A Nonlocal Conservation Law from a Model of Granular Flow

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Abstract

In this paper we study the well-posedness for a scalar conservation law in which the flux term is non-local in space.

This equation represents a reduced model for slow erosion in granular flow ([1, 6]) and describes roughly the evolution of a profile of stationary matter, under the effect of a thin moving layer of granular matter on the top of it.

We show that the present equation admits weak solutions existing globally in time and prove their stability w.r.t the initial data. These properties are related to the assumption on the erosion flux. Different assumptions may lead to significantly different behaviors, see [9].

1 Introduction

We consider the scalar integro-differential equation

$$q_t + (K(q)\, f(q))_x = 0\,, \tag{1.1}$$

where

$$K(q(t,\cdot))(x) = \exp\left\{\int_x^0 f(q(t,\xi))\, d\xi\right\}\,, \tag{1.2}$$

for $(t,x) \in [0,T] \times \mathbb{R}_-$ and

$$\begin{cases} f : (-1,+\infty) \to \mathbb{R}\,, \quad f(0) = 0\,, \\ f' > 0\,, \quad f'' < 0 \\ \lim\limits_{q\to -1} f(q) = -\infty\,, \qquad \lim\limits_{q\to +\infty} \dfrac{f(q)}{q} = 0\,. \end{cases} \tag{1.3}$$

We are interested in the initial-boundary value problem for (1.1) on the domain $\mathbb{R}_+ \times \mathbb{R}_-$, with initial condition

$$q(0,x) = q_0(x) > -1\,, \qquad\qquad x < 0 \tag{1.4}$$

and no boundary condition at $x = 0$.

This equation arises as a *slow erosion limit* in a model of granular flow, studied in [1]. In more detail, let h be the height of the moving layer, u the height of the standing layer and $p = u_x$ be its slope. Assume that $p > 0$. The granular flow in one space dimension is described by the following 2×2 system of balance laws (originally proposed in [6])

$$\begin{cases} h_t - (hp)_x &= (p-1)h\,, \\ p_t + \big((p-1)h\big)_x &= 0\,. \end{cases} \tag{1.5}$$

As the moving layer becomes very thin, i.e., as $\|h\|_{\mathbf{L}^\infty} \to 0$, we proved in [1] that the solution for the slope p in (1.5) provides the weak solution of the following scalar integro-differential equation

$$p_\mu + \left(\frac{p-1}{p} \cdot \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\right)_x \;=\; 0\,.$$

Here, the new time variable $\mu(t) = \int_0^t p(\tau,0)h(\tau,0)\,d\tau$ accounts for the total mass of granular material being poured from above. Recognizing $p = 1$ as the equilibrium slope, we introduce $q \doteq p-1$. Using the variable t in place of μ, we obtain the equation (1.1) with

$$f(q) \;=\; \frac{q}{q+1}$$

that clearly satisfies (1.3).

In this paper we establish the well-posedness result for equation (1.1). In order to define entropy weak solutions, we recall some results in [1]. From Theorem 2 in [1], for h sufficiently small, we proved that the solution for p of (1.5) satisfies the following bounds for all $t \geq 0$, uniformly in h:

$$p(t,x) \geq p_0 > 0, \qquad \text{TV }p(\cdot) \leq M, \qquad \|p-1\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq M\,. \tag{1.6}$$

This leads to the state of the solutions $q = p-1$ for (1.1), on a domain $[0,T] \times \mathbb{R}_-$, for a given $T > 0$. To be precise, let C_0, p_0 be some positive constants and define $\mathcal{D}_{C_0,p_0}$ as

$$\mathcal{D}_{C_0,p_0} = \left\{ q(x) : \inf_{x<0} q(x) + 1 \geq p_0\,,\ \text{TV }q \leq C_0\,,\ \|q\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq C_0 \right\}. \tag{1.7}$$

Note that for any $q \in \mathcal{D}$, the *local* term $f(q)$ of the flux satisfies $Kf' > 0$, i.e., the characteristic speed of equation (1.1) is positive. Therefore no boundary condition is posed at $x = 0$.

We now state a natural definition of an entropy weak solution of (1.1)–(1.4) on a domain $[0,T] \times \mathbb{R}_-$, with $T > 0$.

Definition 1.1. For some $C_0 > 0$ and $p_0 > 0$, let $q_0 \in \mathcal{D}_{C_0,p_0}$ (see (1.7)). We say that q provides an **entropy weak solution** to (1.1)–(1.2) on $[0,T] \times \mathbb{R}_-$ with initial condition (1.4), if the following conditions are satisfied:

(H1) $q : [0,T] \to \mathcal{D}_{C_1,p_1}$ for some $C_1 > 0$, $p_1 > 0$; the map $[0,T] \ni t \mapsto q(t)$ is Lipschitz in $\mathbf{L}^1(\mathbb{R}_-)$.

(H2) q is a weak entropy solution of the scalar conservation law

$$\begin{cases} q_t + (k(t,x)\, f(q))_x = 0\,, \\ q(0,x) = q_0(x) \end{cases} \tag{1.8}$$

where the coefficient k is defined by

$$k(t,x) = K(q(t,\cdot))(x) \;=\; \exp\left\{\int_x^0 f(q(t,\xi))\,d\xi\right\}. \tag{1.9}$$

The main result of the paper is the uniqueness and well-posedness of the solution for (1.1), as stated in the following Theorem.

Theorem 1.2. *Let C_0, p_0 be given positive constants. Then for any initial data $q_0 \in \mathcal{D}_{C_0,p_0}$ there exists a solution $q(t,x)$ to the initial-boundary value problem (1.1) on $x < 0$, $t \geq 0$ with the following properties:*

(i) for all $t \geq 0$, $\inf_{x<0} q(t,x) + 1 \geq p_0 > 0$;

(ii) $q \in \mathbf{L}^\infty([0,+\infty) \times \mathbb{R}_-)$;

(iii) for all $t \geq 0$, $q(t,\cdot) \in \mathbf{L}^1(\mathbb{R}_-) \cap BV(\mathbb{R}_-)$.

Moreover, consider two solutions $q_0(t,\cdot)$, $q_1(t,\cdot)$ of the integro-differential equation (1.1), corresponding to the initial data $\bar{q}_0$, $\bar{q}_1 \in \mathcal{D}_{C_0,p_0}$ respectively. Then for any $T > 0$ there exists $L = L(T, C_0, p_0) > 0$ such that, for $t \in [0,T]$

$$\|q_0(t,\cdot) - q_1(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq \mathrm{e}^{Lt}\, \|\bar{q}_0 - \bar{q}_1\|_{\mathbf{L}^1(\mathbb{R}_-)}\,. \tag{1.10}$$

Recalling that $q = p - 1 = u_x - 1$, the solution q established by Theorem 1.2 allows us to reconstruct the profile u of the standing layer:

$$u(t,x) - x = \int_{-\infty}^x q(t,y)\,dy\,. \tag{1.11}$$

Moreover an equation for u is deduced as follows. From (1.2) we have

$$K_x = -Kf(q(t,x)),$$

so that equation (1.1) can be rewritten as $q_t - K_{xx} = 0$.

By integrating in space, using (1.11) and noticing that $K_x(q(t,\cdot)) \in \mathbf{L}^1(\mathbb{R}_-)$, we arrive at

$$u_t - K_x = u_t + Kf\,(u_x - 1) \;=\; 0\,.$$

This is a nonlocal Hamilton-Jacobi equation for the unknown u. An analysis for this type of equation has been carried out in [9] for a different class of functions f, that allows for singularities on the standing profile.

More precisely, the main difference between Theorem 1.2 and the results in [9] is due to the different behavior of f at $+\infty$. Indeed a slower increase of f as

$$\frac{f(q)}{q} \to 0 \qquad \text{as } q \to +\infty\,, \tag{1.12}$$

assumed in (1.3), yields to globally bounded solutions. On the other hand, the faster behavior assumed in [9] requires that the above limit is positive and leads to blow-up of the solutions, that renders more convenient to study the equation in terms of the antiderivative u.

To be more specific, for both problems it is possible to derive an apriori, global bound on the quantity

$$\frac{q(t,x)}{f(q(t,x))}\,.$$

Thanks to (1.12), the function $q/f(q)$ diverges as $q \to +\infty$. From this one deduces that $q(t,x)$ must be globally bounded.

Other problems involving a nonlocal term in the flux have been studied in [5, 3, 4]. In the rest of the paper we review the proof of Theorem 1.2, whose details can be found in [2].

2 Well-posedness for the equation with a given $k(t,x)$

Towards the proof of Theorem 1.2, we study the equation with a given local coefficient $k(t,x)$

$$u_t + \Big(k(t,x)f(u)\Big)_x = 0\,, \qquad x \le 0, \quad t \ge 0 \tag{2.1}$$

where $k(t,x)$ satisfies the following assumptions, for some given $T > 0$:

(K1) $k(t,x) \in \mathbf{L}^\infty([0,T]\times\mathbb{R}_-)$, is Lipschitz continuous and $\inf_{t,x} k > 0$;

(K2) $\mathrm{TV}\, k(t,\cdot)$, $\mathrm{TV}\, k_x(t,\cdot)$ are bounded uniformly in time;

(K3) $[0,T] \ni t \to k_x(t,\cdot) \in \mathbf{L}^1(\mathbb{R}_-)$ is Lipschitz continuous.

The local existence and well-posedness for (2.1) is stated in the following Theorem.

Theorem 2.1. *Let f satisfy* (1.3) *and k satisfy* **(K1)**–**(K3)**. *Let $C_0 > 0$, $p_0 > 0$ be given. Then there exist two positive constants $C_1 > C_0$, $p_1 < p_0$ and an operator $P_t : [0,T]\times\mathcal{D}_{C_0,p_0} \to \mathcal{D}_{C_1,p_1}$ such that:*

1) for all $\bar u_0$, $\bar u_1 \in \mathcal{D}_{C_0,p_0}$ one has

$$\|P_t(\bar u_0) - P_t(\bar u_1)\|_{\mathbf{L}^1(\mathbb{R}_-)} \le \|\bar u_0 - \bar u_1\|_{\mathbf{L}^1(\mathbb{R}_-)}\,; \tag{2.2}$$

2) the function $u(t,x) = P_t(\bar u_0)$ is a weak entropy solution of (2.1) *with initial data $u(0,\cdot) = \bar u_0 \in \mathcal{D}_{C_0,p_0}$.*

Moreover, the solutions of (2.1) depend continuously in $\mathbf{L}^1$ on the coefficient k (see also [7]).

Theorem 2.2. *Let k, $\tilde k$ satisfy the assumptions* **(K1)**–**(K3)**. *Let u, $\tilde u$ be the solutions of the conservation laws*

$$\begin{aligned} u_t + \Big(k(t,x)f(u)\Big)_x &= 0\,, \qquad x\le 0, \quad t\ge 0\,,\\ u_t + \Big(\tilde k(t,x)f(u)\Big)_x &= 0\,, \qquad x\le 0, \quad t\ge 0\,, \end{aligned}$$

respectively, with the same initial data $\bar u_0 \in \mathcal{D}_{C_0,p_0}$, on the time interval $[0,T]$. Then, the following estimate holds

$$\begin{aligned} &\frac{1}{t}\;\|u(t,\cdot) - \tilde u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)}\\ &\qquad\le\; \hat C_1 \|k - \tilde k\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}_-)} \;+\; \hat C_2 \sup_{t\in[0,T]} \mathrm{TV}\,\Big(k(t,\cdot) - \tilde k(t,\cdot)\Big) \end{aligned}$$

where $\hat C_1$, $\hat C_2$ depend on $\sup_t \mathrm{TV}\, u(t,\cdot)$, $\sup_t \mathrm{TV}\, \tilde u(t,\cdot)$ and on $\sup|f|$, $\sup|f'|$ taken over the range of the solutions.

3 Sketch of the proof of Theorem 1.2

The existence of solutions for (1.1), in the case of $f(q) = q/(q+1)$, is a consequence of the slow erosion limit studied in [1]. In the more general assumptions (1.3) for f, a proof can be obtained through a time step approximation of (1.1). Below we give some formal arguments for smooth solutions, while rigorous analysis can be found in [2].

(a). Bound on the $\mathbf{L}^1$ norm. Because $\mathrm{sign}(q) = \mathrm{sign}(f(q))$, the conservation law (1.1) implies that, for all $t \geq 0$, we have

$$\|q(t,\cdot)\|_{L^1(\mathbb{R}_-)} \leq \|q(0,\cdot)\|_{L^1(\mathbb{R}_-)} - \int_0^t |f(q(\tau,0))|\, d\tau\,. \tag{3.1}$$

(b). Lower bound on q. Along a characteristic curve $t \to x(t)$ we have

$$\begin{cases} x'(t) = k(t,x)f'(q) \\ q'(t) = -k_x(t,x)f(q) = kf(q)^2 \geq 0\,. \end{cases} \tag{3.2}$$

Then the solution q is non-decreasing along any characteristics, and therefore $\inf_x q(t,x) \geq \inf q_0(x) \geq p_0 - 1 > -1$ for all $t \geq 0$.

(c). Bounds on f, f', k. As a consequence of *(a)*, *(b)* we immediately deduce apriori bounds on $\|f(q(t,\cdot))\|_\infty$, $\|f(q(t,\cdot))\|_{\mathbf{L}^1}$, $\|f'(q(t,\cdot))\|_\infty$, $\|k(t,\cdot)\|_\infty$, $\mathrm{TV}\,(k)$ and the characteristic speed kf'.

(d). Upper bound on q. The physical meaning of the function $f(q)$ is the amount of erosion per unit length in x. Then the function $q/f(q)$ is a measurement for the change in height per unit erosion. Therefore, the function

$$\phi(t,x) \doteq \int_{-\infty}^x q(t,y)\, dy + \frac{q}{f(q)}$$

would remain constant along characteristics. In fact, we can easily check that

$$\dot{\phi}(t,x(t)) = \phi_t + \dot{x}\phi_x = [qf'(q) - f(q)]k + \frac{f(q) - qf'(q)}{f^2(q)} f^2(q)k = 0.$$

For any $q(0) > 0$, for any time $t > 0$, along the characteristic we have

$$\frac{q(t)}{f(q(t))} = \frac{q(0)}{f(q(0))} + \int_{-\infty}^{x(0)} q(0,y)\, dy - \int_{-\infty}^{x(t)} q(t,y)\, dy \leq M \tag{3.3}$$

which is bounded thanks to *(a)*. Recalling the last assumption in (1.3), the function $q \to q/f(q)$ is monotone increasing and approaches $+\infty$ as $q \to +\infty$. Therefore (3.3) provides an upper bound for q for all $t > 0$.

(e). BV bound. For smooth solutions, the equation (1.1) gives

$$(q_x)_t + (kf'(q)q_x)_x = (kf^2(q))_x = -f^3(q)k + 2f(q)f'(q)q_x k\,.$$

Formally, the $\mathbf{L}^1$ norm of q_x provides the total variation for q. One has

$$\begin{aligned} \frac{d}{dt}\mathrm{TV}\,(q(t,\cdot)) \leq\ & \|k\|_\infty \cdot \|f(q(t,\cdot))\|_\infty^2 \cdot \|f(q(t,\cdot))\|_{\mathbf{L}^1} \\ &+\ 2\|k(t,\cdot)\|_\infty \cdot \|f(q(t,\cdot)\|_\infty \cdot \mathrm{TV}\,(f(q(t,\cdot))). \end{aligned}$$

Then the total variation of q can grow exponentially in t, but remains bounded for finite t.

Finally, to see that the flow generated by equation (1.1) is Lipschitz continuous, we consider two solutions $q_0(t,\cdot)$, $q_1(t,\cdot)$ of the integro-differential equation (1.1) with initial data

$$q_0(0,x) = \bar{q}_0(x)\,, \qquad\qquad q_1(0,x) = \bar{q}_1(x) \qquad\qquad x < 0\,,$$

and satisfying (1.7) for $t \in [0,T]$. We have the following estimate

$$\begin{aligned}\|q_0(t,\cdot) - q_1(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} &\leq \|\bar{q}_0 - \bar{q}_1\|_{\mathbf{L}^1(\mathbb{R}_-)} \\ &\quad + L\int_0^t \|q_0(s,\cdot) - q_1(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)}\,ds \end{aligned} \tag{3.4}$$

for a suitable constant L. By Gronwall lemma, this yields (1.10), hence the Lipschitz continuous dependence of solutions of (1.1) on the initial data.

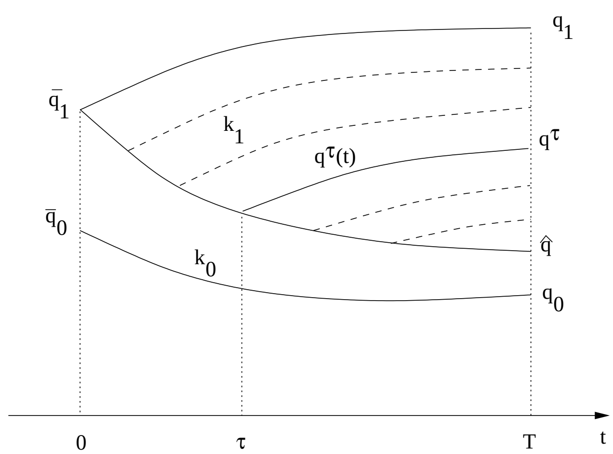

Figure 3.1: The flow of solutions $q_0, \hat{q}, q^\tau, q_1$ for the integro-differential equation.

To prove the estimate (3.4), we define the functions $k_0(t,x)$, $k_1(t,x)$ as in (1.9), corresponding to $q_0(t,x)$, $q_1(t,x)$ respectively. Let $\hat{q}$ be the solution of

$$q_t + (k_0(t,x)\,f(q))_x = 0\,,$$

with initial data $\hat{q}(0,x) = \bar{q}_1(x)$ (see Figure 3.1). Since q_0 and $\hat{q}$ satisfy the *same* equation (with coefficient k_0), we have

$$\|q_0(t,\cdot) - \hat{q}(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \;\leq\; \|\bar{q}_0 - \bar{q}_1\|_{\mathbf{L}^1(\mathbb{R}_-)} \qquad \forall\, t \in [0,T]\,. \tag{3.5}$$

Now we evaluate $\|\hat{q}(t,\cdot) - q_1(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)}$ as follows:

$$\|\hat{q}(T,\cdot) - q_1(T,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \le \int_0^T E(\tau)\, d\tau\,, \tag{3.6}$$

where

$$E(\tau) \doteq \limsup_{h\to 0+} \frac{\|q^\tau(\tau+h,\cdot) - \hat{q}(\tau+h,\cdot)\|_{\mathbf{L}^1}}{h}$$

and $q^\tau(t,\cdot)$ is the solution, for $t \ge \tau$, of

$$q_t + (k_1(t,x)\, f(q))_x = 0\,, \qquad q(\tau,x) = \hat{q}(\tau,x)\,.$$

By Theorems 2.1 and 2.2 we get the following estimate for E:

$$E(\tau) \le M\, \mathrm{TV}\,\{k_0(\tau,\cdot) - k_1(\tau,\cdot)\} \le L \cdot \|q_0(\tau,\cdot) - q_1(\tau,\cdot)\|_{\mathbf{L}^1}$$

for some suitable constants M, L. By inserting this estimate into (3.6) and using (3.5) one finally obtains (3.4).

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