Finiteness property of pairs of 2×2 sign-matrices via real extremal polytope norms

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Abstract

This paper deals with the joint spectral radius of a finite set of matrices. We say that a set of matrices has the *finiteness property* if the maximal rate of growth, in the multiplicative semigroup it generates, is given by the powers of a finite product.

Here we address the problem of establishing the finiteness property of pairs of 2×2 sign-matrices. Such problem is related to the conjecture that pairs of sign-matrices fulfil the finiteness property for any dimension. This would imply, by a recent result by Blondel and Jungers, that finite sets of rational matrices fulfil the finiteness property, which would be very important in terms of the computation of the *joint spectral radius*. The technique used in this paper could suggest an extension of the analysis to $n \times n$ sign-matrices, which still remains an open problem.

As a main tool of our proof we make use of a procedure to find a so-called *real extremal polytope norm* for the set. In particular, we present an algorithm which, under some suitable assumptions, is able to check if a certain product in the multiplicative semigroup is spectrum maximizing.

For pairs of sign-matrices we develop the computations exactly and hence are able to prove analytically the finiteness property. On the other hand, the algorithm can be used in a floating point arithmetic and provide a general tool for approximating the joint spectral radius of a set of matrices.

Key words: Joint spectral radius, extremal norm, real polytope norm, finiteness property, sign-matrices.

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1 Framework

Let $\mathscr{F} = \{A^{(i)}\}_{i \in \mathscr{I}}$ be a family of $n \times n$-matrices, $\mathscr{I}$ being a set of indices. Then, for each $k = 0, 1, \ldots$, consider the set $\Sigma_k(\mathscr{F})$ of all possible products of length k whose factors are elements of $\mathscr{F}$, that is $\Sigma_k(\mathscr{F}) = \{A^{(i_1)} \cdots A^{(i_k)} \mid i_1, \ldots, i_k \in \mathscr{I}\}$ and set $\Sigma(\mathscr{F}) = \bigcup_{k \geq 0} \Sigma_k(\mathscr{F})$ (with $\Sigma_0(\mathscr{F}) = I$) the multiplicative semigroup associated with $\mathscr{F}$. Let $\rho(\cdot)$ denote the spectral radius of an $n \times n$-matrix. Then consider

$$\rho_k(\mathscr{F}) = \sup_{P \in \Sigma_k(\mathscr{F})} \rho(P), \qquad k = 0, 1, \ldots$$

and define the *generalized spectral radius* of $\mathscr{F}$ (see [8]) as

$$\rho(\mathscr{F}) = \limsup_{k \to \infty} \rho_k(\mathscr{F})^{1/k}.$$

Recently it has been shown (see [2], [9], [28] and [27]) that $\rho(\mathscr{F})$ is equal to the joint spectral radius defined in [26]. This allows to simply call ρ the *spectral radius* of the family of matrices $\mathscr{F}$. We introduce now a further characterization of the joint spectral radius. Given a norm $\|\cdot\|$ on the vector space $\mathbb{C}^n$ and the corresponding induced $n \times n$-matrix norm, we use the same notation to define

$$\|\mathscr{F}\| = \sup_{i \in \mathscr{I}} \|A^{(i)}\|,$$

where we assume that $\sup_{i \in \mathscr{I}} \|A^{(i)}\| < +\infty$, that is the family $\mathscr{F}$ is bounded. The following result can be found, for example, in [26] and [9].

Theorem 1.1 *The spectral radius of a bounded family $\mathscr{F}$ satisfies the equality*

$$\rho(\mathscr{F}) = \inf_{\|\cdot\| \in \mathscr{N}} \|\mathscr{F}\|, \tag{1}$$

where $\mathscr{N}$ denotes the set of all possible induced $n \times n$-matrix norms.

The actual computation of $\rho(\mathscr{F})$ is an important problem in several applications (see e.g. [1,13,14,22,23]). The problem, however, appears quite difficult in general (see e.g. [29]). We are interested in using Theorem 1.1 as an actual computational tool. For this we need that the inf in (1) is a min. This is true, for example, for irreducible families of matrices (see [1,9]). Irreducibility means that the matrices of the family $\mathscr{F}$ do not admit any non-trivial common invariant subspace. A norm $\|\cdot\|_*$ satisfying the condition

$$\|\mathscr{F}\|_* = \rho(\mathscr{F})$$

is said to be *extremal* for the family $\mathscr{F}$ (for an extended discussion see [31]). A family which admits an extremal norm is said to be *non-defective* (see, e.g., [12]). We are interested in establishing whether a family $\mathscr{F}$ fulfils the following property.

Definition 1.1 *A finite family $\mathscr{F}$ has the finiteness property if there exists a product $P \in \Sigma_k(\mathscr{F})$ such that*

$$\rho(P) = \rho(\mathscr{F})^k.$$

Such a product is called spectrum maximizing product (in short s.m.p.)

Although it was conjectured to be valid in all cases (see [21]), the finiteness property does not hold for every finite family (see [7], [3], [18]) but has been conjectured to be true for some classes of matrices. Some approaches for the approximation of the joint spectral radius have been recently considered (see for example [5,6] and [24]). The problem we handle here, however, is that of an exact computation.

A way to compute exactly the joint spectral radius is based on the following property. If $\alpha > 0$ then

$$\rho(\mathscr{F}) = \alpha\rho\left(\frac{1}{\alpha}\mathscr{F}\right).$$

So, if $Q \in \Sigma_k(\mathscr{F})$ is a certain product and $\alpha = \rho(Q)^{1/k}$, then we have that $\rho\left(\frac{1}{\alpha}\mathscr{F}\right) \geq 1$. Therefore, if we are able to find a norm such that $\|\frac{1}{\alpha}\mathscr{F}\| = 1$, then we have that

$$\alpha \leq \rho(\mathscr{F}) \leq \alpha \quad \Longrightarrow \quad \rho(\mathscr{F}) = \alpha = \rho(Q)^{1/k}.$$

That would mean that the finiteness property holds and the product Q is an s.m.p. The key point is the search for an extremal norm.

The summary of the paper is the following. In Section 2, after recalling some definitions and results on real polytope norms, we introduce the main ideas of a procedure able to find an extremal norm in this class. It is obtained by applying the product semigroup to a suitable initial vector. Then, in Section 3, we prove the finiteness property for pairs of 2×2 sign-matrices on a case-by-case basis. Finally, in Section 5, we outline some conclusions. Section 4 contains an appendix with the details of the analysis presented in Section 3.

2 Finding real extremal polytope norms

In this section we are concerned with the possible construction of the unit ball of an extremal norm for a finite family. We focus our attention on a special class of norms. Following [15], where the more general complex case has been treated, we say that a bounded set $\mathscr{P} \subset \mathbb{R}^n$ is a *balanced real polytope* (b.r.p.) if there exists a finite set of vectors $\mathscr{X} = \{x_i\}_{1 \leq i \leq m}$ (with $m \geq n$) such that $\mathrm{span}(\mathscr{X}) = \mathbb{R}^n$ and

$$\mathscr{P} = \mathrm{co}(\mathscr{X}, -\mathscr{X}), \tag{2}$$

where co denotes the convex hull. Therefore

$$\mathscr{P} = \left\{ x = \sum_{i=1}^{m} \lambda_i x_i + \mu_i(-x_i) \quad \text{with} \quad \lambda_i, \mu_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m} (\lambda_i + \mu_i) \leq 1 \right\}.$$

Moreover, if $\text{co}(\mathscr{X}', -\mathscr{X}') \subsetneq \text{co}(\mathscr{X}, -\mathscr{X}) \ \forall \ \mathscr{X}' \subsetneq \mathscr{X}$, then the set $\mathscr{X}$ is called an *essential system of vertices* for $\mathscr{P}$ and any vector x_i is called a *vertex* of $\mathscr{P}$. Clearly, the set $\mathscr{P}$ is the unit ball of a norm $\|\cdot\|_{\mathscr{P}}$ on $\mathbb{R}^n$, which we call a *real polytope norm* and is characterized as follows.

Lemma 2.1 *Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a set of vectors spanning $\mathbb{R}^n$ and $\mathscr{P} = \text{co}(\mathscr{X}, -\mathscr{X})$. Set $\|\cdot\|_{\mathscr{P}}$ the corresponding real polytope norm. Then, $\forall z \in \mathbb{R}^n$, we have*

$$\|z\|_{\mathscr{P}} = \min_{\lambda_i \geq 0, \mu_i \geq 0} \left\{ \sum_{i=1}^{m} (\lambda_i + \mu_i) \;\middle|\; z = \sum_{i=1}^{m} \lambda_i x_i + \mu_i (-x_i) \right\}. \tag{3}$$

Note that (3) is a linear programming problem, which can be solved efficiently (see e.g. [30]).

After choosing $Q \in \Sigma_k(\mathscr{F})$ such that $\alpha = \rho(Q)^{1/k} > 0$, it is convenient to consider a scaling of the original family $\mathscr{F} = \{A^{(i)}\}_{i \in \mathscr{I}}$ by the scalar α so as to obtain

$$\mathscr{F}^* = \left\{ \alpha^{-1} A^{(i)} \right\}_{i \in \mathscr{I}}.$$

In such a way we automatically have $\rho(\mathscr{F}^*) \geq 1$, an assumption which will be useful in the forthcoming Theorem 2.1. Let us consider a (scaled) family $\mathscr{F}^*$ with $\rho(\mathscr{F}^*) \geq 1$. Then, for any vector $x \in \mathbb{R}^n$, we define the *tajectory*

$$\mathscr{T}[\mathscr{F}^*, x] = \{Px \ \mid P \in \Sigma(\mathscr{F}^*)\}, \tag{4}$$

i.e., the set obtained by applying all the products $P \in \Sigma(\mathscr{F}^*)$ to the vector x. The following theorem, which is a slight variant of a result proved by Protasov [25], illustrates the possible use of the trajectory to construct an extremal norm. For its proof see also [16], where the more general case of complex matrices is considered.

Theorem 2.1 *Let $\mathscr{F}^*$ be a family of real $n \times n$-matrices such that $\rho(\mathscr{F}^*) \geq 1$ and, for a given $x \in \mathbb{R}^n$, let the trajectory $\mathscr{T}[\mathscr{F}^*, x]$ be a bounded subset of $\mathbb{R}^n$ such that* $\text{span}\Big(\mathscr{T}[\mathscr{F}^*, x]\Big) = \mathbb{R}^n$. *Then $\mathscr{F}^*$ is non-defective and $\rho(\mathscr{F}^*) = 1$. Furthermore,*

$$\mathscr{S}[\mathscr{F}^*, x] = \overline{\text{co}\left(\mathscr{T}[\mathscr{F}^*, x], -\mathscr{T}[\mathscr{F}^*, x]\right)} \tag{5}$$

is the unit ball of an extremal norm $\|\cdot\|$ for $\mathscr{F}^*$ (that is, $\|\mathscr{F}^*\| = 1$).

When $\rho(\mathscr{F}^*) = 1$, building the trajectory provides a tool for the construction of the unit ball of an extremal norm and, hence, for the computation of the spectral radius. Assume that the hypotheses of Theorem 2.1 hold. The possibility of actually determining an extremal polytope norm, if any, is based on the search for a *suitable initial vector* x to which it corresponds a trajectory such that the set $\mathscr{S}[\mathscr{F}^*, x]$ is a balanced real polytope. Such a choice is suggested by the recent result in [11] and is related to the knowledge (or the guess) of a spectrum maximizing product.

In [11] it has been proved that under some suitable conditions, which we do not discuss here, if a finite family $\mathscr{F}^*$, such that $\rho(\mathscr{F}^*) = 1$, of real $n \times n$-matrices has an s.m.p. P having a unique leading eigenvector x, then it admits an extremal polytope norm. More specifically the set $\partial \mathscr{S}[\mathscr{F}^*, x] \bigcap \mathscr{T}[\mathscr{F}^*, x]$ is finite (see (4), (5)). Hence there exists a finite number of products $\{P_k^*\}_{k=1}^s \in \Sigma(\mathscr{F}^*)$ such that

$$\mathscr{S}[\mathscr{F}^*, x] = \text{co}\Big(\mathscr{X}, -\mathscr{X}\Big), \qquad \text{with } \mathscr{X} = \{P_k^* x\}_{k=1}^s.$$

Although the existence of an s.m.p. does not imply the existence of an extremal polytope norm (see [20]), such implication is true in several cases (see [11]).

2.1 A procedure for finding a real extremal polytope norm

We assume that $\mathscr{F}$ is finite and irreducible (for the non-defective although reducible case we can proceed as in [16] and still make use of the method we propose). The following procedure is derived by a suitable development (restricted to the real case) of previous algorithms (see [16], [17] and [25]). For a suitable initial vector x, the idea is that of computing iteratively the trajectory $\mathscr{T}[\mathscr{F}^*, x]$. While iterating, check whether $\mathscr{F}^*$ maps the convex hull of the balanced trajectory $\mathscr{T}[\mathscr{F}^*, x]$ into itself. The idea of the following algorithm is that of applying recursively the scaled family $\mathscr{F}^*$ in order to construct the trajectory step-by-step starting from an initial vector.

Algorithm 2.1 *(for the construction of the unit ball of a real extremal polytope norm for $\mathscr{F} = \{A^{(1)}, \ldots, A^{(\ell)}\}$)*

1. Let $\mathscr{F} = \{A^{(i)}\}_{i \in \{1,2,\ldots,\ell\}}$ be a *finite* family; choose a candidate s.m.p. $P \in \Sigma_k(\mathscr{F})$. Let v_0 be the leading eigenvector of P.
2. Set $\vartheta = \rho(P)^{1/k}$ and define the scaled family

$$\mathscr{F}^* = \{\vartheta^{-1} A^{(i)}\}_{i \in \{1,2,\ldots,\ell\}} \quad \text{s.t.} \quad \rho(\mathscr{F}^*) \geq 1.$$

3. Compute recursively the set $\mathscr{T}[\mathscr{F}^*, v_0]$, that is define

$$\mathscr{T}^{(s+1)} = \mathscr{F}^* \mathscr{T}^{(s)}, \qquad s \geq 0 \qquad \text{with} \quad \mathscr{T}^{(0)} = \{v_0\}.$$

4. Let $\mathscr{P}^{(s)} = \text{absco}(\mathscr{T}^{(s)})$. Check at any step if $\mathscr{P}^{(s)}$ is an invariant set for $\mathscr{F}^*$.

If the procedure halts for some s, then, due to the irreducibility assumption, $\mathscr{P}^{(s-1)}$ determines the unit ball of an extremal real polytope norm for $\mathscr{F}^*$. We remark that the initial choice of the product P may be obtained, for example, by means of the algorithm of Gripenberg [10], which provides candidate s.m.p.'s of progressively higher length.

A stopping criterion

A useful criterion to stop the iteration and eventually discard the candidate s.m.p. P is given by the following theorem (again, see also [25]).

Theorem 2.2 *Let $\mathscr{F}$ be a finite irreducible family of matrices. If, at some step s of Algorithm 2.1, v_0 lies strictly inside $\mathscr{P}^{(s)}$, that is*

$$v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s)}, \tag{6}$$

then $\rho(\mathscr{F}^*) > 1$. Viceversa, if $\rho(\mathscr{F}^*) > 1$, then there exists s such that (6) holds.

Proof. Assume that, at some step s, $v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s)}$. This would mean that there exists $x_s \in \partial \mathscr{P}^{(s)}$ such that $x_s = \beta_s v_0$ with $\beta_s > 1$. Let $\mathscr{V}^{(s)} = \{v_i\}_{i=1}^m$ be such that $\{\mathscr{V}^{(s)}, -\mathscr{V}^{(s)}\}$ is an essential system of vertices of $\mathscr{P}^{(s)}$. Thus we can write

$$x_s = \sum_{i=1}^m \lambda_i v_i + \mu_i(-v_i) \qquad \text{with} \quad \sum_{i=1}^m (\lambda_i + \mu_i) = 1, \quad \lambda_i \geq 0,\ \mu_i \geq 0.$$

Since, by construction, for all i there exists a finite product $P^{(i)} \in \Sigma(\mathscr{F}^*)$ such that $v_i = P^{(i)} v_0$, there must exist at least a product $P \in \Sigma(\mathscr{F}^*)$ such that $\|P v_0\|_{\mathscr{P}^{(s)}} = 1$. Using the fact that $1 = \|x_s\|_{\mathscr{P}^{(s)}} = \beta_s \|v_0\|_{\mathscr{P}^{(s)}}$, we have

$$\|P v_0\|_{\mathscr{P}^{(s)}} = \beta_s \|v_0\|_{\mathscr{P}^{(s)}} > \|v_0\|_{\mathscr{P}^{(s)}} \implies \|P\|_{\mathscr{P}^{(s)}} \geq \beta_s > 1.$$

Thus $\|\mathscr{F}^*\|_{\mathscr{P}^{(s)}} > 1$. Since $\mathscr{P}^{(s)} \subseteq \mathscr{P}^{(s+1)}$, we would still have $v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s+1)}$ and the previous condition would occur for all subsequent values of s, with $\beta_{s+1} \geq \beta_s$. If $\rho(\mathscr{F}^*) = 1$, by the irreducibility assumption, $\mathscr{P}^{(s)}$ would converge to some centrally symmetric convex set as $s \to \infty$. As a consequence there would exist $\hat{s}$ such that $\|P\|_{\mathscr{P}^{(r)}} < \beta_s$ for all $r > \hat{s}$, which is not possible. Consequently $\rho(\mathscr{F}^*) > 1$. Viceversa, by the irreducibility assumption, if $\rho(\mathscr{F}^*) > 1$ then

$$\lim_{s \to \infty} \mathscr{P}^{(s)} = \mathbb{R}^n.$$

This implies that there exists s such that $v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s)}$. ■

3 Finiteness property of pairs of matrices in $M_2(\mathbb{S})$

Now we pass to consider the finiteness property of pairs of sign-matrices. We denote by $M_n(\mathbb{S})$ the set of $n \times n$ matrices with entries in $\mathbb{S} = \{-1, 0, +1\}$. We recall the following conjecture by Blondel and Jungers and Protasov (see [4] and [19]).

Conjecture 3.1 *Every pair of $n \times n$ sign-matrices has the finiteness property.*

We consider here the case of a family $\mathscr{F} = \{A, B\}$ where $A, B \in M_2(\mathbb{S})$. The number of ordered pairs $N_o = (3^4 - 3)(3^4 - 5) = 5928$ (obtained discarding the zero matrix, the identity and its opposite from the set and the cases where the second matrix is equal to the first one or its opposite) is very large, but the number of cases to examine is immediately reduced to $N_e = N_o/8$, since the joint spectral radius of the sets $\{\pm A, \pm B\}$ does not change as well as it does not depend of the ordering of the two matrices. Hence $N_e = 741$, which is still a quite large number of cases. By using suitable properties, we shall see that the actual number of essential cases to examine is much lower. Mainly, the properties we shall use are based on suitable similarity transformations, which do not change the joint spectral radius.

As in [19], in order to analyze the essential cases, we separate them into classes (n_0, n_1), where n_0 is the number of non-zero entries of A and n_1 is the number of non-zero entries of B. By symmetry, we can assume $n_0 \geq n_1$. Our approach consists in showing the finiteness property of every considered case by determining explicitly the associated s.m.p., in most cases through the construction of the unit ball of a suitable real extremal polytope norm. This does not allow a unified proof but, instead, requires to treat most of the essential cases separately.

Although all the pairs of binary matrices have already been considered in [19], here we reconsider the most difficult cases because our procedure is quite different from that used in [19] and does not rely on the possible non-negativity of the matrices.

The set of *representative* matrices (we exclude $-A$ if we consider A) with a single non-zero entry which has to be considered is given by $\mathbf{C} = \{C_i\}_{i=1}^4$ with

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The set of representative matrices with two non-zero entries which has to be considered is given by $\mathbf{D} = \{D_i\}_{i=1}^{11}$ with

$$D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},$$

$$D_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_6 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_7 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad D_8 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

$$D_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The set of representative matrices with three non-zero entries which has to be considered is given by $\mathbf{E} = \{E_i\}_{i=1}^{16}$ with

$$E_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad E_8 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad E_{11} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$E_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_{14} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad E_{15} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad E_{16} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

The set of representative matrices with four non-zero entries which has to be considered is given by $\mathbf{F} = \{F_i\}_{i=1}^{8}$ with

$$F_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad F_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$F_5 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad F_6 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad F_7 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad F_8 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now consider the similarity transformations associated with the following matrices:

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which are such that $P_1^2 = I$, $P_2^2 = I$, $P_3^2 = -I$. Clearly, for $k = 1,2,3$ we have that

$$P_k C_i P_k^{-1} \in \pm \mathbf{C}, \quad P_k D_i P_k^{-1} \in \pm \mathbf{D}, \quad P_k E_i P_k^{-1} \in \pm \mathbf{E}, \quad P_k F_i P_k^{-1} \in \pm \mathbf{F}, \tag{7}$$

so that these similarities do not change the finiteness property, nor the fact that the matrices are sign-matrices. In detail, denoting by $\sim$ a similarity relation, we get

$$D_1 \sim D_2 \sim D_3 \sim D_4, \quad D_5 \sim D_6 \sim D_7 \sim D_8, \tag{8}$$

$$\begin{aligned} &E_1 \sim E_4 \sim E_{13} \sim E_{16}, \quad E_2 \sim E_3 \sim E_{14} \sim E_{15}, \\ &E_5 \sim E_6 \sim E_9 \sim E_{10}, \quad E_7 \sim E_8 \sim E_{11} \sim E_{12}, \end{aligned} \tag{9}$$

$$F_1 \sim F_4, \quad F_2 \sim F_3, \quad F_5 \sim F_6, \quad F_7 \sim F_8. \tag{10}$$

As we have mentioned, in the sequel we shall denote by

$$\mathscr{F}^* = (1/\rho(P))^{1/k}\, \mathscr{F} \qquad \text{for some } P \in \Sigma_k(\mathscr{F}) \quad \text{s.t. } \rho(P) \neq 0$$

and call it the scaled family. Our aim will be that to prove that $\mathscr{F}$ has joint spectral radius equal to 1 (which implies that P is an s.m.p.). In several cases we shall observe that one of the following standard norms, $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, is extremal. Some other cases are easily treated by observing that the real polytope norm $\|\cdot\|_*^+$, associated with the b.r.p. $\mathscr{P}^+ = \operatorname{co}(V,-V)$ with $V = \{v_0, v_1, v_2\}$, where

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

or $\|\cdot\|_*^-$, associated with the b.r.p. $\mathscr{P}^- = \operatorname{co}(V,-V)$ with $V = \{v_0, v_1, v_2\}$, where

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

is extremal. The following lemma is also useful to treat some cases.

Lemma 3.1 *Let $|\mathscr{F}|$ be the family of matrices obtained from $\mathscr{F}$ as follows:*

$$A = \{a_{ij}\} \in \mathscr{F} \quad \longrightarrow \quad |A| = \{|a_{ij}|\} \in |\mathscr{F}|.$$

If $P \in \Sigma_k(\mathscr{F})$ is such that $\rho(P)^{1/k} = \rho(|\mathscr{F}|)$ then P is an s.m.p. for $\mathscr{F}$.

All the other cases are treated by using Algorithm 2.1. Before summarizing the results through different tables, we give an extensive proof of an illustrative case.

3.1 Illustrative case

Consider the case $A = E_2$, $B = D_{11}$.

We want to prove that $P = ABA^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(\frac{3+\sqrt{5}}{2}\right)^{1/5}$ and a real extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = B^* v_5$. To this aim, set $\gamma = \frac{1}{\rho(P)^{1/5}} \approx 0.825$. Then we get

$$v_0 = \begin{pmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{pmatrix},\; v_1 = \gamma \begin{pmatrix} \frac{2}{1+\sqrt{5}} \\ -1 \end{pmatrix},\; v_2 = \gamma^2 \begin{pmatrix} \frac{3+\sqrt{5}}{1+\sqrt{5}} \\ \frac{2}{1+\sqrt{5}} \end{pmatrix},$$
$$v_3 = \gamma^3 \begin{pmatrix} 1 \\ \frac{3+\sqrt{5}}{1+\sqrt{5}} \end{pmatrix},\; v_4 = \gamma^3 \begin{pmatrix} \frac{2}{1+\sqrt{5}} \\ -\frac{3+\sqrt{5}}{1+\sqrt{5}} \end{pmatrix},\; v_5 = \gamma^4 \begin{pmatrix} \frac{3+\sqrt{5}}{1+\sqrt{5}} \\ -1 \end{pmatrix}.$$

As illustrated in Figure 1, we analyze the transformed vectors $\mathscr{F}^*(V)$. Some of them are vertices themselves by construction of $\mathscr{P}$ and, hence, do not need to be analyzed. Here we report such vectors together with the minimizing convex combinations of vertices of $\mathscr{P}$ which determine their norms (see (3)):

$$A^* v_0 = \gamma \begin{pmatrix} \frac{\sqrt{5}-1}{1+\sqrt{5}} \\ 1 \end{pmatrix} = \lambda v_3 + \mu(-v_4), \quad \lambda = \frac{2(3+\sqrt{5})}{\gamma^2(11+5\sqrt{5})},\; \mu = \frac{2}{\gamma^2(7+3\sqrt{5})},$$
$$\|A^* v_0\|_{\mathscr{P}} = \lambda + \mu \approx 0.90;$$
$$A^* v_1 = v_2;$$
$$A^* v_2 = v_3;$$
$$A^* v_3 = \lambda(-v_1), \qquad \lambda = \gamma^3, \quad \|A^* v_3\|_{\mathscr{P}} = \lambda \approx 0.56;$$
$$A^* v_4 = \gamma^4 \begin{pmatrix} \frac{5+\sqrt{5}}{1+\sqrt{5}} \\ \frac{2}{1+\sqrt{5}} \end{pmatrix} = \lambda v_2 + \mu v_5, \quad \lambda = \frac{4(2+\sqrt{5})}{7+3\sqrt{5}}\gamma^2,\; \mu = \frac{2}{7+3\sqrt{5}},$$
$$\|A^* v_4\|_{\mathscr{P}} = \lambda + \mu \approx 0.98;$$
$$A^* v_5 = v_0;$$

$$B^* v_0 = v_1;$$
$$B^* v_1 = \lambda(-v_0), \qquad \lambda = \gamma^2, \quad \|B^* v_0\|_{\mathscr{P}} = \lambda \approx 0.68;$$
$$B^* v_2 = v_4;$$
$$B^* v_3 = v_5;$$
$$B^* v_4 = \lambda(-v_2), \qquad \lambda = \gamma^2, \quad \|B^* v_4\|_{\mathscr{P}} = \lambda \approx 0.68;$$
$$B^* v_5 = \lambda v_3, \qquad \lambda = \gamma^2, \quad \|B^* v_5\|_{\mathscr{P}} = \lambda \approx 0.68.$$

This proves the extremality of $\| \cdot \|_{\mathscr{P}}$ and that $P = ABA^2B$ is an s.m.p..

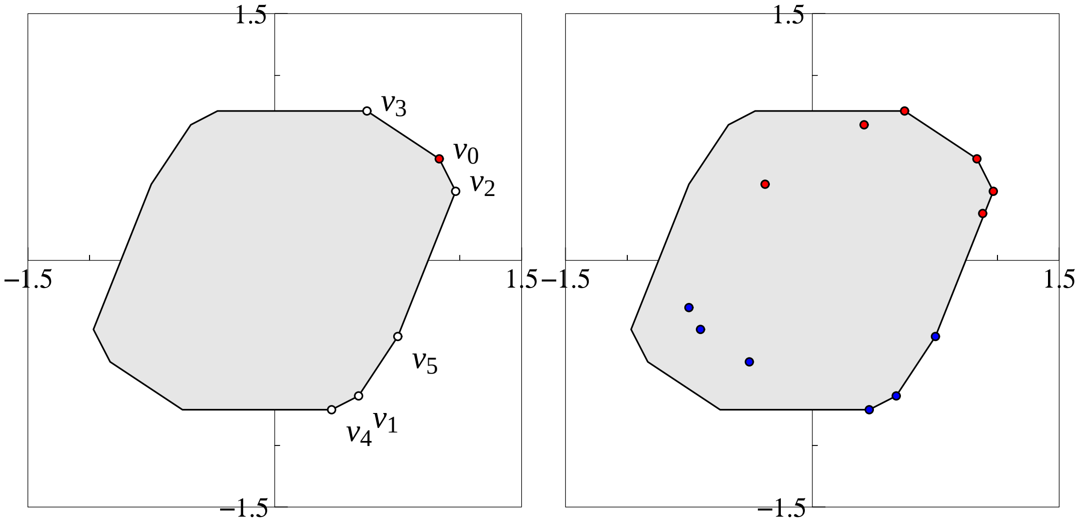

Fig. 1. Polytope norm for the pair $\{A = E_2, B = D_{11}\}$ (left) and the set $\mathscr{F}^*(V)$ (right). Red points indicate the vectors $\{A^* v_i\}_{i=0}^5$ and blue points indicate the vectors $\{B^* v_i\}_{i=0}^5$.

3.2 Summary of results

We show in the subsequent tables the s.m.p. (s.m.p.'s) for all significant cases, that is for those matrix pairs whose analysis cannot be reduced to that of another matrix pair appearing in the tables. In the column we indicate the matrix A in the pair $\mathscr{F} = \{A, B\}$ while in the row we indicate the matrix B. For a detailed analysis of specific cases we refer the reader to the appendix in Section 4.

The case $n_0 = 1$ (families of the type $\mathscr{F} = \{C_i, C_j\}$).

Recall that we suppose $n_0 \geq n_1$. The only possibility is $(n_0, n_1) = (1, 1)$, corresponding to families of the type $\mathscr{F} = \{C_i, C_j\}$ $(i < j)$. The analysis is always trivial. In fact, it is very easy to see that $\rho(\mathscr{F}) = 1$ and any among $\| \cdot \|_1$, $\| \cdot \|_2$ and $\| \cdot \|_\infty$ is an extremal norm. Moreover, if $i = 1$ or $j = 4$ an s.m.p. is $P = C_1$ or C_4, respectively. Only if $(i, j) = (2, 3)$ an s.m.p. is $P = C_2C_3$.

$A \backslash B$	C_2	C_3	C_4
C_1	A	A	A
C_2		AB	B
C_3			B

The case $n_0 = 2$.

The subcase $(n_0, n_1) = (2, 1)$ (families of the type $\mathscr{F} = \{D_i, C_j\}$).

Since $\|C_j\|_1 = \|C_j\|_\infty = 1$, $\rho(D_i) = 1$ and either $\|D_i\|_1 = 1$ or $\|D_i\|_\infty = 1$, we have that $\rho(\mathscr{F}) = 1$ and that an s.m.p. is $P = D_i$.

The subcase $(n_0, n_1) = (2, 2)$ (families of the type $\mathscr{F} = \{D_i, D_j\}$).

In view of (7) and (8), we can restrict the choice of the first matrix A to the set $\mathbf{D}' = \{D_1, D_5, D_9, D_{10}, D_{11}\}$ and let the choice of B be free in $\mathbf{D}$.

In the sequel we mark by an asterisk (*) or two asterisks (**) equivalent columns.

$A \backslash B$	D_2	D_3^*	D_4^*	D_5	D_6	D_7^*	D_8^*	D_9	D_{10}	D_{11}
D_1	A,B	A,B	A,B	AB	B	A,B	A,B	A,B	A,B	A,B
D_5	B	A,B	A,B		A,B	A,B	A,B	A,B	A,B	A,B
D_9	A,B	A,B	A,B		A,B	A,B	A,B		A,B	A,B
D_{10}	A,B	A,B	A,B		A,B	A,B	A,B			A,B
D_{11}	A,B	A,B	A,B		A,B	A,B	A,B			

The case $n_0 = 3$.

In view of (7) and (9), we can restrict the choice of the first matrix A to the set $\mathbf{E}' = \{E_1, E_2, E_5, E_7\}$ and let the choice of B to be free.

The subcase $(n_0, n_1) = (3, 1)$ (families of the type $\mathscr{F} = \{E_i, C_j\}$).

$A \backslash B$	C_1^*	C_2	C_3	C_4^*
E_1	A	A	A	A
E_2	A,B	A	A	A,B
E_5	A,B	A	A^4B	A,B
E_7	A,B	A	A	A,B

The subcase $(n_0,n_1)=(3,2)$ *(families of the type* $\mathscr{F}=\{E_i,D_j\}$*).*

$A\backslash B$	D_1	D_2	D_3^*	D_4	D_5	D_6^*	D_7	D_8	D_9	D_{10}	D_{11}
E_1	A	A	A	A	A	A	A	A	A	A	A
E_2	AB	A^2B	A,B	A,B	A,B	A,B	AB	A^2B	AB	A,B	ABA^2B
E_5	A,B	A^2B	A,B	A^5B	A^2B	A,B	A^5B	A,B	A,B	A^3B	A^4B
E_7	A,B	A,B	A,B	AB	AB	A,B	A,B	A,B	A,B	A,B	AB

The subcase $(n_0,n_1)=(3,3)$ *(families of the type* $\mathscr{F}=\{E_i,E_j\}$*).*

$A\backslash B$	E_2	E_3	E_4^*	E_5	E_6	E_7	E_8	E_9
E_1	A	A	A,B	A	A	A	A	A
E_2		AB	B	AB^3	A^2B^3	AB	A,B	A^2B^3
E_5		A^3B^2	B		A,B	A,B	A,B	AB
E_7		A,B	B		A,B		A,B	AB^3

$A\backslash B$	E_{10}	E_{11}	E_{12}	E_{13}^*	E_{14}	E_{15}	E_{16}^*
E_1	A	A	A	A,B	A	A	A,B
E_2	AB^3	A,B	AB	B	$ABA(AB)^2B$	A,B	B
E_5	A^4B^4	A^3B	A^3B	B	A^3B	A^3B^2	B
E_7	AB^3	AB	A,B	B	A,B	AB	B

We remark that in this case we find the longest spectrum maximizing products, of length $\ell=8$, namely for $\mathscr{F}=\{E_5,E_{10}\}$, where $P=E_5^4E_{10}^4$ and for $\mathscr{F}=\{E_2,E_{14}\}$, where $P=E_2E_{14}E_2(E_2E_{14})^2E_{14}$.

The case $n_0=4$.

In view of (7) and (10), we can restrict the choice of the first matrix A to the set $\mathbf{F}'=\{F_1,F_3,F_5,F_8\}$ and let the choice of B be free.

The subcase $(n_0,n_1)=(4,1)$ *(families of the type* $\mathscr{F}=\{F_i,C_j\}$*).*

$A\backslash B$	C_1^*	C_2^{**}	C_3^{**}	C_4^*
F_1	A	A	A	A
F_3	A	A	A	A
F_5	B	AB	AB	B
F_8	A	A	A	A

The subcase $(n_0,n_1)=(4,2)$ *(families of the type* $\mathscr{F}=\{F_i,D_j\}$*).*

$A\backslash B$	D_1^*	D_2^*	D_3^{**}	D_4^{**}	D_5^{**}	D_6^{**}	D_7^*	D_8^*	D_9^{**}	D_{10}^*	D_{11}^{**}
F_1	A	A	A	A	A	A	A	A	A	A	A
F_3	A	A	A	A	A	A	A	A	A	A	A
F_5	B	B	AB	AB	AB	AB	B	B	AB	B	AB
F_8	A	A	A	A	A	A	A	A	A	A	A

The subcase $(n_0,n_1)=(4,3)$ *(families of the type* $\mathscr{F}=\{F_i,E_j\}$*).*

It is useful observing that $P_3F_1P_3^{-1}=-F_1$, $P_3F_3P_3^{-1}=F_3$, $P_1F_5P_1^{-1}=-F_5$ and that both the similarity transformations associated with P_1 and P_3 are one-to-one applications between the sets of matrices $\mathbf{E}''=\{E_j \mid 1\le j\le 8\}$ and $\mathbf{E}'''=\{E_j \mid 9\le j\le 16\}$. Consequently, when $A=F_i$ $(i=1,3,5)$, we can restrict the choice of the matrix B within the set $\mathbf{E}''$.

$A\backslash B$	E_1^*	E_2	E_3	E_4^*	E_5	E_6	E_7	E_8	E_9-E_{16}
F_1	B	A	A	B	A	A	A	AB	
F_3	B	$(AB)^2A^2B$	A^2BA^3B	B	A^3B^2	AB^2	A^2B	A^2B	
F_5	B	AB	B	B	AB^4	AB^4	B	AB	
F_8	A	A	A	A	A	A	A	A	A

The subcase $(n_0,n_1)=(4,4)$ *(families of the type* $\mathscr{F}=\{F_i,F_j\}$*).*

If $A=F_8$, $B\in\mathbf{F}$ then A is an s.m.p..

Now it is useful to observe that $P_3F_1P_3^{-1} = -F_1$, $P_3F_3P_3^{-1} = F_3$, $P_3F_5P_3^{-1} = -F_6$ and $P_3F_8P_3^{-1} = F_7$. Consequently, when $A = F_i$ ($i = 1,3$), we can restrict the choice of the matrix B within the set $\mathbf{F}'' = \{F_2, F_3, F_4, F_5\}$.

$A\backslash B$	F_2	F_3	F_4	F_5	F_6
F_1	A,B	A,B	A,B	AB	
F_3	A,B		A,B	A^2B	
F_5	AB^2		AB		AB

4 Appendix: detailed analysis of specific cases.

In this section we provide a case-by-case analysis of the matrix pairs tabulated in Section 3.2. In particular we provide explicitly the computed extremal polytope norm in those cases where they have been used to determine an s.m.p..

The case $n_0 = 2$

The subcase $(n_0, n_1) = (2,2)$ (families of the type $\mathscr{F} = \{D_i, D_j\}$).

- $A = D_1$ and $B = D_j$ ($j = 2,3,4,9,10,11$).
  Since $\rho(A) = \rho(B) = \|A\|_1 = \|B\|_1 = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.
- $A = D_1$ and $B = D_5$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$.
- $A = D_1$ and $B = D_6$.
  We find that $P = B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = 1$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$.
- $A = D_1$ and $B = D_j$ ($j = 7,8$).
  Since $A^2 = A$, $B^2 = B$, $\rho(AB) = \rho(BA) = 0$ and $\rho(A) = \rho(B) = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.
- $A = D_5$ and $B \in \mathbf{D}$.
  Since $D_5 = D_1^{\mathrm{T}}$ and $\mathbf{D}^T \subseteq \pm\mathbf{D}$ and since, if P is an s.m.p. of the family $\mathscr{F} = \{A, B\}$, then P^{T} is an s.m.p. of the family $\mathscr{F}^T = \{A^{\mathrm{T}}, B^{\mathrm{T}}\}$, we are led again to the previous cases.
- $A = D_j$ and $B = D_k$ ($j = 9,10,11,\ k = 2,3,4,6,7,8$).
  Since $P_1D_9P_1^{-1} = -D_9$, $P_2D_9P_2^{-1} = D_9$, $P_3D_9P_3^{-1} = -D_9$, $P_1D_{10}P_1^{-1} = D_{10}$, $P_2D_{10}P_2^{-1} = -D_{10}$, $P_3D_{10}P_3^{-1} = -D_{10}$, $P_1D_{11}P_1^{-1} = -D_{11}$, $P_2D_{11}P_2^{-1} = -D_{11}$,

$P_3D_{11}P_3^{-1} = D_{11}$ and since $P_1D_2P_1^{-1} = D_1$, $P_2D_3P_2^{-1} = D_1$, $P_3D_4P_3^{-1} = -D_1$, $P_1D_6P_1^{-1} = D_5$, $P_2D_7P_2^{-1} = D_5$, $P_3D_8P_3^{-1} = -D_5$, by using the similarity transformations associated with P_1, P_2 and P_3 we are led to the previous cases.

- $A = D_j$ and $B = D_k$ ($j,k = 9,10,11$).
  Since $\rho(A) = \rho(B) = \|A\|_\infty = \|B\|_\infty = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.

The case $n_0 = 3$

The subcase $(n_0,n_1) = (3,1)$ (families of the type $\mathscr{F} = \{E_i, C_j\}$).

- $A = E_1$, $B \in \mathbf{C}$.
  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$.
- $A = E_5$, $B = C_j$ ($j = 1,2,4$).
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and A is an s.m.p..
- $A = E_5$, $B = C_3$.
  We find that $P = A^4B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = 4^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = A^*v_2$, $v_4 = A^*v_3$, $v_5 = A^*v_4$.
- $A = E_2$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A is an s.m.p..
- $A = E_7$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A is an s.m.p..

The subcase $(n_0,n_1) = (3,2)$ (families of the type $\mathscr{F} = \{E_i, D_j\}$).

- $A = E_1$, $B \in \mathbf{D}$.
  Since $\rho(A) = \|A\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|B\|_2 \leq \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that A is an s.m.p..
- $A = E_2$, $B = D_1$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = A^*v_0$.
- $A = E_2$, $B = D_j$ ($j = 2,8$).
  We find that $P = A^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 2^{1/3}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = A^*v_1$.
- $A = E_2$, $B = D_j$ ($j = 3,4,5,6,10$).
  Since $\rho(A) = \rho(B) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.

- $A = E_2$, $B = D_7$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$.
- $A = E_2$, $B = D_9$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_0$.
- $A = E_2$, $B = D_{11}$.
  See the illutrative example in Section 3.1.
- $A = E_5$, $B = D_j$ ($j = 1, 3, 6, 8, 9$).
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p.s.
- $A = E_5$, $B = D_j$ ($j = 2, 5$).
  We find that $P = A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 3^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = A^* v_2$.
- $A = E_5$, $B = D_j$ ($j = 4, 7$).
  We find that $P = A^5 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/6} = 2^{1/3}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = A^* v_5$.
- $A = E_5$, $B = D_{10}$.
  We find that $P = A^3 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = B^* v_1$, $v_5 = A^* v_3$.
- $A = E_5$, $B = D_{11}$.
  See the illutrative example in Section 3.2. We find that $P = A^4 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = (2 + \sqrt{3})^{1/5}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = B^* v_5$.
- $A = E_7$, $B = D_j$ ($j = 1, 3, 6, 8, 9$).
  The family $\mathscr{F}$ is upper triangular with $\rho(\mathscr{F}) = 1$ and A and B are both s.m.p.s.
- $A = E_7$, $B = D_j$ ($j = 2, 7, 10$).
  Since $\rho(A) = \rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.
- $A = E_7$, $B = D_j$ ($j = 4, 5$).
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = E_7$, $B = D_{11}$.

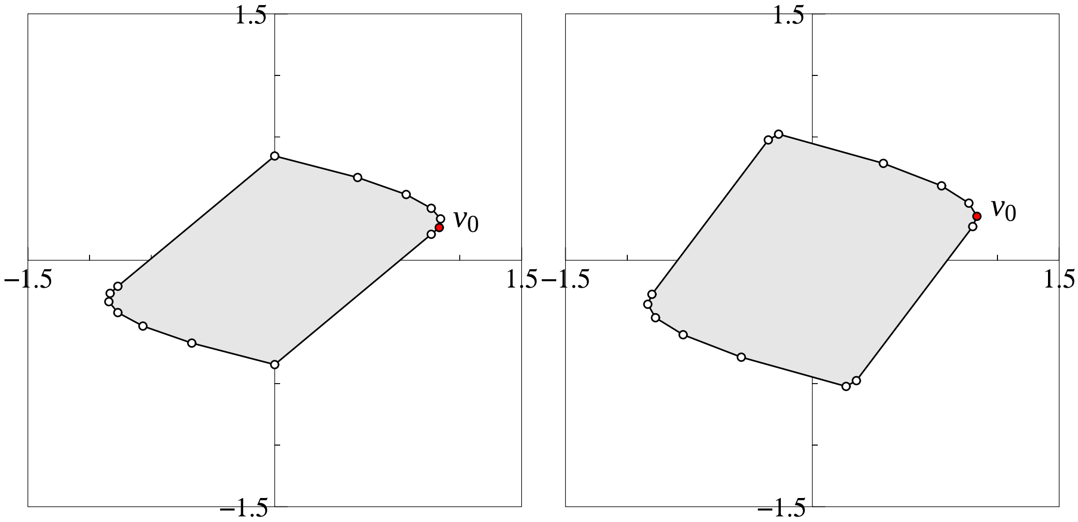

Fig. 2. Polytope norm for the pairs $\{A = E_5, B = D_4\}$ (left) and $\{A = E_5, B = D_{11}\}$ (right).

We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.

The subcase $(n_0, n_1) = (3, 3)$ (families of the type $\mathscr{F} = \{E_i, E_j\}$).

- $A = E_1$, $B \in \mathbf{E}$.
  Since $\rho(A) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and A is an s.m.p..
- $A = E_2$, $B = E_3$.
  Using Lemma 3.1 we find that $P = AB$ is an s.m.p. and $\rho(\mathscr{F}) = \rho(P)^{1/2} = \frac{1+\sqrt{5}}{2}$.
- $A = E_2$, $B = E_j$ ($j = 4, 13, 16$). Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = E_2$, $B = E_j$ ($j = 5, 10$).
  We find that $P = AB^3$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = (2+\sqrt{3})^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = B^* v_2$, $v_5 = A^* v_4$, $v_6 = B^* v_4$, $v_7 = B^* v_6$.
- $A = E_2$, $B = E_j$ ($j = 6, 9$).
  We find that $P = A^2 B^3$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = (2+\sqrt{3})^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = A^* v_3$, $v_5 = B^* v_3$, $v_6 = A^* v_5$.
- $A = E_2$, $B = E_j$ ($j = 7, 12$).
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = (1+\sqrt{2})^{1/2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$.
- $A = E_2$, $B = E_j$ ($j = 8, 11, 15$).

Since $\rho(A) = \rho(B) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.

- $A = E_2$, $B = E_{14}$.
  We find that $P = ABA^2BAB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/8} = (7+4\sqrt{3})^{1/8}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = B^*v_1$, $v_3 = A^*v_2$, $v_4 = B^*v_3$, $v_5 = A^*v_4$, $v_6 = A^*v_5$, $v_7 = B^*v_6$.

  Observe that this is the first of the two cases with the largest number of factors in the s.m.p.. The essential vertices of $\mathscr{P}$ are just the leading eigenvectors of $\mathscr{F}$, that is, the eigenvectors of all the cyclic permutations of P.

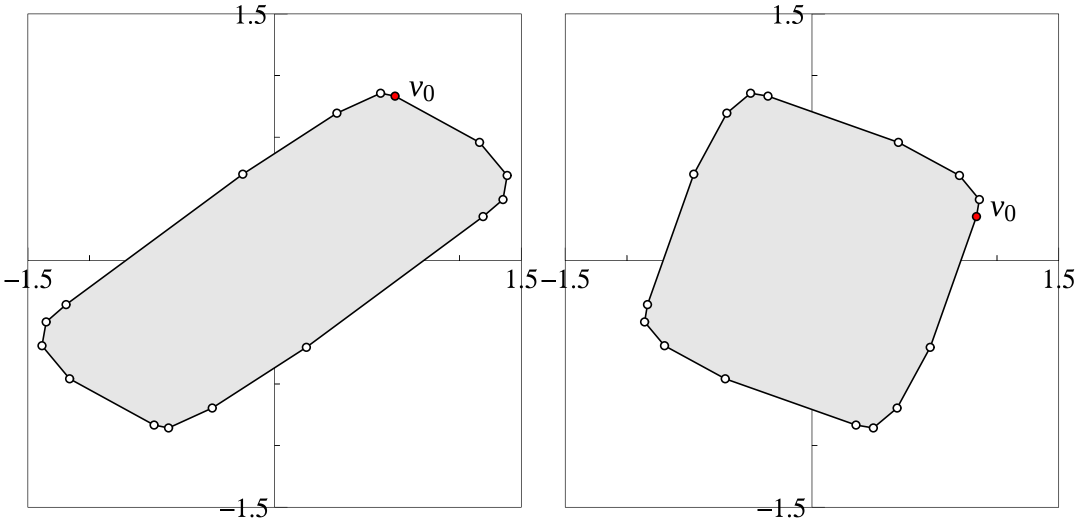

Fig. 3. Polytope norm for the pairs $\{A = E_2, B = E_5\}$ (left) and $\{A = E_2, B = E_{10}\}$ (right).

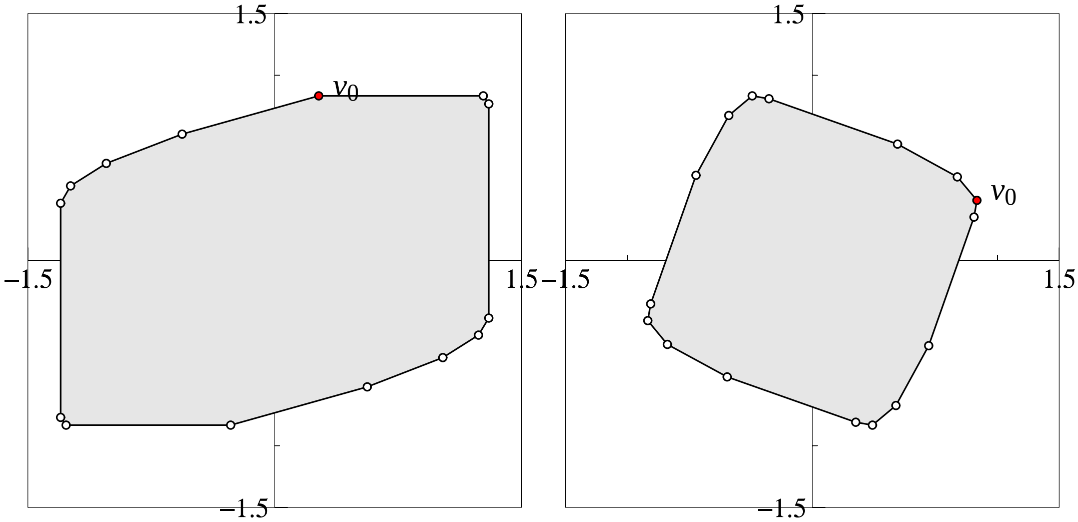

Fig. 4. Polytope norm for the pairs $\{A = E_2, B = E_6\}$ (left) and $\{A = E_2, B = E_{14}\}$ (right).

- $A = E_5$, $B = E_j$ ($j = 3, 15$).
  We find that $P = A^3B^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(2+\sqrt{3}\right)^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$,

where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = B^* v_1$, $v_4 = B^* v_2$, $v_5 = A^* v_4$, $v_6 = A^* v_5$.

- $A = E_5$, $B = E_j$ $(j = 4, 13, 16)$.
  Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = E_5$, $B = E_j$ $(j = 6, 7, 8)$.
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p's.
- $A = E_5$, $B = E_9$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4\}$, where v_0 is the leading eigenvector of P and $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = B^* v_2$.
- $A = E_5$, $B = E_{10}$.
  We find that $P = A^4 B^4$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/8} = \left(7 + 4\sqrt{3}\right)^{1/8}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = B^* v_3$, $v_5 = A^* v_4$, $v_6 = A^* v_5$, $v_7 = A^* v_6$.

  This is the second of the two cases with the largest number of factors in the s.m.p.. Again, the essential vertices of $\mathscr{P}$ are just the leading eigenvectors of $\mathscr{F}$.

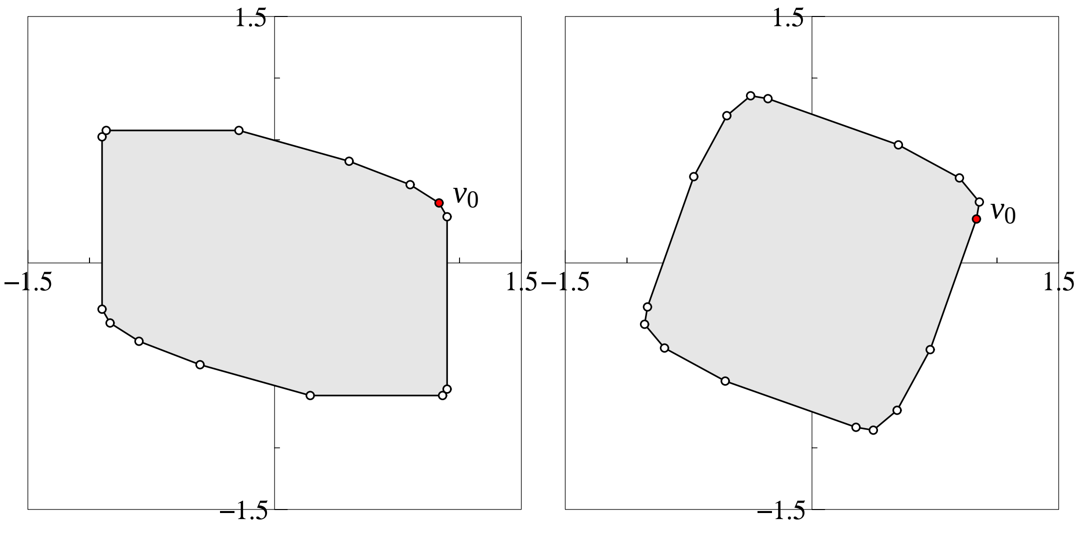

Fig. 5. Polytope norm for the pairs $\{A = E_5, B = E_3\}$ (left) and $\{A = E_5, B = E_{10}\}$ (right).

- $A = E_5$, $B = E_j$ $(j = 11, 12)$.
  We find that $P = A^3 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = B^* v_1$, $v_4 = A^* v_2$, $v_5 = A^* v_4$.
- $A = E_5$, $B = E_{14}$.
  We find that $P = A^3 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(2 + \sqrt{3}\right)^{1/4}$ and an ex-

tremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = B^* v_2$, $v_5 = A^* v_3$, $v_6 = B^* v_3$, $v_7 = B^* v_5$.

- $A = E_7$, $B = E_j$ $(j = 3, 12, 14)$.
  Since $\rho(A) = \rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.

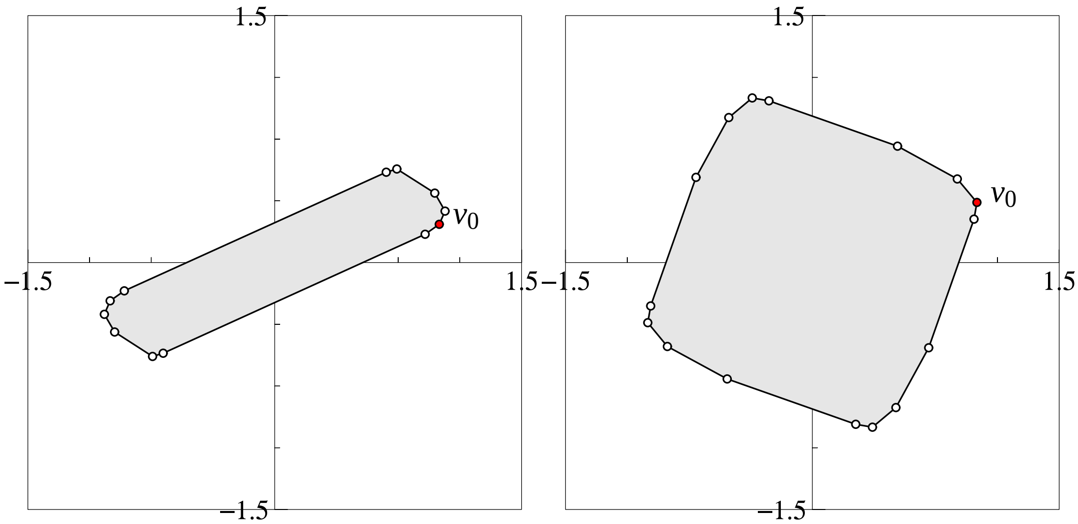

Fig. 6. Polytope norm for the pairs $\{A = E_5, B = E_{11}\}$ (left) and $\{A = E_5, B = E_{14}\}$ (right).

- $A = E_7$, $B = E_j$ $(j = 4, 13, 16)$.
  Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = E_7$, $B = E_j$ $(j = 6, 8)$.
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p's.
- $A = E_7$, $B = E_j$ $(j = 9, 10)$.
  We find that $P = AB^3$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = B^* v_3$, $v_5 = A^* v_4$.
- $A = E_7$, $B = E_{11}$.
  Using Lemma 3.1 we find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \frac{1+\sqrt{5}}{2}$.
- $A = E_7$, $B = E_{15}$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \left(1 + \sqrt{2}\right)^{1/2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$.

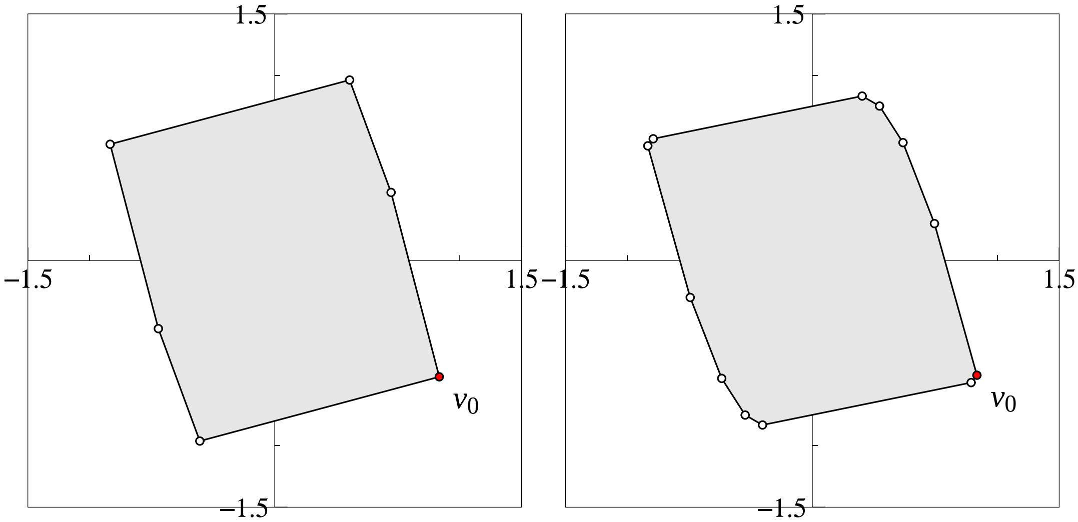

Fig. 7. Polytope norm for the pairs $\{A = E_7, B = E_{15}\}$ (left) and $\{A = E_7, B = E_9\}$ (right).

The case $n_0 = 4$

The subcase $(n_0, n_1) = (4,1)$ (families of the type $\mathscr{F} = \{F_i, C_j\}$).

- $A = F_i$ $(i = 1,3)$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_2 = \sqrt{2}$ and $\|B\|_2 = 1$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that A is an s.m.p..
- $A = F_5$, $B = C_j$ $(j = 1,4)$.
  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p..
- $A = F_5$, $B = C_j$ $(j = 2,3)$.
  Since $\rho(AB) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that $P = AB$ is an s.m.p..
- $A = F_8$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_1 = 2$ and $\|B\|_1 = 1$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..

The subcase $(n_0, n_1) = (4,2)$ (families of the type $\mathscr{F} = \{F_i, D_j\}$).

- $A = F_i$ $(i = 1,3)$, $B \in \mathbf{D}$.
  Since $\rho(A) = \|A\|_2 = \sqrt{2}$ and $\|B\|_2 \le \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that A is an s.m.p..
- $A = F_5$, $B = D_j$ $(j = 1,2,7,8,10)$.
  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p..
- $A = F_5$, $B = D_j$ $(j = 3,4,5,6,9,11)$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_8$, $B \in \mathbf{D}$.

Since $\rho(A) = \|A\|_2 = 2$ and $\|B\|_2 \le \sqrt{2}$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..

The subcase $(n_0, n_1) = (4,3)$ *(families of the type* $\mathscr{F} = \{F_i, E_j\}$*).*

- $A = F_1$, $B = E_j$ $(j = 1,4)$.
  Since $\rho(B) = \|B\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|A\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = F_1$, $B = E_j$ $(j = 2,3,7)$.
  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$.
- $A = F_1$, $B = E_j$ $(j = 5,6)$.
  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$, $v_3 = B^* v_1$, $v_4 = A^* v_3$, $v_5 = B^* v_3$, $v_6 = A^* v_5$.
- $A = F_1$, $B = E_8$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_3$, $B = E_j$ $(j = 1,4)$.
  Since $\rho(B) = \|B\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|A\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = F_3$, $B = E_2$.
  We find that $P = (AB)^2 A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/7} = \left(4\left(2+\sqrt{3}\right)\right)^{1/7}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = A^* v_2$, $v_5 = A^* v_4$, $v_6 = A^* v_5$, $v_7 = B^* v_5$, $v_8 = A^* v_7$, $v_9 = A^* v_8$, $v_{10} = B^* v_9$.
- $A = F_3$, $B = E_3$.
  We find that $P = A^2 B A^3 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/7} = \left(4\left(2+\sqrt{2}\right)\right)^{1/7}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = B^* v_5$, $v_7 = A^* v_6$.
- $A = F_3$, $B = E_5$.
  We find that $P = A^3 B^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(2\left(2+\sqrt{2}\right)\right)^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$, $v_3 = B^* v_1$, $v_4 = A^* v_2$, $v_5 = A^* v_3$, $v_6 = A^* v_4$, $v_7 = A^* v_5$.
- $A = F_3$, $B = E_6$.

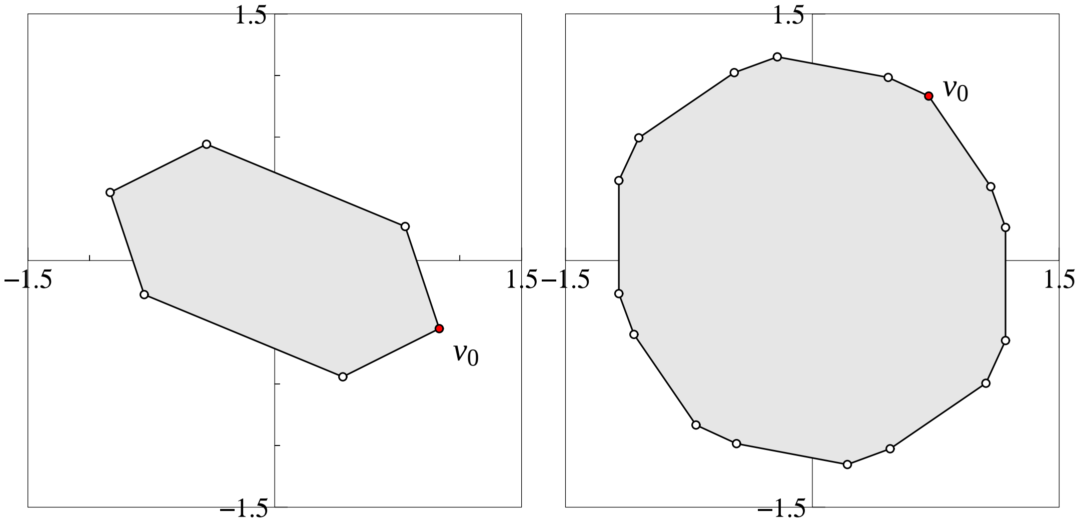

Fig. 8. Polytope norm for the pairs $\{A = F_1, B = E_3\}$ (left) and $\{A = F_3, B = E_3\}$ (right).

We find that $P = AB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = \left(2+\sqrt{2}\right)^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = A^* v_2$, $v_5 = B^* v_2$, $v_6 = A^* v_4$, $v_7 = A^* v_6$.

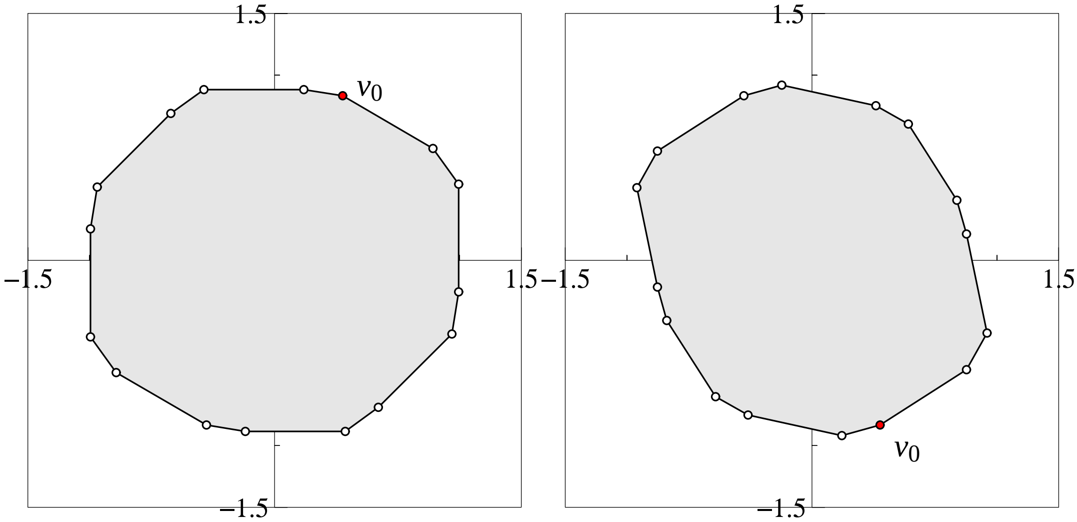

Fig. 9. Polytope norm for the pairs $\{A = F_3, B = E_5\}$ (left) and $\{A = F_3, B = E_6\}$ (right).

- $A = F_3$, $B = E_j$ ($j = 7, 8$).

  We find that $P = A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = \left(1+\sqrt{5}\right)^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$.
- $A = F_5$, $B = E_j$ ($j = 1, 4$).

  We find that $P = B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = A^* v_0$.

- $A = F_5$, $B = E_j$ $(j = 2,8)$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_5$, $B = E_j$ $(j = 3,7)$.
  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p..
- $A = F_5$, $B = E_j$ $(j = 5,6)$.
  We find that $P = AB^4$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = 4^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = B^* v_3$, $v_5 = B^* v_4$.

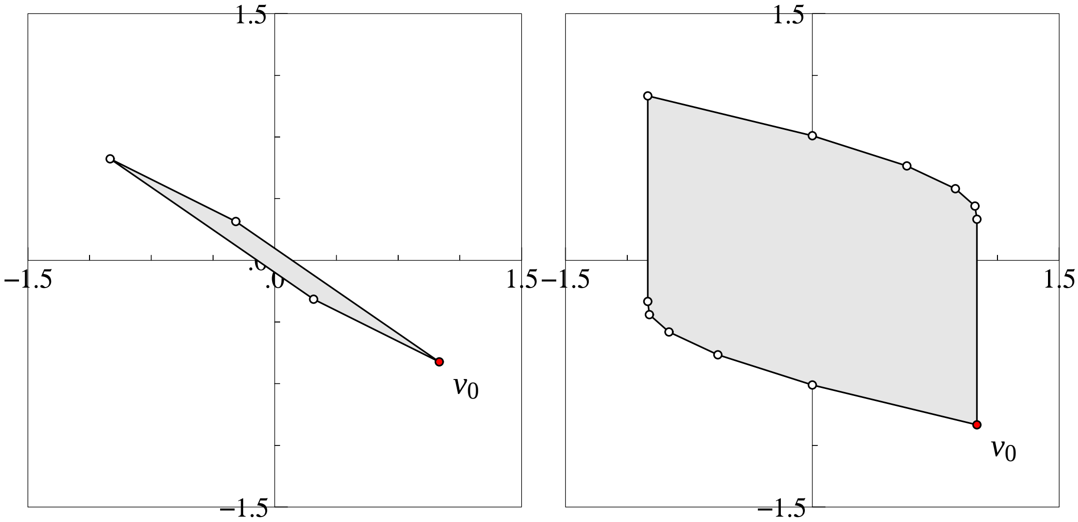

Fig. 10. Polytope norm for the pairs $\{A = F_5, B = E_4\}$ (left) and $\{A = F_5, B = E_5\}$ (right).

- $A = F_8$, $B \in \mathbf{E}$.
  Since $\rho(A) = \|A\|_1 = \|B\|_1 = 2$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..

The subcase $(n_0, n_1) = (4,4)$ *(families of the type* $\mathscr{F} = \{F_i, F_j\}$*).*

- $A = F_8$, $B \in \mathbf{F}$.
  Since $\rho(A) = \|A\|_1 = \|B\|_1 = 2$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..
- $A = F_1$, $B = F_j$ $(j = 2,3,4)$.
  Since $\rho(A) = \rho(B) = \|A\|_2 = \|B\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that both A and B are s.m.p.s.
- $A = F_1$, $B = F_5$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_3$, $B = F_j$ $(j = 2,4)$.
  Since $\rho(A) = \rho(B) = \|A\|_2 = \|B\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that both A and B are s.m.p.s.

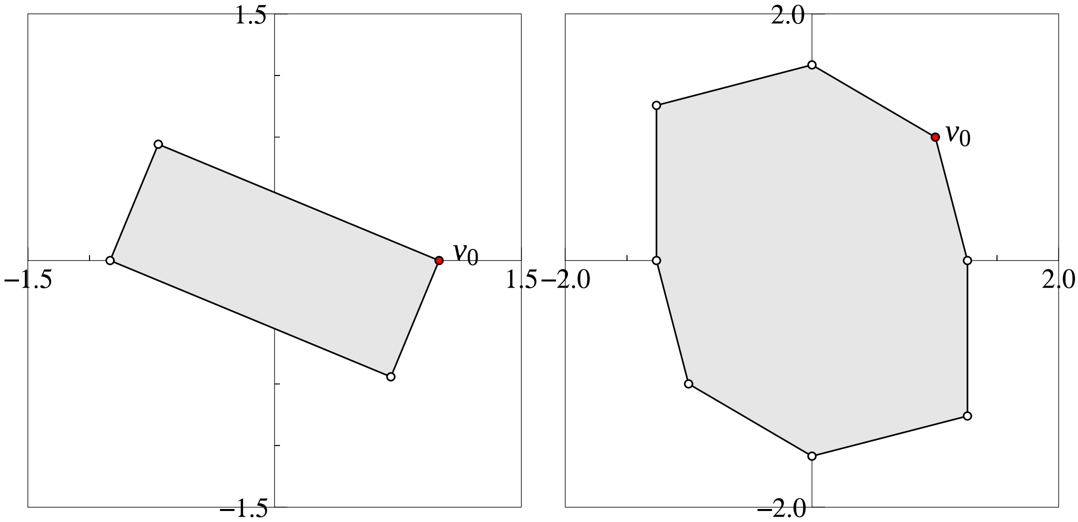

Fig. 11. Polytope norm for the pairs $\{A = F_1, B = F_5\}$ (left) and $\{A = F_3, B = F_5\}$ (right).

- $A = F_3$, $B = F_5$.
  We find that $P = A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 4^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_0, v_3 = A^* v_1$.
- $A = F_5$, $B = F_2$.
  We find that $P = AB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 4^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1, v_3 = B^* v_2$.
- $A = F_5$, $B = F_4$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \text{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_5$, $B = F_6$.
  Since $\|A\|_1 = \|B\|_1 = 2$ and $\rho(AB) = 4$, we have that $\rho(\mathscr{F}) = 2$ and that $P = AB$ is an s.m.p..

5 Conclusions and future work

We have proved the finiteness property for any pair of 2×2 sign-matrices. In most non-trivial cases, this has been made possible by detecting an extremal real polytope norm for the family constituted by the two sign-matrices. The finite convergence of the procedure for constructing the unit ball of such a norm, carried out on a case-by-case basis, implies the finiteness property. An algorithm for the construction of the unit ball is also provided and made publically available. Unfortunately, it seems clear that such an approach can hardly be extended to the general case of a pair of sign-matrices of arbitrary dimension. The use of an induction argument on the dimension seems difficult. Nevertheless, we plan to explore it in future.

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A Matlab version of Algorithm 2.1 is available on the webpage of Nicola Guglielmi,

http://univaq.it/∼guglielm .

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