TEORIA DEI SISTEMI (Systems Theory)

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Problem 1. Consider a feedback control system with unit feedback gain, with the following transfer function in open-loop

$$W(s) = K \frac{s+1}{s^2(s-2)^2}.$$

1. Draw the amplitude and phase Bode diagrams, and the polar diagram for $K = 1$;
2. Compute the denominator of the closed-loop transfer function;
3. Compute the number of poles with negative real part of the closed loop transfer function as a function of the gain $K \in (-\infty, \infty)$, using both the Nyquist criterion and the Routh criterion.

Solution of problem 1. Let $\widetilde{W}(s)$ denote $W(s)$ for $K = 1$:

$$\widetilde{W}(s) = \frac{s+1}{s^2(s-2)^2}.$$

We consider the Bode plots and the polar plot of $\widetilde{W}(s)$. In Bode form we have

$$W(s) = K\,\widetilde{W}(s) = K \left(\frac{1}{4}\right) \frac{1+s}{s^2\left(1-\frac{s}{2}\right)^2}$$

The low frequency gain, epurated from the monomial terms, is

$$K_0 = \lim_{s\to 0} s^2\,\widetilde{W}(s) = \frac{1}{4} \quad \Rightarrow \quad |K_0|_{dB} = 20\log_{10}\left(\frac{1}{4}\right) = 20\log_{10} 2^{-2} = -2\,(20\log_{10} 2) = -12 \text{ dB}.$$

Thus

$$|\widetilde{W}(j\omega)| = \frac{1}{4}\frac{|1+j\omega|}{\omega^2\left|1-j\omega/2\right|^2}$$

$$\langle \widetilde{W}(j\omega)\rangle = \langle 1+j\omega\rangle - 2\langle 1-j\omega/2\rangle.$$

(the Bode plots and the Nyquist plot of the open loop transfer function are in the enclosed file).

The closed-loop transfer function is

$$W_{CL}(s) = \frac{K\,\widetilde{W}(s)}{1+K\,\widetilde{W}(s)} = \frac{K\frac{s+1}{s^2(s-2)^2}}{1+K\frac{s+1}{s^2(s-2)^2}} = \frac{k(s+1)}{s^2(s-2)^2+K(s+1)}$$

and the denominator of $W_{CL}(s)$ is:

$$\begin{aligned} d_{CL}(s) &= s^2(s-2)^2 + K(s+1) = s^2(s^2-4\,s+4) + K\,s + K) \\ &= s^4 - 4s^3 + 4s^2 + Ks + K \end{aligned}$$

NYQUIST ANALYSIS

We see that the Nyquist plot of $\widetilde{W}(j\omega)$ intersects the positive real axis at some frequency ω^* (actually, at a pair of frequencies $\pm\omega^*$). Let us compute the intersection point $\widetilde{W}(j\omega^*)$, by finding the frequency ω^* where the imaginary part of $\widetilde{W}(j\omega)$ is equal to 0. First thing, let us decompose $\widetilde{W}(j\omega)$ as $\Re\big(\widetilde{W}(j\omega)\big) + j\Im\big(\widetilde{W}(j\omega)\big)$:

$$\begin{aligned} \widetilde{W}(j\omega) &= \frac{j\omega+1}{-\omega^2(j\omega-2)^2} = \frac{j\omega+1}{-\omega^2(4-\omega^2-j4\omega)}\frac{(4-\omega^2+j4\omega)}{(4-\omega^2+j4\omega)} \\ &= \frac{4-\omega^2-4\omega^2+j\big(\omega(4-\omega^2)+4\omega\big)}{-\omega^2\big((4-\omega^2)^2+14\omega^2\big)} = \frac{4-5\omega^2+j\omega(8-\omega^2)}{-\omega^2\big(\omega^4+8\omega^2+16\big)} \\ &= \frac{4-5\omega^2+j\omega(8-\omega^2)}{-\omega^2(\omega^2+4)^2} = \frac{4-5\omega^2}{-\omega^2(\omega^2+4)^2} - j\frac{(8-\omega^2)}{\omega(\omega^2+4)^2}. \end{aligned}$$

From this, we have

$$\Im\big(\widetilde{W}(j\omega)\big) = 0 \quad \Leftrightarrow \quad 8 - \omega^2 = 0 \quad \Leftrightarrow \quad \omega^* = \pm\sqrt{8}.$$

Thus $\Im\big(\widetilde{W}(j\sqrt{8})\big) = 0$ and $\widetilde{W}\big(j\sqrt{8}\big)$ is real, and its computation gives

$$\widetilde{W}\big(j\sqrt{8}\big) = \left(\frac{4-5\omega^2}{-\omega^2(\omega^2+4)^2}\right)_{\omega=\sqrt{8}} = \frac{4-5\cdot 8}{-8(8+4)^2} = \frac{1}{32}.$$

Thus, the intersection of $W(j\omega) = K\widetilde{W}(j\omega)$ with the real axis is $K/32$.

Let N be the number of times that the Nyquist plot of $W(j\omega)$ encircles the -1 point in the counterclockwise direction. From the plot it is clear that

- For $K > 0$ we have $N = 0$: the Nyquist plot does not encircle the point -1;
- For $K < 0$ and $-1 < K/32$ we have $N = -1$: the Nyquist plot encircles one time the point -1 in the clockwise (negative) direction;
- For $K < 0$ and $K/32 < -1$ we have $N = 1$: the Nyquist plot encircles one time the point -1 in the counterclockwise (positive) direction;

Recall the Nyquist formula in the form

$$p_{CL} = p_{OL} - N$$

where, p_{CL} is the number of poles with positive real part of the Closed Loop (CL) system, and p_{OL} is the number of poles with positive real part of the Open Loop (OL) system. Since for the given $W(s)$ we have $p_{OL} = 2$ (the unstable pole in $s = 2$ is a double-pole) we have $p_{CL} = 2 - N$, and therefore

- For $K > 0$ we have $N = 0$, and thus $p_{CL} = 2$ (unstable closed loop system);
- For $K \in (-32, 0)$ we have $N = -1$, and thus $p_{CL} = 3$ (unstable closed loop system);
- For $K < -32$ we have $N = 1$, and thus $p_{CL} = 1$ (unstable closed loop system)

Note that for $K = -32$ the denominator of $W_{CL}(s)$ is 0 for $s = \pm j\sqrt{8}$, and thus $|W_{CL}(\pm j\sqrt{8})| = \infty$, and that means that $\pm j\sqrt{8}$ is a pair of imaginary poles of $W_{CL}(s)$ (zero real part).

ROUTH ANALYSIS

The case $K = 0$ will be not analyzed because it corresponds to the trivial case where the open-loop transefer function is zero.

The characteristic polynomial of the closed-loop system is the denominator of $W_{CL}(s)$:

$$d_{CL}(s) = s^4 - 4s^3 + 4s^2 + Ks + K.$$

The first two rows (rows 4 and 3) of the Routh table are:

$$\begin{array}{c|ccc} 4 & 1 & 4 & K \\ 3 & -4 & K & \end{array}$$

The computation of the elements in the third row (row number 2) gives

$$a_{2,1} = \frac{1}{-(-4)}\begin{vmatrix} 1 & 4 \\ -4 & K \end{vmatrix} = \frac{K+16}{4}, \qquad a_{2,2} = \frac{1}{-(-4)}\begin{vmatrix} 1 & K \\ -4 & 0 \end{vmatrix} = \frac{4\,K}{4} = K$$

Thus we have

$$\begin{array}{c|ccc} 4 & 1 & 4 & K \\ 3 & -4 & K & \\ 2 & \frac{K+16}{4} & K & \end{array}$$

The computation of the element in the fourth row (row number 1) gives

$$a_{1,1} = \frac{1}{-(K+16)}\begin{vmatrix} -4 & K \\ \frac{K+16}{4} & K \end{vmatrix} = \frac{K(K+32)}{K+16},$$

and the last element, $a_{0,1}$ is K

$$
\begin{array}{c|ccc}
4 & 1 & 4 & K \\
3 & -4 & K & \\
2 & \frac{K+16}{4} & K & \\
1 & \frac{K(K+32)}{K+16} & & \\
0 & K & &
\end{array}
$$

Analyzing the signs of the first column we have:

- For $K > 0$ we have two sign variation ($4 \to 3$ and $3 \to 2$), so that $p_{CL} = 2$;
- For $K < 0$ and $K + 16 > 0$ (so that also $K + 32 > 0$), we have three sign variations ($4 \to 3$, $3 \to 2$ and $2 \to 1$), so that $p_{CL} = 3$;
- For $K < 0$, $K + 16 < 0$ and $K + 32 > 0$, we have three sign variations ($4 \to 3$, $2 \to 1$ and $1 \to 0$), so that $p_{CL} = 3$;
- For $K + 32 < 0$ (so that also $K < 0$ and $K + 16 < 0$) we have one sign variation ($4 \to 3$) so that $p_{CL} = 1$.

For the particular case of $K = -32$, the polynomial is

$$d_{CL}(s) = s^4 - 4s^3 + 4s^2 - 32s + -32,$$

and the Routh table has the row number 1 equal to zero

$$
\begin{array}{c|ccc}
4 & 1 & 4 & -32 \\
3 & -4 & -32 & \\
2 & -4 & -32 & \\
1 & 0 & &
\end{array}
$$

In this case it is known that the characteristic polynomial can be divided by a polynomial of only even powers whose coefficients are those in the row just over the row of zero. In our case, the coefficients are given by the row number 2, and the polynomial is, $p(s) = -4s^2 - 32$, which we can rewrite by dividing it with -4 ad

$$p(s) = s^2 + 8.$$

Thus we know that $d_{CL}(s)$ can be factorized as

$$d_{CL}(s) = s^4 - 4s^3 + 4s^2 - 32s + -32 = f(s)(s^2 + 8)$$

where $f(s)$ is a polynomial of degree 2, and the sign of its roots can be studied by analyzing the sign of the elemnts in the first column of the Routh table obtained up to now. We see that there is only on sign variation ($4 \to 3$) and therefore $f(s)$ has one root with positive real part (unstable pole). Then, from the factorization, we see that he roots of $p(s) = s^2 + 8$ are also roots of $d_{CL}(s)$, and we conclude that, for $K = -32$, the closed loop transfer function has 1 unstable pole, a pair of imaginary poles and, obviously, a pole wth negative real part (stable).

These results coincide with those obtained from the Nyquist analysis.

Problem 2. Given the systems

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t),
\end{aligned}
\qquad \text{dove} \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 \end{bmatrix}
$$

1. Discuss the propeties of the natural modes;
2. Compute the state-transition matrix e^{At};
3. Compute the impulse response and the input-output transfer function.

Solution of problem 2. The characteristic polynomial and the eigenvalues of the system are

$$|\lambda I_2 - A| = \left| \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} \right| = \lambda^2 - 1, \quad \Rightarrow \quad \lambda_1 = -1, \ \lambda_2 = 1.$$

The right eigenvectors r_1 and r_2 are column vectors that solve $(\lambda_1 I_2 - A)r_1 = 0$ and $(\lambda_2 I_2 - A)r_2 = 0$, while the left-egenvectors ℓ_1 and ℓ_2 are row vectors that solve $\ell_1(\lambda_1 I_2 - A) = 0$ and $\ell_2(\lambda_2 I_2 - A) = 0$. Among the infinite choices of right and left eigenvectors, we must choose those such that $\ell_1 r_1 = \ell_2 r_2 = 1$ (normality condition). Remember that the orthogonality conditions $\ell_1 r_2 = \ell_2 r_1 = 0$ are automatically satisfied.

Thus we proceed as follows: we first compute r_1 and r_2 that solve $(\lambda_1 I_2 - A)r_k = 0$, $k = 1, 2$, and then $\hat{\ell}_1$ and $\hat{\ell}_2$ that solve $\hat{\ell}_k(\lambda_1 I_2 - A) = 0$ (check that $\ell_h r_k = 0$ if $k \neq h$). Then we set $\ell_1 = \hat{\ell}_1/(\hat{\ell}_1 r_1)$ and $\ell_2 = \hat{\ell}_2/(\hat{\ell}_2 r_2)$ (normalization), so that $\ell_1 r_1 = 1$ and $\ell_2 r_2 = 1$. The computations are reported below

$$\lambda_1 = -1, \quad \lambda_1 I_2 - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad r_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \hat{\ell}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\lambda_2 = 1, \quad \lambda_2 I_2 - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad r_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \hat{\ell}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The normaliziation provides the left-eigenvectors

$$\ell_1 = \frac{1}{\hat{\ell}_1 r_1}\hat{\ell}_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \ell_2 = \frac{1}{\hat{\ell}_2 r_2}\hat{\ell}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}$$

(*Note that in this problem we have $\ell_1 = r_1^T$ and $\ell_2 = r_2^T$. This particular situaton happens because the matrix A is symmetric. For symmetric matries it is aways true that: 1) all the eigenvalues are real; 2) the transpose of a right-eigenvector is a left-eigenvector, both associated to the same eigenvalue.*)

Discussion of the properties of the natural modes:

- The mode associated to $\lambda_1 = -1$ is asymptotically stable ($\Re(\lambda_1) < 0$), can be excited by impulsive inputs ($\ell_1 B \neq 0$) and is observable at the output ($C r_1 \neq 0$);
- The mode associated to $\lambda_2 = 1$ is unstable ($\Re(\lambda_2) > 0$), can be excited by impulsive inputs ($\ell_2 B \neq 0$) and is unobservable at the output ($C r_2 = 0$)

(*Note that the presence of unobservable unstable natural modes is a dangerous situation in dynamic systems, because the instability, and the consequent divergence of the state variable $x(t)$, is not detected by the output.*)

The transition matrix can be computed as $e^{At} = e^{\lambda_1 t} r_1 \ell_1 + e^{\lambda_2 t} r_2 \ell_2$:

$$e^{At} = e^{-t}\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} + e^{t}\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = e^{-t}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + e^{t}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

From this

$$e^{At} = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}.$$

We can easily check that $e^{A0} = I_2$, as expected. As an exercise, verify that $(de^{At}/dt)_{t=0} = A$.

The impulse response is easily computed as $w(t) = Ce^{At}B = e^{\lambda_1 t} C r_1 \ell_1 B + e^{\lambda_2 t} C r_2 \ell_2 B$. Recalling that $C r_2 = 0$, we have $w(t) = Ce^{At}B = e^{\lambda_1 t} C r_1 \ell_1 B = -e^{-t}$. The transfer function $W(s)$ is easily computed as the Laplace transform of $w(t)$, i.e. $W(s) = \mathcal{L}\big(w(t)\big)$:

$$w(t) = -e^{-t} \quad \Rightarrow \quad W(s) = -\frac{1}{s+1}.$$

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Problem 3. Given the discrete-time system characterized by the following impulse response

$$w(t) = 0.2^t - 0.5^t,$$

compute the transfer function and the harmonic response to the input $u(t) = \cos\big((\pi/2)t\big)$.

Solution of problem 3. For discrete-time systems the transfer function is the Z-transform of the impulse response. Recalling the basic Z-transform: $Z(a^t) = z/(z-a)$, we have

$$W(z) = Z\big(w(t)\big) = Z\big(0.2^t\big) - Z\big(0.5^t\big) = \frac{z}{z-0.2} - \frac{z}{z-0.5} = \frac{z\big(z - 0.5 - (z-0.2)\big)}{(z-0.2)(z-0.5)} = \frac{-0.3\,z}{(z-0.2)(z-0.5)}$$

Note that the system is asymptotically stable (the magnitude of the poles of $W(z)$ is strictly less than one). Thus, the harmonic response $y_h(t)$ exists. Recall that, for discrete-time systems, the harmonic response to a generic harmonic (or sinusoidal) input $u(t) = m\cos\big(\omega t + \varphi\big)$, can be computed using the following formula:

$$y_h(t) = m\,|W(e^{j\omega})|\cos\big(\omega t + \varphi + \langle W(e^{j\omega})\rangle\big),$$

where $W(e^{j\omega})$ is the transfer function $W(z)$ computed at $z = e^{j\omega}$, where ω is the angular frequency (pulsazione) of the input function. The symbols $|W|$ and $\langle W\rangle$ denote the magnitude and phase of the complex number W, respectively.

In our problem $u(t) = \cos\big((\pi/2)t\big)$, thus $m = 1$, $\omega = \pi/2$ and $\varphi = 0$. The first step to compute the harmonic response is to compute $W(e^{j\omega})$ for $\omega = \pi/2$. Note that $e^{j\pi/2} = \cos(\pi/2) + j\,\sin(\pi/2) = j$. Thus

$$W(e^{j\pi/2}) = W(j) = \frac{-j\,0.3}{(j-0.2)(j-0.5)}$$

There are two ways of computing the magnitude and phase of a complex number that is the ratio of products of some complex numers, like $W(j)$. For instance, consider three complex numbers z_1, z_2 and z_3, and the complex number $y = z_1/(z_1 z_2)$. The magnitude and the phase of y can be computed as follows

$$\left|\frac{z_1}{z_2\,z_3}\right| = \frac{|z_1|}{|z_2|\,|z_3|}, \qquad \left\langle\frac{z_1}{z_2\,z_3}\right\rangle = \langle z_1\rangle - \langle z_2\rangle - \langle z_3\rangle,$$

and

$$|W(j)| = \left|\frac{-j\,0.3}{(j-0.2)(j-0.5)}\right| = \frac{|-j\,0.3|}{|j-0.2|\,|j-0.5|} = \frac{0.3}{\sqrt{1+0.04}\sqrt{1+0.25}} = \frac{0.3}{1.0198\cdot 1.1180} = 0.2631.$$

Thus

$$\langle W(j)\rangle = \left\langle\frac{-j\,0.3}{(j-0.2)(j-0.5)}\right\rangle = \langle -j\,0.3\rangle - \langle j-0.2\rangle - \langle j-0.5\rangle$$

Note that both $j-0.2$ and $j-0.5$ are in the second quadrant (they have negative real part and positive imaginary part), and therefore their phases are in the interval $(\pi/2,\pi)$ or, equivalently, $(-3\pi/2,-\pi)$. Recall that the computation of the phase of a complex number $z = \alpha + j\beta$ using the inverse tangent function $\arctan(\cdot)$ must be made with some care. First of all, when $\alpha = 0$ the function $\arctan(\cdot)$ can not be used, of course (when $\alpha = 0$ we have $\langle z\rangle = \mathit{sign}(\beta)\pi/2$). When $\alpha \neq 0$ the formula $\langle z\rangle = \arctan(\alpha/\beta)$ gives the correct result if an only if $\alpha > 0$. When $\alpha < 0$ the result of $\arctan(\alpha/\beta)$ must be *corrected* by adding π (or $-\pi$, equivalently). We can ue the formula $\langle z\rangle = \arctan(\alpha/\beta) + \pi(\mathit{sign}(\alpha)-1)/2$, where $\mathit{sign}(\alpha) = 1$ if $\alpha > 0$, $\mathit{sign}(\alpha) = -1$ if $\alpha < 0$, $\mathit{sign}(\alpha) = 0$ if $\alpha = 0$. Thus

$$\begin{aligned}
\langle -j\,0.3\rangle &= -\frac{\pi}{2},\\
\langle j-0.2\rangle &= \arctan(-1/0.2) + \pi = -\arctan(1/0.2) + \pi,\\
\langle j-0.5\rangle &= \arctan(-1/0.5) + \pi = -\arctan(1/0.5) + \pi,
\end{aligned}$$

and

$$\langle W(j)\rangle = -\frac{\pi}{2} + \arctan(1/0.2) - \pi + \arctan(1/0.5) - \pi = -1.5708 + 1.3734 + 1.1071 - 2\pi\ \mathit{rad}.$$

Taking the solution modulo 2π, that means that 2π is not considered, we have

$$\langle W(j)\rangle = 0.9097\ \mathit{rad}.$$

An alternative computation of $\langle W(j)\rangle$ consists in computing the real and imginary parts of $W(j)$, $\Re\big(W(j)\big)$ and $\Im\big(W(j)\big)$, and then computing $\arctan\big(\Im\big(W(j)\big)/\Re\big(W(j)\big)\big)$ and adding the π correction, if necessary. This is made as follows

$$\begin{aligned}
W(j) &= \frac{-j\,0.3}{(-0.2+j)(-0.5+j)}\,\frac{(-0.2-j)(-0.5-j)}{(-0.2-j)(-0.5-j)} = 0.3\frac{-j(0.2+j)(0.5+j)}{|-0.2-j|^2\,|-0.5-j|^2}\\
&= 0.3\frac{-j(0.1-1+j(0.2+0.5)}{(0.04+1)(0.25+1)} = \frac{0.3}{1.3}(0.7+j\,0.9)
\end{aligned}$$

From this

$$\langle W(j)\rangle = \langle 0.9 - j\,0.7\rangle = -\arctan(0.7/0.9) = 0.9097\ \mathit{rad}.$$

Thus, the answers to the probelm 3 are:

transfer function: $\quad W(z) = \dfrac{-z\,0.3}{(z-0.2)(z-0.5)},$

harmonic response: $\quad y_h(t) = 0.2631\ \mathit{cos}((\pi/2)t + 0.9097).$

Problem 4. Consider the Linear Time-Invariant (LTI) continuous time system represented by the following matrices

$$A = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix}$$

Find a basis for the space of reachable states and a basis for the space of unobservable states. Moreover, define the 4 subspaces $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$ of the structural Kalman decomposition

Solution of problem 4. The computation of the reachability matrix P_4 and of the obsevability matrix Q_4 gives:

$$P_4 = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q_4 = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 4 & 4 & 8 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

(*note that for the computation of P_4 and Q_4 it is not necessary, nor wise, to compute the powers A^k, for $k = 2$ and $k = 3$, and then multiply the resulting matrice bu B and by C. The fastest and reliable way to compute P_4 is to recursively compute the columns of $P_4 = [P_{:,1}\ P_{:,2}\ P_{:,3}\ P_{:,4}]$ as $P_{:,k+1} = AP_{:,k}$, starting with $P_{:,1} = B$. Similarly, for the computation of Q_4, we can recursively compute the four rows of Q_4, denoted $Q_{k,:}$, as $Q_{k+1,:} = Q_{:,k}A$, starting with $C_{:,1} = C$.*)

Looking at P_4, it is clear that the rank is 1, because only the first column (the vector B) is nonzero. Thus the *range* (or *image space*) of P_4, denoted $\mathcal{P} = \mathcal{R}(P_4)$, has dimension 1, and its first column forms its basis: $\mathcal{P} = span(B)$.

Consider now matrix Q_4. Note that $\rho(Q_4) = 2$, because the first and third rows are proportional, and the second and fourth rows are proportional. Thus, only two rows are independent, and the rank is 2. Thus we know that the null-space of Q_4 has dimension 2 (in general, the dimension of the null-space of Q_n is $n - \rho(Q_n)$). Thus, we must find two independent vectors v_1 and v_2 in the null-space of Q_4 (that we denote $\mathcal{N}(Q_4)$ or $\mathcal{Q}$), that means such that $Q_4 v_1 = 0$ and $Q_4 v_2 = 0$.

Noting that the first two columns of Q_4 are equal, we see that the vector $v_1 = [1\ -1\ 0\ 0]^T$ is such that $Q_4 v_1 = 0$ (the product $Q_4 v_1$ performs the subtraction of the second column from the first column of Q_4). Noting that the third column is twice the first column, we see that the vector $v_2 = [2\ \ 0\ -1\ 0]^T$ is such that $Q_4 v_2 = 0$ (the product $Q_4 v_2$ performs the subtraction of the third column from the double of the first column of Q_4). Thus we have

$$\mathcal{P} = \mathcal{R}(P_4) = span\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{Q} = \mathcal{N}(Q_4) = span\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Obviously, other basis can be chosen for the subspaces $\mathcal{P}$ and $\mathcal{Q}$. (*A very common –and bad– error made by students is to define $\mathcal{Q} = \mathcal{R}(Q_4)$*).

Now, let's find the bases for the 4 subspaces $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ and $\mathcal{X}_4$. By definition $\mathcal{X}_1 = \mathcal{P} \cap \mathcal{Q}$. A common error made by (unprepared) students is to look for a common vector in the bases of $\mathcal{P}$ and $\mathcal{Q}$. If the unprepared student doesn't find any common vector in the two bases, he draws the (wrong) conclusion that the intersection $\mathcal{P} \cap \mathcal{Q}$ is empty ($\mathcal{X}_1 = \emptyset$).

Thus, in our problem, the unprepared student would claim that $\mathcal{X}_1$ is empty, so that there are no reachable states that are unobservable. Instead, a simple test shows that *all* reachable states are unobservable, i.e. $\mathcal{P} \cap \mathcal{Q} = \mathcal{P}$. The test consists in multiplying Q_4 by B, obtaining $Q_4 B = 0$. This means that $B \in \mathcal{N}(Q_4)$, equivalent to $span(B) \subset \mathcal{Q}$ or $\mathcal{P} \subset \mathcal{Q}$, from which $\mathcal{X}_1 = \mathcal{P} \cap \mathcal{Q} = \mathcal{P}$ (the student can verify that B can be obtained as a linear combination of the two vectors v_1 and v_2: $B = -v_1 + v_2$). Thus, let $x_1^{(1)} = B$, so that $\mathcal{X}_1 = span(x_1^{(1)})$.

Now we have to find a subspace $\mathcal{X}_2$ such that $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{P}$. Clearly, $\mathcal{X}_2 = \emptyset$, because $\mathcal{X}_1 = \mathcal{P}$.

Now we have to find a subspace $\mathcal{X}_3$ such that $\mathcal{X}_1 \oplus \mathcal{X}_3 = \mathcal{Q}$. The simplest way to find a basis of $\mathcal{X}_3$ is to transform the basis of $\mathcal{Q}$ so that $x_1^{(1)}$ is an vector of the basis. Recalling that $\mathcal{Q}$ has dimension 2, we have that $\mathcal{X}_3$ must have dimension 1 (one vector in the basis). The other vector in the basis of $\mathcal{Q}$ can be chosen as the basis of $\mathcal{X}_3$. This operation can be made in many ways. One way is simply to replace the vector v_2 of the basis of $\mathcal{Q}$ with $x_1^{(1)}$, i.e. $\mathcal{Q} = span\{v_1, x_1^{(1)}\}$. It is easy to check that the vectors of the *old* basis $\{v_1, v_2\}$ can be obtained as a linear combination of the *new* basis $\{v_1, x_1^{(1)}\}$ (we easily see that $v_2 = v_1 + x_1^{(1)}$). Now that we have $\mathcal{Q} = span\{v_1, x_1^{(1)}\}$, we can choose $x_1^{(3)} = v_1$.

Now we have to find a subspace $\mathcal{X}_3$ such that $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 = \mathbb{C}^4$ (or $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_3 = \mathbb{C}^4$, since $\mathcal{X}_2 = \emptyset$). Since $\mathcal{X}_1 \oplus \mathcal{X}_3 = span\{x_1^{(1)}, x_1^{(3)}\}$ has dimension 2, we need two vector for completing a basis of $\mathbb{C}^4$. Being

$$\mathcal{X}_1 \oplus \mathcal{X}_3 = span\{x_1^{(1)}, x_1^{(3)}\} = span \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

a straightforward choice is

$$\mathcal{X}_4 = span\{x_1^{(4)}, x_2^{(4)}\} = span \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Indeed, it is easy to see that the matrix $T = \begin{bmatrix} x_1^{(1)} & x_1^{(3)} & x_1^{(4)} & x_2^{(4)} \end{bmatrix}$ is nonsingular, and therefore $\mathcal{R}(T) = \mathbb{C}^4$.

The matrix T defines a change of coordinates that transforms the system in the Kalman canonical form.

Problem 5. Consider a continuous time system $\dot{x}(t) = Ax(t) + Bu(t)$ and a quadratic function $V(x) = x^T P x$, with

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \qquad \text{e} \qquad P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Verify that the origin is an asymptotically stable equilibrium, and that $V(x)$ is a Lyapunov function. (*Suggestion: in order to check whether a given matrix is positive definite, use the Sylvester criterion.*)

Solution of problem 5.
Following the Sylvester criterion, P is positive definite (the determinants of the two principal minors of P are both positive):

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad \Rightarrow \quad |3| = 3, \quad \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5.$$

Note that for linear systems the origin is *always* and equilibrium point.

Now consider the function $V = x^T P x$ (Lypaunov function candidate), along a generic system trajectory $x(t)$: $V(x(t)) = x^T(t) P x(t)$. Differentiating $V(x(t))$ with respect to time we have

$$\frac{dV(x(t))}{dt} = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t)$$

Replace $\dot{x}(t) = Ax(t)$ and $\dot{x}^T(t) = A^T x^T(t)$ to get

$$\frac{dV(x(t))}{dt} = x^T(t) A^T P x(t) + x^T(t) P A x(t) = x^T(t)(A^T P + PA) x(t)$$

The problem is solved simply by computing the matrix $A^T P + PA$ and verifying that it is negative definite (that means, defining $Q = -(A^T P + PA)$, verify that Q is positive definite).

Strightforward computations give

$$Q = -(A^T P + PA) = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}.$$

The determinants of the principal minors are positive (4 and 20) and therefore Q is positive definite. It follows that $V = x^T P x$ is a Lyapunov function and the origin is n asymptotically stable equilibrium for the given system.

Problem 6. Given the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) - \big(x_1(t) - 1\big)^3 \\ \dot{x}_2(t) = \alpha\big(x_1(t) + x_2^2(t) - 1\big) - x_2(t) - \big(x_1(t) - 1\big)^3 \end{cases}$$

study the stability of the equilibrium point $x_e = (1, 0)$ for all the values of the parameter α in $(-\infty, \infty)$, using the method of linear approximation at the equilibrium point, and the Lyapunov method, if necessary. (*Suggestion for the Lyapunov function:* $V(x) = (x_1 - x_{e,1})^4 + \beta(x_2 - x_{e,2})^2$, with suitable $\beta > 0$.)

Solution of problem 6.
The system considered is of the form $\dot{x}(t) = f\big(x(t); \alpha\big)$, where $x(t)$ is the state and α is a constant parameter. The vector function $f(x; \alpha) = [f_1(x)\ f_2(x; \alpha)]^T$ is as follows

$$\begin{cases} f_1(x) = x_2 - \big(x_1 - 1\big)^3 \\ f_2(x; \alpha) = \alpha\big(x_1 + x_2^2 - 1\big) - x_2 - \big(x_1 - 1\big)^3 \end{cases}$$

The Jacobian is

$$J(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix} = \begin{bmatrix} -3(x_1 - 1)^2 & 1 \\ \alpha - 3(x_1 - 1)^2 & 2\alpha\, x_2 - 1 \end{bmatrix}$$

The value of the Jacobian at the equilibrum point $x_e = (1, 0)$ and the characteristic polynomial are

$$J(x_e) = \begin{bmatrix} 0 & 1 \\ \alpha & -1 \end{bmatrix}, \qquad |\lambda I_2 - J(x_e)| = \left|\begin{bmatrix} \lambda & -1 \\ -\alpha & \lambda + 1 \end{bmatrix}\right| = \lambda^2 + \lambda - \alpha.$$

It is known that the roots of a second-degree polynomial have strictly negative real part *if and only if* all coefficients have the same sign. Thus, in our problem, the two eigenvalues of $J(x_e)$ have strictly negative real part *if and only if* $\alpha < 0$. As a consequence, if $\alpha < 0$ then x_e is an asymptotically stable (A.S.) equilibrium point. On the other hand, if $\alpha > 0$, then there exists at least one eigenvalue with positive real part. As a consequence, if $\alpha > 0$ then x_e is an unstable equilibrium point.

When $\alpha = 0$ then the characteristic polynomial is $\lambda^2 + \lambda = \lambda(\lambda + 1)$, and therefore the two roots are $\lambda_1 = 0$ and $\lambda_2 = -1$. Thus, the origin is a simply stable equilibrium point *of the linear approximation* of the nonlinear system.

However, this **does not** imply the simple stability of the original nonlinear system (only asymptotic stability of the linear approximation implies the asymptotic stability of the nonlinear system). Thus, we must study the stability of the point $x_e = (1, 0)$ when $\alpha = 0$ by using a suitable Lyapunov function. Following the suggestion, for the system $\dot{x}(t) = f\big(x(t); 0\big)$ we consider the Lyapunov function $V(x) = (x_1 - x_{e,1})^4 + \beta(x_2 - x_{e,2})^2$, which is positive definite for any $\beta > 0$. According to the Lyapunov theorem, if the derivative $\dot{V}(x) = (dV/dx)f(x; \alpha)$ is semidefinite negative, then the equilibrium x_e is (simply) stable, while if $\dot{V}(x)$ is definite negative, then the equilibrium is asymptotically stable. The computation of $\dot{V}(x)$ gives

$$\begin{aligned} \dot{V}(x) &= \frac{dV}{dx} f(x; 0) = (\partial_{x_1} V) f_1(x) + (\partial_{x_2} V) f_2(x; 0) \\ &= 4(x_1 - 1)^3\big(x_2 - (x_1 - 1)^3\big) + 2\beta\, x_2\big(-x_2 - (x_1 - 1)^3\big) \\ &= 4x_2(x_1 - 1)^3 - 4(x_1 - 1)^6 - 2\beta x_2^2 - 2\beta x_2(x_1 - 1)^3. \end{aligned}$$

Note that the two terms $-4(x_1 - 1)^6$ and $-2\beta x_2^2$ are strictly negative when $x \neq x_e$, while the sign of the terms $4x_2(x_1 - 1)^3$ and $-2\beta x_2(x_1 - 1)^3$ is indefinite. If we choose $\beta = 2$ we can cancel the two terms with indefinite sign, and get the following derivative of the Lyapunov function

$$\dot{V}(x) = -4(x_1 - 1)^6 - 4x_2^2,$$

which is definite negative. This proves that when $\alpha = 0$ the equilibrium x_e is asymptotically stable.

In conclusion, for $\alpha \in (-\infty, 0]$ the equilibrium $x_e = (1, 0)$ is asymptotically stable, and is unstable when $\alpha \in (0, \infty)$.

Moreover, note that the Lyapunov function $V(x)$ is *radially unbounded*, i.e.

$$\|x\| \to \infty \qquad \Longrightarrow \qquad V(x) \to \infty,$$

and therefore when $\alpha = 0$ the equlibrium is *globally asymptotically stable* (G.A.S.).