A Compact Survey of Modern Fourier Analysis

Fourier Analysis as found in most mathematical books has little to do with applications of Fourier Analysis as described in engineering or physics books. The course will provide (in 8 units) a fast lane to modern Time-Frequency Analysis (TFA), thus providing mathematical tools which allow the description not only classical questions of Fourier Analysis, but provide versatile tools useful for many applied settings.

The key object of TFA is the so-called Short-Time Fourier Transform (STFT). It provides an isometry from the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ into the range of the STFT, which is a closed subspace of $\boldsymbol{L}^2(\mathbb{R}^{2d})$, but also a space of continuous functions over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (phase space). The subspace of all functions with an integrable STFT (known as Feichtinger's algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$) is a Banach algebra of continuous, integrable functions on $\mathbb{R}^d$, which is also invariant under the (ordinary) Fourier transform (using just Riemann integrals). The dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, nowadays known as Banach space of mild distributions, can be viewed as a model for all possible *signals*, characterized by the boundedness of the STFT. All the function spaces used in the engineering context (including discrete an continuous, periodic and non-periodic signals) can be viewed as subspaces of $\boldsymbol{S}_0'(\mathbb{R}^d)$. Any mild distribution can be approximated by bounded continuous functions and has a natural Fourier transform in $\boldsymbol{S}_0'(\mathbb{R}^d)$, which coincides with the classical version in each case, but goes far beyond. Together with the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ these two spaces form the so-called *Banach Gelfand Triple* (BGT) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which behaves in some sense like the triple $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ of fields, forming a natural chain.

This setting allows the correct mathematical treatment of Dirac measures and Dirac combs. Translation invariant linear systems (TILS) are characterized as convolution operators or alternatively as Fourier multipliers (via some transfer function). Gabor Expansions (atomic series representations of signals as double sums using time-frequency shifted copies of a Gabor atom, such as the Gaussian) allows to characterize the BGT by coefficients in $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$. Also the foundations of Gabor Analysis can be well described with these tools, including Gabor multipliers or Anti-Wick operators.

In addition there is a kernel theorem, which allows to characterize linear operators by their kernel, but also as a superposition of time-frequency shifts, the so-called spreading representation, which is equivalent to the characterization of time-variant systems (as used for mobile communication) using the Kohn-Nirenberg characterization.

As time permits also questions of discretization and approximation via realizable, constructive methods will be shortly discussed. Most of the terms explained in the course have already been implemented using MATLAB and the speaker has long-standing experience with this tool.

Overall the course will be more like a survey of problems, methods and results offered by this new approach than a discussion of technical details. Some material is found here:

http://www.nuhag.eu/ETH20
https://nuhagphp.univie.ac.at/home/feitalks.php
https://nuhagphp.univie.ac.at/home/feipub_db.php