A hyperbolic model of multi-phase flow

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1 Introduction

We consider the following model for the flow of an inviscid fluid admitting liquid and vapor phases:

$$\begin{cases} v_t - u_x &= 0 \\ u_t + p(v,\lambda)_x &= 0 \\ \lambda_t &= 0\,. \end{cases} \tag{1}$$

Here $t > 0$ and $x \in \mathbb{R}$; moreover $v > 0$ is the specific volume, u the velocity, λ the mass density fraction of vapor in the fluid. Then $\lambda \in [0,1]$, with $\lambda = 0$ characterizes the liquid and $\lambda = 1$ the vapor phase; intermediate values of λ model mixtures of the two pure phases. The pressure is $p = p(v,\lambda)$; under natural assumptions the system is strictly hyperbolic. We refer to [4, 3] for more information on the model. System (1) has close connections to a system considered by Peng [6]. A comparison of the two models is done in [1].

We consider and prove here, as a preliminary study for a forthcoming paper, the basic features of system (1): wave curves, Riemann problem, wave interactions. The results improve some of those in [6]; the proofs are different. We refer to [1] for more details as well as, for instance, refined interaction estimates and a simple but complete proof of Glimm estimates.

2 Assumptions, wave curves and the Riemann problem

We consider a pressure law of the form

$$p(v,\lambda) = \frac{A(\lambda)}{v} \tag{2}$$

where $A(\lambda) = a^2(\lambda)$ is a smooth function defined on $[0,1]$ satisfying for every $\lambda \in [0,1]$

$$A(\lambda) > 0, \qquad A'(\lambda) > 0\,.$$

We denote $U = (v, u, \lambda) \in \Omega = (0, +\infty) \times \mathbb{R} \times [0,1]$ and by $\tilde{U} = (v, u)$ the projection of U onto the plane vu. Under the above assumptions on the pressure the system (1) is strictly hyperbolic in the whole Ω: the eigenvalues are $\pm\sqrt{-p_v(v,\lambda)}$, both genuinely nonlinear, and 0, which is linearly degenerate.

The direct shock-rarefaction curves through $U_o = (v_o, u_o, \lambda_o)$ for (1) are

$$\Phi_1(v, U_o) = \begin{cases} \left(v, u_o + a(\lambda_o)\dfrac{v - v_o}{\sqrt{vv_o}}, \lambda_o\right) & v < v_o \text{ shock} \\ \left(v, u_o + a(\lambda_o)\log\dfrac{v}{v_o}, \lambda_o\right) & v > v_o \text{ rarefaction,} \end{cases} \tag{3}$$

$$\Phi_2(\lambda, U_o) = \left(v_o\frac{a^2(\lambda)}{a^2(\lambda_o)}, u_o, \lambda\right) \quad \lambda \in [0,1] \quad \text{contact discontinuity,} \tag{4}$$

$$\Phi_3(v, U_o) = \begin{cases} \left(v, u_o - a(\lambda_o)\log\dfrac{v}{v_o}, \lambda_o\right) & v < v_o \text{ rarefaction} \\ \left(v, u_o - a(\lambda_o)\dfrac{v - v_o}{\sqrt{vv_o}}, \lambda_o\right) & v > v_o \text{ shock.} \end{cases} \tag{5}$$

Remark that the pressure is constant along contact discontinuities. The curves Φ_1, Φ_2 and Φ_3 are *plane* curves; as for states we denote by $\tilde{\Phi}_i$ the projection of these curves on the plane vu. We denote the u-component of the 1- (3-) shock-rarefaction curves by $\phi_1(v, U_o)$ (resp. $\phi_3(v, U_o)$) so that $\Phi_i(v, U_o) = (v, \phi_i(v, U_o), \lambda_o)$ for $i = 1, 3$.

Lemma 1. *Fix any $(v_o, u_o, \lambda_o) \in \Omega$. Then for $i = 1, 3$ and any $\alpha > 0$, $\lambda_1, \lambda_2 \in [0,1]$, $u_1, u_2 \in \mathbb{R}$:*

$$\phi_i\left(\alpha v, (\alpha v_o, u_o, \lambda_o)\right) = \phi_i\left(v, (v_o, u_o, \lambda_o)\right) \tag{6}$$

$$\frac{\phi_i\left(v, (v_o, u_1, \lambda_1)\right) - u_1}{a(\lambda_1)} = \frac{\phi_i\left(v, (v_o, u_2, \lambda_2)\right) - u_2}{a(\lambda_2)}\,. \tag{7}$$

Moreover, for $i, j = 1, 3$, $i \neq j$, and $\bar{v} = \frac{v_o^2}{v}$,

$$\phi_i(v, U_o) = \phi_j(\bar{v}, U_o)\,. \tag{8}$$

Remark that property (8) exchanges shocks of the first family with shocks of the third one, and analogously for rarefactions. The proof of the lemma follows from the definition of the functions ϕ_i; equality (6) is a consequence of the property of congruence of shock curves by rigid motions for fixed λ [5].

Definition 1. *With the notation (3)–(5) we define the strength ε_i of a i-wave by*

$$\varepsilon_1 = \frac{1}{2}\log\left(\frac{v}{v_o}\right), \quad \varepsilon_2 = 2\,\frac{a(\lambda) - a(\lambda_o)}{a(\lambda) + a(\lambda_o)}, \quad \varepsilon_3 = \frac{1}{2}\log\left(\frac{v_o}{v}\right)\,. \tag{9}$$

We define the function h as $h(\varepsilon) = 2\varepsilon$ if $\varepsilon \geq 0$ and $h(\varepsilon) = 2\sinh\varepsilon$ if $\varepsilon < 0$. Then $|h(\varepsilon)| \geq 2|\varepsilon|$ and $\phi_i(v, U_o) = u_o + a(\lambda_o) \cdot h(\varepsilon_i)$ for $i = 1, 3$.

We consider the Riemann problem, i.e., the initial-value problem for (1) under the piecewise constant initial conditions

$$(v, u, \lambda)(0, x) = \begin{cases} (v_\ell, u_\ell, \lambda_\ell) = U_\ell & \text{if } x < 0 \\ (v_r, u_r, \lambda_r) = U_r & \text{if } x > 0 \end{cases} \tag{10}$$

for states U_ℓ and U_r in Ω. We denote $A_r = A(\lambda_r)$, $A_\ell = A(\lambda_\ell)$, $A_{r\ell} = A(\lambda_r)/A(\lambda_\ell)$; analogous notations are used for the function a.

If $\lambda_r = \lambda_\ell$ then the solution to the Riemann problem is classical, [7]; otherwise we proceed as follows. For any pair of states U_ℓ, U_r in Ω we introduce

$$U^*_{\ell r} = \Phi_2(\lambda_r, U_\ell) = (A_{r\ell} v_\ell, u_\ell, \lambda_r) \ .$$

The state $U^*_{\ell r}$ is the unique state with mass density fraction λ_r that can be connected to U_ℓ by a 2-contact discontinuity; both states have then the same pressure. It lies on the right, resp. on the left, of $\tilde{U}_\ell$ according to either $\lambda_\ell < \lambda_r$ or $\lambda_\ell > \lambda_r$. Consider for $v > 0$ the curves

$$\Phi_2\left(\lambda_r, \Phi_1(v, U_\ell)\right) = (A_{r\ell} v, \phi_1(v, U_\ell), \lambda_r) \ , \tag{11}$$

$$\Phi_2\left(\lambda_r, \Phi_3(v, U_\ell)\right) = (A_{r\ell} v, \phi_3(v, U_\ell), \lambda_r) \ . \tag{12}$$

The first curve is the composition of the 1-curve through U_ℓ with the 2-curve to λ_r and passes through the point $U^*_{\ell r}$ when $v = v_\ell$. Similarly, the second is the composition of the 3-curve through U_ℓ with the 2-curve to λ_r and passes through the point $U^*_{\ell r}$ when $v = v_\ell$. We change parametrization $v \to v/A_{r\ell}$ in (11), (12) and define for $i = 1, 3$

$$\Phi_{i2}(v, U_\ell, \lambda_r) = \left(v, \phi_i\left(\frac{v}{A_{r\ell}}, U_\ell\right), \lambda_r\right) = (v, \phi_{i2}(v, U_\ell, \lambda_r), \lambda_r) \ . \tag{13}$$

The point $U^*_{\ell r}$ corresponds now to $v = A_{r\ell} v_\ell$. In an analogous way we define for $v > 0$ and $i = 1, 3$ the curves Φ_{2i}, ϕ_{2i}, by

$$\Phi_{2i}(v, U_\ell, \lambda_r) = \Phi_i\left(v, \Phi_2(\lambda_r, U_\ell)\right) = (v, \phi_i(v, U^*_{\ell r}), \lambda_r) = (v, \phi_{2i}(v, U_\ell, \lambda_r), \lambda_r) \ .$$

The curves defined above are the composition of the 2-curve from U_ℓ to $U^*_{\ell r}$ with the i-curve through $U^*_{\ell r}$. Both Φ_{21} and Φ_{23} pass through the point $U^*_{\ell r}$ when $v = A_{r\ell} v_\ell$.

Lemma 2 (Commutation of curves). *For $i = 1, 3$, the $i2$- and the $2i$-curves from U_ℓ are related by*

$$\frac{\phi_{i2}(v, U_\ell, \lambda_r) - u_\ell}{a_\ell} = \frac{\phi_{2i}(v, U_\ell, \lambda_r) - u_\ell}{a_r} \ . \tag{14}$$

Proof. Using first (6) and then (7), we have

$$\frac{\phi_{i2}(v,U_\ell,\lambda_r)-u_\ell}{a_\ell} = \frac{\phi_i\left(\frac{v}{A_{r\ell}},U_\ell\right)-u_\ell}{a_\ell} = \frac{\phi_i\left(v,(A_{r\ell}v_\ell,u_\ell,\lambda_\ell)\right)-u_\ell}{a_\ell}$$
$$= \frac{\phi_i\left(v,(A_{r\ell}v_\ell,u_\ell,\lambda_r)\right)-u_\ell}{a_r} = \frac{\phi_{2i}(v,U_\ell,\lambda_r)-u_\ell}{a_r}\,.$$

□

A consequence of the lemma is that if $\lambda_r \neq \lambda_\ell$ then the curves $\Phi_{i2}(v,U_\ell,\lambda_r)$ and $\Phi_{2i}(v,U_\ell,\lambda_r)$, $i=1,3$, meet only for $v=v^*_{\ell r}$. More precisely remark that $\phi'_{i2}(v,U_\ell,\lambda_r)=\frac{1}{a_{r\ell}}\,\phi'_{2i}(v,U_\ell,\lambda_r)$ for every $v>0$. Therefore if $\lambda_r>\lambda_\ell$ then $\phi'_{23}<\phi'_{32}<0<\phi'_{12}<\phi'_{21}$, while if $\lambda_r<\lambda_\ell$ then $\phi'_{32}<\phi'_{23}<0<\phi'_{21}<\phi'_{12}$. The mutual position of the four curves $\tilde\Phi_{ij}$ is as in Figure 1.

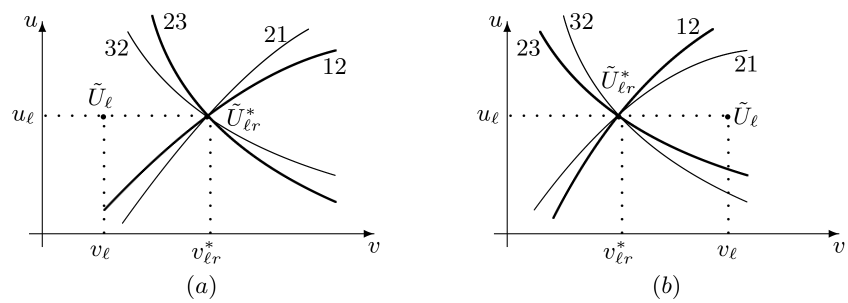

Fig. 1. The curves $\tilde\Phi_{i2}(v,U_\ell,\lambda_r)$, $\tilde\Phi_{2i}(v,U_\ell,\lambda_r)$, $i=1,3$. (a): $\lambda_r>\lambda_\ell$; (b): $\lambda_r<\lambda_\ell$.

Theorem 1. *For any pair of states U_ℓ, U_r in Ω the Riemann problem (1) (10) has a unique Ω-valued solution in the class of solutions consisting of simple Lax waves. If ε_i is the strength of the i-wave, $i=1,2,3$, then*

$$\varepsilon_3-\varepsilon_1=\frac{1}{2}\log\left(\frac{A_{r\ell}v_\ell}{v_r}\right) \tag{15}$$

$$a_\ell h\left(\varepsilon_1\right)+a_r h(\varepsilon_3)=u_r-u_\ell\,. \tag{16}$$

Moreover, let $\underline{v}>0$ be a fixed number. There exists a constant $C_1>0$ depending on $\underline{v}$ and $a(\lambda)$ such that if $v_l,v_r>\underline{v}$ then

$$|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|\le C_1|U_\ell-U_r|\,. \tag{17}$$

Proof. The case $\lambda_r=\lambda_l$ is solved as in [7]. We consider the case $\lambda_r>\lambda_\ell$. Then $\tilde U^*_{\ell r}$ lies on the right of $\tilde U_\ell$: $v_\ell<v^*_{\ell r}=A_{r\ell}\cdot v_\ell$. The curves $\tilde\Phi_{12}(v,U_\ell,\lambda_r)$ and $\tilde\Phi_{23}(v,U_\ell,\lambda_r)$ divide the plane into four regions, see Figure 2. The different patterns of the solution are classified as in the case of the p-system, [7], according to $\tilde U_r$ belongs to the regions RS, SS, SR, RR.

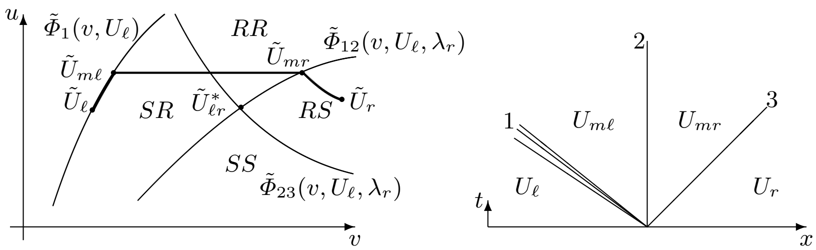

Fig. 2. Solution to the Riemann problem.

Assume that $\tilde{U}_r$ lies in region RS. Then by a continuity and transversality argument, [7], there exists a unique point $U_{mr} = (v_{mr}, u_{mr}, \lambda_r)$ on the curve $\Phi_{12}(v, U_\ell, \lambda_r)$ such that the 3-curve through U_{mr} passes by U_r. The curve $\phi_{12}(v, U_\ell, \lambda_r)$ is strictly monotone and surjective on $\mathbb{R}$; then we find a unique state $U_{m\ell} = (v_{m\ell}, u_{m\ell}, \lambda_\ell)$ on the 1-curve through U_ℓ, with $u_{m\ell} = u_{mr}$. The solution to the Riemann problem is then given by a 1-wave from U_ℓ to $U_{m\ell}$, a 2-wave to U_{mr} and a 3-wave to U_r. The other cases are treated analogously.

Formula (15) follows from the definition of strengths (9) as well as (16). Finally, let us prove (17). Concerning ε_2 remark that $|\varepsilon_2| \leq \frac{\max a'}{\min a} \cdot |\lambda_r - \lambda_\ell|$. If $\varepsilon_1 \varepsilon_3 \leq 0$, from (15) we get

$$\begin{aligned} |\varepsilon_1| + |\varepsilon_3| &= \frac{1}{2} \left|\log v_\ell - \log v_r\right| + \left|\log a_\ell - \log a_r\right| \\ &\leq \frac{1}{2\underline{v}} |v_r - v_\ell| + \frac{\max a'}{\min a} \cdot |\lambda_r - \lambda_\ell| \,. \end{aligned}$$

If $\varepsilon_1 \varepsilon_3 > 0$, from (16) we get $|u_r - u_\ell| = a_\ell \left|h\left(\varepsilon_1\right)\right| + a_r \left|h(\varepsilon_3)\right| \geq 2a_\ell \left|\varepsilon_1\right| + 2a_r \left|\varepsilon_3\right|$. Then $|\varepsilon_1| + |\varepsilon_3| \leq \frac{1}{2 \min a} \left|u_r - u_\ell\right|$. Then (17) follows for a suitable C_1. $\square$

3 Interactions

We focus on interactions involving contact discontinuities, the interactions of 1- and 3-waves being treated as in [5, 7], see also [2].

Proposition 1. *Let λ_ℓ, λ_r be the side values of λ along a 2-wave. The interactions of 1- or 3-waves with the 2-wave give rise to the following pattern of solutions:*

interaction	*outcome*	
	$\lambda_\ell < \lambda_r$	$\lambda_\ell > \lambda_r$
$2 \times 1R$	$1R + 2 + 3R$	$1R + 2 + 3S$
$2 \times 1S$	$1S + 2 + 3S$	$1S + 2 + 3R$
$3R \times 2$	$1S + 2 + 3R$	$1R + 2 + 3R$
$3S \times 2$	$1R + 2 + 3S$	$1S + 2 + 3S.$

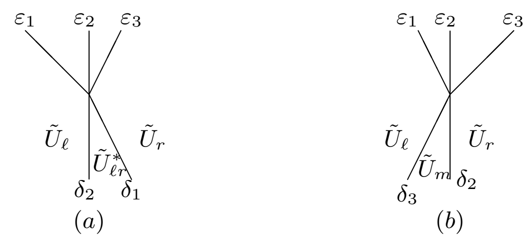

Fig. 3. Interactions. (a): from the right; (b): from the left.

Proof. The interactions are solved in a geometric way by referring to Figure 1. Thick lines there split the plane according to different patterns of solution of the Riemann problem, thin lines describe the interaction.

Case 2×1. Consider the case of the interaction of the 2-wave with a 1-wave coming from the right, see Figure 3. The state U_r lies on the curve $\Phi_{21}(v, U_\ell, \lambda_r)$. If the incoming wave is a rarefaction, then $v_r > v^*_{\ell r}$, if it is a shock, then U_r lies on the left branch of Φ_{21}: $v_r < v^*_{\ell r}$. Assume $\lambda_r > \lambda_\ell$. Then the curve $\tilde\Phi_{21}(v, U_\ell, \lambda_r)$ is contained either in the region RR if $v > v^*_{\ell r}$ or in SS if $v < v^*_{\ell r}$. The interaction is then solved according to Theorem 1 either by a rarefaction of the first family, a contact discontinuity and a rarefaction of the third family or by a shock of the first family, a contact discontinuity and a shock of the third family. If $\lambda_r < \lambda_\ell$ then the curve $\tilde\Phi_{21}(v, U_\ell, \lambda_r)$ is contained either in the region RS if $v > v^*_{\ell r}$ (incoming rarefaction) or in SR if $v < v^*_{\ell r}$ (incoming shock). The interaction pattern is solved again by Theorem 1.

Case 3×2. Consider now the case of the interaction of the 2-wave with a 3-wave coming from the left. The state U_r lies now on the curve $\Phi_{32}(v, U_\ell, \lambda_r)$. One can check that if the incoming wave is a rarefaction then $v_r < v^*_{\ell r}$, while if it is a shock then $v_r > v^*_{\ell r}$. If $\lambda_r > \lambda_\ell$ the curve $\tilde\Phi_{32}(v, U_\ell, \lambda_r)$ is contained either in the region SR if $v < v^*_{\ell r}$ or in RS if $v > v^*_{\ell r}$. The interaction is then solved either by a shock of the first family, a contact discontinuity and a rarefaction of the third family or by a rarefaction of the first family, a contact discontinuity and a shock of the third family. If $\lambda_r < \lambda_\ell$ then the curve $\tilde\Phi_{21}(v, U_\ell, \lambda_r)$ is contained either in the region RR if $v < v^*_{\ell r}$ (incoming rarefaction) or in SS if $v > v^*_{\ell r}$ (incoming shock). The interaction is solved consequently. $\square$

Theorem 2. *Assume that a* 1*-wave of strength* δ_1 *or a* 3*-wave of strength* δ_3 *interacts with a* 2*-wave of strength* δ_2*. Then the strengths* ε_i *of the outgoing waves satisfy* $\varepsilon_2 = \delta_2$ *and, for* $[\delta_2]_+ = \max\{\delta_2, 0\}$, $[\delta_2]_- = \max\{-\delta_2, 0\}$,

$$|\varepsilon_i - \delta_i| = |\varepsilon_j| \le \frac{1}{2} |\delta_2| \cdot |\delta_i| \qquad i, j = 1, 3,\ i \neq j\,. \tag{18}$$

$$|\varepsilon_1| + |\varepsilon_3| \le \begin{cases} |\delta_1| + |\delta_1| [\delta_2]_+ & \text{if } 1 \text{ interacts} \\ |\delta_3| + |\delta_3| [\delta_2]_- & \text{if } 3 \text{ interacts}\,. \end{cases} \tag{19}$$

Proof. We prove (18) when $i = 1$, $j = 3$. Assume that a 1 wave of strength δ_1 interacts with a 2 wave of strength $\delta_2 = 2\frac{a_r - a_\ell}{a_r + a_\ell}$. From (15) and comparing the velocities before and after the interactions we have

$$\varepsilon_3 - \varepsilon_1 = -\delta_1 \tag{20}$$

$$a_\ell h(\varepsilon_1) + a_r h(\varepsilon_3) = a_r \cdot h(\delta_1)\,. \tag{21}$$

Using (20), (21) we get

$$a_\ell h\left(\delta_1 + \varepsilon_3\right) + a_r h(\varepsilon_3) = a_r h(\delta_1)\,. \tag{22}$$

Recall that $\varepsilon_1 = \delta_1 + \varepsilon_3$ and δ_1 have the same sign; observe that $\varepsilon_3\delta_1\delta_2 > 0$. We consider the four possible types of interaction, as listed in Proposition 1.

• $2 \times 1R$, $a_r > a_\ell$. The identity (22) gives $a_\ell\left(\delta_1 + \varepsilon_3\right) + a_r\varepsilon_3 = a_r\delta_1$ from which we obtain $(a_\ell + a_r)\varepsilon_3 = (a_r - a_\ell)\delta_1$. Then (18) follows.

• $2 \times 1R$, $a_r < a_\ell$. Here (22) gives $a_\ell\left(\delta_1 + \varepsilon_3\right) - a_r\delta_1 = -a_r \sinh\varepsilon_3 \geq -a_r\varepsilon_3$ from which we obtain $-(a_r + a_\ell)\varepsilon_3 = (a_r + a_\ell)|\varepsilon_3| \leq (a_\ell - a_r)\delta_1$.

• $2 \times 1S$, $a_r > a_\ell$. From (22) we have $a_\ell \sinh\left(|\delta_1| + |\varepsilon_3|\right) + a_r \sinh(|\varepsilon_3|) = a_r \sinh(|\delta_1|)$. Denote $x = |\delta_1|$, $y = |\varepsilon_3|$, $k = a_r/a_\ell > 1$. Then the previous identity is written for $x \geq 0$, $y \geq 0$ as $F(x,y) = \sinh\left(x+y\right) + k\sinh(y) - k\sinh(x) = 0$. We have $F(x,0) < 0$ and $F(0,y) > 0$ for $x > 0$, $y > 0$; moreover $\partial F/\partial y > 0$. Therefore the implicit equation above is solved globally by $y = y(x)$.

The estimate (18) writes now $y(x) \leq \frac{k-1}{k+1}x$. To prove this it is sufficient to show that $F(x, \frac{k-1}{k+1}x) \geq 0$. The Mac Laurin expansion of $F(x, \frac{k-1}{k+1}x)$ is

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!(k+1)^{2n+1}} \left[(2k)^{2n+1} + k(k-1)^{2n+1} - k(k+1)^{2n+1}\right].$$

Consider the term in brackets; we claim that for every $n \geq 0$ and $k > 1$

$$(2k)^{2n+1} + k(k-1)^{2n+1} - k(k+1)^{2n+1} \geq 0\,. \tag{23}$$

Now $(k+1)^{2n+1} - (k-1)^{2n+1} = [(k+1)-(k-1)]\sum_{j=0}^{2n}(k-1)^j(k+1)^{2n-j}$ and $(2k)^{2n+1} = 2k\cdot[(k+1)+(k-1)]^{2n}$. Then the left side of (23) equals

$$2k\Big\{\sum_{j=0}^{2n}\binom{2n}{j}(k-1)^j(k+1)^{2n-j} - \sum_{j=0}^{2n}(k-1)^j(k+1)^{2n-j}\Big\}$$

which is always positive. That proves the claim and hence (18).

• $2\times 1S$, $a_r < a_\ell$. Here $0 < \varepsilon_3 < |\delta_1|$, and (22) gives $a_\ell \sinh\left(|\delta_1| - \varepsilon_3\right) - a_r\varepsilon_3 = a_r \sinh(|\delta_1|)$. As in the previous case set $x = |\delta_1|$, $y = \varepsilon_3$, $k = a_\ell/a_r > 1$ so that for $0 \leq y \leq x$ $F(x,y) = k\sinh\left(x-y\right) - y - \sinh(x) = 0$. Since $F(x,x) < 0$, $F(x,0) > 0$ for $x > 0$ and $\partial F/\partial y < 0$ we solve the above implicit equation globally with $y = y(x)$. In order to prove $y(x) \leq \frac{k-1}{k+1}x$ we show that $F\left(x, \frac{k-1}{k+1}x\right) \leq 0$. We have that $F\left(x, \frac{k-1}{k+1}x\right)$ equals

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!(k+1)^{(2n+1)}} \left[k2^{2n+1} - (k+1)^{2n+1}\right] - \frac{k-1}{k+1}x\,.$$

The first term in the sum in the right side is precisely $\frac{k-1}{k+1}x$, so we need to prove that for every $n \geq 1$, $k > 1$ we have $(k+1)^{2n+1} \geq k2^{2n+1}$, i.e., $(\frac{k+1}{2})^{2n+1} \geq k$. This follows by Bernoulli inequality: $(\frac{k+1}{2})^{2n+1} = (1 + \frac{k-1}{2})^{2n+1} \geq 1 + (2n+1)\frac{k-1}{2} \geq k$. Then (18) follows.
Finally, we prove (19), the first line. The case $\delta_2 < 0$ corresponds to $\lambda_\ell > \lambda_r$. From Proposition 1 the outcoming waves have different signs: $\varepsilon_1\varepsilon_3 < 0$. Hence we apply (20) to get $|\varepsilon_1| + |\varepsilon_3| = |\varepsilon_1 - \varepsilon_3| = |\delta_1|$ and we just get the equality in the first line of (19), for $\delta_2 < 0$. On the other hand, if $\delta_2 > 0$, the inequality simply follows by (18). In the other case, when a 3-wave interacts, one has $\varepsilon_1\varepsilon_3 < 0$ if $\lambda_\ell < \lambda_r$, that is $\delta_2 > 0$; the rest of the proof is as above. □

The inequalities (19) improve the inequality (3.3) in [6] in the case of two interacting wave fronts, one of them being of the second family. Under the notations of [6] we find a term $1/(a_r + a_\ell)$ instead of $1/\min\{a_r, a_\ell\}$. The proof differs from Peng's. Our estimates are sharp: in some cases (19) reduces to an identity.

We finally remark that, with the choice of the size of the wave-fronts in Definition 1, the total size of the strengths does not increase across any interaction of waves belonging only to the families 1 or 3, [1]. On the other hand, if a 2-wave is involved in the interaction, as in Theorem 2, the variation $|\varepsilon_1| + |\varepsilon_3| - |\delta_i|$ of the sizes of the strengths may be positive if and only if the incoming and the reflected waves are of the same type; this happens if and only if the colliding wave is moving toward a more liquid phase.

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