Glimm estimates for a model of multiphase flow

Debora Amadori*

Department of Pure & Applied Mathematics, University of L'Aquila, 67010 L'Aquila, Italy

Andrea Corli

Department of Mathematics, University of Ferrara, 44121 Ferrara, Italy

Abstract

We prove Glimm interaction estimates for a 3×3 hyperbolic system of conservation laws arising in the modeling of multi-phase flows. No smallness of the interacting waves is assumed. Our proof simplifies and improves a previous result by Y.-J. Peng [9].

Key words: Hyperbolic systems of conservation laws, phase transitions, Glimm interaction estimates

1. Introduction

We consider the following simple model for the flow of an inviscid fluid, where liquid and vapor phases coexist:

$$\begin{cases} v_t - u_x &= 0, \\ u_t + p(v,\lambda)_x &= 0, \\ \lambda_t &= 0\,. \end{cases} \tag{1.1}$$

Here $t > 0$ and $x \in \mathbb{R}$, $v > 0$ is the specific volume, u the velocity, $\lambda \in [0,1]$ the mass density fraction of vapor in the fluid. The pressure p is defined by

$$p(v,\lambda) = \frac{a^2(\lambda)}{v}, \tag{1.2}$$

where a is a smooth function defined on $[0,1]$ satisfying $a(\lambda) > 0$. We refer to Fan [6] for more information on this model.

If λ is constant, (1.1) reduces to the system of isothermal gasdynamics. The existence of global solution for this system was proved in [8], with data having arbitrarily large total variation.

In [1, 2] we proved the global existence of weak solutions to (1.1), for a wide class of initial data with large total variation, by means of a front-tracking scheme. Another proof, again by a front-tracking scheme, was provided in [4]. A system close to (1.1)

*Corresponding author

was previously considered by Peng [9], where the author proves the Glimm interaction estimates, but the proof of the main theorem seems incomplete. We refer to [5, 10, 7] for definitions and more information on systems of conservation laws and on the Glimm scheme.

In this note we prove the Glimm interaction estimates by a procedure different from that of [9, Lemma 3, p. 529]; such proof allows us to improve the quadratic interaction estimate. The technique used below consists in letting waves interact one at a time and shows precisely when the variation of the solution increases. From this analysis, a convenient choice of the interaction potential emerges. Also, our method completely avoids the splitting of the proof into many cases (in [9], *fifteen* cases have to be considered for just one subcase). In a sense, then, the spirit of this note is close to [11]. In particular, the ideas exploited here are possibly applicable to systems more general than (1.1). We point out that the existence proof via the Glimm scheme is not considered here since [2, 4] already provide global existence results.

We now state the main result. We denote $U = (v, u, \lambda) \in \Omega = (0, +\infty) \times \mathbb{R} \times [0, 1]$. Under the assumption (1.2), the system (1.1) is strictly hyperbolic in Ω with eigenvalues $e_1 = -c$, $e_2 = 0$, $e_3 = c$, for $c = \sqrt{-p_v} = a(\lambda)/v$. The eigenvalues e_1, e_3 are genuinely nonlinear while e_2 is linearly degenerate. For $i = 1, 3$ the Lax curves through the point $U_o = (v_o, u_o, \lambda_o)$ are

$$v \mapsto \big(v, u_o + a(\lambda_o) \cdot 2h(\varepsilon_i), \lambda_o\big) \ ,$$

where

$$\varepsilon_1 = \frac{1}{2} \log \left(\frac{v}{v_o}\right), \qquad \varepsilon_3 = \frac{1}{2} \log \left(\frac{v_o}{v}\right),$$

and

$$h(\varepsilon) = \begin{cases} \varepsilon & \text{if } \varepsilon \geq 0, \\ \sinh \varepsilon & \text{if } \varepsilon < 0, \end{cases}$$

see [1], [9]. The Lax curve for $i = 2$ is characterized by steady solutions of (1.1), across which pressure and velocity are constant:

$$\frac{a^2(\lambda)}{v} = \frac{a^2(\lambda_o)}{v_o}, \qquad u = u_o.$$

The strength ε_2 of a 2-wave (contact discontinuity) is defined as

$$\varepsilon_2 = 2\,\frac{a(\lambda) - a(\lambda_o)}{a(\lambda) + a(\lambda_o)}\,.$$

Rarefaction (shock) waves have positive (negative) strengths. The Riemann problem for (1.1) has a unique solution for any initial data valued in Ω, see Proposition 2.1. We notice that in [1, 6] the function $a(\lambda)$ is assumed to be increasing, on the ground of physical motivations; such an assumption is not needed in the forthcoming analysis.

We deal now with Glimm estimates. Consider the interaction problem displayed in Figure 1. Since contact discontinuities are stationary, exactly *one* contact discontinuity can be present in any interval of random sampling of the Glimm scheme. Therefore, in the estimates below we only take into account the case when only one 2-wave is present in the previous pattern, that is, either $\alpha_2 = 0$ or $\beta_2 = 0$.

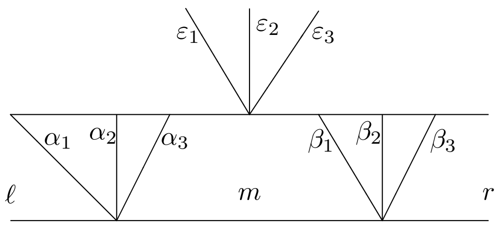

Figure 1: Glimm interactions.

We denote $[x]_\pm = \max\{\pm x, 0\}$ and define the interaction potential

$$q(\alpha, \beta) = [\alpha_2]_+ \cdot |\beta_1| + |\alpha_3| \cdot [\beta_2]_- . \tag{1.3}$$

Since we only deal with one 2-wave at a time, then $q(\alpha, \beta) = [\alpha_2]_+ \cdot |\beta_1|$ if $\beta_2 = 0$ and $q(\alpha, \beta) = |\alpha_3| \cdot [\beta_2]_-$ if $\alpha_2 = 0$. Here follows our main result.

Theorem 1.1. *Consider the interaction pattern in Figure 1, with either $\alpha_2 = 0$ or $\beta_2 = 0$. Then*

$$|\varepsilon_1| + |\varepsilon_3| \leq |\alpha_1| + |\alpha_3| + |\beta_1| + |\beta_3| + q(\alpha, \beta) . \tag{1.4}$$

We notice that (1.4) and (1.3) improve the inequality (3.3) in [9], where the usual Glimm potential was considered. Indeed, such a full quadratic potential is not needed: our potential q vanishes according to the signs of α_2 and β_2. This fact has a possible consequence in the definition of the Glimm functional and permits to improve the conditions on the initial data that guarantee the global existence of solutions.

2. Proof of Theorem 1.1

We begin this section with a result on the Riemann problem. Then, we consider two cases of simple interactions. First, we deal with the interaction of a 1- or 3-wave with a 2-wave, where the total variation *can* increase (Proposition 2.2); second, we study the case of $q = 0$ (Lemma 2.3). They are both special cases of Theorem 1.1. Finally, the full proof of Theorem 1.1 is deduced by a decoupling technique.

We consider the Riemann problem for (1.1) with initial condition

$$(v, u, \lambda)(0, x) = \begin{cases} (v_\ell, u_\ell, \lambda_\ell) = U_\ell & \text{if } x < 0, \\ (v_r, u_r, \lambda_r) = U_r & \text{if } x > 0, \end{cases} \tag{2.1}$$

for U_ℓ and U_r in Ω. We denote $a_r = a(\lambda_r)$, $p_r = a_r^2/v_r$ and similarly a_ℓ, p_ℓ.

Proposition 2.1 ([1])**.** *For any pair of states U_ℓ, U_r in Ω, the Riemann problem (1.1), (2.1) has a unique Ω-valued solution in the class of solutions consisting of simple Lax waves. If ε_i is the strength of the i-wave, $i = 1, 2, 3$, then*

$$\varepsilon_3 - \varepsilon_1 = \frac{1}{2} \log\left(\frac{p_r}{p_\ell}\right), \qquad 2\Big(a_\ell h(\varepsilon_1) + a_r h(\varepsilon_3)\Big) = u_r - u_\ell .$$

The interactions between shocks and rarefactions, with λ constant, are analyzed in [3]. The following result analyzes the interaction of a single wave with a 2-wave, which is a special case of the interaction pattern in Figure 1.

Proposition 2.2 ([1]). *Assume that either a 1-wave of strength δ_1 or a 3-wave of strength δ_3 interacts with a 2-wave of strength δ_2. Then, the strengths ε_i of the outgoing waves satisfy $\varepsilon_2 = \delta_2$ and*

$$\begin{array}{rcl} |\varepsilon_i - \delta_i| = |\varepsilon_j| & \le & \dfrac{1}{2}|\delta_2| \cdot |\delta_i|, \qquad i,j = 1,3,\ i \neq j\,, \\ |\varepsilon_1| + |\varepsilon_3| & \le & \begin{cases} |\delta_1| + |\delta_1|[\delta_2]_+ & \text{if } 1 \text{ interacts,} \\ |\delta_3| + |\delta_3|[\delta_2]_- & \text{if } 3 \text{ interacts.} \end{cases} \end{array} \tag{2.2}$$

In the next Lemma we prove that waves "outgoing" from a Riemann solution, see Figure 2, do not increase the total variation of the solution. The following result is contained also in [9]; however, the proof is different and we obtain some sharper estimates.

Figure 2: Special interactions: $\beta_1 = \beta_2 = 0$ (left) and $\alpha_2 = \alpha_3 = 0$ (right).

Lemma 2.3. *Consider the interaction patterns in Figure 2. Then*

$$|\varepsilon_1| + |\varepsilon_3| \le |\alpha_1| + |\alpha_3| + |\beta_1| + |\beta_3|\,. \tag{2.3}$$

Proof. We assume that $\beta_1 = \beta_2 = 0$; the other case is analogous. We must prove that

$$|\varepsilon_1| + |\varepsilon_3| \le |\alpha_1| + |\alpha_3| + |\beta_3|\,. \tag{2.4}$$

We recall the elementary identities about the interaction pattern in Figure 1, see [9],

$$\begin{array}{rcl} \varepsilon_3 - \varepsilon_1 & = & \alpha_3 + \beta_3 - \alpha_1 - \beta_1, \\ a_\ell h(\varepsilon_1) + a_r h(\varepsilon_3) & = & a_\ell h(\alpha_1) + a_m h(\alpha_3) + a_m h(\beta_1) + a_r h(\beta_3)\,, \end{array}$$

that here reduce to

$$\varepsilon_3 - \varepsilon_1 = \alpha_3 + \beta_3 - \alpha_1, \tag{2.5}$$

$$a_\ell \big[h(\varepsilon_1) - h(\alpha_1)\big] + a_r \big[h(\varepsilon_3) - h(\alpha_3) - h(\beta_3)\big] = 0\,. \tag{2.6}$$

If $\varepsilon_1 \cdot \varepsilon_3 \le 0$, then (2.4) holds thanks to (2.5). Then we focus on the case $\varepsilon_1 \cdot \varepsilon_3 > 0$.
(a) If $\varepsilon_1, \varepsilon_3 > 0$, then $h(\varepsilon_i) = \varepsilon_i$ and (2.5), (2.6) give a 2×2 linear system, whose solutions ε_i can be explicitly computed. We show that

$$\varepsilon_1 \le \alpha_1\,, \qquad \varepsilon_3 \le \alpha_3 + \beta_3\,, \tag{2.7}$$

that lead to (2.4). Indeed, using that $h(x) \le x$ for all x, we easily find

$$\varepsilon_1 = \frac{1}{a_\ell + a_r} \big\{ a_\ell h(\alpha_1) + a_r h(\alpha_3) + a_r h(\beta_3) - a_r(\alpha_3 + \beta_3 - \alpha_1) \big\} \le \alpha_1 \,.$$

The inequality on the right in (2.7) is proved in an entirely similar way.

(b) If ε_1, $\varepsilon_3 < 0$, then formulas (2.5), (2.6) can be written as

$$|\varepsilon_1| + \alpha_1 = |\varepsilon_3| + \alpha_3 + \beta_3, \tag{2.8}$$
$$a_\ell \big(\sinh(|\varepsilon_1|) + h(\alpha_1)\big) + a_r \big(\sinh(|\varepsilon_3|) + h(\alpha_3) + h(\beta_3)\big) = 0\,. \tag{2.9}$$

Then $|\varepsilon_1| + |\varepsilon_3| = 2|\varepsilon_1| - \alpha_3 - \beta_3 + \alpha_1 = 2|\varepsilon_3| + \alpha_3 + \beta_3 - \alpha_1$. Therefore, inequality (2.4) is equivalent to any one of

$$|\varepsilon_3| \le [\alpha_1]_+ + [\alpha_3]_- + [\beta_3]_-, \qquad |\varepsilon_1| \le [\alpha_1]_- + [\alpha_3]_+ + [\beta_3]_+.$$

It is sufficient to prove that

$$|\varepsilon_3| \le [\alpha_3]_- + [\beta_3]_-. \tag{2.10}$$

If $|\varepsilon_3| + \alpha_3 + \beta_3 \le 0$ then (2.10) holds. On the other hand, if the quantity in (2.8) is positive, then $\sinh(|\varepsilon_1|) + h(\alpha_1) > 0$ and from (2.8), (2.9) we deduce

$$|\varepsilon_3| + \alpha_3 + \beta_3 > 0, \tag{2.11}$$
$$\sinh(|\varepsilon_3|) + h(\alpha_3) + h(\beta_3) < 0\,. \tag{2.12}$$

By (2.12) we see that α_3 and β_3 cannot be both positive. Assume that they are both negative; then $|\varepsilon_3| > -\alpha_3 - \beta_3 = |\alpha_3| + |\beta_3|$ from (2.11). By (2.12) we would obtain

$$\sinh\big(|\alpha_3| + |\beta_3|\big) < \sinh(|\varepsilon_3|) < \sinh(|\alpha_3|) + \sinh(|\beta_3|)\,.$$

This contradicts the elementary inequality $\sinh(x + y) \ge \sinh(x) + \sinh(y)$, which holds for $x \ge 0$, $y \ge 0$. Therefore α_3 and β_3 have different signs. Assume that $\alpha_3 > 0$, $\beta_3 < 0$, the other case being symmetric. We rewrite (2.12) as

$$\sinh(|\varepsilon_3|) + \alpha_3 < \sinh(|\beta_3|)\,. \tag{2.13}$$

From (2.13) and since $\alpha_3 > 0$, we find that $|\varepsilon_3| < |\beta_3|$ and therefore (2.10). □

Proof of Theorem 1.1. We focus on the case $\beta_2 = 0$, the case $\alpha_2 = 0$ being symmetric. If also $\beta_1 = 0$, then Lemma 2.3 applies; we are left with the case when $\beta_1 \ne 0$. The proof consists of some steps that reduce the estimate (1.4) to special cases, see Figure 3.

(1): $\alpha_1 = \alpha_3 = \beta_2 = \beta_3 = 0$. In this case Proposition 2.2 applies and (2.2) gives

$$|\varepsilon_1| + |\varepsilon_3| \le |\beta_1| + [\alpha_2]_+ \cdot |\beta_1|\,.$$

In the same way we treat the symmetric case $\alpha_1 = \alpha_2 = \beta_1 = \beta_3 = 0$.

(2): $\alpha_3 = \beta_2 = \beta_3 = 0$. Let U^* be the state between α_1 and α_2. The Riemann problem of states (U^*, U_r) is solved by waves ε_i^*, $i = 1, 2, 3$; by step *(1)* we have $|\varepsilon_1^*| + |\varepsilon_3^*| \le |\beta_1| + [\alpha_2]_+ \cdot |\beta_1|$. Then, consider the interaction pattern given by the wave α_1 on the left

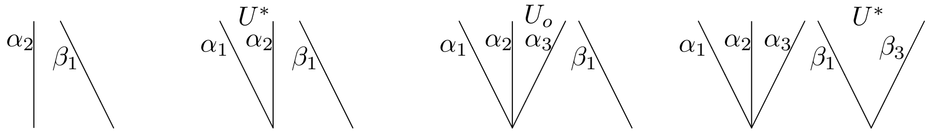

Figure 3: Special cases of Glimm interactions.

and the Riemann solution $(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*)$ on the right. This pattern gives rise to a solution $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of the Riemann problem of states (U_l, U_r); Lemma 2.3 applies and we obtain

$$|\varepsilon_1| + |\varepsilon_3| \le |\alpha_1| + |\varepsilon_1^*| + |\varepsilon_3^*| \le |\alpha_1| + |\beta_1| + [\alpha_2]_+ \cdot |\beta_1| \,.$$

(3): $\beta_2 = \beta_3 = 0$. Let U_o be the state between α_2 and α_3. The Riemann problem of states (U_o, U_r) is solved by (β_1, α_3), [10]; let U^* be the state between these waves. The Riemann problem of states (U_l, U^*) is then solved by $(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*)$ and, because of step *(2)*, $|\varepsilon_1^*| + |\varepsilon_3^*| \le |\alpha_1| + |\beta_1| + [\alpha_2]_+ \cdot |\beta_1|$. The Riemann problem of states (U_l, U_r) is solved by $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and, because of Lemma 2.3,

$$|\varepsilon_1| + |\varepsilon_3| \le |\varepsilon_1^*| + |\varepsilon_3^*| + |\alpha_3| \le |\alpha_1| + |\alpha_3| + |\beta_1| + [\alpha_2]_+ \cdot |\beta_1| \,.$$

(4): $\beta_2 = 0$. Let U^* be the state between β_1 and β_3. The Riemann problem of states (U_l, U^*) is solved by $(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*)$ and, because of step *(3)*, $|\varepsilon_1^*| + |\varepsilon_3^*| \le |\alpha_1| + |\alpha_3| + |\beta_1| + [\alpha_2]_+ \cdot |\beta_1|$. The Riemann problem of states (U_l, U_r) is solved by $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and, again by Lemma 2.3,

$$|\varepsilon_1| + |\varepsilon_3| \le |\varepsilon_1^*| + |\varepsilon_3^*| + |\beta_3| \le |\alpha_1| + |\alpha_3| + |\beta_1| + |\beta_3| + [\alpha_2]_+ \cdot |\beta_1| \,.$$

The cases symmetric to cases *(2)*, *(3)*, *(4)*, are dealt analogously. □

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