Esercizio 1 [redacted]

$$W(s) = k\,\frac{50(s+1)}{(s+5)(s^2+100)}\Bigg|_{k=1} = \frac{\cancel{50}^{10}(1+s)}{\cancel{5}\left(1+\frac{s}{5}\right)10\cancel{0}\left(1+\frac{s^2}{100}\right)} = \frac{1}{10}\,\frac{(1+s)}{\left(1+\frac{s}{5}\right)\left(1+\frac{s^2}{100}\right)}$$

Guadagno $k_W = \frac{1}{10}$ $\Rightarrow$ Moduli: $|k_W|_{dB} = 20\log|k_W| = -20$ dB [redacted]

Fasi: $\langle k_W \rangle = 0$

Binomio numeratore: $1+s$ $\Rightarrow$ Moduli: $\omega \in (1, +\infty)$ 20 dB/dec

Fasi: $\omega \in (0.1, 10)$ $\frac{\pi}{4}$ rad/dec

Binomio denominatore: $1+\frac{s}{5}$ $\Rightarrow$ Moduli: $\omega \in (5, +\infty)$ -20 dB/dec

Fasi: $\omega \in (0.5, 50)$ $-\frac{\pi}{4}$ rad/dec

Trinomio denominatore: $1+\frac{s^2}{100}$, $\zeta = 0$ $\Rightarrow$ Moduli: $\omega \in (10, +\infty)$ -40 dB/dec con correzione in $\omega = 10$

Fasi: salto di $-\pi$ in $\omega = 10$

$\Rightarrow$ Moduli: $\omega \in (1, 5)$ 20 dB/dec

$\omega \in (5, 10)$ 0 dB/dec

$\omega \in (10, +\infty)$ -40 dB/dec

Dove non ho definito gli intervalli la pendenza è nulla.

Fase asintotica: $\left\langle \frac{50j\omega + 50}{(j\omega+5)(-\omega^2+100)} \right\rangle \approx$

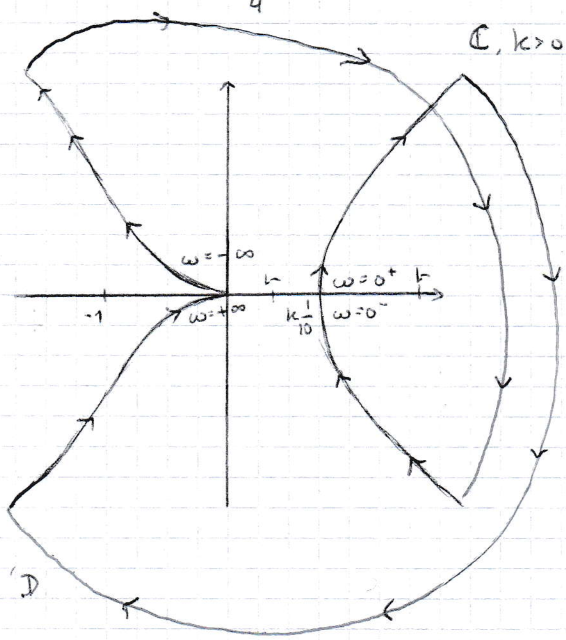
$\approx \left\langle \frac{50\cancel{j\omega}}{-\cancel{j}\omega^{\cancel{2}}} \right\rangle = \pm\pi$

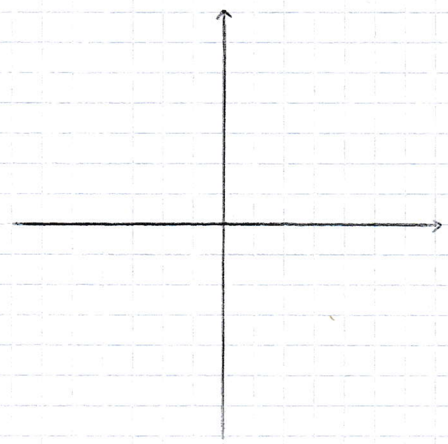
Fasi: $\omega \in (0.1, 0.5)$ $\frac{\pi}{4}$ rad/dec

$\omega \in (0.5, 10)$ 0 rad/dec

$\omega = 10$ salto di $-\pi$ rad

$\omega \in (10, 50)$ $-\frac{\pi}{4}$ rad/dec





$D_{CH}(s) = N_{AP}(s) + D_{AP}(s) = 50ks + 50k + (s+5)(s^2+100) =$

$= 50ks + 50k + s^3 + 100s + 5s^2 + 500 =$

$= s^3 + 5s^2 + 50(k+2)s + 50(k+10)$

3	1	50(k+2)
2	5	50(k+10)
1	40k	
0	50(k+10)	

Con Routh

$$\frac{50(k+10) - 250(k+2)}{-5} = -10(k+10) + 50(k+2) =$$

$= +40k - 100 + 100 = 40k$

		-10	0
1	+	+	+
5	+	+	+
40k	-	-	+
50(k+10)	-	+	+
	1V	2V	0V

$k+10 > 0 \iff k > -10$

$\Rightarrow$ Per $k>0$: non ci sono poli a parte reale positiva $\Rightarrow$ Stabile

Per $-10 < k < 0$ ci sono 2 poli a parte reale positiva $\Rightarrow$ Instabile

Per $k < -10$ c'è un polo a parte reale positiva $\Rightarrow$ Instabile

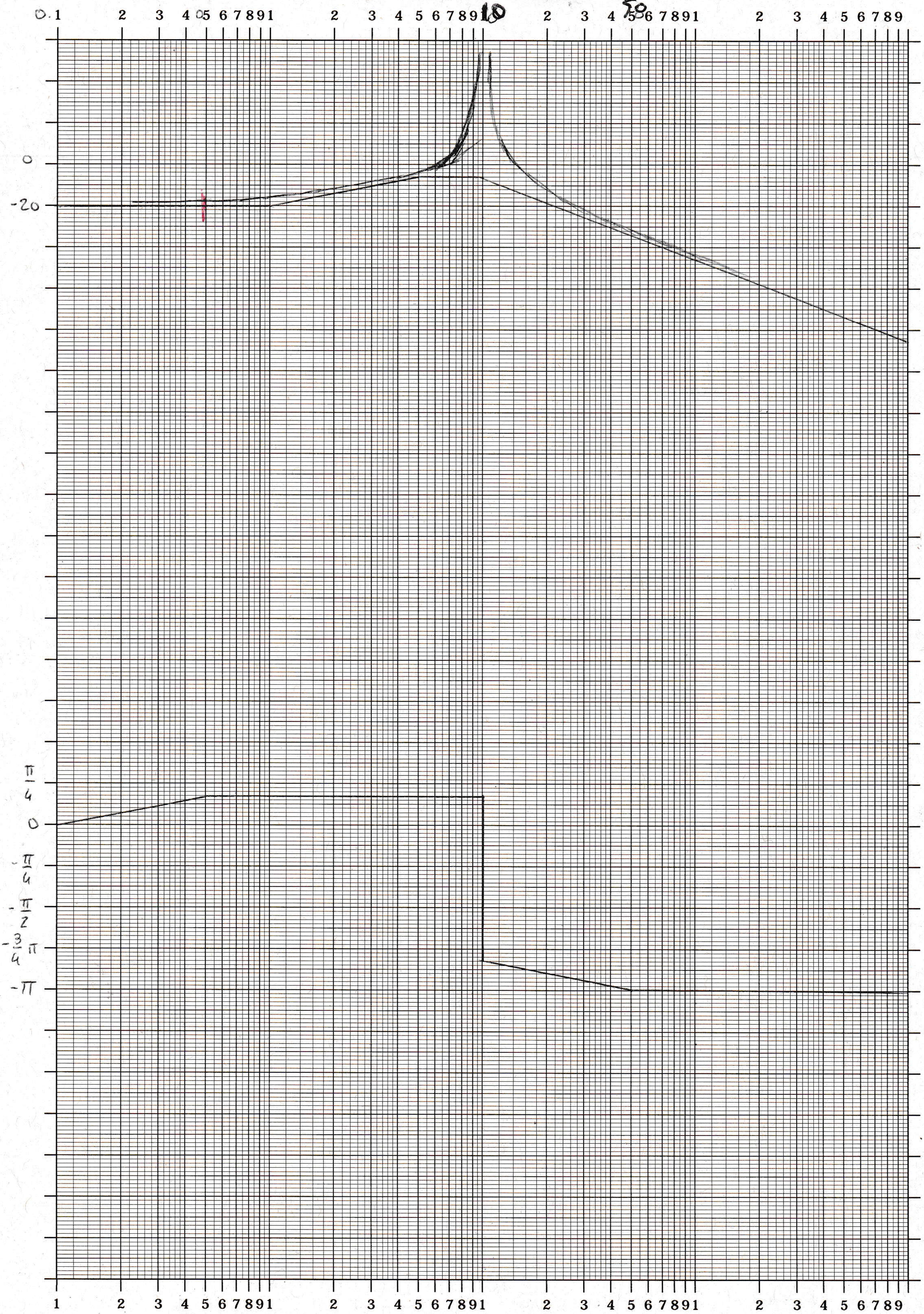
Con Nyquist

$n_{CH}^+ = n_{AP}^+ - N = -N$

Per $k > 0$: $N = 0 \Rightarrow n_{CH}^+ = 0$

Per $k<0$: se $\frac{1}{10}k < -1 \Rightarrow k < -10$: $N = -1 \Rightarrow n_{CH}^+ = 1$

se $\frac{1}{10}k > -1 \Rightarrow k > -10$: $N = -2 \Rightarrow n_{CH}^+ = 2$


0.1
10
50
0
-20
$\frac{\pi}{4}$
0
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
$-\frac{3}{4}\pi$
$-\pi$

Esercizio 2

$$A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} \lambda - 2 & 2 \\ -2 & \lambda - 2 \end{vmatrix} = \lambda^2 - 4\lambda + 4 + 4 = \lambda^2 - 4\lambda + 8 = 0 \iff \lambda_{1/2} = \frac{4 \pm \sqrt{16 - 32}}{2} = 2 \pm \frac{4j}{2} = 2 \pm 2j$$

$\lambda_1 = 2 + 2j$, $\lambda_2 = 2 - 2j$ sono entrambi instabili $(|Re(\lambda_1)| > 1, |Re(\lambda_2)| > 1)$

$$r_1 : (\lambda_1 I - A) r_1 = 0 \Rightarrow \begin{bmatrix} 2j & 2 \\ -2 & 2j \end{bmatrix} r_1 = 0 \Rightarrow \begin{cases} 2ja + 2b = 0 \\ -2a + 2jb = 0 \end{cases} \Rightarrow \begin{cases} a = jb \\ a = jb \end{cases} \Rightarrow r_1 = \begin{bmatrix} j \\ 1 \end{bmatrix}$$

$$r_2 = r_1^* = \begin{bmatrix} -j \\ 1 \end{bmatrix}$$

$$L = R^{-1} = \begin{bmatrix} j & -j \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2j} \begin{bmatrix} 1 & j \\ -1 & j \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}j & \frac{1}{2} \\ \frac{1}{2}j & \frac{1}{2} \end{bmatrix} \begin{matrix} = \ell_1^T \\ = \ell_2^T \end{matrix}$$

$\ell_1^T B \neq 0$ eccitabile

$\ell_2^T B \neq 0$ eccitabile

$C r_1 \neq 0$ osservabile

$C r_2 \neq 0$ osservabile

$$A = \lambda_1 \begin{bmatrix} j \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}j & \frac{1}{2} \end{bmatrix} + \lambda_2 \begin{bmatrix} -j \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}j & \frac{1}{2} \end{bmatrix} = (2+2j) \begin{bmatrix} \frac{1}{2} & \frac{1}{2}j \\ -\frac{1}{2}j & \frac{1}{2} \end{bmatrix} + (2-2j) \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}j \\ \frac{1}{2}j & \frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 1+j+1-j & j-1-j-1 \\ -j+1+j+1 & 1+j+1-j \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \quad \text{per verifica}$$

$$\sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\varphi = \frac{\pi}{4}$$

$$\phi(t) = A^t = \lambda_1^t r_1 \ell_1^T + \lambda_2^T r_2 \ell_2^T = 2Re\left((2+2j)^t \begin{bmatrix} \frac{1}{2} & \frac{1}{2}j \\ -\frac{1}{2}j & \frac{1}{2} \end{bmatrix}\right) = \quad , \quad \Rightarrow (2+2j)^t = 8^{\frac{t}{2}} e^{j\frac{\pi}{4}t}$$

$$= 2Re\left(8^{\frac{t}{2}} e^{j\frac{\pi}{4}t} \begin{bmatrix} \frac{1}{2} & \frac{1}{2}j \\ -\frac{1}{2}j & \frac{1}{2} \end{bmatrix}\right) = 2Re\left(8^{\frac{t}{2}} \left(\cos\left(\frac{\pi}{4}t\right) + j\sin\left(\frac{\pi}{4}t\right)\right) \begin{bmatrix} \frac{1}{2} & \frac{1}{2}j \\ -\frac{1}{2}j & \frac{1}{2} \end{bmatrix}\right) =$$

$$= 2 \cdot 8^{\frac{t}{2}} \begin{bmatrix} \frac{1}{2}\cos\left(\frac{\pi}{4}t\right) & -\frac{1}{2}\sin\left(\frac{\pi}{4}t\right) \\ \frac{1}{2}\sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} = 8^{\frac{t}{2}} \begin{bmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{bmatrix}$$

$$\phi(0) = I$$

$$\phi(1) = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = A$$

$$W(t) = CA^{t-1}B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 8^{\frac{t-1}{2}}\cos(\frac{\pi}{4}(t-1)) & -8^{\frac{t-1}{2}}\sin(\frac{\pi}{4}(t-1)) \\ 8^{\frac{t-1}{2}}\sin(\frac{\pi}{4}(t-1)) & 8^{\frac{t-1}{2}}\cos(\frac{\pi}{4}(t-1)) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$= 8^{\frac{t-1}{2}} \left(\cos\left(\frac{\pi}{4}(t-1)\right) + \sin\left(\frac{\pi}{4}(t-1)\right) \right)$$

$$w(1) = \cos(0) + \sin(0) = 1$$

$$w(2) = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right) = 4$$

$$W(z) = C(zI-A)^{-1}B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z-2 & 2 \\ -2 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{z^2-4z+8} \begin{bmatrix} z-2 & -2 \\ 2 & z-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$= \frac{z}{z^2-4z+8}$$

Esercizio 3

$$w(t) = e^{-t} + e^{-4t} \Rightarrow W(s) = \frac{1}{s+1} + \frac{1}{s+4} = \frac{s+4+s+1}{(s+1)(s+4)} = \frac{2s+5}{(s+1)(s+4)}$$

$$u(t) = e^{2t} \Rightarrow U(s) = \frac{1}{s-2}$$

$$Y_f(s) = \underbrace{\frac{2s+5}{(s+1)(s+4)(s-2)}}_{= W(s)U(s)} = \frac{R_1}{(s+1)} + \frac{R_2}{s+4} + \frac{R_3}{s-2}$$

$$R_1 = \lim_{s\to -1} \frac{2s+5}{(s+4)(s-2)} = -\frac{1}{3}$$

$$R_2 = \lim_{s\to -4} \frac{2s+5}{(s+1)(s-2)} = -\frac{1}{6}$$

$$R_3 = \lim_{s\to 2} \frac{2s+5}{(s+1)(s+4)} = \frac{1}{2}$$

$$Y_f(s) = -\frac{1}{3}\frac{1}{s+1} - \frac{1}{6}\frac{1}{s+4} + \frac{1}{2}\frac{1}{s-2}$$

$$y_f(t) = -\frac{1}{3}e^{-t} - \frac{1}{6}e^{-4t} + \frac{1}{2}e^{2t}$$

La risposta armonica esiste perché entrambi gli autovalori (che si osservano sottoforma di poli nella funzione di trasferimento) sono a parte reale negativa $\Rightarrow$ stabili

$u(t) = \sin(2t + \pi)$

$\Rightarrow y_A(t) = |W(j2)| \sin(2t + \pi + \langle W(j2) \rangle)$

$$|W(j2)| = \frac{|4j+5|}{|(2j+1)(2j+4)|} = \frac{\sqrt{16+25}}{|-4 + 10j + 4|} = \frac{\sqrt{41}}{10}$$

$$\langle W(j2) \rangle = \langle 4j+5 \rangle - \langle 10j \rangle = \arctan\left(\frac{4}{5}\right) - \frac{\pi}{2}$$

$$\Rightarrow y_A(t) = \frac{\sqrt{41}}{10} \sin\left(2t + \frac{\pi}{2} + \arctan\left(\frac{4}{5}\right)\right)$$

Esercizio 4

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\det(P) = -(\cancel{4} + \cancel{6} + \cancel{2} - \cancel{4} - \cancel{6} - 2) - (6 + 2 + 4 - 2 - 4 - 6) = 0$ ma si vedeva perché

ci sono due righe uguali

$\rho(P) = 3$

$$Im(P) = R = span\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right\} = span\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}\right\} =$$

$$= span\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \rho(Q) = 2 \Rightarrow \dim(\mathcal{Q}) = \dim(N(Q)) = 4-2 = 2$$

$$Qx = 0 \Rightarrow \begin{cases} a = 0 \\ b + d = 0 \Rightarrow b = -d \\ \text{—} \\ \cancel{b + d = 0} \Rightarrow \end{cases}$$

$$\mathcal{Q} = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ +1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{R} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{X}_1 = \mathcal{R} \cap \mathcal{Q} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{X}_2 : \mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{R} \Rightarrow \mathcal{X}_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{X}_3 : \mathcal{X}_1 \oplus \mathcal{X}_3 = \mathcal{Q} \Rightarrow \mathcal{X}_3 = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{X}_4 = \{0\}$$

$x_a = \begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix}^T$ è irraggiungibile e inosservabile

$x_b = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T$ è raggiungibile e osservabile

Esercizio 5

$$f(x) = \begin{bmatrix} x_1(k-1+x_2-x_1^2) \\ 1-x_2-\frac{1}{2}x_1^2 \end{bmatrix}$$, il punto di equilibrio è $x_b(0,1)$

infatti: $f(x_b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

$\cancel{\xi = x - x_b \Rightarrow \xi_1 = x_1 \quad \xi_2 = x_2 - 1 \Leftrightarrow x_2 = \xi_2 + 1}$

$\Rightarrow \cancel{f(\xi) = \begin{bmatrix} \xi_1(k-1+\xi_2-\xi_1^2) \\ 1-\xi_2 \end{bmatrix}}$

$$\xi = \begin{bmatrix} x_1 \\ x_2 - 1 \end{bmatrix}$$

$$f(\xi) = \begin{bmatrix} \xi_1(k-\cancel{1}+\xi_2+\cancel{1}-\xi_1^2) \\ \cancel{} -\xi_2-\frac{1}{2}\xi_1^2 \end{bmatrix} = \begin{bmatrix} \xi_1(k+\xi_2-\xi_1^2) \\ -\xi_2-\frac{1}{2}\xi_1^2 \end{bmatrix} = 0 \text{ in } (0,0)$$

$$\left.\frac{df}{d\xi}\right|_{\xi=0} = \begin{bmatrix} k+\xi_2-3\xi_1^2 & \xi_1 \\ -\xi_1 & -1 \end{bmatrix}_{\xi=0} = \begin{bmatrix} k & 0 \\ 0 & -1 \end{bmatrix}$$

$\lambda_1 = k$, $\lambda_2 = -1$

Se $k<0 \Rightarrow x_b$ è stabile

Se $k>0 \Rightarrow x_b$ è instabile

Se $k=0$ uso Lyapunov

$V(\xi) = \frac{\alpha}{2}\xi_1^2 + \frac{1}{2}\xi_2^2 > 0$ per $\alpha > 0$

$$\frac{dV}{d\xi}f(\xi) = \begin{bmatrix} \alpha\xi_1 & \xi_2 \end{bmatrix} \begin{bmatrix} \xi_1(k+\xi_2-\xi_1^2) \\ -\xi_2-\frac{1}{2}\xi_1^2 \end{bmatrix} = \alpha\xi_1^2(\cancel{k}+\xi_2-\xi_1^2) - \xi_2^2 - \frac{1}{2}\xi_1^2\xi_2 =$$

$$= \cancel{k\alpha\xi_1^2} + \alpha\xi_1^2\xi_2 - \alpha\xi_1^4 - \xi_2^2 - \frac{1}{2}\xi_1^2\xi_2$$

Per $\alpha = \frac{1}{2}$:

- $V(\xi) = \frac{1}{4}\xi_1^2 + \frac{1}{2}\xi_2^2$

$$\frac{dV}{d\xi} F(\xi) = -\frac{1}{2}\xi_1^4 - \xi_2^2 \le 0$$

$\Rightarrow$ Per $k=0$, x_b è asintoticamente stabile