# Homogenization of conservation laws with oscillatory source and non-oscillatory data

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## 1 Introduction

We consider the equation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \frac{1}{\varepsilon} V'\left(\frac{x}{\varepsilon}\right), \qquad x \in \mathbb{R},\ t > 0,\ \varepsilon > 0\,. \tag{1}$$

We assume that

**(H1)** $f : \mathbb{R} \to \mathbb{R}$ is $\mathbf{C}^2$, $f(u) \to +\infty$ as $|u| \to \infty$;
**(H2)** $f'' > 0$;
**(H3)** $V \in \mathbf{C}^2(\mathbb{R})$, periodic with period 1;
**(H4)** $V$ attains its minimum value at a single point in $\mathbb{R}/\mathbb{Z}$.

Without loss of generality, we can assume that $\min V(x) = 0$ and $f(0) = \min_{\mathbb{R}} f = 0$. As a consequence of **(H2)**, one has

**(H2)$'$** $uf'(u) > 0$ if $u \neq 0\,.$

Given the initial data $u^\varepsilon(x,0) = u_o(x)$, we will analyze the limiting behavior of the sequence $u^\varepsilon$. To describe the problem, we first recall the results obtained in [AS06], [Ama06]. Let us write the equation (1) for $\varepsilon = 1$:

$$u_t + f(u)_x = V'(x)\,. \tag{2}$$

We introduce the family $\mathcal{S}$ of 1-periodic, steady solutions to (2):

$$\mathcal{S} = \{\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}; \qquad \psi = \psi(x) \text{ is a weak entropy solution to (2)}\}\ ;$$

the properties of $\mathcal{S}$, resulting by the assumptions above, will be described in Sect. 2. We denote by $\langle\cdot\rangle$ the average over the period of a periodic function. The following theorem is concerned with the large time behavior of the periodic solutions to (2).

**Theorem 1.** *([AS06], [E92]) Assume* **(H1)**, **(H2)**, **(H3)**. *Let* $u_o \in \mathbf{L}^\infty(\mathbb{R})$ *be 1–periodic. Let* $u$ *be the entropic solution to the Cauchy problem* $u_t + f(u)_x = V'(x)$ , $u(x,0) = u_o(x)$ . *Then there exists a steady state solution* $\psi \in \mathcal{S}$ *such that*

$$\|u(t,\cdot) - \psi\|_{\mathbf{L}^1(\mathbb{R}/\mathbb{Z})} \to 0 \qquad as \quad t \to +\infty\,.$$

*The limit state* $\psi$ *has the property* $\langle\psi\rangle = \langle u_o\rangle$ . *If moreover* **(H4)** *holds, then* $\psi$ *is uniquely identified by its mean value.*

Now let us consider the Cauchy problem for (1) with initial data

$$u^\varepsilon(x,0) = u_o\left(x, \frac{x}{\varepsilon}\right) \tag{3}$$

where $u_o \in \mathbf{L}^\infty(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}))$. After [LPV87], it is known that the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ converges to a function $\bar u$ weak $^*$ in $\mathbf{L}^\infty{}_{loc}(\mathbb{R}\times\mathbb{R}_+)$, where $\bar u$ is the unique entropy solution of

$$\bar u_t + \bar f(\bar u)_x = 0\,, \qquad \bar u(x,0) = \int_0^1 u_o(x,y)\,dy \tag{4}$$

and $\bar f$, the so called *effective flux*, is a well defined function on $\mathbb{R}$, see Sect. 2.

**Theorem 2.** *([Ama06]) Assume* **(H1)**, **(H2)**$'$, **(H3)**, **(H4)** *and that* $f$ *is convex in an arbitrarily small neighborhood of* $0$. *Let* $u^\varepsilon$ *be the unique solution to (1), (3) and assume that the initial data* $u_o \in \mathbf{L}^\infty(\mathbb{R}\times(\mathbb{R}/\mathbb{Z}))$ *satisfy*

$$y \mapsto u_o(x,y) \in \mathcal{S}, \qquad \text{for a.e. } x\,. \tag{5}$$

*Let* $\bar u$ *be the entropy solution to (4) and let* $U : \mathbb{R}\times\mathbb{R}_+\times(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ *be uniquely defined, for a.e.* $(x,t)$, *as follows:*

$$y \mapsto U(x,t,y) \in \mathcal{S}, \qquad \langle U(x,t,\cdot)\rangle = \bar u(x,t)\,. \tag{6}$$

*Then, as* $\varepsilon \to 0$,

$$u^\varepsilon(x,t) - U\left(x,t,\frac{x}{\varepsilon}\right) \to 0 \qquad in \quad \mathbf{L}^2_{loc}(\mathbb{R}\times(0,+\infty))\,. \tag{7}$$

In this note, we are going to extend the result of Theorem 2, by relaxing the assumption (5) on the initial data. We focus on the special case of $u^\varepsilon(x,0) = u_o(x)$: the initial data do not depend on $\varepsilon$. In particular, they are not prepared, in the sense that do not satisfy the strong assumption (5).

Without assuming the strict convexity of $f$, we may expect that the asymptotic profile $U$ depends on both $x/\varepsilon$, $t/\varepsilon$, as can be seen by simple counterexamples with linear flux (see [E92]). Observe that the (global) convexity is not required in Theorem 2, but the assumption on the initial data prevents the formation of initial layers.

Here we will assume that the flux is uniformly convex, **(H2)**. The main result is the following.

**Theorem 3.** *Assume* **(H1)**, **(H2)**, **(H3)**, **(H4)** *and that* $u_o \in \mathbf{BV}_{loc}(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$. *Let* $u^\varepsilon$ *be the entropy solution to the Cauchy problem (1),* $u^\varepsilon(x,0) = u_o(x)$ *and* $\bar u$ *be the entropy solution to*

$$\bar u_t + \bar f(\bar u)_x = 0\,, \qquad \bar u(x,0) = u_o(x)\,. \tag{8}$$

*Let* $U : \mathbb{R} \times \mathbb{R}_+ \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ *be defined by (6). Then, as* $\varepsilon \to 0$ *one has (7).*

We remark that, since $u^\varepsilon, U$ are uniformly bounded in $\mathbf{L}^\infty$, the convergence (7) actually holds in $\mathbf{L}^p_{loc}$ for every $p \geq 1$.

After some preliminary arguments, object of Sect. 2, the proof of Theorem 3 will be given in Sect. 3. It makes use of both Theorems 1, 2. The key point is the introduction of a sequence of solutions to (1) with suitably prepared initial data, with the same asymptotic representation as $u^\varepsilon$. Then we proceed by using a localization argument, as done in [EE93] for conservation laws with smooth oscillatory data.

See [TT97] for related problems and [Tar86], plus references therein, as a general reference on the study of oscillations in nonlinear partial differential equations, with an application to one-dimensional scalar conservation laws.

## 2 Preliminaries

We start by reporting some properties of $\mathcal{S}$. Clearly, if $\psi \in \mathcal{S}$, then $\psi$ satisfies $f(\psi(y)) - V(y) = C$ for some constant $C$; $\psi^\varepsilon(x) = \psi(x/\varepsilon)$ is a steady solution to (1); $\psi$ is bounded because of **(H1)**.

*The effective flux.* We recall the definition of $\bar f$ ([E92], [LPV87]). If there exists a 1-periodic function $w$ such that

$$f(p + w'(y)) - V(y) = const.$$

independently of $y$, then $\bar f(p)$ is defined as the constant value on the right hand side. This procedure leads to a well defined function $\bar f$; an explicit formula in terms of $f$, $V$ is given by (2.8) in [E92]. See Fig. 1 for an example.

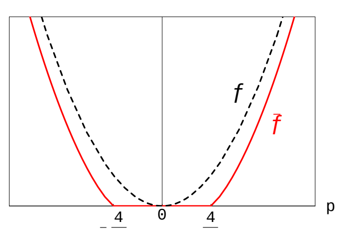


**Fig. 1.** Graph of $\bar f$ in the case $f(u) = u^2/2$, $V(y) = 1 + \sin(2\pi y)$

**Proposition 1.** *(a) Assume* **(H1)**, **(H2)′**, **(H3)**. *Then, for every* $p \in \mathbb{R}$, *there exists a map* $\psi_p(y) \in \mathcal{S}$ *such that*

$$\langle \psi_p \rangle = p\,, \qquad f(\psi_p(y)) - V(y) = \bar{f}(p) \qquad \forall\, y \in \mathbb{R}/\mathbb{Z}\,. \tag{9}$$

*(b) In addition, if* **(H4)** *holds, there exists a unique* $\psi_p$ *satisfying (9), hence the map* $\mathcal{S} \ni \psi \mapsto \langle \psi \rangle \in \mathbb{R}$ *is bijective. Moreover, the following monotonicity property holds:*

$$p_1 \le p_2 \quad \Rightarrow \quad \psi_{p_1}(y) \le \psi_{p_2}(y), \qquad \forall y\,.$$

See [Ama06] for details.

*Modified initial data.* From now on, assume **(H1)**, **(H2)′**, **(H3)**, **(H4)**. Given $u_o \in \mathbf{L}^\infty(\mathbb{R})$ we can define

$$w_o(x,y) \doteq \psi_{u_o(x)}(y)\,. \tag{10}$$

That is, $w_o$ is defined by the two following properties: $y \to w_o(x,y) \in \mathcal{S}$, for a.e. $x$, and

$$\langle w_o(x,\cdot) \rangle = \int_0^1 \psi_{u_o(x)}(y)\,dy = u_o(x)\,. \tag{11}$$

Thanks to assumption **(H4)**, $w_o$ is well defined in $\mathbf{L}^\infty(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}))$. Denote by $w^\varepsilon(x,t)$ the solution to (1) with initial data $w_o\left(x, \frac{x}{\varepsilon}\right)$:

$$w^\varepsilon_t + f(w^\varepsilon)_x = \frac{1}{\varepsilon} V'\left(\frac{x}{\varepsilon}\right) \qquad w^\varepsilon(x,0) = w_o\left(x, \frac{x}{\varepsilon}\right)\,. \tag{12}$$

Observe that $w_o(x, x/\varepsilon)$ is a prepared initial data, with pointwise average $u_o$.

*Uniform* $\mathbf{L}^\infty$ *bounds.* As in [ES92, Ama06], we can prove that $u^\varepsilon$, $w^\varepsilon$ are bounded independently of $\varepsilon$ in $\mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$, by bounding the initial data from above and below with suitable steady solutions (which are bounded because of **(H1)**) and then by using a comparison argument.

Also, the map $U(x,t,y)$ defined by (6) is bounded in $\mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}_+ \times (\mathbb{R}/\mathbb{Z}))$: indeed, $y \mapsto U(x,t,y) \in \mathcal{S}$ and $\langle U(x,t,\cdot) \rangle = \bar{u}(x,t)$, solution to (8), whose values belong to a bounded set.

*An approximation lemma.* Let $I = (0,1)$, $N \in \mathbb{N}$, $h = 1/N$ and define

$$\begin{aligned} A_j &= (jh, (j+1)h)\,, & j &= 0, \dots, N-1 \\ u_o^h(x) &= \frac{1}{h} \int_{A_j} u_o(y)\,dy\,, & x &\in A_j \\ w_o^h(x,y) &= \psi_{u_o^h(x)}(y)\,, & x &\in A_j,\ y \in \mathbb{R}\,. \end{aligned}$$

**Lemma 1.** *Let $u_o \in \mathbf{BV}_{loc}(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ and $w_o$ defined by (10). Then, for every $\delta > 0$ the following holds. There exist $N_o \in \mathbb{N}$ and $\varepsilon_o > 0$ such that, for all $N = 1/h \geq N_o$ and $0 < \varepsilon \leq \varepsilon_o$ one has*

$$\int_{\cup A_j} \left| w_o\left(x, \frac{x}{\varepsilon}\right) - w_o^h\left(x, \frac{x}{\varepsilon}\right)\right| \, dx \ < \ \delta \, .$$

*Proof.* Set $M_j = \sup_{A_j} u_o$, $m_j = \inf_{A_j} u_o$ (we choose the right continuous representative of $u_o$ to avoid ambiguity). Thanks to the monotonicity property of $\psi_p$ w.r.t. $p$, we have for $x \in A_j$

$$\begin{aligned} \left| w_o\left(x, \frac{x}{\varepsilon}\right) - w_o^h\left(x, \frac{x}{\varepsilon}\right)\right| &= \psi_{\max\{u_o(x), u_o^h(x)\}}\left(\frac{x}{\varepsilon}\right) - \psi_{\min\{u_o(x), u_o^h(x)\}}\left(\frac{x}{\varepsilon}\right) \\ &\leq \psi_{M_j}\left(\frac{x}{\varepsilon}\right) - \psi_{m_j}\left(\frac{x}{\varepsilon}\right) \end{aligned}$$

and hence

$$\begin{aligned} \int_{A_j} \left| w_o\left(x, \frac{x}{\varepsilon}\right) - w_o^h\left(x, \frac{x}{\varepsilon}\right)\right| \, dx \ &\leq \ \int_{A_j} \psi_{M_j}\left(\frac{x}{\varepsilon}\right) - \psi_{m_j}\left(\frac{x}{\varepsilon}\right) \, dx \\ &\leq \varepsilon \left( \left[\frac{h}{\varepsilon}\right] + 1 \right) \int_I \left[\psi_{M_j}(y) - \psi_{m_j}(y)\right] \, dy \\ &\leq (h + \varepsilon)(M_j - m_j) \ \ \leq (h+\varepsilon) \, \mathrm{Tot.Var.}\{u_o, A_j\} \end{aligned}$$

so that

$$\int_{\cup A_j} \left| w_o\left(x, \frac{x}{\varepsilon}\right) - w_o^h\left(x, \frac{x}{\varepsilon}\right)\right| \, dx \leq (h+\varepsilon) \, \mathrm{Tot.Var.}\{u_o, I\} < \delta$$

for $h$, $\varepsilon$ sufficiently small. This concludes the proof of the Lemma. □

## 3 Proof of the main theorem

In this section we prove theorem 3. Since $u^\varepsilon$, $U$ are uniformly bounded, it is enough to prove that the convergence (7) holds in $\mathbf{L}^1_{loc}(\mathbb{R} \times \mathbb{R}_+)$.

*Step 1: Constant initial data.* Assume that $u_o(x) = \bar{u}_o \in \mathbb{R}$. Then one has $u^\varepsilon(x,t) = v\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$ where $v$ is the solution of $u_t + f(u)_x = V'(x)$, $u(x,0) = \bar{u}_o$. Applying Theorem 1, there exists a limit state $\psi$ such that

$$v(y,\tau) - \psi(y) \to 0 \qquad \text{in } \mathbf{L}^1(\mathbb{R}/\mathbb{Z}) \qquad \text{as } \tau \to \infty \, . \tag{13}$$

Observe that $\langle \psi \rangle = \bar{u}_o$ and that the solution to (8) is constant: $\bar{u}(x,t) \equiv \bar{u}_o$. Hence the asymptotic profile $U$, introduced at (6), is given here by $U = U(y) \doteq \psi(y)$.

Then, let $K > 0$, $t_2 > t_1 > 0$ and define $M(\varepsilon) = \left[\frac{K}{\varepsilon}\right] + 1$, so that $K < \varepsilon M(\varepsilon) < K + 1$ as $\varepsilon \to 0$. We easily get

$$\begin{aligned}
&\iint_{[t_1,t_2]\times[-K,K]} |u^\varepsilon(x,t)-\psi(x/\varepsilon)|dxdt \\
&\quad\le 2\varepsilon M(\varepsilon)\int_{[t_1,t_2]}\int_{[0,1]} |v(y,t/\varepsilon)-\psi(y)|dydt \\
&\quad\le 2\varepsilon M(\varepsilon)\,(t_2-t_1)\int_{[0,1]} |v(y,t_1/\varepsilon)-\psi(y)|dy \to 0 \qquad \text{as } \varepsilon\to 0\,.
\end{aligned}$$

Remark that, for this part of the proof, **(H4)** is not required: the profile $\psi$ is that one, among the ones having mean value $\bar{u}_o$, that satisfies (13) and whose existence is guaranteed by Theorem 1. In other words, it is the limit profile *chosen* by the constant initial data $\bar{u}_o$. In a similar way, one proves also that

$$\int_I |u^\varepsilon(x,t)-\psi(x/\varepsilon)|\,dx \to 0 \qquad \text{as } \varepsilon\to 0\,, \tag{14}$$

for every closed interval $I=[a,b]$ and every $t>0$.

*Step 2: Modified initial data.* In general, with $u_o\in \mathbf{BV}_{loc}(\mathbb{R})\cap\mathbf{L}^\infty(\mathbb{R})$, we consider the solution $w^\varepsilon$ of the Cauchy problem (12).

By [LPV87], and thanks to (11), the sequences $u^\varepsilon$, $w^\varepsilon$ have the same weak limit $\bar{u}$, solution to (8). Also, the corresponding asymptotic profile $U$ is the same, see (6), since it depends only on the limit $\bar{u}$. Then, by applying Theorem 2, we get that $w^\varepsilon(x,t)-U(x,t,x/\varepsilon)\to 0$ in $\mathbf{L}^1_{loc}$.

Now we evaluate $u^\varepsilon - w^\varepsilon$. We proceed similarly to [EE93], proof of Theorem 4.1, and show that for any fixed $t>0$

$$u^\varepsilon(\cdot,t)-w^\varepsilon(\cdot,t)\to 0 \qquad \text{in } \mathbf{L}^1_{loc}(\mathbb{R})\,. \tag{15}$$

By using (15) and the $\mathbf{L}^1$-contraction, we can easily prove that $u^\varepsilon - w^\varepsilon\to 0$ in $\mathbf{L}^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$.

*Step 3: Proof of (15).* Let $I$ be a closed, bounded interval. Without loss of generality in the following arguments, we can assume $I=[0,1]$.

Fix a $\delta>0$. Let $t_o>0$, $N\in\mathbb{N}$ to be chosen later, $h=1/N$. Define, for $j=0,\dots,N-1$

$$\begin{aligned}
A_j &= (jh,(j+1)h)\,, \qquad B_j=(jh+Lt_o,(j+1)h-Lt_o)\\
A &= \cup_{j=0}^{N-1}A_j\,, \qquad B=\cup_{j=0}^{N-1}B_j\\
u_o^h(x) &= \frac{1}{h}\int_{A_j} u_o(\xi)\,d\xi \qquad w_o^h(x,y)=\psi_{u_o^h(x)}(y) \quad \text{if } x\in A_j\,,
\end{aligned}$$

being $L\doteq\sup_{|z|\le M}|f'(z)|$, $M$ a bound on $\|u^\varepsilon\|_\infty$, $\|w^\varepsilon\|_\infty$. Similarly, define

$$\begin{aligned}
j&=0,\dots,N\,, \qquad x_j=\left(j+\frac{1}{2}\right)h\\
C_j&=(x_{j-1},x_j)\,, \qquad E_j=(x_{j-1}+Lt_o,x_j-Lt_o)
\end{aligned}$$

$$C = \cup_{j=0}^{N} C_j \,, \qquad E = \cup_{j=0}^{N} E_j$$

$$v_o^h(x) = \frac{1}{h}\int_{C_j} u_o(\xi)\,d\xi \qquad W_o^h(x,y) = \psi_{v_o^h(x)}(y) \quad \text{if } x \in C_j \,.$$

By Lemma 1, we can choose $N = 1/h$ so large that the following holds: there exists a $\varepsilon_o > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_o$, one has

$$\int_A |u_o(x) - u_o^h(x)|\,dx \;+\; \int_A \left| w_o\left(x, \frac{x}{\varepsilon}\right) - w_o^h\left(x, \frac{x}{\varepsilon}\right)\right| dx \;<\; \delta \,, \quad (16)$$

$$\int_C |u_o(x) - v_o^h(x)|\,dx \;+\; \int_C \left| w_o\left(x, \frac{x}{\varepsilon}\right) - W_o^h\left(x, \frac{x}{\varepsilon}\right)\right| dx < \delta \,.$$

Then, we choose $t_o$ such that: $Lt_o < h/4$. With this choice of $t_o$, one has $I \subset B \cup E$.

Now, let $u^{\varepsilon,h}(x,t)$ and $w^{\varepsilon,h}(x,t)$ be the solutions to (1) with initial data, respectively, $u_o^h(x)$ and $w_o^h(x, x/\varepsilon)$. Using (16) and the $\mathbf{L}^1$-contraction property, one has

$$\int_B \left|u^\varepsilon(x,t_o) - u^{\varepsilon,h}(x,t_o)\right| dx \;+\; \int_B \left|w^\varepsilon(x,t_o) - w^{\varepsilon,h}(x,t_o)\right| dx$$

$$\leq \int_A \left|u_o(x) - u_o^h(x)\right| dx \;+\; \int_A \left| w_o\left(x, \frac{x}{\varepsilon}\right) - w_o^h\left(x, \frac{x}{\varepsilon}\right)\right| dx < \delta \,. (17)$$

Now, observe that

$$w^{\varepsilon,h}(x,t_o) = w_o^h\left(x, \frac{x}{\varepsilon}\right) = \psi_{u_o(x_j)}\left(\frac{x}{\varepsilon}\right) \qquad \text{if } x \in B_j$$

because the initial data $w_o^h$ coincides, on $A_j$, with a steady state, hence $w^{\varepsilon,h}$ coincides on $B_j$ with the same steady state. Nevertheless, $u^{\varepsilon,h}(x,t_o)$ coincides, on $B_j$, with the solution to the equation (1) corresponding to the constant initial data $u_o(x_j)$. Then, we can use (14) and obtain

$$\int_{B_j} \left|u^{\varepsilon,h}(x,t_o) - w^{\varepsilon,h}(x,t_o)\right| dx \;=$$

$$= \int_{B_j} \left|u^{\varepsilon,h}(x,t_o) - \psi_{u_o(x_j)}\left(\frac{x}{\varepsilon}\right)\right| dx \to 0 \;\; \text{as } \varepsilon \to 0 \,.$$

Summing up over $j$, one can conclude that there exists $\varepsilon_o > 0$ such that, for all $0 < \varepsilon < \varepsilon_o$,

$$\int_B \left|u^{\varepsilon,h}(x,t_o) - w^{\varepsilon,h}(x,t_o)\right| dx < \delta \,. \quad (18)$$

In conclusion, using (17), (18), one gets that $\int_B |u^\varepsilon(x,t_o) - w^\varepsilon(x,t_o)|\,dx \;\leq\; 2\delta$ for any $\varepsilon$ sufficiently small.

We repeat the same argument to estimate $\int_E |u^\varepsilon(x,t_o) - w^\varepsilon(x,t_o)| dx$ and finally get that, for any $\varepsilon$ sufficiently small

$$\int_I |u^\varepsilon(x,t_o) - w^\varepsilon(x,t_o)|\, dx \;\le\; \le \int_B |u^\varepsilon(x,t_o) - w^\varepsilon(x,t_o)|\, dx \;+\; \int_E |u^\varepsilon(x,t_o) - w^\varepsilon(x,t_o)|\, dx \;\le\; 4\delta\,.$$

To conclude the proof of (15), let $I = [a,b]$ be a given interval and $t > 0$ a given time. Define $J = [a - Lt, b + Lt]$. By the previous argument, for any positive $\delta$ there exist $\varepsilon_1 > 0$, $t_o > 0$ (that can be chosen less than $t$) such that

$$\int_J |u^\varepsilon(x,t_o) - w^\varepsilon(x,t_o)|\, dx \;\le\; \delta \qquad \forall\, 0 < \varepsilon < \varepsilon_1\,.$$

Since $\int_I |u^\varepsilon(x,t) - w^\varepsilon(x,t)|\, dx \;\le\; \int_J |u^\varepsilon(x,t_o) - w^\varepsilon(x,t_o)|\, dx$, the conclusion follows. □

## References

[Ama06] Amadori, D.: On the homogenization of conservation laws with resonant oscillatory source. Asymptotic Anal., **46**, 53–79 (2006)

[AS06] Amadori, D., Serre, D.: Asymptotic behavior of solutions to conservation laws with periodic forcing. J. Hyperbolic Differ. Equ., **3**, 387–401 (2006)

[E92] E, W.: Homogenization of scalar conservation laws with oscillatory forcing terms. SIAM J. Appl. Math., **52**, 959–972 (1992)

[ES92] E, W., Serre, D.: Correctors for the homogenization of conservation laws with oscillatory forcing terms. Asymptotic Anal., **5**, 311–316 (1992)

[EE93] Engquist, B., E, W.: Large time behavior and homogenization of solutions of two-dimensional conservation laws. Comm. Pure Appl. Math., **46**, 1–26 (1993)

[LPV87] Lions, P.L., Papanicolaou, G., Varadhan, S.R.S.: Homogenization of Hamilton–Jacobi equations. Preprint (1987)

[TT97] Tadmor, E., Tassa, T.: On the homogenization of oscillatory solutions to nonlinear convection-diffusion equations. Adv. Math. Sci. Appl., **7**, 93–117 (1997)

[Tar86] Tartar, L.: Oscillations in nonlinear partial differential equations: compensated compactness and homogenization. In: Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math. **23** (1986)