$$\sum_{i=1}^h v_{[i]} \leq \sum_{i=1}^h u_{[i]} \qquad h = 1, \dots, n-1.$$

The following theorem is a useful characterization of majorization.

Theorem 3.2 (Hardy-Littlewood-Polya)**.** *Let $v, u \in \mathbb{R}^n$. Then, v is majorized by u if and only if there exists a doubly stochastic matrix A such that $v = Au$.*

Lemma 3.3. *Let $v, u \in \mathbb{R}^n$. If v is majorized by u and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, then*

$$\sum_{i=1}^n \phi(v_i) \leq \sum_{j=1}^n \phi(u_i). \tag{26}$$

The following corollary is an easy consequence of these results, and it proves that $L_\pm$ is non-increasing in time.

Corollary 1. *Denote σ_j^n the j^{th} component of $\boldsymbol{\sigma}(t^n+)$. Then,*

$$\sum_{j=1}^{2N} |\sigma_j^{n+1}| \le \sum_{j=1}^{2N} |\sigma_j^n| \,, \qquad n \ge 0\,.$$

Proof. Since $\boldsymbol{\sigma}(t^{n+1}+) = B^{(n)}\boldsymbol{\sigma}(t^n+)$ and $B^{(n)}$ is doubly stochastic, we have that $\boldsymbol{\sigma}(t^{n+1}+)$ is majorized by $\boldsymbol{\sigma}(t^n+)$. Then, we can conclude by applying the previous lemma to $\phi(\cdot) = |\cdot|$ and $v = \boldsymbol{\sigma}(t^{n+1}+)$, $u = \boldsymbol{\sigma}(t^n+)$. □

4. **A discrete representation formula.** The proof of Theorem (1.1) is given in [1] (with a slight improvement in the condition (10) given in [2]). Here we provide some key points.

First, a proposition which relates the L^∞-norm of $J(\cdot, t^n)$, $\rho(\cdot, t^n)$ as $n \to \infty$ to the evolution of the ℓ_1–norm of the operator $\mathcal{B}_n$:

$$\mathcal{B}_n \doteq \left[B^{(n)}B^{(n-1)}\cdots B^{(2)}B^{(1)}\right]\,, \qquad B^{(n)} = B(\boldsymbol{c}^n) \in M_{2N}\,, \qquad n \ge 1 \tag{27}$$

on the eigenspace $E_- \doteq \, < e, v_- >^\perp$, see (23).

Proposition 1. *For some constant values $\widetilde{C}_j > 0$, $j = 1, 2, 3$, independent on Δx one has that for every $t \in (t^n, t^{n+1})$*

$$\|J^{\Delta x}(\cdot, t)\|_\infty \le \widetilde{C}_1 \Delta x + \|\mathcal{B}_n \widetilde{\boldsymbol{\sigma}}(0+)\|_{\ell^1}$$
$$\|\rho^{\Delta x}(\cdot, t)\|_\infty \le \widetilde{C}_2 \Delta x + \widetilde{C}_3\|\mathcal{B}_n \widetilde{\boldsymbol{\sigma}}(0+)\|_{\ell^1}$$

where $\widetilde{\boldsymbol{\sigma}}(0+)$ is the projection of $\boldsymbol{\sigma}(0+)$ onto E_-.

Next, the goal is to prove that $\|\mathcal{B}_n \widetilde{\boldsymbol{\sigma}}(0+)\|_{\ell^1}$ decays exponentially fast as $n \to \infty$, uniformly as $\Delta x = N^{-1} \to 0$. We focus our analysis on the iteration of the matrices $B = B(\boldsymbol{c}^n)$ up to time

$$t^N = N\Delta t = N\Delta x = 1\,.$$

Recalling (25), we get the following inequality:

$$\mathcal{B}_N \le \left(1 + \frac{d_1}{N}\right)^{-N} \left[B_0 + \frac{d_2}{N} B_1\right]^N\,, \qquad B_0 \doteq B(\mathbf{0}). \tag{28}$$

It is clear that

$$\left(1 + \frac{d_1}{N}\right)^{-N} \to \mathrm{e}^{-d_1} \qquad \text{as } N \to \infty\,,$$

while it takes a bigger effort to estimate the second factor

$$[B_0 + \gamma B_1]^N = \sum_{k=0}^{N} \gamma^k S_k(B_0, B_1), \qquad \gamma = \frac{d_2}{N} \tag{29}$$

since the matrices B_0, $B_1 \in M_{2N}$ **do not commute**. Each term $S_k(B_0, B_1)$ is the sum of all possible products of $2N$ matrices of size $2N$ equal to either B_1 or B_0 (and in which B_1 appears exactly k times). In particular,

$$S_0 = B_0^N\,, \qquad S_1 = \sum_{j=0}^{N-1} B_0^{2j} \doteq \widehat{P}\,.$$

In the following theorem we provide an estimate of the sum in (29) for the terms with $k \ge 2$.

Theorem 4.1. *Let $N \in 2\mathbb{N}$. Then,*

$$\left[B_0 + \frac{d}{N}B_1\right]^N = B_0^N + \frac{d}{N}\widehat{P} + \sum_{j=0}^{N-1}\zeta_{j,N}B_0^{2j+1}B_1 + \sum_{j=1}^{N-1}\eta_{j,N}B_0^{2j}\,, \tag{30}$$

where the scalar coefficients in the sums are bounded by:

$$0 \leq \sum_{j=0}^{N-1}\zeta_{j,N} \leq \sinh(d) - d + \frac{f_0(d)}{N}\,,$$

$$0 \leq \sum_{j=1}^{N-1}\eta_{j,N} \leq \cosh(d) - 1 + \frac{f_1(d)}{N}\,,$$

with terms $f_0(d)$ and $f_1(d)$ containing modified Bessel functions of the first type.

Thanks to (30) we can prove the following contraction property:

$$\|\mathcal{B}_N\widetilde{\boldsymbol{\sigma}}(0+)\|_{\ell^1} \leq C_N(d_1,d_2)\|\widetilde{\boldsymbol{\sigma}}(0+)\|_{\ell^1} \tag{31}$$

where

$$C_N(d_1,d_2) \to \mathrm{e}^{-d_1}(\mathrm{e}^{d_2} - d_2) \doteq C(d_1,d_2) < 1\,, \qquad N \to \infty\,.$$

The last inequality follows from the assumption (10). By iterating the estimate (31), recalling Prop. (1) and sending $N \to \infty$, it is possible to prove the L^∞ decay stated in (11).

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