Chapter 7

Minimal Realizations and Coprime Fractions

7.1 Introduction

This chapter studies further the realization problem discussed in Section 4.4. Recall that a transfer matrix $\hat{\mathbf{G}}(s)$ is said to be realizable if there exists a state-space equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

that has $\hat{\mathbf{G}}(s)$ as its transfer matrix. This is an important problem for the following reasons. First, many design methods and computational algorithms are developed for state equations. In order to apply these methods and algorithms, transfer matrices must be realized into state equations. As an example, computing the response of a transfer function in MATLAB is achieved by first transforming the transfer function into a state equation. Second, once a transfer function is realized into a state equation, the transfer function can be implemented using op-amp circuits, as discussed in Section 2.3.1.

If a transfer function is realizable, then it has infinitely many realizations, not necessarily of the same dimension, as shown in Examples 4.6 and 4.7. An important question is then raised: What is the smallest possible dimension? Realizations with the smallest possible dimension are called *minimal-dimensional* or *minimal* realizations. If we use a minimal realization to implement a transfer function, then the number of integrators used in an op-amp circuit will be minimum. Thus minimal realizations are of practical importance.

In this chapter, we show how to obtain minimal realizations. We will show that a realization of $\hat{g}(s) = N(s)/D(s)$ is minimal if and only if it is controllable and observable, or if and only

if its dimension equals the degree of $\hat{g}(s)$. The degree of $\hat{g}(s)$ is defined as the degree of $D(s)$ if the two polynomials $D(s)$ and $N(s)$ are coprime or have no common factors. Thus the concept of coprimeness is essential here. In fact, coprimeness in the fraction $N(s)/D(s)$ plays the same role of controllability and observability in state-space equations.

This chapter studies only linear time-invariant systems. We study first SISO systems and then MIMO systems.

7.2 Implications of Coprimeness

Consider a system with proper transfer function $\hat{g}(s)$. We decompose it as

$$\hat{g}(s) = \hat{g}(\infty) + \hat{g}_{sp}(s)$$

where $\hat{g}_{sp}(s)$ is strictly proper and $\hat{g}(\infty)$ is a constant. The constant $\hat{g}(\infty)$ yields the D-matrix in every realization and will not play any role in what will be discussed. Therefore we consider in this section only strictly proper rational functions. Consider

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \tag{7.1}$$

To simplify the discussion, we have assumed that the denominator $D(s)$ has degree 4 and is monic (has 1 as its leading coefficient). In Section 4.4, we introduced for (7.1) the realization in (4.41) without any discussion of its state variables. Now we will redevelop (4.41) by first defining a set of state variables and then discussing the implications of the coprimeness of $D(s)$ and $N(s)$.

Consider

$$\hat{y}(s) = N(s)D^{-1}(s)\hat{u}(s) \tag{7.2}$$

Let us introduce a new variable $v(t)$ defined by $\hat{v}(s) = D^{-1}(s)\hat{u}(s)$. Then we have

$$D(s)\hat{v}(s) = \hat{u}(s) \tag{7.3}$$

$$\hat{y}(s) = N(s)\hat{v}(s) \tag{7.4}$$

Define state variables as

$$\mathbf{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} := \begin{bmatrix} v^{(3)}(t) \\ \ddot{v}(t) \\ \dot{v}(t) \\ v(t) \end{bmatrix} \quad \text{or} \quad \hat{\mathbf{x}}(s) = \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \\ \hat{x}_3(s) \\ \hat{x}_4(s) \end{bmatrix} = \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \hat{v}(s) \tag{7.5}$$

Then we have

$$\dot{x}_2 = x_1, \quad \dot{x}_3 = x_2, \quad \text{and} \quad \dot{x}_4 = x_3 \tag{7.6}$$

They are independent of (7.1) and follow directly from the definition in (7.5). In order to develop an equation for $\dot{x}_1$, we substitute (7.5) into (7.3) or

$$(s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4)\hat{v}(s) = \hat{u}(s)$$

to yield

$$s\hat{x}_1(s) = -\alpha_1\hat{x}_1(s) - \alpha_2\hat{x}_2(s) - \alpha_3\hat{x}_3(s) - \alpha_4\hat{x}_4(s) + \hat{u}(s)$$

which becomes, in the time domain,

$$\dot{x}_1(t) = [-\alpha_1 \ \ -\alpha_2 \ \ -\alpha_3 \ \ -\alpha_4]\mathbf{x}(t) + 1\cdot u(t) \tag{7.7}$$

Substituting (7.5) into (7.4) yields

$$\begin{aligned}\hat{y}(s) &= (\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4)\hat{v}(s)\\ &= \beta_1\hat{x}_1(s) + \beta_2\hat{x}_2(s) + \beta_3\hat{x}_3(s) + \beta_4\hat{x}_4(s)\\ &= [\beta_1 \ \ \beta_2 \ \ \beta_3 \ \ \beta_4]\hat{\mathbf{x}}(s)\end{aligned}$$

which becomes, in the time domain,

$$y(t) = [\beta_4 \ \ \beta_3 \ \ \beta_2 \ \ \beta_1]\mathbf{x}(t) \tag{7.8}$$

Equations (7.6), (7.7), and (7.8) can be combined as

$$\begin{aligned}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u &= \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u \\ y = \mathbf{c}\mathbf{x} &= [\beta_1 \ \ \beta_2 \ \ \beta_3 \ \ \beta_4]\mathbf{x}\end{aligned} \tag{7.9}$$

This is a realization of (7.1) and was developed in (4.41) by direct verification.

Before proceeding, we mention that if $N(s)$ in (7.1) is 1, then $y(t) = v(t)$ and the output $y(t)$ and its derivatives can be chosen as state variables. However, if $N(s)$ is a polynomial of degree 1 or higher and if we choose the output and its derivatives as state variables, then its realization will be of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u\\ y &= \mathbf{c}\mathbf{x} + du + d_1\dot{u} + d_2\ddot{u} + \cdots\end{aligned}$$

This equation requires differentiations of u and is not used. Therefore, in general, we cannot select the output y and its derivatives as state variables.[1] We must define state variables by using $v(t)$. Thus $v(t)$ is called a *pseudo state*.

Now we check the controllability and observability of (7.9). Its controllability matrix can readily be computed as

$$\mathcal{C} = \begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 & -\alpha_1^3 + 2\alpha_1\alpha_2 - \alpha_3 \\ 0 & 1 & -\alpha_1 & \alpha_1^2 - \alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.10}$$

1. See also Example 2.16, in particular, (2.47).

Its determinant is 1 for any α_i. Thus the controllability matrix $\mathcal{C}$ has full row rank and the state equation is always controllable. This is the reason that (7.9) is called a *controllable canonical form*.

Next we check its observability. It turns out that it depends on whether or not $N(s)$ and $D(s)$ are *coprime*. Two polynomials are said to be *coprime* if they have no common factor of degree at least 1. More specifically, a polynomial $R(s)$ is called a common factor or a common divisor of $D(s)$ and $N(s)$ if they can be expressed as $D(s) = \bar{D}(s)R(s)$ and $N(s) = \bar{N}(s)R(s)$, where $\bar{D}(s)$ and $\bar{N}(s)$ are polynomials. A polynomial $R(s)$ is called a *greatest common divisor* (gcd) of $D(s)$ and $N(s)$ if (1) it is a common divisor of $D(s)$ and $N(s)$ and (2) it can be divided without remainder by every other common divisor of $D(s)$ and $N(s)$. Note that if $R(s)$ is a gcd, so is $\alpha R(s)$ for any nonzero constant α. Thus greatest common divisors are not unique.[2] In terms of the gcd, the polynomials $D(s)$ and $N(s)$ are coprime if their gcd $R(s)$ is a nonzero constant, a polynomial of degree 0; they are not coprime if their gcd has degree 1 or higher.

Theorem 7.1

The controllable canonical form in (7.9) is observable if and only if $D(s)$ and $N(s)$ in (7.1) are coprime.

Proof: We first show that if (7.9) is observable, then $D(s)$ and $N(s)$ are coprime. We show this by contradiction. If $D(s)$ and $N(s)$ are not coprime, then there exists a λ_1 such that

$$N(\lambda_1) = \beta_1\lambda_1^3 + \beta_2\lambda_1^2 + \beta_3\lambda_1 + \beta_4 = 0 \tag{7.11}$$

$$D(\lambda_1) = \lambda_1^4 + \alpha_1\lambda_1^3 + \alpha_2\lambda_1^2 + \alpha_3\lambda_1 + \alpha_4 = 0 \tag{7.12}$$

Let us define $\mathbf{v} := [\lambda_1^3 \ \ \lambda_1^2 \ \ \lambda_1 \ \ 1]'$; it is a 4×1 nonzero vector. Then (7.11) can be written as $N(\lambda_1) = \mathbf{c}\mathbf{v} = 0$, where $\mathbf{c}$ is defined in (7.9). Using (7.12) and the shifting property of the companion form, we can readily verify

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^4 \\ \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \end{bmatrix} = \lambda_1\mathbf{v} \tag{7.13}$$

Thus we have $\mathbf{A}^2\mathbf{v} = \mathbf{A}(\mathbf{A}\mathbf{v}) = \lambda_1\mathbf{A}\mathbf{v} = \lambda_1^2\mathbf{v}$ and $\mathbf{A}^3\mathbf{v} = \lambda_1^3\mathbf{v}$. We compute, using $\mathbf{c}\mathbf{v} = 0$,

$$\mathcal{O}\mathbf{v} = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \mathbf{c}\mathbf{A}^2 \\ \mathbf{c}\mathbf{A}^3 \end{bmatrix}\mathbf{v} = \begin{bmatrix} \mathbf{c}\mathbf{v} \\ \mathbf{c}\mathbf{A}\mathbf{v} \\ \mathbf{c}\mathbf{A}^2\mathbf{v} \\ \mathbf{c}\mathbf{A}^3\mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{c}\mathbf{v} \\ \lambda_1\mathbf{c}\mathbf{v} \\ \lambda_1^2\mathbf{c}\mathbf{v} \\ \lambda_1^3\mathbf{c}\mathbf{v} \end{bmatrix} = \mathbf{0}$$

which implies that the observability matrix does not have full column rank. This contradicts the hypothesis that (7.9) is observable. Thus if (7.9) is observable, then $D(s)$ and $N(s)$ are coprime.

Next we show the converse; that is, if $D(s)$ and $N(s)$ are coprime, then (7.9) is observable. We show this by contradiction. Suppose (7.9) is not observable, then Theorem 6.O1 implies that there exists an eigenvalue λ_1 of $\mathbf{A}$ and a nonzero vector $\mathbf{v}$ such that

2. If we require $R(s)$ to be monic, then the gcd is unique.

$$\begin{bmatrix} \mathbf{A} - \lambda_1 \mathbf{I} \\ \mathbf{c} \end{bmatrix} \mathbf{v} = \mathbf{0}$$

or

$$\mathbf{A}\mathbf{v} = \lambda_1 \mathbf{v} \quad \text{and} \quad \mathbf{c}\mathbf{v} = 0$$

Thus $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated with eigenvalue λ_1. From (7.13), we see that $\mathbf{v} = [\lambda_1^3 \ \lambda_1^2 \ \lambda_1 \ 1]'$ is an eigenvector. Substituting this $\mathbf{v}$ into $\mathbf{c}\mathbf{v} = 0$ yields

$$N(\lambda_1) = \beta_1 \lambda_1^3 + \beta_2 \lambda_1^2 + \beta_3 \lambda_1 + \beta_4 = 0$$

Thus λ_1 is a root of $N(s)$. The eigenvalue of $\mathbf{A}$ is a root of its characteristic polynomial, which, because of the companion form of $\mathbf{A}$, equals $D(s)$. Thus we also have $D(\lambda_1) = 0$, and $D(s)$ and $N(s)$ have the same factor $s - \lambda_1$. This contradicts the hypothesis that $D(s)$ and $N(s)$ are coprime. Thus if $D(s)$ and $N(s)$ are coprime, then (7.9) is observable. This establishes the theorem. Q.E.D.

If (7.9) is a realization of $\hat{g}(s)$, then we have, by definition,

$$\hat{g}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

Taking its transpose yields

$$\hat{g}'(s) = \hat{g}(s) = [\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}]' = \mathbf{b}'(s\mathbf{I} - \mathbf{A}')^{-1}\mathbf{c}'$$

Thus the state equation

$$\dot{\mathbf{x}} = \mathbf{A}'\mathbf{x} + \mathbf{c}'u = \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} u \tag{7.14}$$

$$y = \mathbf{b}'\mathbf{x} = [1 \ 0 \ 0 \ 0]\mathbf{x}$$

is a different realization of (7.1). This state equation is always observable and is called an *observable canonical form*. Dual to Theorem 7.1, Equation (7.14) is controllable if and only if $D(s)$ and $N(s)$ are coprime.

We mention that the equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{7.15}$$

will transform (7.9) into

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\beta_4 \ \beta_3 \ \beta_2 \ \beta_1]\mathbf{x}$$

This is also called a controllable canonical form. Similarly, (7.15) will transform (7.14) into

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & -\alpha_4 \\ 1 & 0 & 0 & -\alpha_3 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \end{bmatrix} u$$

$$y = \mathbf{c}\mathbf{x} = [0 \ 0 \ 0 \ 1]\mathbf{x}$$

This is a different observable canonical form.

7.2.1 Minimal Realizations

We first define a degree for proper rational functions. We call $N(s)/D(s)$ a *polynomial fraction* or, simply, a *fraction*. Because

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{N(s)Q(s)}{D(s)Q(s)}$$

for any polynomial $Q(s)$, fractions are not unique. Let $R(s)$ be a greatest common divisor (gcd) of $N(s)$ and $D(s)$. That is, if we write $N(s) = \bar{N}(s)R(s)$ and $D(s) = \bar{D}(s)R(s)$, then the polynomials $\bar{N}(s)$ and $\bar{D}(s)$ are coprime. Clearly every rational function $\hat{g}(s)$ can be reduced to $\hat{g}(s) = \bar{N}(s)/\bar{D}(s)$. Such an expression is called a *coprime fraction*. We call $\bar{D}(s)$ a *characteristic polynomial* of $\hat{g}(s)$. The degree of the characteristic polynomial is defined as the *degree* of $\hat{g}(s)$. Note that characteristic polynomials are not unique; they may differ by a nonzero constant. If we require the polynomial to be monic, then it is unique.

Consider the rational function

$$\hat{g}(s) = \frac{s^2 - 1}{4(s^3 - 1)}$$

Its numerator and denominator contain the common factor $s - 1$. Thus its coprime fraction is $\hat{g}(s) = (s+1)/4(s^2+s+1)$ and its characteristic polynomial is $4s^2+4s+4$. Thus the rational function has degree 2. Given a proper rational function, if its numerator and denominator are coprime—as is often the case—then its denominator is a characteristic polynomial and the degree of the denominator is the degree of the rational function.

Theorem 7.2

A state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is a minimal realization of a proper rational function $\hat{g}(s)$ if and only if $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{A}, \mathbf{c})$ is observable or if and only if

$$\dim \mathbf{A} = \deg \hat{g}(s)$$

Proof: If $(\mathbf{A}, \mathbf{b})$ is not controllable or if $(\mathbf{A}, \mathbf{c})$ is not observable, then the state equation can be reduced to a lesser dimensional state equation that has the same transfer function (Theorems 6.6 and 6.O6). Thus $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is not a minimal realization. This shows the necessity of the theorem.

To show the sufficiency, consider the n-dimensional controllable and observable state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$
$$y = \mathbf{c}\mathbf{x} + du \tag{7.16}$$

Clearly its $n \times n$ controllability matrix

$$C = [\mathbf{b}\ \ \mathbf{A}\mathbf{b}\ \ \cdots\ \ \mathbf{A}^{n-1}\mathbf{b}] \tag{7.17}$$

and its $n \times n$ observability matrix

$$O = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \vdots \\ \mathbf{c}\mathbf{A}^{n-1} \end{bmatrix} \tag{7.18}$$

both have rank n. We show that (7.16) is a minimal realization by contradiction. Suppose the $\bar{n}$-dimensional state equation, with $\bar{n} < n$,

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u$$
$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} + \bar{d}u \tag{7.19}$$

is a realization of $\hat{g}(s)$. Then Theorem 4.1 implies $d = \bar{d}$ and

$$\mathbf{c}\mathbf{A}^m\mathbf{b} = \bar{\mathbf{c}}\bar{\mathbf{A}}^m\bar{\mathbf{b}} \qquad \text{for } m = 0,\ 1,\ 2,\ \ldots \tag{7.20}$$

Let us consider the product

$$OC = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \vdots \\ \mathbf{c}\mathbf{A}^{n-1} \end{bmatrix} [\mathbf{b}\ \mathbf{A}\mathbf{b}\ \cdots \mathbf{A}^{n-1}\mathbf{b}]$$
$$= \begin{bmatrix} \mathbf{c}\mathbf{b} & \mathbf{c}\mathbf{A}\mathbf{b} & \mathbf{c}\mathbf{A}^2\mathbf{b} & \cdots & \mathbf{c}\mathbf{A}^{n-1}\mathbf{b} \\ \mathbf{c}\mathbf{A}\mathbf{b} & \mathbf{c}\mathbf{A}^2\mathbf{b} & \mathbf{c}\mathbf{A}^3\mathbf{b} & \cdots & \mathbf{c}\mathbf{A}^{n}\mathbf{b} \\ \mathbf{c}\mathbf{A}^2\mathbf{b} & \mathbf{c}\mathbf{A}^3\mathbf{b} & \mathbf{c}\mathbf{A}^4\mathbf{b} & \cdots & \mathbf{c}\mathbf{A}^{n+1}\mathbf{b} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}\mathbf{A}^{n-1}\mathbf{b} & \mathbf{c}\mathbf{A}^{n}\mathbf{b} & \mathbf{c}\mathbf{A}^{n+1}\mathbf{b} & \cdots & \mathbf{c}\mathbf{A}^{2(n-1)}\mathbf{b} \end{bmatrix} \tag{7.21}$$

Using (7.20), we can replace every $\mathbf{c}\mathbf{A}^m\mathbf{b}$ by $\bar{\mathbf{c}}\bar{\mathbf{A}}^m\bar{\mathbf{b}}$. Thus we have

$$OC = \bar{O}_n\bar{C}_n \tag{7.22}$$

where $\bar{O}_n$ is defined as in (6.21) for the $\bar{n}$-dimensional state equation in (7.19) and $\bar{C}_n$ is defined similarly. Because (7.16) is controllable and observable, we have $\rho(O) = n$ and $\rho(C) = n$. Thus (3.62) implies $\rho(OC) = n$. Now $\bar{O}_n$ and $\bar{C}_n$ are, respectively, $n \times \bar{n}$ and $\bar{n} \times n$; thus (3.61) implies that the matrix $\bar{O}_n\bar{C}_n$ has rank at most $\bar{n}$. This contradicts $\rho(\bar{O}_n\bar{C}_n) = \rho(OC) = n$. Thus $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is minimal. This establishes the first part of the theorem.

The realization in (7.9) is controllable and observable if and only if $\hat{g}(s) = N(s)/D(s)$ is a coprime fraction (Theorem 7.1). In this case, we have dim $\mathbf{A}$ = deg $D(s)$ = deg $\hat{g}(s)$. Because all minimal realizations are equivalent, as will be established immediately, we conclude that every realization is minimal if and only if dim $\mathbf{A}$ = deg $\hat{g}(s)$. This establishes the theorem. Q.E.D.

To complete the proof of Theorem 7.2, we need the following theorem.

Theorem 7.3

All minimal realizations of $\hat{g}(s)$ are equivalent.

Proof: Let $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ and $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{d})$ be minimal realizations of $\hat{g}(s)$. Then we have $\bar{d} = d$ and, following (7.22),

$$OC = \bar{O}\bar{C} \tag{7.23}$$

Multiplying OAC out explicitly and then using (7.20), we can show

$$OAC = \bar{O}\bar{A}\bar{C} \tag{7.24}$$

Note that the controllability and observability matrices are all nonsingular square matrices. Let us define

$$\mathbf{P} := \bar{O}^{-1}O$$

Then (7.23) implies

$$\mathbf{P} = \bar{O}^{-1}O = \bar{C}C^{-1} \quad \text{and} \quad \mathbf{P}^{-1} = O^{-1}\bar{O} = C\bar{C}^{-1} \tag{7.25}$$

From (7.23), we have $\bar{C} = \bar{O}^{-1}OC = \mathbf{P}C$. The first columns on both side of the equality yield $\bar{\mathbf{b}} = \mathbf{P}\mathbf{b}$. Again from (7.23), we have $\bar{O} = OC\bar{C}^{-1} = O\mathbf{P}^{-1}$. The first rows on both sides of the equality yield $\bar{\mathbf{c}} = \mathbf{c}\mathbf{P}^{-1}$. Equation (7.24) implies

$$\bar{\mathbf{A}} = \bar{O}^{-1}OAC\bar{C}^{-1} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$$

Thus $(\mathbf{A}, \mathbf{b}, \mathbf{c}\, d)$ and $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{d})$ meet the conditions in (4.26) and, consequently, are equivalent. This establishes the theorem. Q.E.D.

Theorem 7.2 has many important implications. Given a state equation, if we compute its transfer function and degree, then the minimality of the state equation can readily be determined without checking its controllability and observability. Thus the theorem provides an alternative way of checking controllability and observability. Conversely, given a rational function, if we compute first its common factors and reduce it to a coprime fraction, then the state equations obtained by using its coefficients as shown in (7.9) and (7.14) will automatically be controllable and observable.

Consider a proper rational function $\hat{g}(s) = N(s)/D(s)$. If the fraction is coprime, then every root of $D(s)$ is a pole of $\hat{g}(s)$ and vice versa. This is not true if $N(s)$ and $D(s)$ are not coprime. Let $(\mathbf{A}, \mathbf{b}.\mathbf{c}.d)$ be a minimal realization of $\hat{g}(s) = N(s)/D(s)$. Then we have

$$\frac{N(s)}{D(s)} = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d = \frac{1}{\det(s\mathbf{I} - \mathbf{A})}\mathbf{c}\left[\mathrm{Adj}(s\mathbf{I} - \mathbf{A})\right]\mathbf{b} + d$$

If $N(s)$ and $D(s)$ are coprime, then $\deg D(s) = \deg \hat{g}(s) = \dim \mathbf{A}$. Thus we have

$$D(s) = k \det(s\mathbf{I} - \mathbf{A})$$

for some nonzero constant k. Note that $k = 1$ if $D(s)$ is monic. This shows that if a state equation is controllable and observable, then every eigenvalue of $\mathbf{A}$ is a pole of $\hat{g}(s)$ and every pole of $\hat{g}(s)$ is an eigenvalue of $\mathbf{A}$. Thus we conclude that if $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is controllable and observable, then we have

$$\text{Asymptotic stability} \Longleftrightarrow \text{BIBO stability}$$

More generally, *controllable and observable state equations and coprime fractions contain essentially the same information and either description can be used to carry out analysis and design.*

7.3 Computing Coprime Fractions

The importance of coprime fractions and degrees was demonstrated in the preceding section. In this section, we discuss how to compute them. Consider a proper rational function

$$\hat{g}(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials. If we use the MATLAB function `roots` to compute their roots and then to cancel their common factors, we will obtain a coprime fraction. The MATLAB function `minreal` can also be used to obtain coprime fractions. In this section, we introduce a different method by solving a set of linear algebraic equations. The method does not offer any advantages over the aforementioned methods for scalar rational functions. However, it can readily be extended to the matrix case. More importantly, the method will be used to carry out design in Chapter 9.

Consider $N(s)/D(s)$. To simplify the discussion, we assume $\deg N(s) \le \deg D(s) = n = 4$. Let us write

$$\frac{N(s)}{D(s)} = \frac{\bar{N}(s)}{\bar{D}(s)}$$

which implies

$$D(s)(-\bar{N}(s)) + N(s)\bar{D}(s) = 0 \tag{7.26}$$

It is clear that $D(s)$ and $N(s)$ are not coprime if and only if there exist polynomials $\bar{N}(s)$ and $\bar{D}(s)$ with $\deg \bar{N}(s) \le \deg \bar{D}(s) < n = 4$ to meet (7.26). The condition $\deg \bar{D}(s) < n$ is crucial; otherwise, (7.26) has infinitely many solutions $\bar{N}(s) = N(s)R(s)$ and $\bar{D}(s) = D(s)R(s)$ for any polynomial $R(s)$. Thus the coprimeness problem can be reduced to solving the polynomial equation in (7.26).

Instead of solving (7.26) directly, we will change it into solving a set of linear algebraic equations. We write

$$\begin{aligned}
D(s) &= D_0 + D_1 s + D_2 s^2 + D_3 s^3 + D_4 s^4 \\
N(s) &= N_0 + N_1 s + N_2 s^2 + N_3 s^3 + N_4 s^4 \\
\bar{D}(s) &= \bar{D}_0 + \bar{D}_1 s + \bar{D}_2 s^2 + \bar{D}_3 s^3 \\
\bar{N}(s) &= \bar{N}_0 + \bar{N}_1 s + \bar{N}_2 s^2 + \bar{N}_3 s^3
\end{aligned} \tag{7.27}$$

where $D_4 \ne 0$ and the remaining D_i, N_i, $\bar{D}_i$, and $\bar{N}_i$ can be zero or nonzero. Substituting these into (7.26) and equating to zero the coefficients associated with s^k, for $k = 0, 1, \ldots, 7$, we obtain

$$\mathbf{Sm} := \begin{bmatrix}
D_0 & N_0 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\
D_1 & N_1 & \vdots & D_0 & N_0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\
D_2 & N_2 & \vdots & D_1 & N_1 & \vdots & D_0 & N_0 & \vdots & 0 & 0 \\
D_3 & N_3 & \vdots & D_2 & N_2 & \vdots & D_1 & N_1 & \vdots & D_0 & N_0 \\
D_4 & N_4 & \vdots & D_3 & N_3 & \vdots & D_2 & N_2 & \vdots & D_1 & N_1 \\
0 & 0 & \vdots & D_4 & N_4 & \vdots & D_3 & N_3 & \vdots & D_2 & N_2 \\
0 & 0 & \vdots & 0 & 0 & \vdots & D_4 & N_4 & \vdots & D_3 & N_3 \\
0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & D_4 & N_4
\end{bmatrix}
\begin{bmatrix} -\bar{N}_0 \\ \bar{D}_0 \\ \cdots \\ -\bar{N}_1 \\ \bar{D}_1 \\ \cdots \\ -\bar{N}_2 \\ \bar{D}_2 \\ \cdots \\ -\bar{N}_3 \\ \bar{D}_3 \end{bmatrix} = \mathbf{0} \tag{7.28}$$

This is a homogeneous linear algebraic equation. The first block column of $\mathbf{S}$ consists of two columns formed from the coefficients of $D(s)$ and $N(s)$ arranged in ascending powers of s. The second block column is the first block column shifted down one position. Repeating the process until $\mathbf{S}$ is a square matrix of order $2n = 8$. The square matrix $\mathbf{S}$ is called the *Sylvester resultant*. If the Sylvester resultant is singular, nonzero solutions exist in (7.28) (Theorem 3.3). This means that polynomials $\bar{N}(s)$ and $\bar{D}(s)$ of degree 3 or less exist to meet (7.26). Thus $D(s)$ and $N(s)$ are not coprime. If the Sylvester resultant is nonsingular, no nonzero solutions exist in (7.28) or, equivalently, no polynomials $\bar{N}(s)$ and $\bar{D}(s)$ of degree 3 or less exist to meet (7.26). Thus $D(s)$ and $N(s)$ are coprime. In conclusion, *$D(s)$ and $N(s)$ are coprime if and only if the Sylvester resultant is nonsingular.*

If the Sylvester resultant is singular, then $N(s)/D(s)$ can be reduced to

$$\frac{N(s)}{D(s)} = \frac{\bar{N}(s)}{\bar{D}(s)}$$

where $\bar{N}(s)$ and $\bar{D}(s)$ are coprime. We discuss how to obtain a coprime fraction directly from (7.28). Let us search linearly independent columns of $\mathbf{S}$ in order from left to right. We call columns formed from D_i D-columns and formed from N_i N-columns. Then every D-column is linearly independent of its left-hand-side (LHS) columns. Indeed, because $D_4 \ne 0$, the first D-column is linearly independent. The second D-column is also linearly independent of its LHS columns because the LHS entries of D_4 are all zero. Proceeding forward, we conclude that all D-columns are linearly independent of their LHS columns. On the other hand, an N-column can be dependent or independent of its LHS columns. Because of the repetitive pattern of S, if an N-column becomes linearly dependent on its LHS columns, then all subsequent N-columns are linearly dependent of their LHS columns. Let μ denote the number of linearly independent N-columns in S. Then the $(\mu+1)$th N-column is the first N-column to become linearly dependent on its LHS columns and will be called the *primary dependent* N-*column*. Let us use $\mathbf{S}_1$ to denote the submatrix of S that consists of the primary dependent N-column and all its LHS columns. That is, $\mathbf{S}_1$ consists of $\mu+1$ D-columns (all of them are linearly independent) and $\mu+1$ N-columns (the last one is dependent). Thus $\mathbf{S}_1$ has $2(\mu+1)$ columns but rank $2\mu+1$. In other words, $\mathbf{S}_1$ has nullity 1 and, consequently, has one independent null vector. Note that if $\bar{\mathbf{n}}$ is a null vector, so is $\alpha\bar{\mathbf{n}}$ for any nonzero α. Although any null vector can be used, we will use exclusively the null vector with 1 as its last entry to develop $\bar{N}(s)$ and $\bar{D}(s)$. For convenience, we call such a null vector a *monic null vector*. If we use the MATLAB function `null` to generate a null vector, then the null vector must be divided by its last entry to yield a monic null vector. This is illustrated in the next example.

EXAMPLE 7.1 Consider

$$\frac{N(s)}{D(s)} = \frac{6s^3+s^2+3s-20}{2s^4+7s^3+15s^2+16s+10} \tag{7.29}$$

We have $n=4$ and its Sylvester resultant S is 8×8. The fraction is coprime if and only if S is nonsingular or has rank 8. We use MATLAB to check the rank of S. Because it is simpler to key in the transpose of S, we type

```
d=[10 16 15 7 2];n=[-20 3 1 6 0];
s=[d 0 0 0;n 0 0 0;0 d 0 0;0 n 0 0;...
  0 0 d 0;0 0 n 0;0 0 0 d;0 0 0 n]';
m=rank(s)
```

The answer is 6; thus $D(s)$ and $N(s)$ are not coprime. Because all four D-columns of S are linearly independent, we conclude that S has only two linearly independent N-columns and $\mu=2$. The third N-column is the primary dependent N-column and all its LHS columns are linearly independent. Let $\mathbf{S}_1$ denote the first six columns of S, an 8×6 matrix. The submatrix $\mathbf{S}_1$ has three D-column (all linearly independent) and two linearly independent N-columns, thus it has rank 5 and nullity 1. Because all entries of the last row of $\mathbf{S}_1$ are zero, they can be skipped in forming $\mathbf{S}_1$. We type

```
s1=[d 0 0;n 0 0;0 d 0;0 n 0;0 0 d;0 0 n]';
z=null(s1)
```

which yields

```
ans z= [ 0.6860  0.3430  -0.5145  0.3430  0.0000  0.1715 ]'
```

This null vector does not have 1 as its last entry. We divide it by the last entry or the sixth entry of z by typing

```
zb=z/z(6)
```


which yields

```
ans zb= [4 2  -3 2 0 1]'
```

This monic null vector equals $[-\bar{N}_0 \quad \bar{D}_0 \quad -\bar{N}_1 \quad \bar{D}_1 \quad -\bar{N}_2 \quad \bar{D}_2]'$. Thus we have

$$\bar{N}(s) = -4+3s+0\cdot s^2 \qquad \bar{D}(s) = 2+2s+s^2$$

and

$$\frac{6s^3+s^2+3s-20}{2s^4+7s^3+15s^2+16s+10} = \frac{3s-4}{s^2+2s+2}$$

Because the null vector is computed from the first linearly dependent N-column, the computed $\bar{N}(s)$ and $\bar{D}(s)$ have the smallest possible degrees to meet (7.26) and, therefore, are coprime. This completes the reduction of $N(s)/D(s)$ to a coprime fraction.

The preceding procedure can be summerized as a theorem.

Theroem 7.4

Consider $\hat{g}(s) = N(s)/D(s)$. We use the coefficients of $D(s)$ and $N(s)$ to form the Sylvester resultant S in (7.28) and search its linearly independent columns in order from left to right. Then we have

$$\deg \hat{g}(s) = \text{ number of linearly independent } N\text{-columns } =: \mu$$

and the coefficients of a coprime fraction $\hat{g}(s) = \bar{N}(s)/\bar{D}(s)$ or

$$[-\bar{N}_0 \quad \bar{D}_0 \quad -\bar{N}_1 \quad \bar{D}_1 \ \cdots \ -\bar{N}_\mu \quad \bar{D}_\mu]'$$

equals the monic null vector of the submatrix that consists of the primary dependent N-column and all its LHS linearly independent columns of S.

We mention that if D- and N-columns in S are arranged in descending powers of s, then it is not true that all D-columns are linearly independent of their LHS columns and that the degree of $\hat{g}(s)$ equals the number of linearly independent N-columns. See Problem 7.6. Thus it is essential to arrange the D- and N-columns in ascending powers of s in S.

7.3.1 QR Decomposition

As discussed in the preceding section, a coprime fraction can be obtained by searching linearly independent columns of the Sylvester resultant in order from left to right. It turns out the widely available QR decomposition can be used to achieve this searching.

Consider an $n\times m$ matrix M. Then there exists an $n\times n$ orthogonal matrix $\bar{\mathbf{Q}}$ such that

$$\bar{\mathbf{Q}}\mathbf{M} = \mathbf{R}$$

where R is an upper triangular matrix of the same dimensions as M. Because $\bar{\mathbf{Q}}$ operates on the rows of M, the linear independence of the columns of M is preserved in the columns of R. In other words, if a column of R is linearly dependent on its left-hand-side (LHS) columns, so is the corresponding column of $\mathbf{M}$. Now because $\mathbf{R}$ is in upper triangular form, its mth column is linearly independent of its LHS columns if and only if its mth entry at the diagonal position is nonzero. Thus using $\mathbf{R}$, the linearly independent columns of $\mathbf{M}$, in order from left to right, can be obtained by inspection. Because $\bar{\mathbf{Q}}$ is orthogonal, we have $\bar{\mathbf{Q}}^{-1} = \bar{\mathbf{Q}}' =: \mathbf{Q}$ and $\bar{\mathbf{Q}}\mathbf{M} = \mathbf{R}$ becomes $\mathbf{M} = \mathbf{QR}$. This is called *QR decomposition*. In MATLAB, $\mathbf{Q}$ and $\mathbf{R}$ can be obtained by typing `[q,r]=qr(m)`.

Let us apply QR decomposition to the resultant in Example 7.1. We type

```
d=[10 16 15 7 2];n=[-20 3 1 6 0];
s=[d 0 0 0;n 0 0 0;0 d 0 0;0 n 0 0;...
   0 0 d 0;0 0 n 0;0 0 0 d;0 0 0 n]';
[q,r]=qr(s)
```

Because $\mathbf{Q}$ is not needed, we show only $\mathbf{R}$:

$$r = \begin{bmatrix} -25.1 & 3.7 & -20.6 & 10.1 & -11.6 & 11.0 & -4.1 & 5.3 \\ 0 & -20.7 & -10.3 & 4.3 & -7.2 & 2.1 & -3.6 & 6.7 \\ 0 & 0 & -10.2 & -15.6 & -20.3 & 0.8 & -16.8 & 9.6 \\ 0 & 0 & 0 & 8.9 & -3.5 & -17.9 & -11.2 & 7.3 \\ 0 & 0 & 0 & 0 & -5.0 & 0 & -12.0 & -15.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the matrix is upper triangular. Because the sixth column has 0 as its sixth entry (diagonal position), it is linearly dependent on its LHS columns. So is the last column. To determine whether a column is linearly dependent, we need to know only whether the diagonal entry is zero or not. Thus the matrix can be simplified as

$$r = \begin{bmatrix} d & x & x & x & x & x & x & x \\ 0 & n & x & x & x & x & x & x \\ 0 & 0 & d & x & x & x & x & x \\ 0 & 0 & 0 & n & x & x & x & x \\ 0 & 0 & 0 & 0 & d & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where d, n, and x denote nonzero entries and d also denotes D-column and n denotes N-column. We see that every D-column is linearly independent of its LHS columns and there are only two linearly independent N-columns. Thus by employing QR decomposition, we obtain immediately μ and the primary dependent N-column. In scalar transfer functions, we can use either `rank` or `qr` to find μ. In the matrix case, using `rank` is very inconvenient; we will use QR decomposition.

7.4 Balanced Realization[3]

Every transfer function has infinitely many minimal realizations. Among these realizations, it is of interest to see which realizations are more suitable for practical implementation. If we use the controllable or observable canonical form, then the A-matrix and b- or c-vector have many zero entries, and its implementation will use a small number of components. However, either canonical form is very sensitive to parameter variations; therefore both forms should be avoided if sensitivity is an important issue. If all eigenvalues of $\mathbf{A}$ are distinct, we can transform $\mathbf{A}$, using an equivalence transformation, into a diagonal form (if all eigenvalues are real) or into the modal form discussed in Section 4.3.1 (if some eigenvalues are complex). The diagonal or modal form has many zero entries in $\mathbf{A}$ and will use a small number of components in its implementation. More importantly, the diagonal and modal forms are least sensitive to parameter variations among all realizations; thus they are good candidates for practical implementation.

We discuss next a different minimal realization, called a balanced realization. However, the discussion is applicable only to stable $\mathbf{A}$. Consider

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}\mathbf{x} \end{aligned} \tag{7.30}$$

It is assumed that $\mathbf{A}$ is stable or all its eigenvalues have negative real parts. Then the controllability Gramian $\mathbf{W}_c$ and the observability Gramian $\mathbf{W}_o$ are, respectively, the unique solutions of

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}' = -\mathbf{b}\mathbf{b}' \tag{7.31}$$

and

$$\mathbf{A}'\mathbf{W}_o + \mathbf{W}_o\mathbf{A} = -\mathbf{c}'\mathbf{c} \tag{7.32}$$

They are positive definite if (7.30) is controllable and observable.

Different minimal realizations of the same transfer function have different controllability and observability Gramians. For example, the state equation, taken from Reference [23],

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -1 & -4/\alpha \\ 4\alpha & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2\alpha \end{bmatrix}u \\ y &= [-1 \;\; 2/\alpha]\mathbf{x} \end{aligned} \tag{7.33}$$

for any nonzero α, has transfer function $\hat{g}(s) = (3s+18)/(s^2+3s+18)$, and is controllable and observable. Its controllability and observability Gramians can be computed as

$$\mathbf{W}_c = \begin{bmatrix} 0.5 & 0 \\ 0 & \alpha^2 \end{bmatrix} \quad \text{and} \quad \mathbf{W}_o = \begin{bmatrix} 0.5 & 0 \\ 0 & 1/\alpha^2 \end{bmatrix} \tag{7.34}$$

We see that different α yields different minimal realization and different controllability and observability Gramians. Even though the controllability and observability Gramians will change, their product remains the same as diag (0.25, 1) for all α.

3. This section may be skipped without loss of continuity.

Theorem 7.5

Let $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ and $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$ be minimal and equivalent and let $\mathbf{W}_c\mathbf{W}_o$ and $\bar{\mathbf{W}}_c\bar{\mathbf{W}}_o$ be the products of their controllability and observability Gramians. Then $\mathbf{W}_c\mathbf{W}_o$ and $\bar{\mathbf{W}}_c\bar{\mathbf{W}}_o$ are similar and their eigenvalues are all real and positive.

Proof: Let $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$, where $\mathbf{P}$ is a nonsingular constant matrix. Then we have

$$\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} \qquad \bar{\mathbf{b}} = \mathbf{P}\mathbf{b} \qquad \bar{\mathbf{c}} = \mathbf{c}\mathbf{P}^{-1} \tag{7.35}$$

The controllability Gramian $\bar{\mathbf{W}}_c$ and observability Gramian $\bar{\mathbf{W}}_o$ of $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$ are, respectively, the unique solutions of

$$\bar{\mathbf{A}}\bar{\mathbf{W}}_c + \bar{\mathbf{W}}_c\bar{\mathbf{A}}' = -\bar{\mathbf{b}}\bar{\mathbf{b}}' \tag{7.36}$$

and

$$\bar{\mathbf{A}}'\bar{\mathbf{W}}_o + \bar{\mathbf{W}}_o\bar{\mathbf{A}} = -\bar{\mathbf{c}}'\bar{\mathbf{c}} \tag{7.37}$$

Substituting $\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ and $\bar{\mathbf{b}} = \mathbf{P}\mathbf{b}$ into (7.36) yields

$$\mathbf{P}\mathbf{A}\mathbf{P}^{-1}\bar{\mathbf{W}}_c + \bar{\mathbf{W}}_c(\mathbf{P}')^{-1}\mathbf{A}'\mathbf{P}' = -\mathbf{P}\mathbf{b}\mathbf{b}'\mathbf{P}'$$

which implies

$$\mathbf{A}\mathbf{P}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1} + \mathbf{P}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1}\mathbf{A}' = -\mathbf{b}\mathbf{b}'$$

Comparing this with (7.31) yields

$$\mathbf{W}_c = \mathbf{P}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1} \quad \text{or} \quad \bar{\mathbf{W}}_c = \mathbf{P}\mathbf{W}_c\mathbf{P}' \tag{7.38}$$

Similarly, we can show

$$\mathbf{W}_o = \mathbf{P}'\bar{\mathbf{W}}_o\mathbf{P} \quad \text{or} \quad \bar{\mathbf{W}}_o = (\mathbf{P}')^{-1}\mathbf{W}_o\mathbf{P}^{-1} \tag{7.39}$$

Thus we have

$$\mathbf{W}_c\mathbf{W}_o = \mathbf{P}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1}\mathbf{P}'\bar{\mathbf{W}}_o\mathbf{P} = \mathbf{P}^{-1}\bar{\mathbf{W}}_c\bar{\mathbf{W}}_o\mathbf{P}$$

This shows that all $\mathbf{W}_c\mathbf{W}_o$ are similar and, consequently, have the same set of eigenvalues.

Next we show that all eigenvalues of $\mathbf{W}_c\mathbf{W}_o$ are real and positive. Note that both $\mathbf{W}_c$ and $\mathbf{W}_o$ are symmetric, but their product may not be. Therefore Theorem 3.6 is not directly applicable to $\mathbf{W}_c\mathbf{W}_o$. Now we apply Theorem 3.6 to $\mathbf{W}_c$:

$$\mathbf{W}_c = \mathbf{Q}'\mathbf{D}\mathbf{Q} = \mathbf{Q}'\mathbf{D}^{1/2}\mathbf{D}^{1/2}\mathbf{Q} =: \mathbf{R}'\mathbf{R} \tag{7.40}$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{W}_c$ on the diagonal. Because $\mathbf{W}_c$ is symmetric and positive definite, all its eigenvalues are real and positive. Thus we can express $\mathbf{D}$ as $\mathbf{D}^{1/2}\mathbf{D}^{1/2}$, where $\mathbf{D}^{1/2}$ is diagonal with positive square roots of the diagonal entries of $\mathbf{D}$ as its diagonal entries. Note that $\mathbf{Q}$ is orthogonal or $\mathbf{Q}^{-1} = \mathbf{Q}'$. The matrix $\mathbf{R} = \mathbf{D}^{1/2}\mathbf{Q}$ is not orthogonal but is nonsingular.

Consider $\mathbf{R}\mathbf{W}_o\mathbf{R}'$; it is clearly symmetric and positive definite. Thus its eigenvalues are all real and positive. Using (7.40) and (3.66), we have

$$\det(\sigma^2\mathbf{I} - \mathbf{W}_c\mathbf{W}_o) = \det(\sigma^2\mathbf{I} - \mathbf{R}'\mathbf{R}\mathbf{W}_o) = \det(\sigma^2\mathbf{I} - \mathbf{R}\mathbf{W}_o\mathbf{R}') \tag{7.41}$$

which implies that $\mathbf{W}_c\mathbf{W}_o$ and $\mathbf{R}\mathbf{W}_o\mathbf{R}'$ have the same set of eigenvalues. Thus we conclude that all eigenvalues of $\mathbf{W}_c\mathbf{W}_o$ are real and positive. Q.E.D.

Let us define

$$\mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{7.42}$$

where σ_i are positive square roots of the eigenvalues of $\mathbf{W}_c\mathbf{W}_o$. For convenience, we arrange them in descending order in magnitude or

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

These eigenvalues are called the *Hankel singular values*. The product $\mathbf{W}_c\mathbf{W}_o$ of any minimal realization is similar to $\mathbf{\Sigma}^2$.

Theorem 7.6

For any n-dimensional minimal state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, there exists an equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ such that the controllability Gramian $\bar{\mathbf{W}}_c$ and observability Gramian $\bar{\mathbf{W}}_o$ of its equivalent state equation have the property

$$\bar{\mathbf{W}}_c = \bar{\mathbf{W}}_o = \mathbf{\Sigma} \tag{7.43}$$

This is called a *balanced realization*.

Proof: We first compute $\mathbf{W}_c = \mathbf{R}'\mathbf{R}$ as in (7.40). We then apply Theorem 3.6 to the real and symmetric matrix $\mathbf{R}\mathbf{W}_o\mathbf{R}'$ to yield

$$\mathbf{R}\mathbf{W}_o\mathbf{R}' = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}'$$

where $\mathbf{U}$ is orthogonal or $\mathbf{U}'\mathbf{U} = \mathbf{I}$. Let

$$\mathbf{P}^{-1} = \mathbf{R}'\mathbf{U}\mathbf{\Sigma}^{-1/2} \quad \text{or} \quad \mathbf{P} = \mathbf{\Sigma}^{1/2}\mathbf{U}'(\mathbf{R}')^{-1}$$

Then (7.38) and $\mathbf{W}_c = \mathbf{R}'\mathbf{R}$ imply

$$\bar{\mathbf{W}}_c = \mathbf{\Sigma}^{1/2}\mathbf{U}'(\mathbf{R}')^{-1}\mathbf{W}_c\mathbf{R}^{-1}\mathbf{U}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}$$

and (7.39) and $\mathbf{R}\mathbf{W}_o\mathbf{R}' = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}'$ imply

$$\bar{\mathbf{W}}_o = \mathbf{\Sigma}^{-1/2}\mathbf{U}'\mathbf{R}\mathbf{W}_o\mathbf{R}'\mathbf{U}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma}$$

This establishes the theorem. Q.E.D.

By selecting a different $\mathbf{P}$, it is possible to find an equivalent state equation with $\bar{\mathbf{W}}_c = \mathbf{I}$ and $\bar{\mathbf{W}}_o = \mathbf{\Sigma}^2$. Such a state equation is called the *input-normal* realization. Similarly, we can have a state equation with $\bar{\mathbf{W}}_c = \mathbf{\Sigma}^2$ and $\bar{\mathbf{W}}_o = \mathbf{I}$, which is called the *output-normal* realization. The balanced realization in Theorem 7.5 can be used in system reduction. More specifically, suppose

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} u \qquad (7.44)$$
$$y = [\mathbf{c}_1 \quad \mathbf{c}_2]\mathbf{x}$$

is a balanced minimal realization of a stable $\hat{g}(s)$ with

$$\mathbf{W}_c = \mathbf{W}_o = \text{diag}(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2)$$

where the A-, b-, and c-matrices are partitioned according to the order of $\mathbf{\Sigma}_i$. If the Hankel singular values of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are disjoint, then the reduced state equation

$$\dot{\mathbf{x}}_1 = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{b}_1 u \qquad (7.45)$$
$$y = \mathbf{c}_1\mathbf{x}_1$$

is balanced and $\mathbf{A}_{11}$ is stable. If the singular values of $\mathbf{\Sigma}_2$ are much smaller than those of $\mathbf{\Sigma}_1$, then the transfer function of (7.45) will be close to $\hat{g}(s)$. See Reference [23].

The MATLAB function `balreal` will transform $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ into a balanced state equation. The reduced equation in (7.45) can be obtained by using `balred`. The results in this section are based on the controllability and observability Gramians. Because the Gramians in the MIMO case are square as in the SISO case; all results in this section apply to the MIMO case without any modification.

7.5 Realizations from Markov Parameters[4]

Consider the strictly proper rational function

$$\hat{g}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{s-2} + \cdots + \alpha_{n-1}s + \alpha_n} \qquad (7.46)$$

We expand it into an infinite power series as

$$\hat{g}(s) = h(0) + h(1)s^{-1} + h(2)s^{-2} + \cdots \qquad (7.47)$$

If $\hat{g}(s)$ is strictly proper as assumed in (7.46), then $h(0) = 0$. The coefficients $h(m)$, $m = 1, 2, \ldots$, are called *Markov parameters*. Let $g(t)$ be the inverse Laplace transform of $\hat{g}(s)$ or, equivalently, the impulse response of the system. Then we have

$$h(m) = \left.\frac{d^{m-1}}{dt^{m-1}} g(t)\right|_{t=0}$$

for $m = 1, 2, 3, \ldots$. This method of computing Markov parameters is impractical because it requires repetitive differentiations, and differentiations are susceptible to noise.[5] Equating (7.46) and (7.47) yields

4. This section may be skipped without loss of continuity.

5. In the discrete-time case, if we apply an impulse sequence to a system, then the output sequence directly yields Markov parameters. Thus Markov parameters can easily be generated in discrete-time systems.

$$\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_n$$
$$= (s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n)(h(1)s^{-1} + h(2)s^{-2} + \cdots)$$

From this equation, we can obtain the Markov parameters recursively as

$$h(1) = \beta_1$$
$$h(2) = -\alpha_1 h(1) + \beta_2$$
$$h(3) = -\alpha_1 h(2) - \alpha_2 h(1) + \beta_3$$
$$\vdots$$
$$h(n) = -\alpha_1 h(n-1) - \alpha_2 h(n-2) - \cdots - \alpha_{n-1}h(1) + \beta_n \qquad (7.48)$$
$$h(m) = -\alpha_1 h(m-1) - \alpha_2 h(m-2) - \cdots - \alpha_{n-1}h(m-n+1)$$
$$- \alpha_n h(m-n) \qquad (7.49)$$

for $m = n+1, n+2, \ldots$.

Next we use the Markov parameters to form the $\alpha \times \beta$ matrix

$$\mathbf{T}(\alpha, \beta) = \begin{bmatrix} h(1) & h(2) & h(3) & \cdots & h(\beta) \\ h(2) & h(3) & h(4) & \cdots & h(\beta+1) \\ h(3) & h(4) & h(5) & \cdots & h(\beta+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(\alpha) & h(\alpha+1) & h(\alpha+2) & \cdots & h(\alpha+\beta-1) \end{bmatrix} \qquad (7.50)$$

It is called a *Hankel matrix*. It is important to mention that even if $h(0) \neq 0$, $h(0)$ does not appear in the Hankel matrix.

Theorem 7.7

A strictly proper rational function $\hat{g}(s)$ has degree n if and only if

$$\rho\mathbf{T}(n, n) = \rho\mathbf{T}(n+k, n+l) = n \quad \text{for every } k, l = 1, 2, \ldots \qquad (7.51)$$

where ρ denotes the rank.

Proof: We first show that if deg $\hat{g}(s) = n$, then $\rho\mathbf{T}(n, n) = \rho\mathbf{T}(n+1, n) = \rho\mathbf{T}(\infty, n)$. If deg $\hat{g}(s) = n$, then (7.49) holds, and n is the smallest integer having the property. Because of (7.49), the $(n+1)$th row of $\mathbf{T}(n+1, n)$ can be written as a linear combination of the first n rows. Thus we have $\rho\mathbf{T}(n, n) = \rho\mathbf{T}(n+1, n)$. Again, because of (7.49), the $(n+2)$th row of $\mathbf{T}(n+2, n)$ depends on its previous n rows and, consequently, on the first n rows. Proceeding forward, we can establish $\rho\mathbf{T}(n, n) = \rho\mathbf{T}(\infty, n)$. Now we claim $\rho\mathbf{T}(\infty, n) = n$. If not, there would be an integer $\bar{n} < n$ having the property (7.49). This contradicts the hypothesis that deg $\hat{g}(s) = n$. Thus we have $\rho\mathbf{T}(n, n) = \rho\mathbf{T}(\infty, n) = n$. Applying (7.49) to the columns of $\mathbf{T}$ yields (7.51).

Now we show that if (7.51) holds, then $\hat{g}(s) = h(1)s^{-1} + h(2)s^{-2} + \cdots$ can be expressed as a strictly proper rational function of degree n. From the condition

$\rho\mathbf{T}(n+1,\infty) = \rho\mathbf{T}(n,\infty) = n$, we can compute $\{\alpha_i,\ i = 1, 2, \ldots, n\}$ to meet (7.49). We then use (7.48) to compute $\{\beta_i,\ i = 1, 2, \ldots, n\}$. Hence we have

$$\hat{g}(s) = h(1)s^{-1} + h(2)s^{-2} + h(3)s^{-3}\cdots$$
$$= \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{s-2} + \cdots + \alpha_{n-1}s + \alpha_n}$$

Because the n is the smallest integer having the property in (7.51), we have $\deg \hat{g}(s) = n$. This completes the proof of the theorem. Q.E.D.

With this preliminary, we are ready to discuss the realization problem. Consider a strictly proper transfer function $\hat{g}(s)$ expressed as

$$\hat{g}(s) = h(1)s^{-1} + h(2)s^{-2} + h(3)s^{-3} + \cdots$$

If the triplet $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is a realization of $\hat{g}(s)$, then

$$\hat{g}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \mathbf{c}[s(\mathbf{I} - s^{-1}\mathbf{A})]^{-1}\mathbf{b}$$

which becomes, using (3.57),

$$\hat{g}(s) = \mathbf{cb}s^{-1} + \mathbf{cAb}s^{-2} + \mathbf{cA}^2\mathbf{b}s^{-3} + \cdots$$

Thus we conclude that $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is a realization of $\hat{g}(s)$ if and only if

$$h(m) = \mathbf{cA}^{m-1}\mathbf{b} \qquad \text{for } m = 1,\ 2,\ \ldots \tag{7.52}$$

Substituting (7.52) into the Hankel matrix $\mathbf{T}(n, n)$ yields

$$\mathbf{T}(n,n) = \begin{bmatrix} \mathbf{cb} & \mathbf{cAb} & \mathbf{cA}^2\mathbf{b} & \cdots & \mathbf{cA}^{n-1}\mathbf{b} \\ \mathbf{cAb} & \mathbf{cA}^2\mathbf{b} & \mathbf{cA}^3\mathbf{b} & \cdots & \mathbf{cA}^{n}\mathbf{b} \\ \mathbf{cA}^2\mathbf{b} & \mathbf{cA}^3\mathbf{b} & \mathbf{cA}^4\mathbf{b} & \cdots & \mathbf{cA}^{n+1}\mathbf{b} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{cA}^{n-1}\mathbf{b} & \mathbf{cA}^{n}\mathbf{b} & \mathbf{cA}^{n+1}\mathbf{b} & \cdots & \mathbf{cA}^{2(n-1)}\mathbf{b} \end{bmatrix}$$

which implies, as shown in (7.21),

$$\mathbf{T}(n,n) = OC \tag{7.53}$$

where O and C are, respectively, the $n \times n$ observability and controllability matrices of $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. Define

$$\tilde{\mathbf{T}}(n,n) = \begin{bmatrix} h(2) & h(3) & h(4) & \cdots & h(n+1) \\ h(3) & h(4) & h(5) & \cdots & h(n+2) \\ h(4) & h(5) & h(6) & \cdots & h(n+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(n+1) & h(n+2) & h(n+3) & \cdots & h(2n) \end{bmatrix} \tag{7.54}$$

It is the submatrix of $\mathbf{T}(n+1, n)$ by deleting the first row or the submatrix of $\mathbf{T}(n, n+1)$ by deleting the first column. Then as with (7.53), we can readily show

$$\tilde{\mathbf{T}}(n,n) = OAC \tag{7.55}$$

Using (7.53) and (7.55), we can obtain many different realizations. We discuss here only a companion-form and a balanced-form realization.

Companion form There are many ways to decompose $\mathbf{T}(n, n)$ into OC. The simplest is to select $O = \mathbf{I}$ or $C = \mathbf{I}$. If we select $O = \mathbf{I}$, then (7.53) and (7.55) imply $C = \mathbf{T}(n, n)$ and $\mathbf{A} = \tilde{\mathbf{T}}(n,n)\mathbf{T}^{-1}(n,n)$. The state equation corresponding to $O = \mathbf{I}$, $C = \mathbf{T}(n, n)$, and $\mathbf{A} = \tilde{\mathbf{T}}(n,n)\mathbf{T}^{-1}(n,n)$ is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(n-1) \\ h(n) \end{bmatrix} u$$
$$y = [1\ \ 0\ \ 0 \cdots 0\ \ 0]\mathbf{x} \tag{7.56}$$

Indeed, the first row of $O = \mathbf{I}$ and the first column of $C = \mathbf{T}(n, n)$ yield the $\mathbf{c}$ and $\mathbf{b}$ in (7.56). Instead of showing $\mathbf{A} = \tilde{\mathbf{T}}(n,n)\mathbf{T}^{-1}(n,n)$, we show

$$\mathbf{AT}(n,n) = \tilde{\mathbf{T}}(n,n) \tag{7.57}$$

Using the shifting property of the companion-form matrix in (7.56), we can readily verify

$$\mathbf{A}\begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(n) \end{bmatrix} = \begin{bmatrix} h(2) \\ h(3) \\ \vdots \\ h(n+1) \end{bmatrix}, \quad \mathbf{A}\begin{bmatrix} h(2) \\ h(3) \\ \vdots \\ h(n+1) \end{bmatrix} = \begin{bmatrix} h(3) \\ h(4) \\ \vdots \\ h(n+2) \end{bmatrix}, \quad \cdots \tag{7.58}$$

We see that the Markov parameters of a column are shifted up one position if the column is premultiplied by $\mathbf{A}$. Using this property, we can readily establish (7.57). Thus $O = \mathbf{I}$, $C = \mathbf{T}(n, n)$, and $\mathbf{A} = \tilde{\mathbf{T}}(n,n)\mathbf{T}^{-1}(n,n)$ generate the realization in (7.56). It is a companion-form realization. Now we use (7.52) to show that (7.56) is indeed a realization. Because of the form of $\mathbf{c}$, $\mathbf{cA}^m\mathbf{b}$ equals simply the top entry of $\mathbf{A}^m\mathbf{b}$ or

$$\mathbf{cb} = h(1), \quad \mathbf{cAb} = h(2), \quad \mathbf{cA}^2\mathbf{b} = h(3), \quad \ldots$$

Thus (7.56) is a realization of $\hat{g}(s)$. The state equation is always observable because $O = \mathbf{I}$ has full rank. It is controllable if $C = \mathbf{T}(n, n)$ has rank n.

Example 7.2 Consider

$$\hat{g}(s) = \frac{4s^2 - 2s - 6}{2s^4 + 2s^3 + 2s^2 + 3s + 1}$$
$$= 0 \cdot s^{-1} + 2s^{-2} - 3s^{-3} - 2s^{-4} + 2s^{-5} + 3.5s^{-6} + \cdots \tag{7.59}$$

We form $\mathbf{T}(4, 4)$ and compute its rank. The rank is 3; thus $\hat{g}(s)$ in (7.59) has degree 3 and its numerator and denominator have a common factor of degree 1. There is no need to cancel first the common factor in the expansion in (7.59). From the preceding derivation, we have

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & -2 \\ -3 & -2 & 2 \\ -2 & 2 & 3.5 \end{bmatrix} \begin{bmatrix} 0 & 2 & -3 \\ 2 & -3 & -2 \\ -3 & -2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -1 & 0 \end{bmatrix} \tag{7.60}$$

and

$$\mathbf{b} = [0 \;\; 2 \;\; -3]' \qquad \mathbf{c} = [1 \;\; 0 \;\; 0]$$

The triplet $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is a minimal realization of $\hat{g}(s)$ in (7.59).

We mention that the matrix $\mathbf{A}$ in (7.60) can be obtained without computing $\tilde{\mathbf{T}}(n, n)$ $\mathbf{T}^{-1}(n, n)$. Using (7.49) we can show

$$\mathbf{T}(3,4)\mathbf{a} := \begin{bmatrix} 0 & 2 & -3 & -2 \\ 2 & -3 & -2 & 2 \\ -3 & -2 & 2 & 3.5 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ 1 \end{bmatrix} = \mathbf{0}$$

Thus $\mathbf{a}$ is a null vector of $\mathbf{T}(3, 4)$. The MATLAB function

```
t=[0 2 -3 -2;2 -3 -2 2;-3 -2 2 3.5];a=null(t)
```

yields a=[−0.3333 − 0.6667 0.0000 − 0.6667]′. We normalize the last entry of a to 1 by typing a/a(4), which yields [0.5 1 0 1]′. The first three entries, with sign reversed, are the last row of $\mathbf{A}$.

Balanced form Next we discuss a different decomposition of $\mathbf{T}(n, n) = OC$, which will yield a realization with the property

$$CC' = O'O$$

First we use singular-value decomposition to express $\mathbf{T}(n, n)$as

$$\mathbf{T}(n, n) = \mathbf{K}\mathbf{\Lambda}\mathbf{L}' = \mathbf{K}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{L}' \tag{7.61}$$

where $\mathbf{K}$ and $\mathbf{L}$ are orthogonal matrices and $\mathbf{\Lambda}^{1/2}$ is diagonal with the singular values of $\mathbf{T}(n, n)$ on the diagonal. Let us select

$$O = \mathbf{K}\mathbf{\Lambda}^{1/2} \quad \text{and} \quad C = \mathbf{\Lambda}^{1/2}\mathbf{L}' \tag{7.62}$$

Then we have

$$O^{-1} = \mathbf{\Lambda}^{-1/2}\mathbf{K}' \quad \text{and} \quad C^{-1} = \mathbf{L}\mathbf{\Lambda}^{-1/2} \tag{7.63}$$

For this selection of C and O, the triplet

$$\mathbf{A} = O^{-1}\tilde{\mathbf{T}}(n, n)C^{-1} \tag{7.64}$$

$$\mathbf{b} = \text{first column of } C \tag{7.65}$$

$$\mathbf{c} = \text{first row of } O \tag{7.66}$$

is a minimal realization of $\hat{g}(s)$. For this realization, we have

$$CC' = \mathbf{\Lambda}^{1/2}\mathbf{L}'\mathbf{L}\mathbf{\Lambda}^{1/2} = \mathbf{\Lambda}$$

and

$$O'O = \mathbf{\Lambda}^{1/2}\mathbf{K}'\mathbf{K}\mathbf{\Lambda}^{1/2} = \mathbf{\Lambda} = CC'$$

Thus it is called a *balanced realization*. This balanced realization is different from the balanced realization discussed in Section 7.4. It is not clear what the relationships between them are.

EXAMPLE 7.3 Consider the transfer function in Example 7.2. Now we will find a balanced realization from Hankel matrices. We type

```
t=[0 2 -3;2 -3 -2;-3 -2 2];tt=[2 -3 -2;-3 -2 2;-2 2 3.5];
[k,s,l]=svd(t);
sl=sqrt(s);
O=k*sl;C=sl*l';
a=inv(O)*tt*inv(C),
b=[C(1,1);C(2,1);C(3,1)],c=[O(1,1) O(1,2) O(1,3)]
```

This yields the following balanced realization:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0.4003 & -1.0024 & -0.4805 \\ 1.0024 & -0.3121 & 0.3209 \\ 0.4805 & 0.3209 & -0.0882 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1.2883 \\ -0.7303 \\ 1.0614 \end{bmatrix} u$$

$$y = [1.2883 \;\; 0.7303 \;\; -1.0614]\mathbf{x} + 0 \cdot u$$

To check the correctness of this result, we type [n,d]=ss2tf(a,b,c,0), which yields

$$\hat{g}(s) = \frac{2s - 3}{s^3 + s + 0.5}$$

This equals $\hat{g}(s)$ in (7.59) after canceling the common factor $2(s + 1)$.

7.6 Degree of Transfer Matrices

This section will extend the concept of degree for scalar rational functions to rational matrices. Given a proper rational matrix $\hat{\mathbf{G}}(s)$, it is assumed that every entry of $\hat{\mathbf{G}}(s)$ is a coprime fraction; that is, its numerator and denominator have no common factors. This will be a standing assumption throughout the remainder of this text.

> ***Definition 7.1*** *The* characteristic polynomial *of a proper rational matrix $\hat{\mathbf{G}}(s)$ is defined as the least common denominator of all minors of $\hat{\mathbf{G}}(s)$. The degree of the characteristic polynomial is defined as the* McMillan degree *or, simply, the* degree *of $\hat{\mathbf{G}}(s)$ and is denoted by $\delta\hat{\mathbf{G}}(s)$.*

Example 7.4 Consider the rational matrices

$$\hat{\mathbf{G}}_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \qquad \hat{\mathbf{G}}_2(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

The matrix $\hat{\mathbf{G}}_1(s)$ has $1/(s+1)$, $1/(s+1)$, $1/(s+1)$, and $1/(s+1)$ as its minors of order 1, and $\det \hat{\mathbf{G}}_1(s) = 0$ as its minor of order 2. Thus the characteristic polynomial of $\hat{\mathbf{G}}_1(s)$ is $s+1$ and $\delta\hat{\mathbf{G}}_1(s) = 1$. The matrix $\hat{\mathbf{G}}_2(s)$ has $2/(s+1)$, $1/(s+1)$, $1/(s+1)$, and $1/(s+1)$ as its minors of order 1, and $\det \hat{\mathbf{G}}_2(s) = 1/(s+1)^2$ as its minor of order 2. Thus the characteristic polynomial of $\hat{\mathbf{G}}_2(s)$ is $(s+1)^2$ and $\delta\hat{\mathbf{G}}_2(s) = 2$.

From this example, we see that the characteristic polynomial of $\hat{\mathbf{G}}(s)$ is, in general, different from the denominator of the determinant of $\hat{\mathbf{G}}(s)$ [if $\hat{\mathbf{G}}(s)$ is square] and different from the least common denominator of all entries of $\hat{\mathbf{G}}(s)$.

Example 7.5 Consider the 2×3 rational matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix}$$

Its minors of order 1 are the six entries of $\hat{\mathbf{G}}(s)$. The matrix has the following three minors of order 2:

$$\frac{s}{(s+1)^2(s+2)} + \frac{1}{(s+1)^2(s+2)} = \frac{s+1}{(s+1)^2(s+2)} = \frac{1}{(s+1)(s+2)}$$

$$\frac{s}{s+1} \cdot \frac{1}{s} + \frac{1}{(s+1)(s+3)} = \frac{s+4}{(s+1)(s+3)}$$

$$\frac{1}{(s+1)(s+2)s} - \frac{1}{(s+1)(s+2)(s+3)} = \frac{3}{s(s+1)(s+2)(s+3)}$$

The least common denominator of all these minors is $s(s+1)(s+2)(s+3)$. Thus the degree of $\hat{\mathbf{G}}(s)$ is 4. Note that $\hat{\mathbf{G}}(s)$ has no minors of order 3 or higher.

In computing the characteristic polynomial, every minor must be reduced to a coprime fraction as we did in the preceding example; otherwise, we will obtain an erroneous result. We discuss two special cases. If $\hat{\mathbf{G}}(s)$ is $1 \times p$ or $q \times 1$, then there are no minors of order 2 or higher. Thus the characteristic polynomial equals the least common denominator of all entries of $\hat{\mathbf{G}}(s)$. In particular, if $\hat{\mathbf{G}}(s)$ is scalar, then the characteristic polynomial equals its denominator. If every entry of $q \times p$ $\hat{\mathbf{G}}(s)$ has poles that differ from those of all other entries, such as

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)} & \frac{s+2}{s^2} \\ \frac{s-2}{s+3} & \frac{s}{(s+5)(s-3)} \end{bmatrix}$$

then its minors contain no poles with multiplicities higher than those of each entry. Thus the characteristic polynomial equals the product of the denominators of all entries of $\hat{\mathbf{G}}(s)$.

To conclude this section, we mention two important properties. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a controllable and observable realization of $\hat{\mathbf{G}}(s)$. Then we have the following:

- Monic least common denominator of all *minors* of $\hat{\mathbf{G}}(s)$ = characteristic polynomial of $\mathbf{A}$.
- Monic least common denominator of all *entries* of $\hat{\mathbf{G}}(s)$ = minimal polynomial of $\mathbf{A}$.

For their proofs, see Reference [4, pp. 302–304].

7.7 Minimal Realizations—Matrix Case

We introduced in Section 7.2.1 minimal realizations for scalar transfer functions. Now we discuss the matrix case.

Theorem 7.M2

A state equation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is a minimal realization of a proper rational matrix $\hat{\mathbf{G}}(s)$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{A}, \mathbf{C})$ is observable or if and only if

$$\dim \mathbf{A} = \deg \hat{\mathbf{G}}(s)$$

Proof: The proof of the first part is similar to the proof of Theorem 7.2. If $(\mathbf{A}, \mathbf{B})$ is not controllable or if $(\mathbf{A}, \mathbf{C})$ is not observable, then the state equation is zero-state equivalent to a lesser-dimensional state equation and is not minimal. If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is of dimension n and is controllable and observable, and if the $\bar{n}$-dimensional state equation $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$, with $\bar{n} < n$, is a realization of $\hat{\mathbf{G}}(s)$, then Theorem 4.1 implies $\mathbf{D} = \bar{\mathbf{D}}$ and

$$\mathbf{C}\mathbf{A}^m\mathbf{B} = \bar{\mathbf{C}}\bar{\mathbf{A}}^m\bar{\mathbf{B}} \qquad \text{for } m = 0, 1, 2, \ldots$$

Thus, as in (7.22), we have

$$OC = \bar{O}_n\bar{C}_n$$

Note that O, C, $\bar{O}_n$, and $\bar{C}_n$ are, respectively, $nq \times n$, $n \times np$, $nq \times \bar{n}$, and $\bar{n} \times np$ matrices. Using the Sylvester inequality

$$\rho(O) + \rho(C) - n \le \rho(OC) \le \min(\rho(O), \rho(C))$$

which is proved in Reference [6, p. 31], and $\rho(O) = \rho(C) = n$, we have $\rho(OC) = n$. Similarly, we have $\rho(\bar{O}_n\bar{C}_n) = \bar{n} < n$. This contradicts $\rho(OC) = \rho(\bar{O}_n\bar{C}_n)$. Thus every controllable and observable state equation is a minimal realization.

To show that $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is minimal if and only if $\dim \mathbf{A} = \deg \hat{\mathbf{G}}(s)$ is much more complex and will be established in the remainder of this chapter. Q.E.D.

Theorem 7.M3

All minimal realizations of $\hat{\mathbf{G}}(s)$ are equivalent.

→ ***Proof:*** The proof follows closely the proof of Theorem 7.3. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$ be any two n-dimensional minimal realizations of a $q \times p$ proper rational matrix $\hat{\mathbf{G}}(s)$. As in (7.23) and (7.24), we have

$$OC = \bar{O}\bar{C} \tag{7.67}$$

and

$$O\mathbf{A}C = \bar{O}\bar{\mathbf{A}}\bar{C} \tag{7.68}$$

In the scalar case, O, C, $\bar{O}$, and $\bar{C}$ are all $n \times n$ nonsingular matrices and their inverses are well defined. Here O and $\bar{O}$ are $nq \times n$ matrices of rank n; C and $\bar{C}$ are $n \times np$ matrices of rank n. They are not square and their inverses are not defined. Let us define the $n \times nq$ matrix

$$O^+ := (O'O)^{-1}O' \tag{7.69}$$

Because O' is $n \times nq$ and O is $nq \times n$, the matrix $O'O$ is $n \times n$ and is, following Theorem 3.8, nonsingular. Clearly, we have

$$O^+O = (O'O)^{-1}O'O = \mathbf{I}$$

Thus O^+ is called the *pseudoinverse* or *left inverse* of O. Note that OO^+ is $nq \times nq$ and does not equal a unit matrix. Similarly, we define

$$C^+ := C'(CC')^{-1} \tag{7.70}$$

It is an $np \times n$ matrix and has the property

$$CC^+ = CC'(CC')^{-1} = \mathbf{I}$$

Thus C^+ is called the *pseudoinverse* or *right inverse* of C. In the scalar case, the equivalence transformation is defined in (7.25) as $\mathbf{P} = \bar{O}^{-1}O = \bar{C}C^{-1}$. Now we replace inverses by pseudoinverses to yield

$$\mathbf{P} := \bar{O}^+O = (\bar{O}'\bar{O})^{-1}\bar{O}'O \tag{7.71}$$

$$= \bar{C}C^+ = \bar{C}C'(CC')^{-1} \tag{7.72}$$

This equality can be verified directly by premultiplying $(\bar{O}'\bar{O})$ and postmultiplying (CC') and then using (7.67). The inverse of $\mathbf{P}$ in the scalar case is $\mathbf{P}^{-1} = O^{-1}\bar{O} = C\bar{C}^{-1}$. In the matrix case, it becomes

$$\mathbf{P}^{-1} := O^+\bar{O} = (O'O)^{-1}O'\bar{O} \tag{7.73}$$

$$= C\bar{C}^+ = C\bar{C}'(\bar{C}\bar{C}')^{-1} \tag{7.74}$$

This again can be verified using (7.67). From $\bar{O}\bar{C} = OC$, we have

$$\bar{C} = (\bar{O}'\bar{O})^{-1}\bar{O}'OC = \mathbf{P}C$$

$$\bar{O} = OC\bar{C}'(\bar{C}\bar{C}')^{-1} = O\mathbf{P}^{-1}$$

Their first p columns and first q rows are $\bar{\mathbf{B}} = \mathbf{PB}$ and $\bar{\mathbf{C}} = \mathbf{CP}^{-1}$. The equation $\bar{O}\bar{\mathbf{A}}\bar{C} = O\mathbf{A}C$ implies

$$\bar{\mathbf{A}} = (\bar{O}'\bar{O})^{-1}\bar{O}'O\mathbf{A}C\bar{C}'(\bar{C}\bar{C}')^{-1} = \mathbf{PAP}^{-1}$$

This shows that all minimal realizations of the same transfer matrix are equivalent. Q.E.D.

We see from the proof of Theorem 7.M3 that the results in the scalar case can be extended directly to the matrix case if inverses are replaced by pseudoinverses. In MATLAB, the function `pinv` generates the pseudoinverse. We mention that minimal realization can be obtained from nonminimal realizations by applying Theorems 6.6 and 6.O6 or by calling the MATLAB function `minreal`, as the next example illustrates.

EXAMPLE 7.6 Consider the transfer matrix in Example 4.6 or

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{1}{(s+2)^2} \end{bmatrix} \tag{7.75}$$

Its characteristic polynomial can be computed as $(2s+1)(s+2)^2$. Thus the rational matrix has degree 3. Its six-dimensional realization in (4.39) and four-dimensional realization in (4.44) are clearly not minimal realizations. They can be reduced to minimal realizations by calling the MATLAB function `minreal`. For example, for the realization in (4.39) typing

```
a=[-4.5 0 -6 0 -2 0;0 -4.5 0 -6 0 -2;1 0 0 0 0 0;...
   0 1 0 0 0 0;0 0 1 0 0 0;0 0 0 1 0 0];
b=[1 0;0 1;0 0;0 0;0 0;0 0];
c=[-6 3 -24 7.5 -24 3;0 1 0.5 1.5 1 0.5];d=[2 0;0 0];
[am,bm,cm,dm]=minreal(a,b,c,d)
```

yields

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.8625 & -4.0897 & 3.2544 \\ 0.2921 & -3.0508 & 1.2709 \\ -0.0944 & 0.3377 & -0.5867 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0.3218 & -0.5305 \\ 0.0459 & -0.4983 \\ -0.1688 & 0.0840 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0 & -0.0339 & 35.5281 \\ 0 & -2.1031 & -0.5720 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{u}$$

Its dimension equals the degree of $\hat{\mathbf{G}}(s)$; thus it is controllable and observable and is a minimal realization of $\hat{\mathbf{G}}(s)$ in (7.75).

7.8 Matrix Polynomial Fractions

The degree of the scalar transfer function

$$\hat{g}(s) = \frac{N(s)}{D(s)} = N(s)D^{-1}(s) = D^{-1}(s)N(s)$$

is defined as the degree of $D(s)$ if $N(s)$ and $D(s)$ are coprime. It can also be defined as the smallest possible denominator degree. In this section, we shall develop similar results for transfer matrices. Because matrices are not commutative, their orders cannot be altered in our discussion.

Every $q \times p$ proper rational matrix $\hat{\mathbf{G}}(s)$ can be expressed as

$$\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) \tag{7.76}$$

where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are $q \times p$ and $p \times p$ polynomial matrices. For example, the 2×3 rational matrix in Example 7.5 can be expressed as

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} s & 1 & s \\ -1 & 1 & s+3 \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & s(s+3) \end{bmatrix}^{-1} \tag{7.77}$$

The three diagonal entries of $\mathbf{D}(s)$ in (7.77) are the least common denominators of the three columns of $\hat{\mathbf{G}}(s)$. The fraction in (7.76) or (7.77) is called a *right polynomial fraction* or, simply, *right fraction*. Dual to (7.76), the expression

$$\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$$

where $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ are $q \times q$ and $q \times p$ polynomial matrices, is called a *left polynomial fraction* or, simply, *left fraction*.

Let $\mathbf{R}(s)$ be any $p \times p$ nonsingular polynomial matrix. Then we have

$$\begin{aligned}\hat{\mathbf{G}}(s) &= [\mathbf{N}(s)\mathbf{R}(s)][\mathbf{D}(s)\mathbf{R}(s)]^{-1} \\ &= \mathbf{N}(s)\mathbf{R}(s)\mathbf{R}^{-1}(s)\mathbf{D}^{-1}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\end{aligned}$$

Thus right fractions are not unique. The same holds for left fractions. We introduce in the following right coprime fractions.

Consider $\mathbf{A}(s) = \mathbf{B}(s)\mathbf{C}(s)$, where $\mathbf{A}(s)$, $\mathbf{B}(s)$, and $\mathbf{C}(s)$ are polynomials of compatible orders. We call $\mathbf{C}(s)$ a *right divisor* of $\mathbf{A}(s)$ and $\mathbf{A}(s)$ a *left multiple* of $\mathbf{C}(s)$. Similarly, we call $\mathbf{B}(s)$ a *left divisor* of $\mathbf{A}(s)$ and $\mathbf{A}(s)$ a *right multiple* of $\mathbf{B}(s)$.

Consider two polynomial matrices $\mathbf{D}(s)$ and $\mathbf{N}(s)$ with the same number of columns p. A $p \times p$ square polynomial matrix $\mathbf{R}(s)$ is called a common *right* divisor of $\mathbf{D}(s)$ and $\mathbf{N}(s)$ if there exist polynomial matrices $\hat{\mathbf{D}}(s)$ and $\hat{\mathbf{N}}(s)$ such that

$$\mathbf{D}(s) = \hat{\mathbf{D}}(s)\mathbf{R}(s) \quad \text{and} \quad \mathbf{N}(s) = \hat{\mathbf{N}}(s)\mathbf{R}(s)$$

Note that $\mathbf{D}(s)$ and $\mathbf{N}(s)$ can have different numbers of rows.

> ***Definition 7.2*** *A square polynomial matrix $\mathbf{M}(s)$ is called a* unimodular matrix *if its determinant is nonzero and independent of s.*

The following polynomial matrices are all unimodular matrices:

$$\begin{bmatrix} 2s & s^2+s+1 \\ 2 & s+1 \end{bmatrix}, \quad \begin{bmatrix} -2 & s^{10}+s+1 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} s & s+1 \\ s-1 & s \end{bmatrix}$$

Products of unimodular matrices are clearly unimodular matrices. Consider

$$\det \mathbf{M}(s) \det \mathbf{M}^{-1}(s) = \det [\mathbf{M}(s)\mathbf{M}^{-1}(s)] = \det \mathbf{I} = 1$$

which implies that if the determinant of $\mathbf{M}(s)$ is a nonzero constant, so is the determinant of $\mathbf{M}^{-1}(s)$. Thus the inverse of a unimodular matrix $\mathbf{M}(s)$ is again a unimodular matrix.

> ***Definition 7.3*** *A square polynomial matrix $\mathbf{R}(s)$ is a* greatest common right divisor *(gcrd) of $\mathbf{D}(s)$ and $\mathbf{N}(s)$ if (i) $\mathbf{R}(s)$ is a common right divisor of $\mathbf{D}(s)$ and $\mathbf{N}(s)$ and (ii) $\mathbf{R}(s)$ is a left multiple of every common right divisor of $\mathbf{D}(s)$ and $\mathbf{N}(s)$. If a gcrd is a unimodular matrix, then $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are said to be* right coprime.

Dual to this definition, a square polynomial matrix $\bar{\mathbf{R}}(s)$ is a *greatest common left divisor* (gcld) of $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ if (i) $\bar{\mathbf{R}}(s)$ is a common left divisor of $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ and (ii) $\bar{\mathbf{R}}(s)$ is a right multiple of every common left divisor of $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$. If a gcld is a unimodular matrix, then $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ are said to be *left coprime*.

> ***Definition 7.4*** *Consider a proper rational matrix $\hat{\mathbf{G}}(s)$ factored as*
>
> $$\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$$
>
> *where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime, and $\bar{\mathbf{N}}(s)$ and $\bar{\mathbf{D}}(s)$ are left coprime. Then the characteristic polynomial of $\hat{\mathbf{G}}(s)$ is defined as*
>
> $$\det \mathbf{D}(s) \quad \text{or} \quad \det \bar{\mathbf{D}}(s)$$
>
> *and the degree of $\hat{\mathbf{G}}(s)$ is defined as*
>
> $$\deg \hat{\mathbf{G}}(s) = \deg \det \mathbf{D}(s) = \deg \det \bar{\mathbf{D}}(s)$$

Consider

$$\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) = [\mathbf{N}(s)\mathbf{R}(s)][\mathbf{D}(s)\mathbf{R}(s)]^{-1} \tag{7.78}$$

where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime. Define $\mathbf{D}_1(s) = \mathbf{D}(s)\mathbf{R}(s)$ and $\mathbf{N}_1(s) = \mathbf{N}(s)\mathbf{R}(s)$. Then we have

$$\det \mathbf{D}_1(s) = \det[\mathbf{D}(s)\mathbf{R}(s)] = \det \mathbf{D}(s) \det \mathbf{R}(s)$$

which implies

$$\deg \det \mathbf{D}_1(s) = \deg \det \mathbf{D}(s) + \deg \det \mathbf{R}(s)$$

Clearly we have $\deg \det \mathbf{D}_1(s) \geq \deg \det \mathbf{D}(s)$ and the equality holds if and only if $\mathbf{R}(s)$ is unimodular or, equivalently, $\mathbf{N}_1(s)$ and $\mathbf{D}_1(s)$ are right coprime. Thus we conclude that if $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ is a coprime fraction, then $\mathbf{D}(s)$ has the smallest possible determinantal degree and the degree is defined as the degree of the transfer matrix. Therefore a coprime fraction can also be defined as a polynomial matrix fraction with the smallest denominator's determinantal degree. From (7.78), we can see that coprime fractions are not unique; they can differ by unimodular matrices.

We have introduced Definitions 7.1 and 7.4 to define the degree of rational matrices. Their equivalence can be established by using the Smith–McMillan form and will not be discussed here. The interested reader is referred to, for example, Reference [3].

7.8.1 Column and Row Reducedness

In order to apply Definition 7.4 to determine the degree of $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$, we must compute the determinant of $\mathbf{D}(s)$. This can be avoided if the coprime fraction has some additional property as we will discuss next.

The degree of a polynomial vector is defined as the highest power of s in all entries of the vector. Consider a polynomial matrix $\mathbf{M}(s)$. We define

$$\delta_{ci}\mathbf{M}(s) = \text{degree of } i\text{th column of } \mathbf{M}(s)$$
$$\delta_{ri}\mathbf{M}(s) = \text{degree of } i\text{th row of } \mathbf{M}(s)$$

and call δ_{ci} the *column degree* and δ_{ri} the *row degree*. For example, the matrix

$$\mathbf{M}(s) = \begin{bmatrix} s+1 & s^3-2s+5 & -1 \\ s-1 & s^2 & 0 \end{bmatrix}$$

has $\delta_{c1} = 1$, $\delta_{c2} = 3$, $\delta_{c3} = 0$, $\delta_{r1} = 3$, and $\delta_{r2} = 2$.

> ***Definition 7.5*** *A nonsingular polynomial matrix* $\mathbf{M}(s)$ *is* column reduced *if*
>
> $$\text{deg det } \mathbf{M}(s) = \text{sum of all column degrees}$$
>
> *It is* row reduced *if*
>
> $$\text{deg det } \mathbf{M}(s) = \text{sum of all row degrees}$$

A matrix can be column reduced but not row reduced and vice versa. For example, the matrix

$$\mathbf{M}(s) = \begin{bmatrix} 3s^2+2s & 2s+1 \\ s^2+s-3 & s \end{bmatrix} \tag{7.79}$$

has determinant $s^3 - s^2 + 5s + 3$. Its degree equals the sum of its column degrees 2 and 1. Thus $\mathbf{M}(s)$ in (7.79) is column reduced. The row degrees of $\mathbf{M}(s)$ are 2 and 2; their sum is larger than 3. Thus $\mathbf{M}(s)$ is not row reduced. A diagonal polynomial matrix is always both column and row reduced. If a square polynomial matrix is not column reduced, then the degree of its determinant is less than the sum of its column degrees. Every nonsingular polynomial matrix can be changed to a column- or row-reduced matrix by pre- or postmultiplying a unimodular matrix. See Reference [6, p. 603].

Let $\delta_{ci}\mathbf{M}(s) = k_{ci}$ and define $\mathbf{H}_c(s) = \text{diag}(s^{k_{c1}}, s^{k_{c2}}, \ldots)$. Then the polynomial matrix $\mathbf{M}(s)$ can be expressed as

$$\mathbf{M}(s) = \mathbf{M}_{hc}\mathbf{H}_c(s) + \mathbf{M}_{lc}(s) \tag{7.80}$$

For example, the $\mathbf{M}(s)$ in (7.79) has column degrees 2 and 1 and can be expressed as

$$\mathbf{M}(s) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2s & 1 \\ s-3 & 0 \end{bmatrix}$$

The constant matrix $\mathbf{M}_{hc}$ is called the *column-degree coefficient matrix*; its ith column is the coefficients of the ith column of $\mathbf{M}(s)$ associated with $s^{k_{ci}}$. The polynomial matrix $\mathbf{M}_{lc}(s)$ contains the remaining terms and its ith column has degree less than k_{ci}. If $\mathbf{M}(s)$ is expressed as in (7.80), then it can be verified that $\mathbf{M}(s)$ is column reduced if and only if its column-degree coefficient matrix $\mathbf{M}_{hc}$ is nonsingular. Dual to (7.80), we can express $\mathbf{M}(s)$ as

$$\mathbf{M}(s) = \mathbf{H}_r(s)\mathbf{M}_{hr} + \mathbf{M}_{lr}(s)$$

where $\delta_{ri}\mathbf{M}(s) = k_{ri}$ and $\mathbf{H}_r(s) = \text{diag}(s^{k_{r1}}, s^{k_{r2}}, \ldots)$. The matrix $\mathbf{M}_{hr}$ is called the *row-degree coefficient matrix*. Then $\mathbf{M}(s)$ is row reduced if and only if $\mathbf{M}_{hr}$ is nonsingular.

Using the concept of reducedness, we now can state the degree of a proper rational matrix as follows. Consider $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$, where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime, $\bar{\mathbf{N}}(s)$ and $\bar{\mathbf{D}}(s)$ are left coprime, $\mathbf{D}(s)$ is column reduced, and $\bar{\mathbf{D}}(s)$ is row reduced. Then we have

$$\begin{aligned} \deg \hat{\mathbf{G}}(s) &= \text{sum of column degrees of } \mathbf{D}(s) \\ &= \text{sum of row degrees of } \bar{\mathbf{D}}(s) \end{aligned}$$

We discuss another implication of reducedness. Consider $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$. If $\hat{\mathbf{G}}(s)$ is strictly proper, then $\delta_{ci}\mathbf{N}(s) < \delta_{ci}\mathbf{D}(s)$, for $i = 1, 2, \ldots, p$; that is, the column degrees of $\mathbf{N}(s)$ are less than the corresponding column degrees of $\mathbf{D}(s)$. If $\hat{\mathbf{G}}(s)$ is proper, then $\delta_{ci}\mathbf{N}(s) \le \delta_{ci}\mathbf{D}(s)$, for $i = 1, 2, \ldots, p$. The converse, however, is not necessarily true. For example, consider

$$\mathbf{N}(s)\mathbf{D}^{-1}(s) = [1 \;\; 2]\begin{bmatrix} s^2 & s-1 \\ s+1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{-2s-1}{1} & \dfrac{2s^2-s+1}{1} \end{bmatrix}$$

Although $\delta_{ci}\mathbf{N}(s) < \delta_{ci}\mathbf{D}(s)$ for $i = 1, 2$, $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ is not strictly proper. The reason is that $\mathbf{D}(s)$ is not column reduced.

Theorem 7.8

Let $\mathbf{N}(s)$ and $\mathbf{D}(s)$ be $q \times p$ and $p \times p$ polynomial matrices, and let $\mathbf{D}(s)$ be column reduced. Then the rational matrix $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ is proper (strictly proper) if and only if

$$\delta_{ci}\mathbf{N}(s) \le \delta_{ci}\mathbf{D}(s) \qquad [\delta_{ci}\mathbf{N}(s) < \delta_{ci}\mathbf{D}(s)]$$

for $i = 1, 2, \ldots, p$.

Proof: The necessity part of the theorem follows from the preceding example. We show the sufficiency. Following (7.80), we express

$$\mathbf{D}(s) = \mathbf{D}_{hc}\mathbf{H}_c(s) + \mathbf{D}_{lc}(s) = [\mathbf{D}_{hc} + \mathbf{D}_{lc}(s)\mathbf{H}_c^{-1}(s)]\mathbf{H}_c(s)$$
$$\mathbf{N}(s) = \mathbf{N}_{hc}\mathbf{H}_c(s) + \mathbf{N}_{lc}(s) = [\mathbf{N}_{hc} + \mathbf{N}_{lc}(s)\mathbf{H}_c^{-1}(s)]\mathbf{H}_c(s)$$

Then we have

$$\hat{\mathbf{G}}(s) := \mathbf{N}(s)\mathbf{D}^{-1}(s) = [\mathbf{N}_{hc} + \mathbf{N}_{lc}(s)\mathbf{H}_c^{-1}(s)][\mathbf{D}_{hc} + \mathbf{D}_{lc}(s)\mathbf{H}_c^{-1}(s)]^{-1}$$

Because $\mathbf{D}_{lc}(s)\mathbf{H}_c^{-1}(s)$ and $\mathbf{N}_{lc}(s)\mathbf{H}_c^{-1}(s)$ both approach zero as $s \to \infty$, we have

$$\lim_{s\to\infty} \hat{\mathbf{G}}(s) = \mathbf{N}_{hc}\mathbf{D}_{hc}^{-1}$$

where $\mathbf{D}_{hc}$ is nonsingular by assumption. Now if $\delta_{ci}\mathbf{N}(s) \le \delta_{ci}\mathbf{D}(s)$, then $\mathbf{N}_{hc}$ is a nonzero constant matrix. Thus $\hat{\mathbf{G}}(\infty)$ is a nonzero constant and $\hat{\mathbf{G}}(s)$ is proper. If $\delta_{ci}\mathbf{N}(s) < \delta_{ci}\mathbf{D}(s)$, then $\mathbf{N}_{hc}$ is a zero matrix. Thus $\hat{\mathbf{G}}(\infty) = \mathbf{0}$ and $\hat{\mathbf{G}}(s)$ is strictly proper. This establishes the theorem. Q.E.D.

We state the dual of Theorem 7.8 without proof.

Corollary 7.8

Let $\bar{\mathbf{N}}(s)$ and $\bar{\mathbf{D}}(s)$ be $q \times p$ and $q \times q$ polynomial matrices, and let $\bar{\mathbf{D}}(s)$ be row reduced. Then the rational matrix $\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ is proper (strictly proper) if and only if

$$\delta_{ri}\bar{\mathbf{N}}(s) \le \delta_{ri}\bar{\mathbf{D}}(s) \qquad [\delta_{ri}\bar{\mathbf{N}}(s) < \delta_{ri}\bar{\mathbf{D}}(s)]$$

for $i = 1, 2, \ldots, q$.

7.8.2 Computing Matrix Coprime Fractions

Given a right fraction $\mathbf{N}(s)\mathbf{D}^{-1}(s)$, one way to reduce it to a right coprime fraction is to compute its gcrd. This can be achieved by applying a sequence of elementary operations. Once a gcrd $\mathbf{R}(s)$ is computed, we compute $\hat{\mathbf{N}}(s) = \mathbf{N}(s)\mathbf{R}^{-1}(s)$ and $\hat{\mathbf{D}}(s) = \mathbf{D}(s)\mathbf{R}^{-1}(s)$. Then $\mathbf{N}(s)\mathbf{D}^{-1}(s) = \hat{\mathbf{N}}(s)\hat{\mathbf{D}}^{-1}(s)$ and $\hat{\mathbf{N}}(s)\hat{\mathbf{D}}^{-1}(s)$ is a right coprime fraction. If $\hat{\mathbf{D}}(s)$ is not column reduced, then additional manipulation is needed. This procedure will not be discussed here. The interested reader is referred to Reference [6, pp. 590–591].

We now extend the method of computing scalar coprime fractions in Section 7.3 to the matrix case. Consider a $q \times p$ proper rational matrix $\hat{\mathbf{G}}(s)$ expressed as

$$\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) \tag{7.81}$$

In this section, we use variables with an overbar to denote left fractions, without an overbar to denote right fractions. Clearly (7.81) implies

$$\bar{\mathbf{N}}(s)\mathbf{D}(s) = \bar{\mathbf{D}}(s)\mathbf{N}(s)$$

and

$$\bar{\mathbf{D}}(s)(-\mathbf{N}(s)) + \bar{\mathbf{N}}(s)\mathbf{D}(s) = \mathbf{0} \tag{7.82}$$

We shall show that given a left fraction $\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$, not necessarily left coprime, we can obtain a right coprime fraction $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ by solving the polynomial matrix equation in (7.82). Instead of solving (7.82) directly, we shall change it into solving sets of linear algebraic equations. As in (7.27), we express the polynomial matrices as, assuming the highest degree to be 4 to simplify writing,

$$\bar{\mathbf{D}}(s) = \bar{\mathbf{D}}_0 + \bar{\mathbf{D}}_1 s + \bar{\mathbf{D}}_2 s^2 + \bar{\mathbf{D}}_3 s^3 + \bar{\mathbf{D}}_4 s^4$$
$$\bar{\mathbf{N}}(s) = \bar{\mathbf{N}}_0 + \bar{\mathbf{N}}_1 s + \bar{\mathbf{N}}_2 s^2 + \bar{\mathbf{N}}_3 s^3 + \bar{\mathbf{N}}_4 s^4$$
$$\mathbf{D}(s) = \mathbf{D}_0 + \mathbf{D}_1 s + \mathbf{D}_2 s^2 + \mathbf{D}_3 s^3$$
$$\mathbf{N}(s) = \mathbf{N}_0 + \mathbf{N}_1 s + \mathbf{N}_2 s^2 + \mathbf{N}_3 s^3$$

where $\bar{\mathbf{D}}_i$, $\bar{\mathbf{N}}_i$, $\mathbf{D}_i$, and $\mathbf{N}_i$ are, respectively, $q \times q$, $q \times p$, $p \times p$, and $q \times p$ constant matrices. The constant matrices $\bar{\mathbf{D}}_i$ and $\bar{\mathbf{N}}_i$ are known, and $\mathbf{D}_i$ and $\mathbf{N}_i$ are to be solved. Substituting these into (7.82) and equating to zero the constant matrices associated with s^k, for $k = 0, 1, \ldots$, we obtain

$$\mathbf{SM} := \begin{bmatrix} \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 & \vdots & \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{D}}_2 & \bar{\mathbf{N}}_2 & \vdots & \bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 & \vdots & \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \vdots & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{D}}_3 & \bar{\mathbf{N}}_3 & \vdots & \bar{\mathbf{D}}_2 & \bar{\mathbf{N}}_2 & \vdots & \bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 & \vdots & \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 \\ \bar{\mathbf{D}}_4 & \bar{\mathbf{N}}_4 & \vdots & \bar{\mathbf{D}}_3 & \bar{\mathbf{N}}_3 & \vdots & \bar{\mathbf{D}}_2 & \bar{\mathbf{N}}_2 & \vdots & \bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 \\ \mathbf{0} & \mathbf{0} & \vdots & \bar{\mathbf{D}}_4 & \bar{\mathbf{N}}_4 & \vdots & \bar{\mathbf{D}}_3 & \bar{\mathbf{N}}_3 & \vdots & \bar{\mathbf{D}}_2 & \bar{\mathbf{N}}_2 \\ \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \bar{\mathbf{D}}_4 & \bar{\mathbf{N}}_4 & \vdots & \bar{\mathbf{D}}_3 & \bar{\mathbf{N}}_3 \\ \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \bar{\mathbf{D}}_4 & \bar{\mathbf{N}}_4 \end{bmatrix} \begin{bmatrix} -\mathbf{N}_0 \\ \mathbf{D}_0 \\ \cdots \\ -\mathbf{N}_1 \\ \mathbf{D}_1 \\ \cdots \\ -\mathbf{N}_2 \\ \mathbf{D}_2 \\ \cdots \\ -\mathbf{N}_3 \\ \mathbf{D}_3 \end{bmatrix} = \mathbf{0} \tag{7.83}$$

This equation is the matrix version of (7.28) and the matrix $\mathbf{S}$ will be called a *generalized resultant*. Note that every $\bar{D}$-block column has q $\bar{D}$-columns and every $\bar{N}$-block column has p $\bar{N}$-columns. The generalized resultant $\mathbf{S}$ as shown has four pairs of $\bar{D}$- and $\bar{N}$-block columns; thus it has a total of $4(q + p)$ columns. It has eight block rows; each block rows has q rows. Thus the resultant has $8q$ rows.

We now discuss some general properties of $\mathbf{S}$ under the assumption that linearly independent columns of $\mathbf{S}$ from left to right have been found. It turns out that every $\bar{D}$-column in every $\bar{D}$-block column is linearly independent of its left-hand-side (LHS) columns. The situation for $\bar{N}$-columns, however, is different. Recall that there are p $\bar{N}$-columns in each $\bar{N}$-block column. We use $\bar{N}i$-column to denote the ith $\bar{N}$-column in each $\bar{N}$-block column. It turns out that if the $\bar{N}i$-column in some $\bar{N}$-block column is linearly dependent on its LHS columns, then all subsequent $\bar{N}i$-columns, because of the repetitive structure of $\mathbf{S}$, will be linearly dependent on its LHS columns. Let μ_i, $i = 1, 2, \ldots, p$, be the number of linearly independent $\bar{N}i$-columns in $\mathbf{S}$. They are called the *column indices* of $\hat{\mathbf{G}}(s)$. The first $\bar{N}i$-column that becomes linearly dependent on its LHS columns is called the *primary dependent $\bar{N}i$-column*. It is clear that the $(\mu_i + 1)$th $\bar{N}i$-column is the primary dependent column.

Corresponding to each primary dependent $\bar{N}i$-column, we compute the monic null vector (its last entry equals 1) of the submatrix that consists of the primary dependent $\bar{N}i$-column and all its LHS linearly independent columns. There are totally p such monic null vectors. From these monic null vectors, we can then obtain a right fraction. This fraction is right coprime because we use the least possible μ_i and the resulting $\mathbf{D}(s)$ has the smallest possible column degrees. In addition, $\mathbf{D}(s)$ automatically will be column reduced. The next example illustrates the procedure.

EXAMPLE 7.7 Find a right coprime fraction of the transfer matrix in (7.75) or

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix} \tag{7.84}$$

First we must find a left fraction, not necessarily left coprime. Using the least common denominator of each row, we can readily obtain

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} (2s+1)(s+2) & 0 \\ 0 & (2s+1)(s+2)^2 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} (4s-10)(s+2) & 3(2s+1) \\ s+2 & (s+1)(2s+1) \end{bmatrix} =: \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$$

Thus we have

$$\bar{\mathbf{D}}(s) = \begin{bmatrix} 2s^2+5s+2 & 0 \\ 0 & 2s^3+9s^2+12s+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 12 \end{bmatrix} s + \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} s^3$$

and

$$\bar{\mathbf{N}}(s) = \begin{bmatrix} 4s^2-2s-20 & 6s+3 \\ s+2 & 2s^2+3s+1 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 1 & 3 \end{bmatrix} s + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} s^3$$

We form the generalized resultant and then use the QR decomposition discussed in Section 7.3.1 to search its linearly independent columns in order from left to right. Because it is simpler to key in the transpose of **S**, we type

```
d1=[2 0 5 0 2 0 0 0];d2=[0 4 0 12 0 9 0 2];
n1=[-20 2 -2 1 4 0 0 0];n2=[2 1 1 3 0 2 0 0];
s=[d1 0 0 0 0;d2 0 0 0 0;n1 0 0 0 0;n2 0 0 0 0;...
   0 0 d1 0 0;0 0 d2 0 0;0 0 n1 0 0;0 0 n2 0 0;...
   0 0 0 0 d1;0 0 0 0 d2;0 0 0 0 n1;0 0 0 0 n2]';
[q,r]=qr(s)
```

We need only `r`; therefore the matrix `q` will not be shown. As discussed in Section 7.3.1, we need to know whether or not entries of `r` are zero in determining linear independence of columns; therefore all nonzero entries are represented by x, di, and ni. The result is

$$r = \begin{bmatrix} d1 & 0 & x & x & x & x & x & x & x & 0 & x & x \\ 0 & d2 & x & x & x & x & x & x & 0 & x & x & x \\ 0 & 0 & n1 & x & x & x & x & x & x & x & x & x \\ 0 & 0 & 0 & n2 & x & x & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & d1 & x & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & d2 & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & n1 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d1 & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that all D-columns are linearly independent of their LHS columns. There are two linearly independent $\bar{N}1$-columns and one linearly independent $\bar{N}2$-column. Thus we have $\mu_1 = 2$ and $\mu_2 = 1$. The eighth column of **S** is the primary dependent $\bar{N}2$-column. We compute a null vector of the submatrix that consists of the primary dependent $\bar{N}2$-column and all its LHS linearly independent columns as

```
z2=null([d1 0 0;d2 0 0;n1 0 0;n2 0 0;...
        0 0 d1;0 0 d2;0 0 n1;0 0 n2]');
```

and then normalize the last entry to 1 by typing

```
z2b=z2/z2(8)
```

which yields the first monic null vector as

```
z2b=[7 -1 1 2 -4 0 2 1]'
```

The eleventh column of **S** is the primary dependent $\bar{N}1$-column. We compute a null vector of the submatrix that consists of the primary dependent $\bar{N}1$-column and all its LHS linearly independent columns (that is, deleting the eighth column) as

```
z1=null([d1 0 0 0 0;d2 0 0 0 0;n1 0 0 0 0;n2 0 0 0 0;...
        0 0 d1 0 0;0 0 d2 0 0;0 0 n1 0 0;...
        0 0 0 0 d1;0 0 0 0 d2;0 0 0 0 n1]');
```

and then normalize the last entry to 1 by typing

```
z1b=z1/z1(10)
```

which yields the second monic null vector as

```
z1b=[10 -0.5 1 0 1 0 2.5 -2 0 1]'
```

Thus we have

$$\begin{bmatrix} -\mathbf{N}_0 \\ \cdots \\ \mathbf{D}_0 \\ \cdots \\ -\mathbf{N}_1 \\ \cdots \\ \mathbf{D}_1 \\ \cdots \\ -\mathbf{N}_2 \\ \cdots \\ \mathbf{D}_2 \end{bmatrix} = \begin{bmatrix} -n_0^{11} & -n_0^{12} \\ -n_0^{21} & -n_0^{22} \\ \cdots & \cdots \\ d_0^{11} & d_0^{12} \\ d_0^{21} & d_0^{22} \\ \cdots & \cdots \\ -n_1^{11} & -n_1^{12} \\ -n_1^{21} & -n_1^{22} \\ \cdots & \cdots \\ d_1^{11} & d_1^{12} \\ d_1^{21} & d_1^{22} \\ \cdots & \cdots \\ -n_2^{11} & -n_2^{12} \\ -n_2^{21} & -n_2^{22} \\ \cdots & \cdots \\ d_2^{11} & d_2^{12} \\ d_2^{21} & d_2^{22} \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ -0.5 & -1 \\ \cdots & \cdots \\ 1 & 1 \\ 0 & 2 \\ \cdots & \cdots \\ 1 & -4 \\ 0 & 0 \\ \cdots & \cdots \\ 2.5 & 2 \\ & 1 \\ \cdots & \cdots \\ -2 & \\ 0 & \\ \cdots & \cdots \\ 1 & \\ & \end{bmatrix} \tag{7.85}$$

where we have written out $\mathbf{N}_i$ and $\mathbf{D}_i$ explicitly with the superscripts ij denoting the ijth entry and the subscript denoting the degree. The two monic null vectors are arranged as shown. The order of the two null vectors can be interchanged, as we will discuss shortly. The empty entries are to be filled up with zeros. Note that the empty entry at the (8×1)th location is due to deleting the second $\bar{N}2$ linearly dependent column in computing the second null vector. By equating the corresponding entries in (7.85), we can readily obtain

$$\mathbf{D}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2.5 & 2 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2$$
$$= \begin{bmatrix} s^2 + 2.5s + 1 & 2s + 1 \\ 0 & s + 2 \end{bmatrix}$$

and

$$\mathbf{N}(s) = \begin{bmatrix} -10 & -7 \\ 0.5 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} s^2$$
$$= \begin{bmatrix} 2s^2 - s - 10 & 4s - 7 \\ 0.5 & 1 \end{bmatrix}$$

Thus $\hat{\mathbf{G}}(s)$ in (7.84) has the following right coprime fraction

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} (2s-5)(s+2) & 4s-7 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} (s+2)(s+0.5) & 2s+1 \\ 0 & s+2 \end{bmatrix}^{-1} \tag{7.86}$$

The $\mathbf{D}(s)$ in (7.86) is column reduced with column degrees $\mu_1 = 2$ and $\mu_2 = 1$. Thus we have $\deg \det \mathbf{D}(s) = 2 + 1 = 3$ and the degree of $\hat{\mathbf{G}}(s)$ in (7.84) is 3. The degree was computed in Example 7.6 as 3 by using Definition 7.1.

In general, if the generalized resultant has μ_i linearly independent $\bar{N}i$-columns, then $\mathbf{D}(s)$ computed using the preceding procedure is column reduced with column degrees μ_i. Thus we have

$$\deg \hat{\mathbf{G}}(s) = \deg \det \mathbf{D}(s) = \sum \mu_i$$
$$= \text{total number of linearly independent } \bar{N}\text{-columns in } \mathbf{S}$$

We next show that the order of column degrees is immaterial. In other words, the order of the columns of $\mathbf{N}(s)$ and $\mathbf{D}(s)$ can be changed. For example, consider the permutation matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and $\hat{\mathbf{D}}(s) = \mathbf{D}(s)\mathbf{P}$ and $\hat{\mathbf{N}}(s) = \mathbf{N}(s)\mathbf{P}$. Then the first, second, and third columns of $\mathbf{D}(s)$ and $\mathbf{N}(s)$ become the third, first, and second columns of $\hat{\mathbf{D}}(s)$ and $\hat{\mathbf{N}}(s)$. However, we have

$$\hat{\mathbf{G}}(s) = \hat{\mathbf{N}}(s)\hat{\mathbf{D}}^{-1}(s) = [\mathbf{N}(s)\mathbf{P}][\mathbf{D}(s)\mathbf{P}]^{-1} = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$

This shows that the columns of $\mathbf{D}(s)$ and $\mathbf{N}(s)$ can arbitrarily be permutated. This is the same as permuting the order of the null vectors in (7.83). Thus the set of column degrees is an intrinsic property of a system just as the set of controllability indices is (Theorem 6.3). What has been discussed can be stated as a theorem. It is an extension of Theorem 7.4 to the matrix case.

Theorem 7.M4

Let $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ be a left fraction, not necessarily left coprime. We use the coefficient matrices of $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ to form the generalized resultant $\mathbf{S}$ shown in (7.83) and search its linearly independent columns from left to right. Let $\mu_i, i = 1, 2, \ldots, p$, be the number of linearly independent $\bar{N}i$-columns. Then we have

$$\deg \hat{\mathbf{G}}(s) = \mu_1 + \mu_2 + \cdots + \mu_p \tag{7.87}$$

and a right coprime fraction $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ can be obtained by computing p monic null vectors of the p matrices formed from each primary dependent $\bar{N}i$-column and all its LHS linearly independent columns.

The right coprime fraction obtained by solving the equation in (7.83) has one additional important property. After permutation, the column-degree coefficient matrix $\mathbf{D}_{hc}$ can always become a unit upper triangular matrix (an upper triangular matrix with 1 as its diagonal entries). Such a $\mathbf{D}(s)$ is said to be in *column echelon form*. See References [6, pp. 610–612; 13, pp. 483–487]. For the $\mathbf{D}(s)$ in (7.86), its column-degree coefficient matrix is

$$\mathbf{D}_{hc} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

It is unit upper triangular; thus the $\mathbf{D}(s)$ is in column echelon form. Although we need only column reducedness in subsequent discussions, if $\mathbf{D}(s)$ is in column echelon form, then the result in the next section will be nicer.

Dual to the preceding discussion, we can compute a left coprime fraction from a right fraction $\mathbf{N}(s)\mathbf{D}^{-1}(s)$, which is not necessarily right coprime. Then similar to (7.83), we form

$$[-\bar{\mathbf{N}}_0\ \bar{\mathbf{D}}_0 \vdots -\bar{\mathbf{N}}_1\ \bar{\mathbf{D}}_1 \vdots -\bar{\mathbf{N}}_2\ \bar{\mathbf{D}}_2 \vdots -\bar{\mathbf{N}}_3\ \bar{\mathbf{D}}_3]\mathbf{T} = 0 \tag{7.88}$$

with

$$\mathbf{T} := \begin{bmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix} \tag{7.89}$$

We search linearly independent rows in order from top to bottom. Then all D-rows are linearly independent. Let the Ni-row denote the ith N-row in each N block-row. If an Ni-row becomes linearly dependent, then all Ni-rows in subsequent N block-rows are linearly dependent on their preceding rows. The first Ni-row that becomes linearly dependent is called a primary dependent Ni-row. Let ν_i, $i = 1, 2, \ldots, q$, be the number of linearly independent Ni-rows. They are called the *row indices* of $\hat{\mathbf{G}}(s)$. Then dual to Theorem 7.M4, we have the following corollary.

Corollary 7.M4

Let $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$ be a right fraction, not necessarily right coprime. We use the coefficient matrices of $\mathbf{D}(s)$ and $\mathbf{N}(s)$ to form the generlized resultant $\mathbf{T}$ shown in (7.89) and search its linearly independent rows in order from top to bottom. Let ν_i, for $i = 1, 2, \ldots, q$, be the number of linearly independent Ni-rows in $\mathbf{T}$. Then we have

$$\deg \hat{\mathbf{G}}(s) = \nu_1 + \nu_2 + \cdots + \nu_q$$

and a left coprime fraction $\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ can be obtained by computing q monic left null vectors of the q matrices formed from each primary dependent Ni-row and all its preceding linearly independent rows.

The polynomial matrix $\bar{\mathbf{D}}(s)$ obtained in Corollary 7.M4 is row reduced with row degrees $\{\nu_i, i = 1, 2, \ldots, q\}$. In fact, it is in *row echelon form*; that is, its row-degree coefficient matrix, after some row permutation, is a unit lower triangular matrix.

7.9 Realizations from Matrix Coprime Fractions

Not to be overwhelmed by notation, we discuss a realization of a 2×2 strictly proper rational matrix $\hat{\mathbf{G}}(s)$ expressed as

$$\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) \tag{7.90}$$

where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime and $\mathbf{D}(s)$ is in column echelon form.[6] We further assume that the column degrees of $\mathbf{D}(s)$ are $\mu_1 = 4$ and $\mu_2 = 2$. First we define

$$\mathbf{H}(s) := \begin{bmatrix} s^{\mu_1} & 0 \\ 0 & s^{\mu_2} \end{bmatrix} = \begin{bmatrix} s^4 & 0 \\ 0 & s^2 \end{bmatrix} \tag{7.91}$$

and

$$\mathbf{L}(s) := \begin{bmatrix} s^{\mu_1 - 1} & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & s^{\mu_2 - 1} \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s^3 & 0 \\ s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix} \tag{7.92}$$

The procedure for developing a realization for

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}\hat{\mathbf{u}}(s)$$

follows closely the scalar case from (7.3) through (7.9). First we introduce a new variable $\mathbf{v}(t)$ defined by $\hat{\mathbf{v}}(s) = \mathbf{D}^{-1}(s)\hat{\mathbf{u}}(s)$. Note that $\hat{\mathbf{v}}(s)$, called a pseudo state, is a 2×1 column vector. Then we have

$$\mathbf{D}(s)\hat{\mathbf{v}}(s) = \hat{\mathbf{u}}(s) \tag{7.93}$$

$$\hat{\mathbf{y}}(s) = \mathbf{N}(s)\hat{\mathbf{v}}(s) \tag{7.94}$$

Let us define state variables as

$$\hat{\mathbf{x}}(s) = \mathbf{L}(s)\hat{\mathbf{v}}(s) = \begin{bmatrix} s^{\mu_1 - 1} & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & s^{\mu_2 - 1} \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{v}_1(s) \\ \hat{v}_2(s) \end{bmatrix} = \begin{bmatrix} s^3\hat{v}_1(s) \\ s^2\hat{v}_1(s) \\ s\hat{v}_1(s) \\ \hat{v}_1(s) \\ s\hat{v}_2(s) \\ \hat{v}_2(s) \end{bmatrix} =: \begin{bmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \\ x_4(s) \\ x_5(s) \\ x_6(s) \end{bmatrix} \tag{7.95}$$

or, in the time domain,

$$x_1(t) = v_1^{(3)}(t) \quad x_2(t) = \ddot{v}_1(t) \quad x_3(t) = \dot{v}_1(t) \quad x_4(t) = v_1(t)$$

$$x_5(t) = \dot{v}_2(t) \quad x_6(t) = v_2(t)$$

6. All discussion is still applicable if $\mathbf{D}(s)$ is column reduced but not in echelon form.

This state vector has dimension $\mu_1 + \mu_2 = 6$. These definitions imply immediately

$$\dot{x}_2 = x_1 \qquad \dot{x}_3 = x_2 \qquad \dot{x}_4 = x_3 \qquad \dot{x}_6 = x_5 \tag{7.96}$$

Next we use (7.93) to develop equations for $\dot{x}_1$ and $\dot{x}_5$. First we express $\mathbf{D}(s)$ as

$$\mathbf{D}(s) = \mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s) \tag{7.97}$$

where $\mathbf{H}(s)$ and $\mathbf{L}(s)$ are defined in (7.91) and (7.92). Note that $\mathbf{D}_{hc}$ and $\mathbf{D}_{lc}$ are constant matrices and the column-degree coefficient matrix $\mathbf{D}_{hc}$ is a unit upper triangular matrix. Substituting (7.97) into (7.93) yields

$$[\mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s)]\hat{\mathbf{v}}(s) = \hat{\mathbf{u}}(s)$$

or

$$\mathbf{H}(s)\hat{\mathbf{v}}(s) + \mathbf{D}_{hc}^{-1}\mathbf{D}_{lc}\mathbf{L}(s)\hat{\mathbf{v}}(s) = \mathbf{D}_{hc}^{-1}\hat{\mathbf{u}}(s)$$

Thus we have, using (7.95),

$$\mathbf{H}(s)\hat{\mathbf{v}}(s) = -\mathbf{D}_{hc}^{-1}\mathbf{D}_{lc}\hat{\mathbf{x}}(s) + \mathbf{D}_{hc}^{-1}\hat{\mathbf{u}}(s) \tag{7.98}$$

Let

$$\mathbf{D}_{hc}^{-1}\mathbf{D}_{lc} =: \begin{bmatrix} \alpha_{111} & \alpha_{112} & \alpha_{113} & \alpha_{114} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{213} & \alpha_{214} & \alpha_{221} & \alpha_{222} \end{bmatrix} \tag{7.99}$$

and

$$\mathbf{D}_{hc}^{-1} =: \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} \tag{7.100}$$

Note that the inverse of a unit upper triangular matrix is again a unit upper triangular matrix. Substituting (7.99) and (7.100) into (7.98), and using $s\hat{x}_1(s) = s^4\hat{v}_1(s)$, and $s\hat{x}_5(s) = s^2\hat{v}_2(s)$, we find

$$\begin{bmatrix} s\hat{x}_1(s) \\ s\hat{x}_5(s) \end{bmatrix} = -\begin{bmatrix} \alpha_{111} & \alpha_{112} & \alpha_{113} & \alpha_{114} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{213} & \alpha_{214} & \alpha_{221} & \alpha_{222} \end{bmatrix}\hat{\mathbf{x}}(s) + \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}\hat{\mathbf{u}}(s)$$

which becomes, in the time domain,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_5 \end{bmatrix} = -\begin{bmatrix} \alpha_{111} & \alpha_{112} & \alpha_{113} & \alpha_{114} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{213} & \alpha_{214} & \alpha_{221} & \alpha_{222} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}\mathbf{u} \tag{7.101}$$

If $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$ is strictly proper, then the column degrees of $\mathbf{N}(s)$ are less than the corresponding column degrees of $\mathbf{D}(s)$. Thus we can express $\mathbf{N}(s)$ as

$$\mathbf{N}(s) = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix}\mathbf{L}(s) \tag{7.102}$$

Substituting this into (7.94) and using $\hat{\mathbf{x}}(s) = \mathbf{L}(s)\hat{\mathbf{v}}(s)$, we have

$$\hat{\mathbf{y}}(s) = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix}\hat{\mathbf{x}}(s) \tag{7.103}$$

Combining (7.96), (7.101), and (7.103) yields the following realization for $\hat{\mathbf{G}}(s)$:

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_{111} & -\alpha_{112} & -\alpha_{113} & -\alpha_{114} & \vdots & -\alpha_{121} & -\alpha_{122} \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha_{211} & -\alpha_{212} & -\alpha_{213} & -\alpha_{214} & \vdots & -\alpha_{221} & -\alpha_{222} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{u} \tag{7.104}$$

$$\mathbf{y} = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \vdots & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \vdots & \beta_{221} & \beta_{222} \end{bmatrix}\mathbf{x}$$

This is a $(\mu_1+\mu_2)$-dimensional state equation. The A-matrix has two companion-form diagonal blocks: one with order $\mu_1 = 4$ and the other with order $\mu_2 = 2$. The off-diagonal blocks are zeros except their first rows. This state equation is a generalization of the state equation in (7.9) to two-input two-output transfer matrices. We can easily show that (7.104) is always controllable and is called a *controllable-form* realization. Furthermore, the controllability indices are $\mu_1 = 4$ and $\mu_2 = 2$. As in (7.9), the state equation in (7.104) is observable if and only if $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are right coprime. For its proof, see Reference [6, p. 282]. Because we start with a coprime fraction $\mathbf{N}(s)\mathbf{D}^{-1}$, the realization in (7.104) is observable as well. In conclusion, the realization in (7.104) is controllable and observable and its dimension equals $\mu_1 + \mu_2$ and, following Theorem 7.M4, the degree of $\hat{\mathbf{G}}(s)$. This establishes the second part of Theorem 7.M2, namely, a state equation is minimal or controllable and observable if and only if its dimension equals the degree of its transfer matrix.

Example 7.8 Consider the transfer matrix in Example 7.6. We gave there a minimal realization that is obtained by reducing the nonminimal realization in (4.39). Now we will develop directly a minimal realization by using a coprime fraction. We first write the transfer matrix as

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix} =: \hat{\mathbf{G}}(\infty) + \hat{\mathbf{G}}_{sp}(s)$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \dfrac{-12}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

As in Example 7.7, we can find a right coprime fraction for the strictly proper part of $\hat{\mathbf{G}}(s)$ as

$$\hat{\mathbf{G}}_{sp}(s) = \begin{bmatrix} -6s-12 & -9 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} s^2+2.5s+1 & 2s+1 \\ 0 & s+2 \end{bmatrix}^{-1}$$

Note that its denominator matrix is the same as the one in (7.86). Clearly we have $\mu_1 = 2$ and $\mu_2 = 1$. We define

$$\mathbf{H}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} \qquad \mathbf{L}(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we have

$$\mathbf{D}(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{H}(s) + \begin{bmatrix} 2.5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{L}(s)$$

and

$$\mathbf{N}(s) = \begin{bmatrix} -6 & -12 & -9 \\ 0 & 0.5 & 1 \end{bmatrix} \mathbf{L}(s)$$

We compute

$$\mathbf{D}_{hc}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbf{D}_{hc}^{-1}\mathbf{D}_{lc} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2.5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2.5 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus a minimal realization of $\hat{\mathbf{G}}(s)$ is

$$\dot{\mathbf{x}} = \begin{bmatrix} -2.5 & -1 & \vdots & 3 \\ 1 & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 1 \end{bmatrix} \mathbf{u} \qquad (7.105)$$

$$\mathbf{y} = \begin{bmatrix} -6 & -12 & \vdots & -9 \\ 0 & 0.5 & \vdots & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

This A-matrix has two companion-form diagonal blocks: one with order 2 and the other order 1. This three-dimensional realization is a minimal realization and is in controllable canonical form.

Dual to the preceding minimal realization, if we use $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$, where $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ are left coprime and $\bar{\mathbf{D}}(s)$ is in row echelon form with row degrees $\{\nu_i,\ i = 1, 2, \ldots, q\}$, then we can obtain an observable-form realization with observability indices $\{\nu_i,\ i = 1, 2, \ldots, q\}$. This will not be repeated.

We summarize the main results in the following. As in the SISO case, an n-dimensional multivariable state equation is controllable and observable if its transfer matrix has degree n. If a proper transfer matrix is expressed as a right coprime fraction with column reducedness, then the realization obtained by using the preceding procedure will automatically be controllable and observable.

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a minimal realization of $\hat{\mathbf{G}}(s)$ and let $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$ be coprime fractions; $\bar{\mathbf{D}}(s)$ is row reduced, and $\mathbf{D}(s)$ is column reduced. Then we have

$$\mathbf{B}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{C} + \mathbf{D} = \mathbf{N}(s)\mathbf{D}^{-1}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$$

which implies

$$\frac{1}{\det(s\mathbf{I}-\mathbf{A})}\mathbf{B}[\mathrm{Adj}(s\mathbf{I}-\mathbf{A})]\mathbf{C} + \mathbf{D} = \frac{1}{\det \mathbf{D}(s)}\mathbf{N}(s)[\mathrm{Adj}(\mathbf{D}(s))]$$
$$= \frac{1}{\det \bar{\mathbf{D}}(s)}[\mathrm{Adj}(\bar{\mathbf{D}}(s))]\bar{\mathbf{N}}(s)$$

Because the three polynomials $\det(s\mathbf{I}-\mathbf{A})$, $\det \mathbf{D}(s)$, and $\det \bar{\mathbf{D}}(s)$ have the same degree, they must denote the same polynomial except possibly different leading coefficients. Thus we conclude the following:

- $\deg \hat{\mathbf{G}}(s) = \deg \det \mathbf{D}(s) = \deg \det \bar{\mathbf{D}}(s) = \dim \mathbf{A}$.
- The characteristic polynomial of $\hat{\mathbf{G}}(s) = k_1 \det \mathbf{D}(s) = k_2 \det \bar{\mathbf{D}}(s) = k_3 \det(s\mathbf{I}-\mathbf{A})$ for some nonzero constant k_i.
- The set of column degrees of $\mathbf{D}(s)$ equals the set of controllability indices of $(\mathbf{A}, \mathbf{B})$.
- The set of row degrees of $\bar{\mathbf{D}}(s)$ equals the set of observability indices of $(\mathbf{A}, \mathbf{C})$.

We see that coprime fractions and controllable and observable state equations contain essentially the same information. Thus either description can be used in analysis and design.

7.10 Realizations from Matrix Markov Parameters

Consider a $q \times p$ strictly proper rational matrix $\hat{\mathbf{G}}(s)$. We expand it as

$$\hat{\mathbf{G}}(s) = \mathbf{H}(1)s^{-1} + \mathbf{H}(2)s^{-2} + \mathbf{H}(3)s^{-3} + \cdots \qquad (7.106)$$

where $\mathbf{H}(m)$ are $q \times p$ constant matrices. Let r be the degree of the least common denominator of all entries of $\hat{\mathbf{G}}(s)$. We form

$$\mathbf{T} = \begin{bmatrix} \mathbf{H}(1) & \mathbf{H}(2) & \mathbf{H}(3) & \cdots & \mathbf{H}(r) \\ \mathbf{H}(2) & \mathbf{H}(3) & \mathbf{H}(4) & \cdots & \mathbf{H}(r+1) \\ \mathbf{H}(3) & \mathbf{H}(4) & \mathbf{H}(5) & \cdots & \mathbf{H}(r+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}(r) & \mathbf{H}(r+1) & \mathbf{H}(r+2) & \cdots & \mathbf{H}(2r-1) \end{bmatrix} \tag{7.107}$$

$$\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{H}(2) & \mathbf{H}(3) & \mathbf{H}(4) & \cdots & \mathbf{H}(r+1) \\ \mathbf{H}(3) & \mathbf{H}(4) & \mathbf{H}(5) & \cdots & \mathbf{H}(r+2) \\ \mathbf{H}(4) & \mathbf{H}(5) & \mathbf{H}(6) & \cdots & \mathbf{H}(r+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}(r+1) & \mathbf{H}(r+2) & \mathbf{H}(r+3) & \cdots & \mathbf{H}(2r) \end{bmatrix} \tag{7.108}$$

Note that $\mathbf{T}$ and $\tilde{\mathbf{T}}$ have r block columns and r block rows and, consequently, have dimension $rq \times rp$. As in (7.53) and (7.55), we have

$$\mathbf{T} = OC \quad \text{and} \quad \tilde{\mathbf{T}} = O\mathbf{A}C \tag{7.109}$$

where O and C are some observability and controllability matrices of order $rq \times n$ and $n \times rp$, respectively. Note that n is not yet determined. Because r equals the degree of the minimal polynomial of any minimal realization of $\hat{\mathbf{G}}(s)$ and because of (6.16) and (6.34), the matrix $\mathbf{T}$ is sufficiently large to have rank n. This is stated as a theorem.

Theorem 7.M7

A strictly proper rational matrix $\hat{\mathbf{G}}(s)$ has degree n if and only if the matrix $\mathbf{T}$ in (7.107) has rank n.

The singular-value decomposition method discussed in Section 7.5 can be applied to the matrix case with some modification. This is discussed in the following. First we use singular-value decomposition to express $\mathbf{T}$ as

$$\mathbf{T} = \mathbf{K}\begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{L}' \tag{7.110}$$

where $\mathbf{K}$ and $\mathbf{L}$ are orthogonal matrices and $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i are the positive square roots of the positive eigenvalues of $\mathbf{T}'\mathbf{T}$. Clearly n is the rank of $\mathbf{T}$. Let $\bar{\mathbf{K}}$ denote the first n columns of $\mathbf{K}$ and $\bar{\mathbf{L}}'$ denote the first n rows of $\mathbf{L}'$. Then we can write $\mathbf{T}$ as

$$\mathbf{T} = \bar{\mathbf{K}}\mathbf{\Lambda}\bar{\mathbf{L}}' = \bar{\mathbf{K}}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\bar{\mathbf{L}}' =: OC \tag{7.111}$$

where

$$O = \bar{\mathbf{K}}\mathbf{\Lambda}^{1/2} \quad \text{and} \quad C = \mathbf{\Lambda}^{1/2}\bar{\mathbf{L}}'$$

Note that O is $nq \times n$ and C is $n \times np$. They are not square and their inverses are not defined. However, their pseudoinverses are defined. The pseudoinverse of O is, as defined in (7.69),

$$O^+ = [(\mathbf{\Lambda}^{1/2})'\bar{\mathbf{K}}'\bar{\mathbf{K}}\mathbf{\Lambda}^{1/2}]^{-1}(\mathbf{\Lambda}^{1/2})'\bar{\mathbf{K}}'$$

Because $\mathbf{K}$ is orthogonal, we have $\bar{\mathbf{K}}'\bar{\mathbf{K}} = \mathbf{I}$ and because $\mathbf{\Lambda}^{1/2}$ is symmetric, the pseudoinverse O^+ reduces to

$$O^+ = \mathbf{\Lambda}^{-1/2}\bar{\mathbf{K}}' \tag{7.112}$$

Similarly, we have

$$C^+ = \bar{\mathbf{L}}\mathbf{\Lambda}^{-1/2} \tag{7.113}$$

Then, as in (7.64) through (7.66), the triplet

$$\mathbf{A} = O^+\tilde{\mathbf{T}}C^+ \tag{7.114}$$

$$\mathbf{B} = \text{first } p \text{ columns of } C \tag{7.115}$$

$$\mathbf{C} = \text{first } q \text{ rows of } O \tag{7.116}$$

is a minimal realization of $\hat{\mathbf{G}}(s)$. This realization has the property

$$O'O = \mathbf{\Lambda}^{1/2}\bar{\mathbf{K}}'\bar{\mathbf{K}}\mathbf{\Lambda}^{1/2} = \mathbf{\Lambda}$$

and, using $\bar{\mathbf{L}}'\bar{\mathbf{L}} = \mathbf{I}$,

$$CC' = \mathbf{\Lambda}^{1/2}\bar{\mathbf{L}}'\bar{\mathbf{L}}\mathbf{\Lambda}^{1/2} = \mathbf{\Lambda}$$

Thus the realization is a balanced realization. The MATLAB procedure in Example 7.3 can be applied directly to the matrix case if the function inverse (`inv`) is replaced by the function pseudoinverse (`pinv`). We see once again that the procedure in the scalar case can be extended directly to the matrix case. We also mention that if we decompose $\mathbf{T} = OC$ in (7.111) differently, we will obtain a different realization. This will not be discussed.

7.11 Concluding Remarks

In addition to a number of minimal realizations, we introduced in this chapter coprime fractions (right fractions with column reducedness and left fractions with row reducedness). These fractions can readily be obtained by searching linearly independent vectors of generalized resultants and then solving monic null vectors of their submatrices. A fundamental result of this chapter is that controllable and observable state equations are essentially equivalent to coprime polynomial fractions, denoted as

controllable and observable state equations

$\Updownarrow$

coprime polynomial fractions

Thus either description can be used to describe a system. We use the former in the next chapter and the latter in Chapter 9 to carry out various designs.

A great deal more can be said regarding coprime polynomial fractions. For example, it is possible to show that all coprime fractions are related by unimodular matrices. Controllability and observability conditions can also be expressed in terms of coprimeness conditions. See References [4, 6, 13, 20]. The objectives of this chapter are to discuss a numerical method to compute coprime fractions and to introduce just enough background to carry out designs in Chapter 9.

In addition to polynomial fractions, it is possible to express transfer functions as stable rational function fractions. See References [9, 21]. Stable rational function fractions can be developed without discussing polynomial fractions; however, polynomial fractions can provide an efficient way of computing rational fractions. Thus this chapter is also useful in studying rational fractions.

PROBLEMS

7.1 Given

$$\hat{g}(s) = \frac{s-1}{(s^2-1)(s+2)}$$

find a three-dimensional controllable realization. Check its observability.

7.2 Find a three-dimensional observable realization for the transfer function in Problem 7.1. Check its controllability.

7.3 Find an uncontrollable and unobservable realization for the transfer function in Problem 7.1. Find also a minimal realization.

7.4 Use the Sylvester resultant to find the degree of the transfer function in Problem 7.1.

7.5 Use the Sylvester resultant to reduce $(2s-1)/(4s^2-1)$ to a coprime fraction.

7.6 Form the Sylvester resultant of $\hat{g}(s) = (s+2)/(s^2+2s)$ by arranging the coefficients of $N(s)$ and $D(s)$ in descending powers of s and then search linearly independent columns in order from left to right. Is it true that all D-columns are linearly independent of their LHS columns? Is it true that the degree of $\hat{g}(s)$ equals the number of linearly independent N-columns?

7.7 Consider

$$\hat{g}(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} =: \frac{N(s)}{D(s)}$$

and its realization

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u \qquad y = [\beta_1 \;\; \beta_2]\mathbf{x}$$

Show that the state equation is observable if and only if the Sylvester resultant of $D(s)$ and $N(s)$ is nonsingular.

7.8 Repeat Problem 7.7 for a transfer function of degree 3 and its controllable-form realization.

7.9 Verify Theorem 7.7 for $\hat{g}(s) = 1/(s+1)^2$.

7.10 Use the Markov parameters of $\hat{g}(s) = 1/(s+1)^2$ to find an irreducible companion-form realization.

7.11 Use the Markov parameters of $\hat{g}(s) = 1/(s+1)^2$ to find an irreducible balanced-form realization.

7.12 Show that the two state equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u \qquad y = [2 \;\; 2]\mathbf{x}$$

and

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u \qquad y = [2 \;\; 0]\mathbf{x}$$

are realizations of $(2s+2)/(s^2-s-2)$. Are they minimal realizations? Are they algebraically equivalent?

7.13 Find the characteristic polynomials and degrees of the following proper rational matrices

$$\hat{\mathbf{G}}_1(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix} \qquad \hat{\mathbf{G}}_2(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \end{bmatrix}$$

and

$$\hat{\mathbf{G}}_3(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+3}{s+2} & \frac{1}{s+5} \\ \frac{1}{(s+3)^2} & \frac{s+1}{s+4} & \frac{1}{s} \end{bmatrix}$$

Note that each entry of $\hat{\mathbf{G}}_3(s)$ has different poles from other entries.

7.14 Use the left fraction

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} s & 1 \\ -s & s \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

to form a generalized resultant as in (7.83), and then search its linearly independent columns in order from left to right. What is the number of linearly independent N-columns? What is the degree of $\hat{\mathbf{G}}(s)$? Find a right coprime fraction of $\hat{\mathbf{G}}(s)$. Is the given left fraction coprime?

7.15 Are all D-columns in the generalized resultant in Problem 7.14 linearly independent of their LHS columns? Now in forming the generalized resultant, the coefficient matrices of $D(s)$ and $N(s)$ are arranged in descending powers of s, instead of ascending powers of s as in Problem 7.14. Is it true that all D-columns are linearly independent of their LHS columns? Does the degree of $\hat{\mathbf{G}}(s)$ equal the number of linearly independent N-columns? Does Theorem 7.M4 hold?

7.16 Use the right coprime fraction of $\hat{\mathbf{G}}(s)$ obtained in Problem 7.14 to form a generalized resultant as in (7.89), search its linearly independent rows in order from top to bottom, and then find a left coprime fraction of $\hat{\mathbf{G}}(s)$.

7.17 Find a right coprime fraction of

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{s^2+1}{s^3} & \dfrac{2s+1}{s^2} \\ \dfrac{s+2}{s^2} & \dfrac{2}{s} \end{bmatrix}$$

and then a minimal realization.

Chapter 8

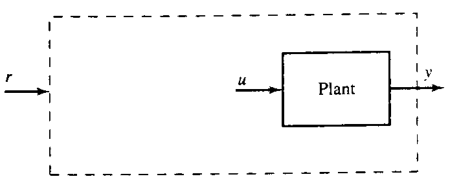
State Feedback and State Estimators

8.1 Introduction

The concepts of controllability and observability were used in the preceding two chapters to study the internal structure of systems and to establish the relationships between the internal and external descriptions. Now we discuss their implications in the design of feedback control systems.

Most control systems can be formulated as shown in Fig. 8.1, in which the *plant* and the reference signal $r(t)$ are given. The input $u(t)$ of the plant is called the *actuating signal* or *control signal*. The output $y(t)$ of the plant is called the *plant output* or *controlled signal*. The problem is to design an overall system so that the plant output will follow as closely as possible the reference signal $r(t)$. There are two types of control. If the actuating signal $u(t)$ depends only on the reference signal and is independent of the plant output, the control is called an *open-loop* control. If the actuating signal depends on both the reference signal and the plant output, the control is called a *closed-loop* or *feedback* control. The open-loop control is, in general, not satisfactory if there are plant parameter variations and/or there are noise and disturbance around the system. A properly designed feedback system, on the other hand,

Figure 8.1 Design of control systems.



can reduce the effect of parameter variations and suppress noise and disturbance. Therefore feedback control is more widely used in practice.

This chapter studies designs using state-space equations. Designs using coprime fractions will be studied in the next chapter. We study first single-variable systems and then multivariable systems. We study only linear time-invariant systems.

8.2 State Feedback

Consider the n-dimensional single-variable state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$
$$y = \mathbf{c}\mathbf{x} \tag{8.1}$$

where we have assumed $d = 0$ to simplify discussion. In state feedback, the input u is given by

$$u = r - \mathbf{k}\mathbf{x} = r - [k_1\ k_2\ \cdots\ k_n]\mathbf{x} = r - \sum_{i=1}^{n} k_i x_i \tag{8.2}$$

as shown in Fig. 8.2. Each feedback gain k_i is a real constant. This is called the *constant gain negative state feedback* or, simply, *state feedback*. Substituting (8.2) into (8.1) yields

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r$$
$$y = \mathbf{c}\mathbf{x} \tag{8.3}$$

Theorem 8.1

The pair $(\mathbf{A} - \mathbf{b}\mathbf{k}, \mathbf{b})$, for any $1 \times n$ real constant vector $\mathbf{k}$, is controllable if and only if $(\mathbf{A}, \mathbf{b})$ is controllable.

Proof: We show the theorem for $n = 4$. Define

$$\mathcal{C} = [\mathbf{b}\ \ \mathbf{A}\mathbf{b}\ \ \mathbf{A}^2\mathbf{b}\ \ \mathbf{A}^3\mathbf{b}]$$

and

$$\mathcal{C}_f = [\mathbf{b}\ \ (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{b}\ \ (\mathbf{A} - \mathbf{b}\mathbf{k})^2\mathbf{b}\ \ (\mathbf{A} - \mathbf{b}\mathbf{k})^3\mathbf{b}]$$

They are the controllability matrices of (8.1) and (8.3). It is straightforward to verify

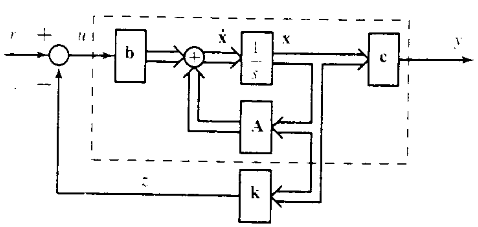


Figure 8.2 State feedback.

$$\mathcal{C}_f = \mathcal{C}\begin{bmatrix} 1 & -\mathbf{k}\mathbf{b} & -\mathbf{k}(\mathbf{A}-\mathbf{b}\mathbf{k})\mathbf{b} & -\mathbf{k}(\mathbf{A}-\mathbf{b}\mathbf{k})^2\mathbf{b} \\ 0 & 1 & -\mathbf{k}\mathbf{b} & -\mathbf{k}(\mathbf{A}-\mathbf{b}\mathbf{k})\mathbf{b} \\ 0 & 0 & 1 & -\mathbf{k}\mathbf{b} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.4}$$

Note that $\mathbf{k}$ is $1 \times n$ and $\mathbf{b}$ is $n \times 1$. Thus $\mathbf{k}\mathbf{b}$ is scalar; so is every entry in the rightmost matrix in (8.4). Because the rightmost matrix is nonsingular for any $\mathbf{k}$, the rank of $\mathcal{C}_f$ equals the rank of $\mathcal{C}$. Thus (8.3) is controllable if and only if (8.1) is controllable.

This theorem can also be established directly from the definition of controllability. Let $\mathbf{x}_0$ and $\mathbf{x}_1$ be two arbitrary states. If (8.1) is controllable, there exists an input u_1 that transfers $\mathbf{x}_0$ to $\mathbf{x}_1$ in a finite time. Now if we choose $r_1 = u_1 + \mathbf{k}\mathbf{x}$, then the input r_1 of the state feedback system will transfer $\mathbf{x}_0$ to $\mathbf{x}_1$. Thus we conclude that if (8.1) is controllable, so is (8.3).

We see from Fig. 8.2 that the input r does not control the state $\mathbf{x}$ directly; it generates u to control $\mathbf{x}$. Therefore, if u cannnot control $\mathbf{x}$, neither can r. This establishes once again the theorem. Q.E.D.

Although the controllability property is invariant under any state feedback, the observability property is not. This is demonstrated by the example that follows.

Example 8.1 Consider the state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$
$$y = [1\ \ 2]\mathbf{x}$$

The state equation can readily be shown to be controllable and observable. Now we introduce the state feedback

$$u = r - [3\ \ 1]\mathbf{x}$$

Then the state feedback equation becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}r$$
$$y = [1\ \ 2]\mathbf{x}$$

Its controllability matrix is

$$\mathcal{C}_f = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

which is nonsingular. Thus the state feedback equation is controllable. Its observability matrix is

$$\mathcal{O}_f = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

which is singular. Thus the state feedback equation is not observable. The reason that the observability property may not be preserved in state feedback will be given later.

We use an example to discuss what can be achieved by state feedback.

Example 8.2 Consider a plant described by

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

The A-matrix has characteristic polynomial

$$\Delta(s) = (s-1)^2 - 9 = s^2 - 2s - 8 = (s-4)(s+2)$$

and, consequently, eigenvalues 4 and -2. It is unstable. Let us introduce state feedback $u = r - [k_1 \;\; k_2]\mathbf{x}$. Then the state feedback system is described by

$$\dot{\mathbf{x}} = \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}\right)\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}r$$

$$= \begin{bmatrix} 1-k_1 & 3-k_2 \\ 3 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}r$$

This new A-matrix has characteristic polynomial

$$\Delta_f(s) = (s - 1 + k_1)(s-1) - 3(3 - k_2)$$
$$= s^2 + (k_1 - 2)s + (3k_2 - k_1 - 8)$$

It is clear that the roots of $\Delta_f(s)$ or, equivalently, the eigenvalues of the state feedback system can be placed in any positions by selecting appropriate k_1 and k_2. For example, if the two eigenvalues are to be placed at $-1 \pm j2$, then the desired characteristic polynomial is $(s+1+j2)(s+1-j2) = s^2 + 2s + 5$. Equating $k_1 - 2 = 2$ and $3k_2 - k_1 - 8 = 5$ yields $k_1 = 4$ and $k_2 = 17/3$. Thus the state feedback gain $[4 \;\; 17/3]$ will shift the eigenvalues from $4, -2$ to $-1 \pm j2$.

This example shows that state feedback can be used to place eigenvalues in any positions. Moreover the feedback gain can be computed by direct substitution. This approach, however, will become very involved for three- or higher-dimensional state equations. More seriously, the approach will not reveal how the controllability condition comes into the design. Therefore a more systematic approach is desirable. Before proceeding, we need the following theorem. We state the theorem for $n = 4$; the theorem, however, holds for every positive integer n.

Theorem 8.2

Consider the state equation in (8.1) with $n = 4$ and the characteristic polynomial

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 \quad (8.5)$$

If (8.1) is controllable, then it can be transformed by the transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{Q} := \mathbf{P}^{-1} = [\mathbf{b} \;\; \mathbf{Ab} \;\; \mathbf{A}^2\mathbf{b} \;\; \mathbf{A}^3\mathbf{b}] \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8.6)$$

into the controllable canonical form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u \quad (8.7)$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} = [\beta_1 \;\; \beta_2 \;\; \beta_3 \;\; \beta_4]\bar{\mathbf{x}}$$

Furthermore, the transfer function of (8.1) with $n = 4$ equals

$$\hat{g}(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \quad (8.8)$$

Proof: Let $\mathcal{C}$ and $\bar{\mathcal{C}}$ be the controllability matrices of (8.1) and (8.7). In the SISO case, both $\mathcal{C}$ and $\bar{\mathcal{C}}$ are square. If (8.1) is controllable or $\mathcal{C}$ is nonsingular, so is $\bar{\mathcal{C}}$. And they are related by $\bar{\mathcal{C}} = \mathbf{P}\mathcal{C}$ (Theorem 6.2 and Equation (6.20)). Thus we have

$$\mathbf{P} = \bar{\mathcal{C}}\mathcal{C}^{-1} \quad \text{or} \quad \mathbf{Q} := \mathbf{P}^{-1} = \mathcal{C}\bar{\mathcal{C}}^{-1}$$

The controllability matrix $\bar{\mathcal{C}}$ of (8.7) was computed in (7.10). Its inverse turns out to be

$$\bar{\mathcal{C}}^{-1} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8.9)$$

This can be verified by multiplying (8.9) with (7.10) to yield a unit matrix. Note that the constant term α_4 of (8.5) does not appear in (8.9). Substituting (8.9) into $\mathbf{Q} = \mathcal{C}\bar{\mathcal{C}}^{-1}$ yields (8.6). As shown in Section 7.2, the state equation in (8.7) is a realization of (8.8). Thus the transfer function of (8.7) and, consequently, of (8.1) equals (8.8). This establishes the theorem. Q.E.D.

With this theorem, we are ready to discuss eigenvalue assignment by state feedback.

Theorem 8.3

If the n-dimensional state equation in (8.1) is controllable, then by state feedback $u = r - \mathbf{kx}$, where $\mathbf{k}$ is a $1 \times n$ real constant vector, the eigenvalues of $\mathbf{A} - \mathbf{bk}$ can arbitrarily be assigned provided that complex conjugate eigenvalues are assigned in pairs.

Proof: We again prove the theorem for $n = 4$. If (8.1) is controllable, it can be transformed into the controllable canonical form in (8.7). Let $\bar{\mathbf{A}}$ and $\bar{\mathbf{b}}$ denote the matrices in (8.7). Then we have $\bar{\mathbf{A}} = \mathbf{PAP}^{-1}$ and $\bar{\mathbf{b}} = \mathbf{Pb}$. Substituting $\bar{\mathbf{x}} = \mathbf{Px}$ into the state feedback yields

$$u = r - \mathbf{kx} = r - \mathbf{kP}^{-1}\bar{\mathbf{x}} =: r - \bar{\mathbf{k}}\bar{\mathbf{x}}$$

where $\bar{\mathbf{k}} := \mathbf{kP}^{-1}$. Because $\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}} = \mathbf{P}(\mathbf{A} - \mathbf{bk})\mathbf{P}^{-1}$, $\mathbf{A} - \mathbf{bk}$ and $\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}}$ have the same set of eigenvalues. From any set of desired eigenvalues, we can readily form

$$\Delta_f(s) = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4 \tag{8.10}$$

If $\bar{\mathbf{k}}$ is chosen as

$$\bar{\mathbf{k}} = [\bar{\alpha}_1 - \alpha_1 \;\; \bar{\alpha}_2 -, \alpha_2 \;\; \bar{\alpha}_3 - \alpha_3 \;\; \bar{\alpha}_4 - \alpha_4] \tag{8.11}$$

the state feedback equation becomes

$$\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\bar{\mathbf{x}} + \bar{\mathbf{b}}r = \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r \tag{8.12}$$

$$y = [\beta_1 \;\; \beta_2 \;\; \beta_3 \;\; \beta_4]\bar{\mathbf{x}}$$

Because of the companion form, the characteristic polynomial of $(\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})$ and, consequently, of $(\mathbf{A} - \mathbf{bk})$ equals (8.10). Thus the state feedback equation has the set of desired eigenvalues. The feedback gain $\mathbf{k}$ can be computed from

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = \bar{\mathbf{k}}\bar{C}C^{-1} \tag{8.13}$$

with $\bar{\mathbf{k}}$ in (8.11), $\bar{C}^{-1}$ in (8.9), and $C = [\mathbf{b} \;\; \mathbf{Ab} \;\; \mathbf{A}^2\mathbf{b} \;\; \mathbf{A}^3\mathbf{b}]$. Q.E.D.

We give an alternative derivation of the formula in (8.11). We compute

$$\begin{aligned}\Delta_f(s) &= \det\,(s\mathbf{I} - \mathbf{A} + \mathbf{bk}) = \det\,\left((s\mathbf{I} - \mathbf{A})[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{bk}]\right) \\ &= \det\,(s\mathbf{I} - \mathbf{A})\det\,[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{bk}]\end{aligned}$$

which becomes, using (8.5) and (3.64),

$$\Delta_f(s) = \Delta(s)[1 + \mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}]$$

Thus we have

$$\Delta_f(s) - \Delta(s) = \Delta(s)\mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \Delta(s)\bar{\mathbf{k}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} \tag{8.14}$$

Let z be the output of the feedback gain shown in Fig. 8.2 and let $\bar{\mathbf{k}} = [\bar{k}_1 \;\; \bar{k}_2 \;\; \bar{k}_3 \;\; \bar{k}_4]$. Because the transfer function from u to y in Fig. 8.2 equals

$$\bar{\mathbf{c}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{\Delta(s)}$$

the transfer function from u to z should equal

$$\bar{\mathbf{k}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} = \frac{\bar{k}_1 s^3 + \bar{k}_2 s^2 + \bar{k}_3 s + \bar{k}_4}{\Delta(s)} \tag{8.15}$$

Substituting (8.15), (8.5), and (8.10) into (8.14) yields

$$(\bar{\alpha}_1 - \alpha_1)s^3 + (\bar{\alpha}_2 - \alpha_2)s^2 + (\bar{\alpha}_3 - \alpha_3)s + (\bar{\alpha}_4 - \alpha_4) = \bar{k}_1 s^3 + \bar{k}_2 s^2 + \bar{k}_3 s + \bar{k}_4$$

This yields (8.11).

Feedback transfer function Consider a plant described by $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. If $(\mathbf{A}, \mathbf{b})$ is controllable, $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ can be transformed into the controllable form in (8.7) and its transfer function can then be read out as, for $n = 4$,

$$\hat{g}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \tag{8.16}$$

After state feedback, the state equation becomes $(\mathbf{A} - \mathbf{bk}, \mathbf{b}, \mathbf{c})$ and is still of the controllable canonical form as shown in (8.12). Thus the feedback transfer function from r to y is

$$\hat{g}_f(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1}\mathbf{b} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4} \tag{8.17}$$

We see that the numerators of (8.16) and (8.17) are the same. In other words, state feedback does not affect the zeros of the plant transfer function. This is actually a general property of feedback: *feedback can shift the poles of a plant but has no effect on the zeros.* This can be used to explain hy a state feedback may alter the observability property of a state quation. If one or more poles are shifted to coincide with zeros of $\hat{g}(s)$, then the numerator and denominator of $\hat{g}_f(s)$ in (8.17) are not coprime. Thus the state equation in (8.12) and, equivalently, $(\mathbf{A} - \mathbf{bk}, \mathbf{c})$ are not observable (Theorem 7.1).

EXAMPLE 8.3 Consider the inverted pendulum studied in Example 6.2. Its state equation is, as derived in (6.11),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u \tag{8.18}$$

$$y = [1 \;\; 0 \;\; 0 \;\; 0]\mathbf{x}$$

It is controllable; thus its eigenvalues can be assigned arbitrarily. Because the A-matrix is block triangular, its characteristic polynomial can be obtained by inspection as

$$\Delta(s) = s^2(s^2 - 5) = s^4 + 0 \cdot s^3 - 5s^2 + 0 \cdot s + 0$$

First we compute $\mathbf{P}$ that will transform (8.18) into the controllable canonical form. Using (8.6), we have

$$\mathbf{P}^{-1} = C\bar{C}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

Its inverse is

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{6} \\ -\frac{1}{3} & 0 & -\frac{1}{6} & 0 \end{bmatrix}$$

Let the desired eigenvalues be $-1.5 \pm 0.5j$ and $-1 \pm j$. Then we have

$$\Delta_f(s) = (s + 1.5 - 0.5j)(s + 1.5 + 0.5j)(s + 1 - j)(s + 1 + j)$$
$$= s^4 + 5s^3 + 10.5s^2 + 11s + 5$$

Thus we have, using (8.11),

$$\bar{\mathbf{k}} = [5 - 0 \quad 10.5 + 5 \quad 11 - 0 \quad 5 - 0] = [5 \quad 15.5 \quad 11 \quad 5]$$

and

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = [-\tfrac{5}{3} \quad -\tfrac{11}{3} \quad -\tfrac{103}{12} \quad -\tfrac{13}{3}] \tag{8.19}$$

This state feedback gain will shift the eigenvalues of the plant from $\{0, 0, \pm j\sqrt{5}\}$ to $\{-1.5 \pm 0.5j, -1 \pm j\}$.

The MATLAB function `place` computes state feedback gains for eigenvalue placement or assignment. For the example, we type

```
a=[0 1 0 0;0 0 -1 0;0 0 0 1;0 0 5 0];b=[0;1;0;-2];
p=[-1.5+0.5j -1.5-0.5j -1+j -1-j];
k=place(a,b,p)
```

which yields $[-1.6667 \quad -3.6667 \quad -8.5833 \quad -4.3333]$. This is the gain in (8.19).

One may wonder at this point how to select a set of desired eigenvalues. This depends on the performance criteria, such as rise time, settling time, and overshoot, used in the design. Because the response of a system depends not only on poles but also on zeros, the zeros of the plant will also affect the selection. In addition, most physical systems will saturate or burn out if the magnitude of the actuating signal is very large. This will again affect the selection of desired poles. As a guide, we may place all eigenvalues inside the region denoted by C in Fig. 8.3(a). The region is bounded on the right by a vertical line. The larger the distance of the vertical line from the imaginary axis, the faster the response. The region is also bounded by two straight lines emanating from the origin with angle θ. The larger the angle, the larger the overshoot. See Reference [7]. If we place all eigenvalues at one point or group them in a very small region, then usually the response will be slow and the actuating signal will be large. Therefore it is better to place all eigenvalues evenly around a circle with radius r inside the sector as shown. The larger the radius, the faster the response; however, the actuating signal will also be larger. Furthermore, the bandwidth of the feedback system will be larger and the resulting system will be more susceptible to noise. Therefore a final selection may involve compromises among many conflicting requirements. One way to proceed is by computer simulation. Another way is to find the state feedback gain $\mathbf{k}$ to minimize the quadratic performance index

$$J = \int_0^\infty [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]\, dt$$

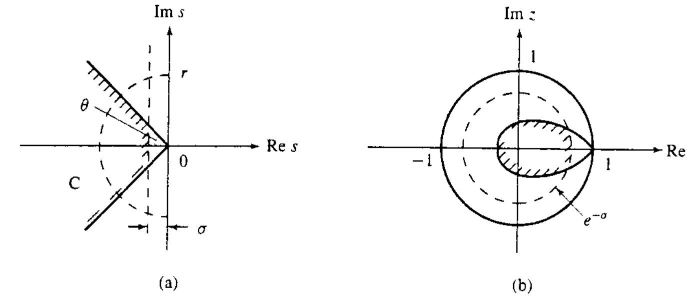


Figure 8.3 Desired eigenvalue location.

See Reference [1]. However, selecting $\mathbf{Q}$ and $\mathbf{R}$ requires trial and error. In conclusion, how to select a set of desired eigenvalues is not a simple problem.

We mention that Theorems 8.1 through 8.3—in fact, all theorems to be introduced later in this chapter—apply to the discrete-time case without any modification. The only difference is that the region in Fig. 8.3(a) must be replaced by the one in Fig. 8.3(b), which is obtained by the transformation $z = e^s$.

8.2.1 Solving the Lyapunov Equation

This subsection discusses a different method of computing state feedback gain for eigenvalue assignment. The method, however, has the restriction that the selected eigenvalues cannot contain any eigenvalues of $\mathbf{A}$.

Procedure 8.1

Consider controllable $(\mathbf{A}, \mathbf{b})$, where $\mathbf{A}$ is $n \times n$ and $\mathbf{b}$ is $n \times 1$. Find a $1 \times n$ real $\mathbf{k}$ such that $(\mathbf{A} - \mathbf{bk})$ has any set of desired eigenvalues that contains no eigenvalues of $\mathbf{A}$.

1. Select an $n \times n$ matrix $\mathbf{F}$ that has the set of desired eigenvalues. The form of $\mathbf{F}$ can be chosen arbitrarily and will be discussed later.
2. Select an arbitrary $1 \times n$ vector $\bar{\mathbf{k}}$ such that $(\mathbf{F}, \bar{\mathbf{k}})$ is observable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}}$.
4. Compute the feedback gain $\mathbf{k} = \bar{\mathbf{k}}\mathbf{T}^{-1}$.

We justify first the procedure. If $\mathbf{T}$ is nonsingular, then $\bar{\mathbf{k}} = \mathbf{kT}$ and the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}}$ implies

$$(\mathbf{A} - \mathbf{bk})\mathbf{T} = \mathbf{TF} \quad \text{or} \quad \mathbf{A} - \mathbf{bk} = \mathbf{TFT}^{-1}$$

Thus $(\mathbf{A} - \mathbf{bk})$ and $\mathbf{T}$ are similar and have the same set of eigenvalues. Thus the eigenvalues of $(\mathbf{A} - \mathbf{bk})$ can be assigned arbitrarily except those of $\mathbf{A}$. As discussed in Section 3.7, if $\mathbf{A}$ and

$\mathbf{F}$ have no eigenvalues in common, then a solution $\mathbf{T}$ exists in $\mathbf{AT}-\mathbf{TF}=\mathbf{b}\bar{\mathbf{k}}$ for any $\bar{\mathbf{k}}$ and is unique. If $\mathbf{A}$ and $\mathbf{F}$ have common eigenvalues, a solution $\mathbf{T}$ may or may not exist depending on $\mathbf{b}\bar{\mathbf{k}}$. To remove this uncertainty, we require $\mathbf{A}$ and $\mathbf{F}$ to have no eigenvalues in common. What remains to be proved is the nonsingularity of $\mathbf{T}$.

Theorem 8.4

If $\mathbf{A}$ and $\mathbf{F}$ have no eigenvalues in common, then the unique solution $\mathbf{T}$ of $\mathbf{AT}-\mathbf{TF}=\mathbf{b}\bar{\mathbf{k}}$ is nonsingular if and only if $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{F}, \bar{\mathbf{k}})$ is observable.

Proof: We prove the theorem for $n=4$. Let the characteristic polynomial of $\mathbf{A}$ be

$$\Delta(s)=s^4+\alpha_1 s^3+\alpha_2 s^2+\alpha_3 s+\alpha_4 \tag{8.20}$$

Then we have

$$\Delta(\mathbf{A})=\mathbf{A}^4+\alpha_1\mathbf{A}^3+\alpha_2\mathbf{A}^2+\alpha_3\mathbf{A}+\alpha_4\mathbf{I}=\mathbf{0}$$

(Cayley–Hamilton theorem). Let us consider

$$\Delta(\mathbf{F}):=\mathbf{F}^4+\alpha_1\mathbf{F}^3+\alpha_2\mathbf{F}^2+\alpha_3\mathbf{F}+\alpha_4\mathbf{I} \tag{8.21}$$

If $\bar{\lambda}_i$ is an eigenvalue of $\mathbf{F}$, then $\Delta(\bar{\lambda}_i)$ is an eigenvalue of $\Delta(\mathbf{F})$ (Problem 3.19). Because $\mathbf{A}$ and $\mathbf{F}$ have no eigenvalues in common, we have $\Delta(\bar{\lambda}_i)\neq 0$ for all eigenvalues of $\mathbf{F}$. Because the determinant of a matrix equals the product of all its eigenvalues, we have

$$\det\Delta(\mathbf{F})=\prod_i \Delta(\bar{\lambda}_i)\neq 0$$

Thus $\Delta(\mathbf{F})$ is nonsingular.

Substituting $\mathbf{AT}=\mathbf{TF}+\mathbf{b}\bar{\mathbf{k}}$ into $\mathbf{A}^2\mathbf{T}-\mathbf{AF}^2$ yields

$$\begin{aligned}\mathbf{A}^2\mathbf{T}-\mathbf{TF}^2&=\mathbf{A}(\mathbf{TF}+\mathbf{b}\bar{\mathbf{k}})-\mathbf{TF}^2=\mathbf{Ab}\bar{\mathbf{k}}+(\mathbf{AT}-\mathbf{TF})\mathbf{F}\\&=\mathbf{Ab}\bar{\mathbf{k}}+\mathbf{b}\bar{\mathbf{k}}\mathbf{F}\end{aligned}$$

Proceeding forward, we can obtain the following set of equations:

$$\begin{aligned}\mathbf{IT}-\mathbf{TI}&=\mathbf{0}\\ \mathbf{AT}-\mathbf{TF}&=\mathbf{b}\bar{\mathbf{k}}\\ \mathbf{A}^2\mathbf{T}-\mathbf{TF}^2&=\mathbf{Ab}\bar{\mathbf{k}}+\mathbf{b}\bar{\mathbf{k}}\mathbf{F}\\ \mathbf{A}^3\mathbf{T}-\mathbf{TF}^3&=\mathbf{A}^2\mathbf{b}\bar{\mathbf{k}}+\mathbf{Ab}\bar{\mathbf{k}}\mathbf{F}+\mathbf{b}\bar{\mathbf{k}}\mathbf{F}^2\\ \mathbf{A}^4\mathbf{T}-\mathbf{TF}^4&=\mathbf{A}^3\mathbf{b}\bar{\mathbf{k}}+\mathbf{A}^2\mathbf{b}\bar{\mathbf{k}}\mathbf{F}+\mathbf{Ab}\bar{\mathbf{k}}\mathbf{F}^2+\mathbf{b}\bar{\mathbf{k}}\mathbf{F}^3\end{aligned}$$

We multiply the first equation by α_4, the second equation by α_3, the third equation by α_2, the fourth equation by α_1, and the last equation by 1, and then sum them up. After some manipulation, we finally obtain

$$\Delta(\mathbf{A})\mathbf{T}-\mathbf{T}\Delta(\mathbf{F})=-\mathbf{T}\Delta(\mathbf{F})$$

$$=[\mathbf{b}\ \mathbf{Ab}\ \mathbf{A}^2\mathbf{b}\ \mathbf{A}^3\mathbf{b}]\begin{bmatrix}\alpha_3 & \alpha_2 & \alpha_1 & 1\\ \alpha_2 & \alpha_1 & 1 & 0\\ \alpha_1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\bar{\mathbf{k}}\\ \bar{\mathbf{k}}\mathbf{F}\\ \bar{\mathbf{k}}\mathbf{F}^2\\ \bar{\mathbf{k}}\mathbf{F}^3\end{bmatrix} \tag{8.22}$$

where we have used $\Delta(\mathbf{A})=\mathbf{0}$. If $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{F}, \bar{\mathbf{k}})$ is observable, then all three matrices after the last equality are nonsingular. Thus (8.22) and the nonsingularity of $\Delta(\mathbf{F})$ imply that $\mathbf{T}$ is nonsingular. If $(\mathbf{A}, \mathbf{b})$ is uncontrollable and/or $(\mathbf{F}, \bar{\mathbf{k}})$ is unobservable, then the product of the three matrices is singular. Therefore $\mathbf{T}$ is singular. This establishes the theorem. Q.E.D.

We now discuss the selection of $\mathbf{F}$ and $\bar{\mathbf{k}}$. Given a set of desired eigenvalues, there are infinitely many $\mathbf{F}$ that have the set of eigenvalues. If we form a polynomial from the set, we can use its coefficients to form a companion-form matrix $\mathbf{F}$ as shown in (7.14). For this $\mathbf{F}$, we can select $\bar{\mathbf{k}}$ as $[1\ 0\ \cdots\ 0]$ and $(\mathbf{F}, \bar{\mathbf{k}})$ is observable. If the desired eigenvalues are all distinct, we can also use the modal form discussed in Section 4.3.1. For example, if $n=5$, and if the five distinct desired eigenvalues are selected as λ_1, $\alpha_1\pm j\beta_1$, and $\alpha_2\pm j\beta_2$, then we can select $\mathbf{F}$ as

$$\mathbf{F}=\begin{bmatrix}\lambda_1 & 0 & 0 & 0 & 0\\ 0 & \alpha_1 & \beta_1 & 0 & 0\\ 0 & -\beta_1 & \alpha_1 & 0 & 0\\ 0 & 0 & 0 & \alpha_2 & \beta_2\\ 0 & 0 & 0 & -\beta_2 & \alpha_2\end{bmatrix} \tag{8.23}$$

It is a block-diagonal matrix. For this $\mathbf{F}$, if $\bar{\mathbf{k}}$ has at least one nonzero entry associated with each diagonal block such as $\bar{\mathbf{k}}=[1\ 1\ 0\ 1\ 0]$, $\bar{\mathbf{k}}=[1\ 1\ 0\ 0\ 1]$, or $\bar{\mathbf{k}}=[1\ 1\ 1\ 1\ 1]$, then $(\mathbf{F}, \bar{\mathbf{k}})$ is observable (Problem 6.16). Thus the first two steps of Procedure 8.1 are very simple. Once $\mathbf{F}$ and $\bar{\mathbf{k}}$ are selected, we may use the MATLAB function `lyap` to solve the Lyapunov equation in Step 3. Thus Procedure 8.1 is easy to carry out as the next example illustrates.

EXAMPLE 8.4 Consider the inverted pendulum studied in Example 8.3. The plant state equation is given in (8.18) and the desired eigenvalues were chosen as $-1\pm j$ and $-1.5\pm 0.5j$. We select $\mathbf{F}$ in modal form as

$$\mathbf{F}=\begin{bmatrix}-1 & 1 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & -1.5 & 0.5\\ 0 & 0 & -0.5 & -1.5\end{bmatrix}$$

and $\bar{\mathbf{k}}=[1\ 0\ 1\ 0]$. We type

```
a=[0 1 0 0;0 0 -1 0;0 0 0 1;0 0 5 0];b=[0;1;0;-2];
f=[-1 1 0 0;-1 -1 0 0;0 0 -1.5 0.5;0 0 -0.5 -1.5];
kb=[1 0 1 0];t=lyap(a,b,-b*kb);
k=kb*inv(t)
```

The answer is $[-1.6667 \ -3.6667 \ -8.5833 \ -4.3333]$, which is the same as the one obtained by using function `place`. If we use a different $\bar{\mathbf{k}} = [1 \ 1 \ 1 \ 1]$, we will obtain the same $\bar{\mathbf{k}}$. Note that the feedback gain is unique for the SISO case.

8.3 Regulation and Tracking

Consider the state feedback system shown in Fig. 8.2. Suppose the reference signal r is zero, and the response of the system is caused by some nonzero initial conditions. The problem is to find a state feedback gain so that the response will die out at a desired rate. This is called a *regulator problem*. This problem may arise when an aircraft is cruising at a fixed altitude H_0. Now, because of turbulance or other factors, the aircraft may deviate from the desired altitude. Bringing the deviation to zero is a regulator problem. This problem also arises in maintaining the liquid level in Fig. 2.14 at equilibrium.

A closely related problem is the tracking problem. Suppose the reference signal r is a constant or $r(t) = a$, for $t \geq 0$. The problem is to design an overall system so that $y(t)$ approaches $r(t) = a$ as t approaches infinity. This is called *asymptotic tracking* of a step reference input. It is clear that if $r(t) = a = 0$, then the tracking problem reduces to the regulator problem. Why do we then study these two problems separately? Indeed, if the same state equation is valid for all r, designing a system to track asymptotically a step reference input will automatically achieve regulation. However, a linear state equation is often obtained by shifting to an operating point and linearization, and the equation is valid only for r very small or zero; thus the study of the regulator problem is needed. We mention that a step reference input can be set by the position of a potentiometer and is therefore often referred to as set point. Maintaining a chamber at a desired temperature is often said to be regulating the temperature; it is actually tracking the desired temperature. Therefore no sharp distinction is made in practice between regulation and tracking a step reference input. Tracking a nonconstant reference signal is called a *servomechanism* problem and is a much more difficult problem.

Consider a plant described by $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. If all eigenvalues of $\mathbf{A}$ lie inside the sector shown in Fig. 8.3, then the response caused by any initial conditions will decay rapidly to zero and no state feedback is needed. If $\mathbf{A}$ is stable but some eigenvalues are outside the sector, then the decay may be slow or too oscillatory. If $\mathbf{A}$ is unstable, then the response excited by any nonzero initial conditions will grow unbounded. In these situations, we may introduce state feedback to improve the behavior of the system. Let $u = r - \mathbf{kx}$. Then the state feedback equation becomes $(\mathbf{A} - \mathbf{bk}, \mathbf{b}, \mathbf{c})$ and the response caused by $\mathbf{x}(0)$ is

$$y(t) = \mathbf{c}e^{(\mathbf{A}-\mathbf{bk})t}\mathbf{x}(0)$$

If all eigenvalues of $(\mathbf{A} - \mathbf{bk})$ lie inside the sector in Fig. 8.3(a), then the output will decay rapidly to zero. Thus regulation can easily be achieved by introducing state feedback.

The tracking problem is slightly more complex. In general, in addition to state feedback, we need a feedforward gain p as

$$u(t) = pr(t) - \mathbf{kx}$$

Then the transfer function from r to y differs from the one in (8.17) only by the feedforward gain p. Thus we have

$$\hat{g}_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = p\frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4} \tag{8.24}$$

If $(\mathbf{A}, \mathbf{b})$ is controllable, all eigenvalues of $(\mathbf{A} - \mathbf{bk})$ or, equivalently, all poles of $\hat{g}_f(s)$ can be assigned arbitrarily, in particular, assigned to lie inside the sector in Fig. 8.3(a). Under this assumption, if the reference input is a step function with magnitude a, then the output $y(t)$ will approach the constant $\hat{g}_f(0) \cdot a$ as $t \to \infty$ (Theorem 5.2). Thus in order for $y(t)$ to track asymptotically any step reference input, we need

$$1 = \hat{g}_f(0) = p\frac{\beta_4}{\bar{\alpha}_4} \quad \text{or} \quad p = \frac{\bar{\alpha}_4}{\beta_4} \tag{8.25}$$

which requires $\beta_4 \neq 0$. From (8.16) and (8.17), we see that β_4 is the numerator constant term of the plant transfer function. Thus $\beta_4 \neq 0$ if and only if the plant transfer function $\hat{g}(s)$ has no zero at $s = 0$. In conclusion, if $\hat{g}(s)$ has one or more zeros at $s = 0$, tracking is not possible. If $\hat{g}(s)$ has no zero at $s = 0$, we introduce a feedforward gain as in (8.25). Then the resulting system will track asymptotically any step reference input.

We summarize the preceding discussion. Given $(\mathbf{A}, \mathbf{b}, \mathbf{c})$; if $(\mathbf{A}, \mathbf{b})$ is controllable, we may introduce state feedback to place the eigenvalues of $(\mathbf{A} - \mathbf{bk})$ in any desired positions and the resulting system will achieve regulation. If $(\mathbf{A}, \mathbf{b})$ is controllable and if $\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ has no zero at $s = 0$, then after state feedback, we may introduce a feedforward gain as in (8.25). Then the resulting system can track asymptotically any step reference input.

8.3.1 Robust Tracking and Disturbance Rejection[1]

The state equation and transfer function developed to describe a plant may change due to change of load, environment, or aging. Thus plant parameter variations often occur in practice. The equation used in the design is often called the *nominal equation*. The feedforward gain p in (8.25), computed for the nominal plant transfer function, may not yield $\hat{g}_f(0) = 1$ for nonnominal plant transfer functions. Then the output will not track asymptotically any step reference input. Such a tracking is said to be *nonrobust*.

In this subsection we discuss a different design that can achieve robust tracking and disturbance rejection. Consider a plant described by (8.1). We now assume that a constant disturbance w with unknown magnitude enters at the plant input as shown in Fig. 8.4(a). Then the state equation must be modified as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{b}u + \mathbf{b}w \\ y &= \mathbf{cx} \end{aligned} \tag{8.26}$$

The problem is to design an overall system so that the output $y(t)$ will track asymptotically any step reference input even with the presence of a disturbance $w(t)$ and with plant parameter variations. This is called *robust tracking and disturbance rejection*. In order to achieve this design, in addition to introducing state feedback, we will introduce an integrator and a unity feedback from the output as shown in Fig. 8.4(a). Let the output of the integrator be denoted by

1. This section may be skipped without loss of continuity.

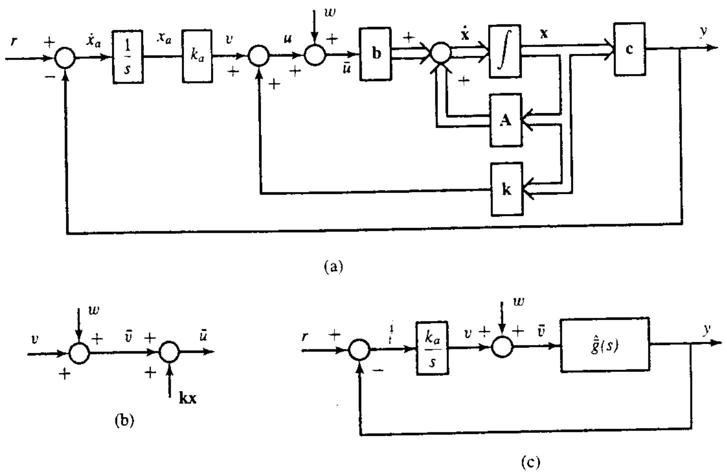


Figure 8.4 (a) State feedback with internal model. (b) Interchange of two summers. (c) Transfer-function block diagram.

$x_a(t)$, an augmented state variable. Then the system has the augmented state vector $[\mathbf{x}' \ x_a]'$. From Fig. 8.4(a), we have

$$\dot{x}_a = r - y = r - \mathbf{c}\mathbf{x} \tag{8.27}$$

$$u = [\mathbf{k} \ \ k_a]\begin{bmatrix}\mathbf{x} \\ x_a\end{bmatrix} \tag{8.28}$$

For convenience, the state is fed back positively to u as shown. Substituting these into (8.26) yields

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{x}_a\end{bmatrix} = \begin{bmatrix}\mathbf{A}+\mathbf{b}\mathbf{k} & \mathbf{b}k_a \\ -\mathbf{c} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ x_a\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ 1\end{bmatrix} r + \begin{bmatrix}\mathbf{b} \\ 0\end{bmatrix} w$$
$$y = [\mathbf{c} \ \ 0]\begin{bmatrix}\mathbf{x} \\ x_a\end{bmatrix} \tag{8.29}$$

This describes the system in Fig. 8.4(a).

Theorem 8.5

If $(\mathbf{A}, \mathbf{b})$ is controllable and if $\hat{g}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ has no zero at $s = 0$, then all eigenvalues of the A-matrix in (8.29) can be assigned arbitrarily by selecting a feedback gain $[\mathbf{k} \ \ k_a]$.

Proof: We show the theorem for $n = 4$. We assume that $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ have been transformed into the controllable canonical form in (8.7) and its transfer function equals (8.8). Then the plant transfer function has no zero at $s = 0$ if and only if $\beta_4 \neq 0$. We now show that the pair

$$\begin{bmatrix}\mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 0\end{bmatrix} \quad \begin{bmatrix}\mathbf{b} \\ 0\end{bmatrix} \tag{8.30}$$

is controllable if and only if $\beta_4 \neq 0$. Note that we have assumed $n = 4$; thus the dimension of (8.30) is five because of the additional augmented state variable x_a. The controllability matrix of (8.30) is

$$\begin{bmatrix}\mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} & \mathbf{A}^4\mathbf{b} \\ 0 & -\mathbf{c}\mathbf{b} & -\mathbf{c}\mathbf{A}\mathbf{b} & -\mathbf{c}\mathbf{A}^2\mathbf{b} & -\mathbf{c}\mathbf{A}^3\mathbf{b}\end{bmatrix}$$
$$= \begin{bmatrix}1 & -\alpha_1 & \alpha_1^2-\alpha_2 & -\alpha_1(\alpha_1^2-\alpha_2)+\alpha_2\alpha_1-\alpha_3 & a_{15} \\ 0 & 1 & -\alpha_1 & \alpha_1^2-\alpha_2 & a_{25} \\ 0 & 0 & 1 & -\alpha_1 & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \\ 0 & -\beta_1 & \beta_1\alpha_1-\beta_2 & -\beta_1(\alpha_1^2-\alpha_2)+\beta_2\alpha_1-\beta_3 & a_{55}\end{bmatrix}$$

where the last column is not written out to save space. The rank of a matrix will not change by elementary operations. Adding the second row multiplied by β_1 to the last row, and adding the third row multiplied by β_2 to the last row, and adding the fourth row multiplied by β_3 to the last row, we obtain

$$\begin{bmatrix}1 & -\alpha_1 & \alpha_1^2-\alpha_2 & -\alpha_1(\alpha_1^2-\alpha_2)+\alpha_2\alpha_1-\alpha_3 & a_{15} \\ 0 & 1 & -\alpha_1 & \alpha_1^2-\alpha_2 & a_{25} \\ 0 & 0 & 1 & -\alpha_1 & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \\ 0 & 0 & 0 & 0 & -\beta_4\end{bmatrix} \tag{8.31}$$

Its determinant is $-\beta_4$. Thus the matrix is nonsingular if and only if $\beta_4 \neq 0$. In conclusion, if $(\mathbf{A}, \mathbf{b})$ is controllable and if $\hat{g}(s)$ has no zero at $s = 0$, then the pair in (8.30) is controllable. It follows from Theorem 8.3 that all eigenvalues of the A-matrix in (8.29) can be assigned arbitrarily by selecting a feedback gain $[\mathbf{k} \ \ k_a]$. Q.E.D.

We mention that the controllability of the pair in (8.30) can also be explained from pole–zero cancellations. If the plant transfer function has a zero at $s = 0$, then the tandem connection of the integrator, which has transfer function $1/s$, and the plant will involve the pole–zero cancellation of s and the state equation describing the connection will not be controllable. On the other hand, if the plant transfer function has no zero at $s = 0$, then there is no pole–zero cancellation and the connection will be controllable.

Consider again (8.29). We assume that a set of $n + 1$ desired stable eigenvalues or, equivalently, a desired polynomial $\Delta_f(s)$ of degree $n + 1$ has been selected and the feedback gain $[\mathbf{k} \ \ k_a]$ has been found such that

$$\Delta_f(s) = \det\begin{bmatrix}s\mathbf{I}-\mathbf{A}-\mathbf{b}\mathbf{k} & -\mathbf{b}k_a \\ \mathbf{c} & s\end{bmatrix} \tag{8.32}$$

Now we show that the output y will track asymptotically and robustly any step reference input $r(t) = a$ and reject any step disturbance with unknown magnitude. Instead of establishing the assertion directly from (8.29), we will develop an equivalent block diagram of Fig. 8.4(a) and then establish the assertion. First we interchange the two summers between v and $\bar{u}$ as shown in Fig. 8.4(b). This is permitted because we have $\bar{u} = v + \mathbf{kx} + w$ before and after the interchange. The transfer function from $\bar{v}$ to y is

$$\hat{\bar{g}}(s) := \frac{\bar{N}(s)}{\bar{D}(s)} := \mathbf{c}(s\mathbf{I} - \mathbf{A} - \mathbf{bk})^{-1}\mathbf{b} \tag{8.33}$$

with $\bar{D}(s) = \det(s\mathbf{I} - \mathbf{A} - \mathbf{bk})$. Thus Fig. 8.4(a) can be redrawn as shown in Fig. 8.4(c). We next establish the relationship between $\Delta_f(s)$ in (8.32) and $\hat{\bar{g}}(s)$ in (8.33). It is straightforward to verify the following equality:

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{c}(s\mathbf{I} - \mathbf{A} - \mathbf{bk})^{-1} & 1 \end{bmatrix} \begin{bmatrix} s\mathbf{I} - \mathbf{A} - \mathbf{bk} & -\mathbf{b}k_a \\ \mathbf{c} & s \end{bmatrix} = \begin{bmatrix} s\mathbf{I} - \mathbf{A} - \mathbf{bk} & -\mathbf{b}k_a \\ 0 & s + \mathbf{c}(s\mathbf{I} - \mathbf{A} - \mathbf{bk})^{-1}\mathbf{b}k_a \end{bmatrix}$$

Taking its determinants and using (8.32) and (8.33), we obtain

$$1 \cdot \Delta_f(s) = \bar{D}(s)\left(s + \frac{\bar{N}(s)}{\bar{D}(s)}k_a\right)$$

which implies

$$\Delta_f(s) = s\bar{D}(s) + k_a\bar{N}(s)$$

This is a key equation.

From Fig. 8.4(c), the transfer function from w to y can readily be computed as

$$\hat{g}_{yw} = \frac{\dfrac{\bar{N}(s)}{\bar{D}(s)}}{1 + \dfrac{k_a\bar{N}(s)}{s\bar{D}(s)}} = \frac{s\bar{N}(s)}{s\bar{D}(s) + k_a\bar{N}(s)} = \frac{s\bar{N}(s)}{\Delta_f(s)}$$

If the disturbance is $w(t) = \bar{w}$ for all $t \geq 0$, where $\bar{w}$ is an unknown constant, then $\hat{w}(s) = \bar{w}/s$ and the corresponding output is given by

$$\hat{y}_w(s) = \frac{s\bar{N}(s)}{\Delta_f(s)}\frac{\bar{w}}{s} = \frac{\bar{w}\bar{N}(s)}{\Delta_f(s)} \tag{8.34}$$

Because the pole s in (8.34) is canceled, all remaining poles of $\hat{y}_w(s)$ are stable poles. Therefore the corresponding time response, for any $\bar{w}$, will die out as $t \to \infty$. The only condition to achieve the disturbance rejection is that $\hat{y}_w(s)$ has only stable poles. Thus the rejection still holds, even if there are plant parameter variations and variations in the feedforward gain k_a and feedback gain $\mathbf{k}$, as long as the overall system remains stable. Thus the disturbance is suppressed at the output both asymptotically and robustly.

The transfer function from r to y is

$$\hat{g}_{yr}(s) = \frac{\dfrac{k_a}{s}\dfrac{\bar{N}(s)}{\bar{D}(s)}}{1 + \dfrac{k_a}{s}\dfrac{\bar{N}(s)}{\bar{D}(s)}} = \frac{k_a\bar{N}(s)}{s\bar{D}(s) + k_a\bar{N}(s)} = \frac{k_a\bar{N}(s)}{\Delta_f(s)}$$

We see that

$$\hat{g}_{yr}(0) = \frac{k_a\bar{N}(0)}{0 \cdot \bar{D}(0) + k_a\bar{N}(0)} = \frac{k_a\bar{N}(0)}{k_a\bar{N}(0)} = 1 \tag{8.35}$$

Equation (8.35) holds even when there are parameter perturbations in the plant transfer function and the gains. Thus asymptotic tracking of any step reference input is robust. Note that this robust tracking holds even for very large parameter perturbations as long as the overall system remains stable.

We see that the design is achieved by inserting an integrator as shown in Fig. 8.4. The integrator is in fact a model of the step reference input and constant disturbance. Thus it is called the *internal model principle*. This will be discussed further in the next chapter.

8.3.2 Stabilization

If a state equation is controllable, all eigenvalues can be assigned arbitrarily by introducing state feedback. We now discuss the case when the state equation is not controllable. Every uncontrollable state equation can be transformed into

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} u \tag{8.36}$$

where $(\bar{\mathbf{A}}_c, \bar{\mathbf{b}}_c)$ is controllable (Theorem 6.6). Because the A-matrix is block triangular, the eigenvalues of the original A-matrix are the union of the eigenvalues of $\bar{\mathbf{A}}_c$ and $\bar{\mathbf{A}}_{\bar{c}}$. If we introduce the state feedback

$$u = r - \mathbf{kx} = r - \bar{\mathbf{k}}\bar{\mathbf{x}} = r - [\bar{\mathbf{k}}_1 \ \bar{\mathbf{k}}_2]\begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix}$$

where we have partitioned $\bar{\mathbf{k}}$ as in $\bar{\mathbf{x}}$, then (8.36) becomes

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c - \bar{\mathbf{b}}_c\bar{\mathbf{k}}_1 & \bar{\mathbf{A}}_{12} - \bar{\mathbf{b}}_c\bar{\mathbf{k}}_2 \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} r \tag{8.37}$$

We see that $\bar{\mathbf{A}}_{\bar{c}}$ and, consequently, its eigenvalues are not affected by the state feedback. Thus we conclude that the controllability condition of $(\mathbf{A}, \mathbf{b})$ in Theorem 8.3 is not only sufficient but also necessary to assign *all* eigenvalues of $(\mathbf{A} - \mathbf{bk})$ to any desired positions.

Consider again the state equation in (8.36). If $\bar{\mathbf{A}}_{\bar{c}}$ is stable, and if $(\bar{\mathbf{A}}_c, \bar{\mathbf{b}}_c)$ is controllable, then (8.36) is said to be *stabilizable*. We mention that the controllability condition for tracking and disturbance rejection can be replaced by the weaker condition of stabilizability. But in this case, we do not have complete control of the rate of tracking and rejection. If the uncontrollable stable eigenvalues have large imaginary parts or are close to the imaginary axis, then the tracking and rejection may not be satisfactory.

8.4 State Estimator

We introduced in the preceding sections state feedback under the implicit assumption that all state variables are available for feedback. This assumption may not hold in practice either because the state variables are not accessible for direct connection or because sensing devices or transducers are not available or very expensive. In this case, in order to apply state feedback, we must design a device, called a *state estimator* or *state observer*, so that the output of the device will generate an estimate of the state. In this section, we introduce full-dimensional state estimators which have the same dimension as the original state equation. We use the circumflex over a variable to denote an estimate of the variable. For example, $\hat{\mathbf{x}}$ is an estimate of $\mathbf{x}$ and $\hat{\dot{\mathbf{x}}}$ is an estimate of $\dot{\mathbf{x}}$.

Consider the n-dimensional state equation

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}\mathbf{x} \end{aligned} \tag{8.38}$$

where $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ are given and the input $u(t)$ and the output $y(t)$ are available to us. The state $\mathbf{x}$, however, is not available to us. The problem is to estimate $\mathbf{x}$ from u and y with the knowledge of $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$. If we know $\mathbf{A}$ and $\mathbf{b}$, we can duplicate the original system as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{b}u \tag{8.39}$$

and as shown in Fig. 8.5. Note that the original system could be an electromechanical system and the duplicated system could be an op-amp circuit. The duplication will be called an *open-loop* estimator. Now if (8.38) and (8.39) have the same initial state, then for any input, we have $\hat{\mathbf{x}}(t) = \mathbf{x}(t)$ for all $t \geq 0$. Therefore the remaining question is how to find the initial state of (8.38) and then set the initial state of (8.39) to that state. If (8.38) is observable, its initial state $\mathbf{x}(0)$ can be computed from u and y over any time interval, say, $[0, t_1]$. We can then compute the state at t_2 and set $\hat{\mathbf{x}}(t_2) = \mathbf{x}(t_2)$. Then we have $\hat{\mathbf{x}}(t) = \mathbf{x}(t)$ for all $t \geq t_2$. Thus if (8.38) is observable, an open-loop estimator can be used to generate the state vector.

There are, however, two disadvantages in using an open-loop estimator. First, the initial state must be computed and set each time we use the estimator. This is very inconvenient. Second, and more seriously, if the matrix $\mathbf{A}$ has eigenvalues with positive real parts, then even for a very small difference between $\mathbf{x}(t_0)$ and $\hat{\mathbf{x}}(t_0)$ for some t_0, which may be caused by disturbance or imperfect estimation of the initial state, the difference between $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ will grow with time. Therefore the open-loop estimator is, in general, not satisfactory.

We see from Fig. 8.5 that even though the input and output of (8.38) are available, we

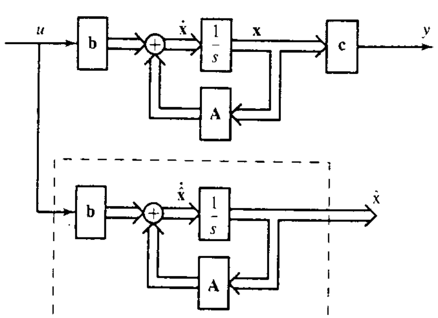
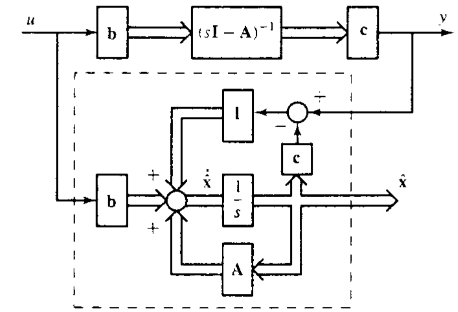


Figure 8.5 Open-loop state estimator.

Figure 8.6 Closed-loop state estimator.



use only the input to drive the open-loop estimator. Now we shall modify the estimator in Fig. 8.5 to the one in Fig. 8.6, in which the output $y(t) = \mathbf{c}\mathbf{x}(t)$ of (8.38) is compared with $\mathbf{c}\hat{\mathbf{x}}(t)$. Their difference, passing through an $n \times 1$ constant gain vector $\mathbf{l}$, is used as a correcting term. If the difference is zero, no correction is needed. If the difference is nonzero and if the gain $\mathbf{l}$ is properly designed, the difference will drive the estimated state to the actual state. Such an estimator is called a *closed-loop* or an *asymptotic* estimator or, simply, an estimator.

The open-loop estimator in (8.39) is now modified as, following Fig. 8.6,

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}(y - \mathbf{c}\hat{\mathbf{x}})$$

which can be written as

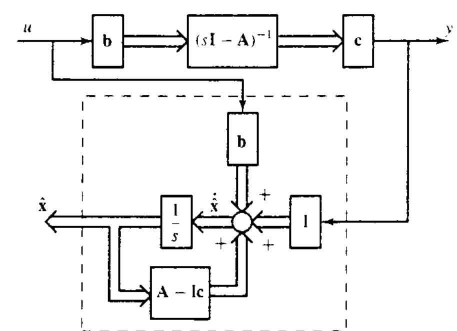
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y \tag{8.40}$$

and is shown in Fig. 8.7. It has two inputs u and y and its output yields an estimated state $\hat{\mathbf{x}}$. Let us define

$$\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

It is the error between the actual state and the estimated state. Differentiating $\mathbf{e}$ and then substituting (8.38) and (8.40) into it, we obtain

Figure 8.7 Closed-loop state estimator.



$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{Ax} + \mathbf{b}u - (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} - \mathbf{b}u - \mathbf{l}(\mathbf{cx})$$
$$= (\mathbf{A} - \mathbf{lc})\mathbf{x} - (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} = (\mathbf{A} - \mathbf{lc})(\mathbf{x} - \hat{\mathbf{x}})$$

or

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{lc})\mathbf{e} \tag{8.41}$$

This equation governs the estimation error. If all eigenvalues of $(\mathbf{A} - \mathbf{lc})$ can be assigned arbitrarily, then we can control the rate for $\mathbf{e}(t)$ to approach zero or, equivalently, for the estimated state to approach the actual state. For example, if all eigenvalues of $(\mathbf{A} - \mathbf{lc})$ have negative real parts smaller than $-\sigma$, then all entries of $\mathbf{e}$ will approach zero at rates faster than $e^{-\sigma t}$. Therefore, even if there is a large error between $\hat{\mathbf{x}}(t_0)$ and $\mathbf{x}(t_0)$ at initial time t_0, the estimated state will approach the actual state rapidly. Thus there is no need to compute the initial state of the original state equation. In conclusion, if all eigenvalues of $(\mathbf{A} - \mathbf{lc})$ are properly assigned, a closed-loop estimator is much more desirable than an open-loop estimator.

As in the state feedback, what constitutes the best eigenvalues is not a simple problem. Probably, they should be placed evenly along a circle inside the sector shown in Fig. 8.3(a). If an estimator is to be used in state feedback, then the estimator eigenvalues should be faster than the desired eigenvalues of the state feedback. Again, saturation and noise problems will impose constraints on the selection. One way to carry out the selection is by computer simulation.

Theorem 8.O3

Consider the pair $(\mathbf{A}, \mathbf{c})$. All eigenvalues of $(\mathbf{A} - \mathbf{lc})$ can be assigned arbitrarily by selecting a real constant vector $\mathbf{l}$ if and only if $(\mathbf{A}, \mathbf{c})$ is observable.

This theorem can be established directly or indirectly by using the duality theorem. The pair $(\mathbf{A}, \mathbf{c})$ is observable if and only if $(\mathbf{A}', \mathbf{c}')$ is controllable. If $(\mathbf{A}', \mathbf{c}')$ is controllable, all eigenvalues of $(\mathbf{A}' - \mathbf{c}'\mathbf{k})$ can be assigned arbitrarily by selecting a constant gain vector $\mathbf{k}$. The transpose of $(\mathbf{A}' - \mathbf{c}'\mathbf{k})$ is $(\mathbf{A} - \mathbf{k}'\mathbf{c})$. Thus we have $\mathbf{l} = \mathbf{k}'$. In conclusion, the procedure for computing state feedback gains can be used to compute the gain $\mathbf{l}$ in state estimators.

Solving the Lyapunov equation We discuss a different method of designing a state estimator for the n-dimensional state equation

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{b}u \\ y &= \mathbf{cx} \end{aligned} \tag{8.42}$$

The method is dual to Procedure 8.1 in Section 8.2.1.

Procedure 8.O1

1. Select an arbitrary $n \times n$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $n \times 1$ vector $\mathbf{l}$ such that $(\mathbf{F}, \mathbf{l})$ is controllable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA} - \mathbf{FT} = \mathbf{lc}$. This $\mathbf{T}$ is nonsingular following the dual of Theorem 8.4.
4. Then the state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{l}y \tag{8.43}$$
$$\hat{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{z} \tag{8.44}$$

generates an estimate of $\mathbf{x}$.

We first justify the procedure. Let us define

$$\mathbf{e} := \mathbf{z} - \mathbf{Tx}$$

Then we have, replacing $\mathbf{TA}$ by $\mathbf{FT} + \mathbf{lc}$,

$$\dot{\mathbf{e}} = \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{lcx} - \mathbf{TAx} - \mathbf{Tb}u$$
$$= \mathbf{Fz} + \mathbf{lcx} - (\mathbf{FT} + \mathbf{lc})\mathbf{x} = \mathbf{F}(\mathbf{z} - \mathbf{Tx}) = \mathbf{Fe}$$

If $\mathbf{F}$ is stable, for any $\mathbf{e}(0)$, the error vector $\mathbf{e}(t)$ approaches zero as $t \to \infty$. Thus $\mathbf{z}$ approaches $\mathbf{Tx}$ or, equivalently, $\mathbf{T}^{-1}\mathbf{z}$ is an estimate of $\mathbf{x}$. All discussion in Section 8.2.1 applies here and will not be repeated.

8.4.1 Reduced-Dimensional State Estimator

Consider the state equation in (8.42). If it is observable, then it can be transformed, dual to Theorem 8.2, into the observable canonical form in (7.14). We see that y equals x_1, the first state variable. Therefore it is sufficient to construct an $(n-1)$-dimensional state estimator to estimate x_i for $i = 2, 3, \ldots, n$. This estimator with the output equation can then be used to estimate all n state variables. This estimator has a lesser dimension than (8.42) and is called a reduced-dimensional estimator.

Reduced-dimensional estimators can be designed by transformations or by solving Lyapunov equations. The latter approach is considerably simpler and will be discussed next. For the former approach, the interested reader is referred to Reference [6, pp. 361–363].

Procedure 8.R1

1. Select an arbitrary $(n-1) \times (n-1)$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $(n-1) \times 1$ vector $\mathbf{l}$ such that $(\mathbf{F}, \mathbf{l})$ is controllable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA} - \mathbf{FT} = \mathbf{lc}$. Note that $\mathbf{T}$ is an $(n-1) \times n$ matrix.
4. Then the $(n-1)$-dimensional state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{l}y \tag{8.45}$$

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} y \\ \mathbf{z} \end{bmatrix} \tag{8.46}$$

is an estimate of $\mathbf{x}$.

We first justify the procedure. We write (8.46) as

$$\begin{bmatrix} y \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix} \hat{\mathbf{x}} =: \mathbf{P}\hat{\mathbf{x}}$$

which implies $y = \mathbf{c}\hat{\mathbf{x}}$ and $\mathbf{z} = \mathbf{T}\hat{\mathbf{x}}$. Clearly y is an estimate of $\mathbf{cx}$. We now show that $\mathbf{z}$ is an estimate of $\mathbf{Tx}$. Define

$$\mathbf{e} = \mathbf{z} - \mathbf{Tx}$$

Then we have

$$\dot{\mathbf{e}} = \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{Fz} + \mathbf{T}\mathbf{b}u + \mathbf{lcx} - \mathbf{TAx} - \mathbf{Tb}u = \mathbf{Fe}$$

Clearly if $\mathbf{F}$ is stable, then $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$. Thus $\mathbf{z}$ is an estimate of $\mathbf{Tx}$.

Theorem 8.6

If $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues, then the square matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}$$

where $\mathbf{T}$ is the unique solution of $\mathbf{TA} - \mathbf{FT} = \mathbf{lc}$, is nonsingular if and only if $(\mathbf{A}, \mathbf{c})$ is observable and $(\mathbf{F}, \mathbf{l})$ is controllable.

Proof: We prove the theorem for $n = 4$. The first part of the proof follows closely the proof of Theorem 8.4. Let

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$$

Then, dual to (8.22), we have

$$-\mathbf{T}\Delta(\mathbf{F}) = [\mathbf{l}\ \mathbf{Fl}\ \mathbf{F}^2\mathbf{l}\ \mathbf{F}^3\mathbf{l}] \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \mathbf{cA}^3 \end{bmatrix} \tag{8.47}$$

and $\Delta(\mathbf{F})$ is nonsingular if $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues. Note that if $\mathbf{A}$ is 4×4, then $\mathbf{F}$ is 3×3. The rightmost matrix in (8.47) is the observability matrix of $(\mathbf{A}, \mathbf{c})$ and will be denoted by O. The first matrix after the equality is the controllability matrix of $(\mathbf{F}, \mathbf{l})$ with one extra column and will be denoted by C_4. The middle matrix will be denoted by $\mathbf{\Lambda}$ and is always nonsingular. Using these notations, we write $\mathbf{T}$ as $-\Delta^{-1}(\mathbf{F})C_4\mathbf{\Lambda}O$ and $\mathbf{P}$ becomes

$$\mathbf{P} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ -\Delta^{-1}(\mathbf{F})C_4\mathbf{\Lambda}O \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}(\mathbf{F}) \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ C_4\mathbf{\Lambda}O \end{bmatrix} \tag{8.48}$$

Note that if $n = 4$, then $\mathbf{P}$, O, and $\mathbf{\Lambda}$ are 4×4; $\mathbf{T}$ and C_4 are 3×4 and $\Delta(\mathbf{F})$ is 3×3. If $(\mathbf{F}, \mathbf{l})$ is not controllable, C_4 has rank at most 2. Thus $\mathbf{T}$ has rank at most 2 and $\mathbf{P}$ is singular. If $(\mathbf{A}, \mathbf{c})$ is not observable, then there exists a nonzero 4×1 vector $\mathbf{r}$ such that

$O\mathbf{r} = \mathbf{0}$, which implies $\mathbf{cr} = 0$ and $\mathbf{Pr} = \mathbf{0}$. Thus $\mathbf{P}$ is singular. This shows the necessity of the theorem.

Next we show the sufficiency by contradiction. Suppose $\mathbf{P}$ is singular. Then there exists a nonzero vector $\mathbf{r}$ such that $\mathbf{Pr} = \mathbf{0}$, which implies

$$\begin{bmatrix} \mathbf{c} \\ C_4\mathbf{\Lambda}O \end{bmatrix} \mathbf{r} = \begin{bmatrix} \mathbf{cr} \\ C_4\mathbf{\Lambda}O\mathbf{r} \end{bmatrix} = \mathbf{0} \tag{8.49}$$

Define $\mathbf{a} := \mathbf{\Lambda}O\mathbf{r} = [a_1\ a_2\ a_3\ a_4]' =: [\bar{\mathbf{a}}\ a_4]'$, where $\bar{\mathbf{a}}$ represents the first three entries of $\mathbf{a}$. Expressing it explicitly yields

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{cr} \\ \mathbf{cAr} \\ \mathbf{cA}^2\mathbf{r} \\ \mathbf{cA}^3\mathbf{r} \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \\ \mathbf{cr} \end{bmatrix}$$

where x denotes entries that are not needed in subsequent discussion. Thus we have $a_4 = \mathbf{cr}$. Clearly (8.49) implies $a_4 = \mathbf{cr} = 0$. Substituting $a_4 = 0$ into the lower part of (8.49) yields

$$C_4\mathbf{\Lambda}O\mathbf{r} = C_4\mathbf{a} = C\bar{\mathbf{a}} = \mathbf{0} \tag{8.50}$$

where C is 3×3 and is the controllability matrix of $(\mathbf{F}, \mathbf{l})$ and $\bar{\mathbf{a}}$ is the first three entries of $\mathbf{a}$. If $(\mathbf{F}, \mathbf{l})$ is controllable, then $C\bar{\mathbf{a}} = \mathbf{0}$ implies $\bar{\mathbf{a}} = \mathbf{0}$. In conclusion, (8.49) and the controllability of $(\mathbf{F}, \mathbf{l})$ imply $\mathbf{a} = \mathbf{0}$.

Consider $\mathbf{\Lambda}O\mathbf{r} = \mathbf{a} = \mathbf{0}$. The matrix $\mathbf{\Lambda}$ is always nonsingular. If $(\mathbf{A}, \mathbf{c})$ is observable, then O is nonsingular and $\mathbf{\Lambda}O\mathbf{r} = \mathbf{0}$ implies $\mathbf{r} = \mathbf{0}$. This contradicts the hypothesis that $\mathbf{r}$ is nonzero. Thus if $(\mathbf{A}, \mathbf{c})$ is observable and $(\mathbf{F}, \mathbf{l})$ is controllable, then $\mathbf{P}$ is nonsingular. This establishes Theorem 8.6. Q.E.D.

Designing state estimators by solving Lyapunov equations is convenient because the same procedure can be used to design full-dimensional and reduced-dimensional estimators. As we shall see in a later section, the same procedure can also be used to design estimators for multi-input multi-output systems.

8.5 Feedback from Estimated States

Consider a plant described by the n-dimensional state equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{b}u$$
$$y = \mathbf{cx} \tag{8.51}$$

If $(\mathbf{A}, \mathbf{b})$ is controllable, state feedback $u = r - \mathbf{kx}$ can place the eigenvalues of $(\mathbf{A} - \mathbf{bk})$ in any desired positions. If the state variables are not available for feedback, we can design a state estimator. If $(\mathbf{A}, \mathbf{c})$ is observable, a full- or reduced-dimensional estimator with arbitrary eigenvalues can be constructed. We discuss here only full-dimensional estimators. Consider the n-dimensional state estimator

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y \tag{8.52}$$

The estimated state in (8.52) can approach the actual state in (8.51) with any rate by selecting the vector $\mathbf{l}$.

The state feedback is designed for the state in (8.51). If $\mathbf{x}$ is not available, it is natural to apply the feedback gain to the estimated state as

$$u = r - \mathbf{k}\hat{\mathbf{x}} \tag{8.53}$$

as shown in Fig. 8.8. The connection is called the *controller-estimator* configuration. Three questions may be raised in this connection: (1) The eigenvalues of $(\mathbf{A} - \mathbf{bk})$ are obtained from $u = r - \mathbf{kx}$. Do we still have the same set of eigenvalues in using $u = r - \mathbf{k}\hat{\mathbf{x}}$? (2) Will the eigenvalues of the estimator be affected by the connection? (3) What is the effect of the estimator on the transfer function from r to y? To answer these questions, we must develop a state equation to describe the overall system in Fig. 8.8. Substituting (8.53) into (8.51) and (8.52) yields

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{bk}\hat{\mathbf{x}} + \mathbf{b}r$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} + \mathbf{b}(r - \mathbf{k}\hat{\mathbf{x}}) + \mathbf{lcx}$$

They can be combined as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{bk} \\ \mathbf{lc} & \mathbf{A} - \mathbf{lc} - \mathbf{bk} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} r$$
$$y = [\mathbf{c} \;\; \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \tag{8.54}$$

This $2n$-dimensional state equation describes the feedback system in Fig. 8.8. It is not easy to answer the posed questions from this equation. Let us introduce the following equivalence transformation:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x} - \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} =: \mathbf{P} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Computing $\mathbf{P}^{-1}$, which happens to equal $\mathbf{P}$, and then using (4.26), we can obtain the following equivalent state equation:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A} - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$
$$y = [\mathbf{c} \;\; \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{8.55}$$

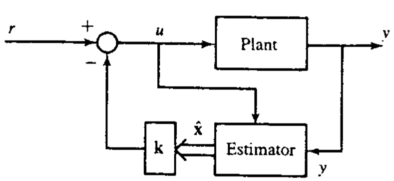


Figure 8.8 Controller-estimator configuration.

The A-matrix in (8.55) is block triangular; therefore its eigenvalues are the union of those of $(\mathbf{A} - \mathbf{bk})$ and $(\mathbf{A} - \mathbf{lc})$. Thus inserting the state estimator does not affect the eigenvalues of the original state feedback; nor are the eigenvalues of the state estimator affected by the connection. Thus the design of state feedback and the design of state estimator can be carried out independently. This is called the *separation property*.

The state equation in (8.55) is of the form shown in (6.40); thus (8.55) is not controllable and the transfer function of (8.55) equals the transfer function of the reduced equation

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{bk})\mathbf{x} + \mathbf{b}r \qquad y = \mathbf{cx}$$

or

$$\hat{g}_f(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A} + \mathbf{bk})^{-1}\mathbf{b}$$

(Theorem 6.6). This is the transfer function of the original state feedback system without using a state estimator. Therefore the estimator is completely canceled in the transfer function from r to y. This has a simple explanation. In computing transfer functions, all initial states are assumed to be zero. Consequently, we have $\mathbf{x}(0) = \hat{\mathbf{x}}(0) = \mathbf{0}$, which implies $\mathbf{x}(t) = \hat{\mathbf{x}}(t)$ for all t. Thus, as far as the transfer function from r to y is concerned, there is no difference whether a state estimator is employed or not.

8.6 State Feedback—Multivariable Case

This section extends state feedback to multivariable systems. Consider a plant described by the n-dimensional p-input state equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} \tag{8.56}$$

In state feedback, the input $\mathbf{u}$ is given by

$$\mathbf{u} = \mathbf{r} - \mathbf{Kx} \tag{8.57}$$

where $\mathbf{K}$ is a $p \times n$ real constant matrix and $\mathbf{r}$ is a reference signal. Substituting (8.57) into (8.56) yields

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{Br}$$
$$\mathbf{y} = \mathbf{Cx} \tag{8.58}$$

Theorem 8.M1

The pair $(\mathbf{A} - \mathbf{BK}, \mathbf{B})$, for any $p \times n$ real constant matrix $\mathbf{K}$, is controllable if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

The proof of this theorem follows closely the proof of Theorem 8.1. The only difference is that we must modify (8.4) as

$$\mathbf{C}_f = \mathbf{C}\begin{bmatrix} \mathbf{I}_p & -\mathbf{KB} & -\mathbf{K}(\mathbf{A}-\mathbf{BK})\mathbf{B} & -\mathbf{K}(\mathbf{A}-\mathbf{BK})^2\mathbf{B} \\ \mathbf{0} & \mathbf{I}_p & -\mathbf{KB} & -\mathbf{K}(\mathbf{A}-\mathbf{BK})\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_p & -\mathbf{KB} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_p \end{bmatrix}$$

where $\mathcal{C}_f$ and $\mathcal{C}$ are $n \times np$ controllability matrices with $n = 4$ and $\mathbf{I}_p$ is the unit matrix of order p. Because the rightmost $4p \times 4p$ matrix is nonsingular, $\mathcal{C}_f$ has rank n if and only if $\mathcal{C}$ has rank n. Thus the controllability property is preserved in any state feedback. As in the SISO case, the observability property, however, may not be preserved. Next we extend Theorem 8.3 to the matrix case

Theorem 8.M3

All eigenvalues of $(\mathbf{A} - \mathbf{BK})$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

If $(\mathbf{A}, \mathbf{B})$ is not controllable, then $(\mathbf{A}, \mathbf{B})$ can be transformed into the form shown in (8.36) and the eigenvalues of $\bar{\mathbf{A}}_{\bar{c}}$ will not be affected by any state feedback. This shows the necessity of the theorem. The sufficiency will be established constructively in the next three subsections.

8.6.1 Cyclic Design

In this design, we change the multi-input problem into a single-input problem and then apply Theorem 8.3. A matrix $\mathbf{A}$ is called *cyclic* if its characteristic polynomial equals its minimal polynomial. From the discussion in Section 3.6, we can conclude that $\mathbf{A}$ is cyclic if and only if the Jordan form of $\mathbf{A}$ has one and only one Jordan block associated with each distinct eigenvalue.

Theorem 8.7

If the n-dimensional p-input pair $(\mathbf{A}, \mathbf{B})$ is controllable and if $\mathbf{A}$ is cyclic, then for almost any $p \times 1$ vector $\mathbf{v}$, the single-input pair $(\mathbf{A}, \mathbf{Bv})$ is controllable.

We argue intuitively the validity of this theorem. Controllability is invariant under any equivalence transformation; thus we may assume $\mathbf{A}$ to be in Jordan form. To see the basic idea, we use the following example:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \quad \mathbf{Bv} = \mathbf{B}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} x \\ x \\ \alpha \\ x \\ \beta \end{bmatrix} \qquad (8.59)$$

There is only one Jordan block associated with each distinct eigenvalue; thus $\mathbf{A}$ is cyclic. The

condition for $(\mathbf{A}, \mathbf{B})$ to be controllable is that the third and the last rows of $\mathbf{B}$ are nonzero (Theorem 6.8).

The necessary and sufficient conditions for $(\mathbf{A}, \mathbf{Bv})$ to be controllable are $\alpha \neq 0$ and $\beta \neq 0$ in (8.59). Because $\alpha = v_1 + 2v_2$ and $\beta = v_1$, either α or β is zero if and only if $v_1 = 0$ or $v_1/v_2 = -2/1$. Thus any $\mathbf{v}$ other than $v_1 = 0$ and $v_1 = -2v_2$ will make $(\mathbf{A}, \mathbf{Bv})$ controllable. The vector $\mathbf{v}$ can assume any value in the two-dimensional real space shown in Fig. 8.9. The conditions $v_1 = 0$ and $v_1 = -2v_2$ constitute two straight lines as shown. The probability for an arbitrarily selected $\mathbf{v}$ to lie on either straight line is zero. This establishes Theorem 8.6. The cyclicity assumption in this theorem is essential. For example, the pair

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

is controllable (Theorem 6.8). However, there is no $\mathbf{v}$ such that $(\mathbf{A}, \mathbf{Bv})$ is controllable (Corollary 6.8).

If all eigenvalues of $\mathbf{A}$ are distinct, then there is only one Jordan block associated with each eigenvalue. Thus a sufficient condition for $\mathbf{A}$ to be cyclic is that all eigenvalues of $\mathbf{A}$ are distinct.

Theorem 8.8

If $(\mathbf{A}, \mathbf{B})$ is controllable, then for almost any $p \times n$ real constant matrix $\mathbf{K}$, the matrix $(\mathbf{A} - \mathbf{BK})$ has only distinct eigenvalues and is, consequently, cyclic.

We show intuitively the theorem for $n = 4$. Let the characteristic polynomial of $\mathbf{A} - \mathbf{BK}$ be

$$\Delta_f(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

where the a_i are functions of the entries of $\mathbf{K}$. The differentiation of $\Delta_f(s)$ with respect to s yields

$$\Delta_f'(s) = 4s^3 + 3a_1 s^2 + 2a_2 s + a_3$$

If $\Delta_f(s)$ has repeated roots, then $\Delta_f(s)$ and $\Delta_f'(s)$ are not coprime. The necessary and sufficient condition for them to be not coprime is that their Sylvester resultant is singular or

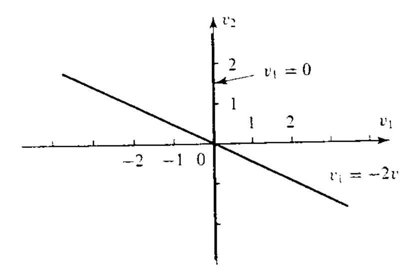


Figure 8.9 Two-dimensional real space.

$$\det \begin{bmatrix} a_4 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 2a_2 & a_4 & a_3 & 0 & 0 & 0 & 0 \\ a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 & 0 & 0 \\ a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 \\ 1 & 0 & a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 \\ 0 & 0 & 1 & 0 & a_1 & 4 & a_2 & 3a_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = b(k_{ij}) = 0$$

See (7.28). It is clear that all possible solutions of $b(k_{ij}) = 0$ constitute a very small subset of all real k_{ij}. Thus if we select an arbitrary **K**, the probability for its entries to meet $b(k_{ij}) = 0$ is 0. Thus all eigenvalues of $(\mathbf{A} - \mathbf{BK})$ will be distinct. This establishes the theorem.

With these two theorems, we can now find a **K** to place all eigenvalues of $(\mathbf{A} - \mathbf{BK})$ in any desired positions. If **A** is not cyclic, we introduce $\mathbf{u} = \mathbf{w} - \mathbf{K}_1\mathbf{x}$, as shown in Fig. 8.10, such that $\bar{\mathbf{A}} := \mathbf{A} - \mathbf{BK}_1$ in

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK}_1)\mathbf{x} + \mathbf{Bw} =: \bar{\mathbf{A}}\mathbf{x} + \mathbf{Bw} \tag{8.60}$$

is cyclic. Because $(\mathbf{A}, \mathbf{B})$ is controllable, so is $(\bar{\mathbf{A}}, \mathbf{B})$. Thus there exists a $p \times 1$ real vector **v** such that $(\bar{\mathbf{A}}, \mathbf{Bv})$ is controllable.[2] Next we introduce another state feedback $\mathbf{w} = \mathbf{r} - \mathbf{K}_2\mathbf{x}$ with $\mathbf{K}_2 = \mathbf{vk}$, where **k** is a $1 \times n$ real vector. Then (8.60) becomes

$$\dot{\mathbf{x}} = (\bar{\mathbf{A}} - \mathbf{BK}_2)\mathbf{x} + \mathbf{Br} = (\bar{\mathbf{A}} - \mathbf{Bvk})\mathbf{x} + \mathbf{Br}$$

Because the single-input pair $(\bar{\mathbf{A}}, \mathbf{Bv})$ is controllable, the eigenvalues of $(\bar{\mathbf{A}} - \mathbf{Bvk})$ can

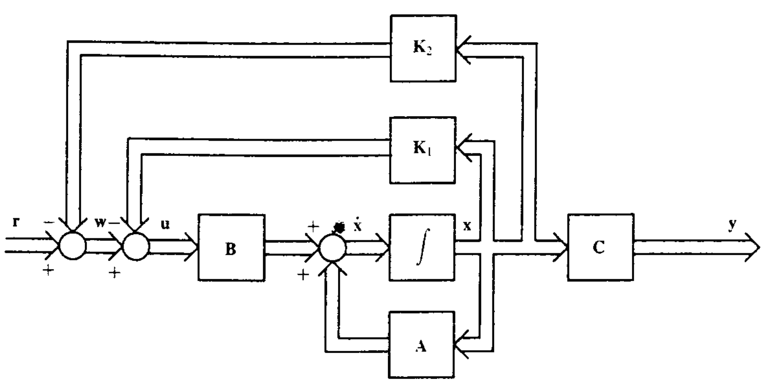


Figure 8.10 State feedback by cyclic design.

2. The choices of $\mathbf{K}_1$ and **v** are not unique. They can be chosen arbitrarily and the probability is 1 that they will meet the requirements. In Theorem 7.5 of Reference [5], a procedure is given to choose $\mathbf{K}_1$ and **v** with no uncertainty. The computation, however, is complicated.

be assigned arbitrarily by selecting a **k** (Theorem 8.3). Combining the two state feedback $\mathbf{u} = \mathbf{w} - \mathbf{K}_1\mathbf{x}$ and $\mathbf{w} = \mathbf{r} - \mathbf{K}_2\mathbf{x}$ as

$$\mathbf{u} = \mathbf{r} - (\mathbf{K}_1 + \mathbf{K}_2)\mathbf{x} =: \mathbf{r} - \mathbf{Kx}$$

we obtain a $\mathbf{K} := \mathbf{K}_1 + \mathbf{K}_2$ that achieves arbitrary eigenvalue assignment. This establishes Theorem 8.M3.

8.6.2 Lyapunov-Equation Method

This section will extend the procedure of computing feedback gain in Section 8.2.1 to the multivariable case. Consider an n-dimensional p-input pair $(\mathbf{A}, \mathbf{B})$. Find a $p \times n$ real constant matrix **K** so that $(\mathbf{A} - \mathbf{BK})$ has any set of desired eigenvalues as long as the set does not contain any eigenvalue of **A**.

Procedure 8.M1

1. Select an $n \times n$ matrix **F** with a set of desired eigenvalues that contains no eigenvalues of **A**.
2. Select an arbitrary $p \times n$ matrix $\bar{\mathbf{K}}$ such that $(\mathbf{F}, \bar{\mathbf{K}})$ is observable.
3. Solve the unique **T** in the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{B}\bar{\mathbf{K}}$.
4. If **T** is singular, select a different $\bar{\mathbf{K}}$ and repeat the process. If **T** is nonsingular, we compute $\mathbf{K} = \bar{\mathbf{K}}\mathbf{T}^{-1}$, and $(\mathbf{A} - \mathbf{BK})$ has the set of desired eigenvalues.

   If **T** is nonsingular, the Lyapunov equation and $\mathbf{KT} = \bar{\mathbf{K}}$ imply

$$(\mathbf{A} - \mathbf{BK})\mathbf{T} = \mathbf{TF} \quad \text{or} \quad \mathbf{A} - \mathbf{BK} = \mathbf{TFT}^{-1}$$

Thus $(\mathbf{A} - \mathbf{BK})$ and **F** are similar and have the same set of eigenvalues. Unlike the SISO case where **T** is always nonsingular, the **T** here may not be nonsingular even if $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{F}, \bar{\mathbf{K}})$ is observable. In other words, the two conditions are necessary but not sufficient for **T** to be nonsingular.

Theorem 8.M4

If **A** and **F** have no eigenvalues in common, then the unique solution **T** of $\mathbf{AT} - \mathbf{TF} = \mathbf{B}\bar{\mathbf{K}}$ is nonsingular only if $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{F}, \bar{\mathbf{K}})$ is observable.

Proof: The proof of Theorem 8.4 applies here except that (8.22) must be modified as, for $n = 4$,

$$-\mathbf{T}\Delta(\mathbf{F}) = [\mathbf{B} \;\; \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\; \mathbf{A}^3\mathbf{B}] \begin{bmatrix} \alpha_3\mathbf{I} & \alpha_2\mathbf{I} & \alpha_1\mathbf{I} & \mathbf{I} \\ \alpha_2\mathbf{I} & \alpha_1\mathbf{I} & \mathbf{I} & \mathbf{0} \\ \alpha_1\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{K}} \\ \bar{\mathbf{K}}\mathbf{F} \\ \bar{\mathbf{K}}\mathbf{F}^2 \\ \bar{\mathbf{K}}\mathbf{F}^3 \end{bmatrix}$$

or

$$-\mathbf{T}\Delta(\mathbf{F}) = C\Sigma O \tag{8.61}$$

where $\Delta(\mathbf{F})$ is nonsingular and C, Σ, and O are, respectively, $n \times np$, $np \times np$, and $np \times n$. If C or O has rank less than n, then $\mathbf{T}$ is singular following (3.61). However, the conditions that C and O have rank n do not imply the nonsingularity of $\mathbf{T}$. Thus the controllability of $(\mathbf{A}, \mathbf{B})$ and observability of $(\mathbf{F}, \bar{\mathbf{K}})$ are only necessary conditions for $\mathbf{T}$ to be nonsingular. This establishes Theorem 8.M4. Q.E.D.

Given a controllable $(\mathbf{A}, \mathbf{B})$, it is possible to construct an observable $(\mathbf{F}, \bar{\mathbf{K}})$ so that the $\mathbf{T}$ in Theorem 8.M4 is singular. However, after selecting $\mathbf{F}$, if $\bar{\mathbf{K}}$ is selected randomly and if $(\mathbf{F}, \bar{\mathbf{K}})$ is observable, it is believed that the probability for $\mathbf{T}$ to be nonsingular is 1. Therefore solving the Lyapunov equation is a viable method of computing a feedback gain matrix to achieve arbitrary eigenvalue assignment. As in the SISO case, we may choose $\mathbf{F}$ in companion form or in modal form as shown in (8.23). If $\mathbf{F}$ is chosen as in (8.23), then we can select $\bar{\mathbf{K}}$ as

$$\bar{\mathbf{K}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \bar{\mathbf{K}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(see Problem 6.16). Once $\mathbf{F}$ and $\bar{\mathbf{K}}$ are chosen, we can then use the MATLAB function `lyap` to solve the Lyapunov equation. Thus the procedure can easily be carried out.

8.6.3 Canonical-Form Method

We introduced in the preceding subsections two methods of computing a feedback gain matrix to achieve arbitrary eigenvalue assignment. The methods are relatively simple; however, they will not reveal the structure of the resulting feedback system. In this subsection, we discuss a different design that will reveal the effect of state feedback on the transfer matrix. We also give a transfer matrix interpretation of state feedback.

In this design, we must transform $(\mathbf{A}, \mathbf{B})$ into a controllable canonical form. It is an extension of Theorem 8.2 to the multivariable case. Although the basic idea is the same, the procedure can become very involved. Therefore we will skip the details and present the final result. To simplify the discussion, we assume that (8.56) has dimension 6, two inputs, and two outputs. We first search linearly independent columns of $C = [\mathbf{B}\ \mathbf{AB}\ \cdots\ \mathbf{A}^5\mathbf{B}]$ in order from left to right. It is assumed that its controllability indices are $\mu_1 = 4$ and $\mu_2 = 2$. Then there exists a nonsingular matrix $\mathbf{P}$ and $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ will transform (8.56) into the controllable canonical form

$$\dot{\bar{\mathbf{x}}} = \left[\begin{array}{cccc|cc} -\alpha_{111} & -\alpha_{112} & -\alpha_{113} & -\alpha_{114} & -\alpha_{121} & -\alpha_{122} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline -\alpha_{211} & -\alpha_{212} & -\alpha_{213} & -\alpha_{214} & -\alpha_{221} & -\alpha_{222} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \bar{\mathbf{x}}$$

$$+ \left[\begin{array}{cc} 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] \mathbf{u} \tag{8.62}$$

$$\mathbf{y} = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix} \bar{\mathbf{x}}$$

Note that this form is identical to the one in (7.104).

We now discuss how to find a feedback gain matrix to achieve arbitrary eigenvalue assignment. From a given set of six desired eigenvalues, we can form

$$\Delta_f(s) = (s^4 + \bar{\alpha}_{111}s^3 + \bar{\alpha}_{112}s^2 + \bar{\alpha}_{113}s + \bar{\alpha}_{114})(s^2 + \bar{\alpha}_{221}s + \bar{\alpha}_{222}) \tag{8.63}$$

Let us select $\bar{\mathbf{K}}$ as

$$\bar{\mathbf{K}} = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\alpha}_{111} - \alpha_{111} & \bar{\alpha}_{112} - \alpha_{112} & \bar{\alpha}_{113} - \alpha_{113} & \bar{\alpha}_{114} - \alpha_{114} & -\alpha_{121} & -\alpha_{122} \\ \bar{\alpha}_{211} - \alpha_{211} & \bar{\alpha}_{212} - \alpha_{212} & \bar{\alpha}_{213} - \alpha_{213} & \bar{\alpha}_{214} - \alpha_{214} & \bar{\alpha}_{221} - \alpha_{221} & \bar{\alpha}_{222} - \alpha_{222} \end{bmatrix} \tag{8.64}$$

Then it is straightforward to verify the following

$$\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}} = \left[\begin{array}{cccc|cc} -\bar{\alpha}_{111} & -\bar{\alpha}_{112} & -\bar{\alpha}_{113} & -\bar{\alpha}_{114} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline -\bar{\alpha}_{211} & -\bar{\alpha}_{212} & -\bar{\alpha}_{213} & -\bar{\alpha}_{214} & -\bar{\alpha}_{221} & -\bar{\alpha}_{222} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{8.65}$$

Because $(\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}})$ is block triangular, for any $\bar{\alpha}_{21i}$, $i = 1, 2, 3, 4$, its characteristic polynomial equals the product of the characteristic polynomials of the two diagonal blocks of orders 4 and 2. Because the diagonal blocks are of companion form, we conclude that the characteristic polynomial of $(\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}})$ equals the one in (8.63). If $\mathbf{K} = \bar{\mathbf{K}}\mathbf{P}$, then $(\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}}) = \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{P}^{-1}$. Thus the feedback gain $\mathbf{K} = \bar{\mathbf{K}}\mathbf{P}$ will place the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K})$ in the desired locations. This establishes once again Theorem 8.M3.

Unlike the single-input case, where the feedback gain is unique, the feedback gain matrix in the multi-input case is not unique. For example, the $\bar{\mathbf{K}}$ in (8.64) yields a lower block-triangular matrix in $(\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}})$. It is possible to select a different $\bar{\mathbf{K}}$ to yield an upper block-triangular matrix or a block-diagonal matrix. Furthermore, a different grouping of (8.63) will again yield a different $\bar{\mathbf{K}}$.

8.6.4 Effect on Transfer Matrices[3]

In the single-variable case, state feedback can shift the poles of a plant transfer function $\hat{g}(s)$ to any positions and yet has no effect on the zeros. Or, equivalently, state feedback can change the denominator coefficients, except the leading coefficient 1, to any values but has no effect on the numerator coefficients. Although we can establish a similar result for the multivariable case from (8.62) and (8.65), it is instructive to do so by using the result in Section 7.9. Following the notation in Section 7.9, we express $\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ as

$$\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) \tag{8.66}$$

or

$$\hat{\mathbf{y}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\hat{\mathbf{u}}(s) \tag{8.67}$$

where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime and $\mathbf{D}(s)$ is column reduced. Define

$$\mathbf{D}(s)\hat{\mathbf{v}}(s) = \hat{\mathbf{u}}(s) \tag{8.68}$$

as in (7.93). Then we have

$$\hat{\mathbf{y}}(s) = \mathbf{N}(s)\hat{\mathbf{v}}(s) \tag{8.69}$$

Let $\mathbf{H}(s)$ and $\mathbf{L}(s)$ be defined as in (7.91) and (7.92). Then the state vector in (8.62) is

$$\hat{\mathbf{x}}(s) = \mathbf{L}(s)\hat{\mathbf{v}}(s)$$

Thus the state feedback becomes, in the Laplace-transform domain,

$$\hat{\mathbf{u}}(s) = \hat{\mathbf{r}}(s) - \mathbf{K}\hat{\mathbf{x}}(s) = \hat{\mathbf{r}}(s) - \mathbf{K}\mathbf{L}(s)\hat{\mathbf{v}}(s) \tag{8.70}$$

and can be represented as shown in Fig. 8.11.

Let us express $\mathbf{D}(s)$ as

$$\mathbf{D}(s) = \mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s) \tag{8.71}$$

Substituting (8.71) and (8.70) into (8.68) yields

$$[\mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s)]\,\hat{\mathbf{v}}(s) = \hat{\mathbf{r}}(s) - \mathbf{K}\mathbf{L}(s)\hat{\mathbf{v}}(s)$$

which implies

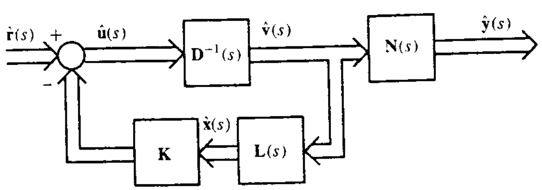


Figure 8.11 Transfer matrix interpretation of state feedback.

3. This subsection may be skipped without loss of continuity. The material in Section 7.9 is needed to study this subsection.

$$[\mathbf{D}_{hc}\mathbf{H}(s) + (\mathbf{D}_{lc} + \mathbf{K})\mathbf{L}(s)]\,\hat{\mathbf{v}}(s) = \hat{\mathbf{r}}(s)$$

Substituting this into (8.69) yields

$$\hat{\mathbf{y}}(s) = \mathbf{N}(s)\,[\mathbf{D}_{hc}\mathbf{H}(s) + (\mathbf{D}_{lc} + \mathbf{K})\mathbf{L}(s)]^{-1}\,\hat{\mathbf{r}}(s)$$

Thus the transfer matrix from $\mathbf{r}$ to $\mathbf{y}$ is

$$\hat{\mathbf{G}}_f(s) = \mathbf{N}(s)\,[\mathbf{D}_{hc}\mathbf{H}(s) + (\mathbf{D}_{lc} + \mathbf{K})\mathbf{L}(s)]^{-1} \tag{8.72}$$

The state feedback changes the plant transfer matrix $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ to the one in (8.72). We see that the numerator matrix $\mathbf{N}(s)$ is not affected by the state feedback. Neither are the column degree $\mathbf{H}(s)$ and the column-degree coefficient matrix $\mathbf{D}_{hc}$ affected by the state feedback. However, all coefficients associated with $\mathbf{L}(s)$ can be assigned arbitrarily by selecting a $\mathbf{K}$. This is similar to the SISO case.

It is possible to extend the robust tracking and disturbance rejection discussed in Section 8.3 to the multivariable case. It is simpler, however, to do so by using coprime fractions; therefore it will not be discussed here.

8.7 State Estimators—Multivariable Case

All discussion for state estimators in the single-variable case applies to the multivariable case; therefore the discussion will be brief. Consider the n-dimensional p-input q-output state equation

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned} \tag{8.73}$$

The problem is to use available input $\mathbf{u}$ and output $\mathbf{y}$ to drive a system whose output gives an estimate of the state $\mathbf{x}$. We extend (8.40) to the multivariable case as

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y} \tag{8.74}$$

This is a full-dimensional state estimator. Let us define the error vector as

$$\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t) \tag{8.75}$$

Then we have, as in (8.41),

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} \tag{8.76}$$

If $(\mathbf{A}, \mathbf{C})$ is observable, then all eigenvalues of $(\mathbf{A} - \mathbf{L}\mathbf{C})$ can be assigned arbitrarily by choosing an $\mathbf{L}$. Thus the convergence rate for the estimated state $\hat{\mathbf{x}}$ to approach the actual state $\mathbf{x}$ can be as fast as desired. As in the SISO case, the three methods of computing state feedback gain $\mathbf{K}$ in Sections 8.6.1 through 8.6.3 can be applied here to compute $\mathbf{L}$.

Next we discuss reduced-dimensional state estimators. The next procedure is an extension of Procedure 8.R1 to the multivariable case.

Procedure 8.MR1

Consider the n-dimensional q-output observable pair $(\mathbf{A}, \mathbf{C})$. It is assumed that $\mathbf{C}$ has rank q.

1. Select an arbitrary $(n-q) \times (n-q)$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $(n-q) \times q$ matrix $\mathbf{L}$ such that $(\mathbf{F}, \mathbf{L})$ is controllable.
3. Solve the unique $(n-q) \times n$ matrix $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA} - \mathbf{FT} = \mathbf{LC}$.
4. If the square matrix of order n

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} \tag{8.77}$$

is singular, go back to Step 2 and repeat the process. If $\mathbf{P}$ is nonsingular, then the $(n-q)$-dimensional state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{TBu} + \mathbf{Ly} \tag{8.78}$$

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \tag{8.79}$$

generates an estimate of $\mathbf{x}$.

We first justify the procedure. We write (8.79) as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} \hat{\mathbf{x}}$$

which implies $\mathbf{y} = \mathbf{C}\hat{\mathbf{x}}$ and $\mathbf{z} = \mathbf{T}\hat{\mathbf{x}}$. Clearly $\mathbf{y}$ is an estimate of $\mathbf{Cx}$. We now show that $\mathbf{z}$ is an estimate of $\mathbf{Tx}$. Let us define

$$\mathbf{e} := \mathbf{z} - \mathbf{Tx}$$

Then we have

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{Fz} + \mathbf{TBu} + \mathbf{LCx} - \mathbf{TAx} - \mathbf{TBu} \\ &= \mathbf{Fz} + (\mathbf{LC} - \mathbf{TA})\mathbf{x} = \mathbf{F}(\mathbf{z} - \mathbf{Tx}) = \mathbf{Fe} \end{aligned}$$

If $\mathbf{F}$ is stable, then $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$. Thus $\mathbf{z}$ is an estimate of $\mathbf{Tx}$.

Theorem 8.M6

If $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues, then the square matrix

$$\mathbf{P} := \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}$$

where $\mathbf{T}$ is the unique solution of $\mathbf{TA} - \mathbf{FT} = \mathbf{LC}$, is nonsingular only if $(\mathbf{A}, \mathbf{C})$ is observable and $(\mathbf{F}, \mathbf{L})$ is controllable.

This theorem can be proved by combining the proofs of Theorems 8.M4 and 8.6. Unlike Theorem 8.6, where the conditions are necessary and sufficient for $\mathbf{P}$ to be nonsingular, the conditions here are only necessary. Given $(\mathbf{A}, \mathbf{C})$, it is possible to construct a controllable pair $(\mathbf{F}, \mathbf{L})$ so that $\mathbf{P}$ is singular. However, after selecting $\mathbf{F}$, if $\mathbf{L}$ is selected randomly and if $(\mathbf{F}, \mathbf{L})$ is controllable, it is believed that the probability for $\mathbf{P}$ to be nonsingular is 1.

8.8 Feedback from Estimated States–Multivariable Case

This section will extend the separation property discussed in Section 8.5 to the multivariable case. We use the reduced-dimensional state estimator; therefore the development is more complex.

Consider the n-dimensional state equation

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned} \tag{8.80}$$

and the $(n-q)$-dimensional state estimator in (8.78) and (8.79). First we compute the inverse of $\mathbf{P}$ in (8.77) and then partition it as $[\mathbf{Q}_1 \ \ \mathbf{Q}_2]$, where $\mathbf{Q}_1$ is $n \times q$ and $\mathbf{Q}_2$ is $n \times (n-q)$; that is,

$$[\mathbf{Q}_1 \ \ \mathbf{Q}_2] \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} = \mathbf{Q}_1\mathbf{C} + \mathbf{Q}_2\mathbf{T} = \mathbf{I} \tag{8.81}$$

Then the $(n-q)$-dimensional state estimator in (8.78) and (8.79) can be written as

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{TBu} + \mathbf{Ly} \tag{8.82}$$

$$\hat{\mathbf{x}} = \mathbf{Q}_1\mathbf{y} + \mathbf{Q}_2\mathbf{z} \tag{8.83}$$

If the original state is not available for state feedback, we apply the feedback gain matrix to $\hat{\mathbf{x}}$ to yield

$$\mathbf{u} = \mathbf{r} - \mathbf{K}\hat{\mathbf{x}} = \mathbf{r} - \mathbf{KQ}_1\mathbf{y} - \mathbf{KQ}_2\mathbf{z} \tag{8.84}$$

Substituting this into (8.80) and (8.82) yields

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{B}(\mathbf{r} - \mathbf{KQ}_1\mathbf{Cx} - \mathbf{KQ}_2\mathbf{z}) \\ &= (\mathbf{A} - \mathbf{BKQ}_1\mathbf{C})\mathbf{x} - \mathbf{BKQ}_2\mathbf{z} + \mathbf{Br} \end{aligned} \tag{8.85}$$

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{Fz} + \mathbf{TB}(\mathbf{r} - \mathbf{KQ}_1\mathbf{Cx} - \mathbf{KQ}_2\mathbf{z}) + \mathbf{LCx} \\ &= (\mathbf{LC} - \mathbf{TBKQ}_1\mathbf{C})\mathbf{x} + (\mathbf{F} - \mathbf{TBKQ}_2)\mathbf{z} + \mathbf{TBr} \end{aligned} \tag{8.86}$$

They can be combined as

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} &= \begin{bmatrix} \mathbf{A} - \mathbf{BKQ}_1\mathbf{C} & -\mathbf{BKQ}_2 \\ \mathbf{LC} - \mathbf{TBKQ}_1\mathbf{C} & \mathbf{F} - \mathbf{TBKQ}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{TB} \end{bmatrix} \mathbf{r} \\ \mathbf{y} &= [\mathbf{C} \ \ \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \end{aligned} \tag{8.87}$$

This $(2n-q)$-dimensional state equation describes the feedback system in Fig. 8.8. As in the SISO case, let us carry out the following equivalence transformation

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{z} - \mathbf{Tx} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{T} & \mathbf{I}_{n-q} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}$$

After some manipulation and using $\mathbf{TA} - \mathbf{FT} = \mathbf{LC}$ and (8.81), we can finally obtain the following equivalent state equation

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BKQ}_2 \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r} \tag{8.88}$$

$$y = [\mathbf{C} \ \ \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

This equation is similar to (8.55) for the single-variable case. Therefore all discussion there applies, without any modification, to the multivariable case. In other words, the design of a state feedback and the design of a state estimator can be carried out independently. This is the *separation property*. Furthermore, all eigenvalues of $\mathbf{F}$ are not controllable from $\mathbf{r}$ and the transfer matrix from $\mathbf{r}$ to $\mathbf{y}$ equals

$$\hat{\mathbf{G}}_f(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}$$

PROBLEMS

8.1 Given

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \qquad y = [1 \ \ 1]\mathbf{x}$$

find the state feedback gain $\mathbf{k}$ so that the state feedback system has -1 and -2 as its eigenvalues. Compute $\mathbf{k}$ directly without using any equivalence transformation.

8.2 Repeat Problem 8.1 by using (8.13).

8.3 Repeat Problem 8.1 by solving a Lyapunov equation.

8.4 Find the state feedback gain for the state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

so that the resulting system has eigenvalues -2 and $-1 \pm j1$. Use the method you think is the simplest by hand to carry out the design.

8.5 Consider a system with transfer function

$$\hat{g}(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}$$

Is it possible to change the transfer function to

$$\hat{g}_f(s) = \frac{s-1}{(s+2)(s+3)}$$

by state feedback? Is the resulting system BIBO stable? Asymptotically stable?

8.6 Consider a system with transfer function

$$\hat{g}(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}$$

Is it possible to change the transfer function to

$$\hat{g}_f(s) = \frac{1}{s+3}$$

by state feedback? Is the resulting system BIBO stable? Asymptotically stable?

8.7 Consider the continuous-time state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [2 \ \ 0 \ \ 0]\mathbf{x}$$

Let $u = pr - \mathbf{kx}$. Find the feedforward gain p and state feedback gain $\mathbf{k}$ so that the resulting system has eigenvalues -2 and $-1 \pm j1$ and will track asymptotically any step reference input.

8.8 Consider the discrete-time state equation

$$\mathbf{x}[k+1] = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = [2 \ \ 0 \ \ 0]\mathbf{x}[k]$$

Find the state feedback gain so that the resulting system has all eigenvalues at $z = 0$. Show that for any initial state, the zero-input response of the feedback system becomes identically zero for $k \geq 3$.

8.9 Consider the discrete-time state equation in Problem 8.8. Let $u[k] = pr[k] - \mathbf{kx}[k]$, where p is a feedforward gain. For the $\mathbf{k}$ in Problem 8.8, find a gain p so that the output will track any step reference input. Show also that $y[k] = r[k]$ for $k \geq 3$. Thus exact tracking is achieved in a finite number of sampling periods instead of asymptotically. This is possible if all poles of the resulting system are placed at $z = 0$. This is called the *dead-beat* design.

8.10 Consider the uncontrollable state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

Is it possible to find a gain $\mathbf{k}$ so that the equation with state feedback $u = r - \mathbf{kx}$ has eigenvalues $-2, -2, -1, -1$? Is it possible to have eigenvalues $-2, -2, -2, -1$? How about $-2, -2, -2, -2$? Is the equation stabilizable?

8.11 Design a full-dimensional and a reduced-dimensional state estimator for the state equation in Problem 8.1. Select the eigenvalues of the estimators from $\{-3, -2 \pm j2\}$.

8.12 Consider the state equation in Problem 8.1. Compute the transfer function from r to y of the state feedback system. Compute the transfer function from r to y if the feedback gain is applied to the estimated state of the full-dimensional estimator designed in Problem 8.11. Compute the transfer function from r to y if the feedback gain is applied to the estimated state of the reduced-dimensional state estimator also designed in Problem 8.11. Are the three overall transfer functions the same?

8.13 Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$$

Find two different constant matrices $\mathbf{K}$ such that $(\mathbf{A} - \mathbf{BK})$ has eigenvalues $-4 \pm 3j$ and $-5 \pm 4j$.

Chapter 9

Pole Placement and Model Matching

9.1 Introduction

We first give reasons for introducing this chapter. Chapter 6 discusses state-space analysis (controllability and observability) and Chapter 8 introduces state-space design (state feedback and state estimators). In Chapter 7 coprime fractions were discussed. Therefore it is logical to discuss in this chapter their applications in design.

One way to introduce coprime fraction design is to develop the Bezout identity and to parameterize all stabilization compensators. See References [3, 6, 9, 13, 20]. This approach is important in some optimization problems but is not necessarily convenient for all designs. See Reference [8]. We study in this chapter only designs of minimum-degree compensators to achieve pole placement and model matching. We will change the problems into solving linear algebraic equations. Using only Theorem 3.2 and its corollary, we can establish all needed results. Therefore we can bypass the Bezout identity and some polynomial theorems and simplify the discussion.

Most control systems can be formulated as shown in Fig. 8.1. That is, given a plant with input u and output y and a reference signal r, design an overall system so that the output y will follow the reference signal r as closely as possible. The plant input u is also called the actuating signal and the plant output y, the controlled signal. If the actuating signal u depends only on the reference signal r as shown in Fig. 9.1(a), it is called an open-loop control. If u depends on r and y, then it is called a closed-loop or feedback control. The open-loop control is, in general, not satisfactory if there are plant parameter variations due to changes of load, environment, or aging. It is also very sensitive to noise and disturbance, which often exist in the real world. Therefore open-loop control is used less often in practice.

There are many possible feedback configurations. The simplest is the unity-feedback configuration shown in Fig. 9.1(b) in which the constant gain p and the compensator with transfer function $C(s)$ are to be designed. Clearly we have

$$\hat{u}(s) = C(s)[p\hat{r}(s) - \hat{y}(s)] \tag{9.1}$$

Because p is a constant, the reference signal r and the plant output y drive essentially the same compensator to generate an actuating signal. Thus the configuration is said to have one degree of freedom. Clearly the open-loop configuration also has one degree of freedom.

The connection of state feedback and state estimator in Fig. 8.8 can be redrawn as shown in Fig. 9.1(c). Simple manipulation yields

$$\hat{u}(s) = \frac{1}{1 + C_1(s)}\hat{r}(s) - \frac{C_2(s)}{1 + C_1(s)}\hat{y}(s) \tag{9.2}$$

We see that r and y drive two independent compensators to generate a u. Thus the configuration is said to have two degrees of freedom.

A more natural two-degree-of-freedom configuration can be obtained by modifying (9.1) as

$$\hat{u}(s) = C_1(s)\hat{r}(s) - C_2(s)\hat{y}(s) \tag{9.3}$$

and is plotted in Fig. 9.1(d). This is the most general control signal because each of r and y drives a compensator, which we have freedom in designing. Thus no configuration has three degrees of freedom. There are many possible two-degree-of-freedom configurations; see, for example, Reference [12]. We call the one in Fig. 9.1(d) the *two-parameter configuration*; the one in Fig. 9.1(c) the *controller-estimator* or *plant-input–output-feedback* configuration. Because the two-parameter configuration seems to be more natural and more suitable for practical application, we study only this configuration in this chapter. For designs using the plant-input–output-feedback configuration, see Reference [6].

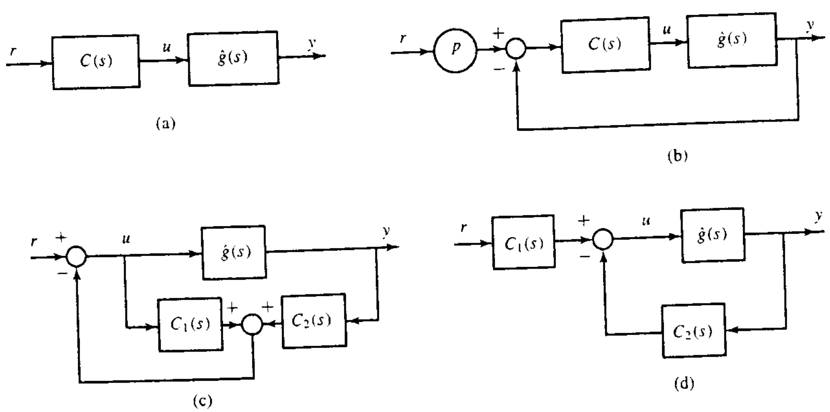


Figure 9.1 Control configurations.

The plants studied in this chapter will be limited to those describable by strictly proper rational functions or matrices. We also assume that every transfer matrix has full rank in the sense that if $\hat{\mathbf{G}}(s)$ is $q \times p$, then it has a $q \times q$ or $p \times p$ submatrix with a nonzero determinant. If $\hat{\mathbf{G}}(s)$ is square, then its determinant is nonzero or its inverse exists. This is equivalent to the assumption that if $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a minimal realization of the transfer matrix, then $\mathbf{B}$ has full column rank and $\mathbf{C}$ has full row rank.

The design to be introduced in this chapter is based on coprime polynomial fractions of rational matrices. Thus the concept of coprimeness and the method of computing coprime fractions introduced in Sections 7.1 through 7.3 and 7.6 through 7.8 are needed for studying this chapter. The rest of Chapter 7 and the entire Chapter 8, however, are not needed here. In this chapter, we will change the design problem into solving sets of linear algebraic equations. Thus the method is called the *linear algebraic* method in Reference [7].

For convenience, we first introduce some terminology. Every transfer function $\hat{g}(s) = N(s)/D(s)$ is assumed to be a coprime fraction. Then every root of $D(s)$ is a pole and every root of $N(s)$ is a zero. A pole is called a stable pole if it has a negative real part; an unstable pole if it has a zero or positive real part. We also define

- Minimum-phase zeros: zeros with negative real parts
- Nonminimum-phase zeros: zeros with zero or positive real parts

Although some texts call them stable and unstable zeros, they have nothing to do with stability. A transfer function with only minimum-phase zeros has the smallest phase among all transfer functions with the same amplitude characteristics. See Reference [7, pp. 284–285]. Thus we use the aforementioned terminology. A polynomial is called a *Hurwitz polynomial* if all its roots have negative real parts.

9.1.1 Compensator Equations—Classical Method

Consider the equation

$$A(s)D(s) + B(s)N(s) = F(s) \tag{9.4}$$

where $D(s)$, $N(s)$, and $F(s)$ are given polynomials and $A(s)$ and $B(s)$ are unknown polynomials to be solved. Mathematically speaking, this problem is equivalent to the problem of solving integer solutions A and B in $AD + BN = F$, where D, N, and F are given integers. This is a very old mathematical problem and has been associated with mathematicians such as Diophantine, Bezout, and Aryabhatta.[1] To avoid controversy, we follow Reference [3] and call it a *compensator equation*. All design problems in this chapter can be reduced to solving compensator equations. Thus the equation is of paramount importance.

We first discuss the existence condition and general solutions of the equation. What will be discussed, however, is not needed in subsequent sections and the reader may glance through this subsection.

1. See Reference [21, last page of Preface].

Theorem 9.1

Given polynomials $D(s)$ and $N(s)$, polynomial solutions $A(s)$ and $B(s)$ exist in (9.4) for any polynomial $F(s)$ if and only if $D(s)$ and $N(s)$ are coprime.

Suppose $D(s)$ and $N(s)$ are not coprime and contain the same factor $s + a$. Then the factor $s + a$ will appear in $F(s)$. Thus if $F(s)$ does not contain the factor, no solutions exist in (9.4). This shows the necessity of the theorem.

If $D(s)$ and $N(s)$ are coprime, there exist polynomials $\bar{A}(s)$ and $\bar{B}(s)$ such that

$$\bar{A}(s)D(s) + \bar{B}(s)N(s) = 1 \tag{9.5}$$

Its matrix version is called the *Bezout identity* in Reference [13]. The polynomials $\bar{A}(s)$ and $\bar{B}(s)$ can be obtained by the Euclidean algorithm and will not be discussed here. See Reference [6, pp. 578–580]. For example, if $D(s) = s^2 - 1$ and $N(s) = s - 2$, then $\bar{A}(s) = 1/3$ and $\bar{B}(s) = -(s+2)/3$ meet (9.5). For any polynomial $F(s)$, (9.5) implies

$$F(s)\bar{A}(s)D(s) + F(s)\bar{B}(s)N(s) = F(s) \tag{9.6}$$

Thus $A(s) = F(s)\bar{A}(s)$ and $B(s) = F(s)\bar{B}(s)$ are solutions. This shows the sufficiency of the theorem.

Next we discuss general solutions. For any $D(s)$ and $N(s)$, there exist two polynomials $\hat{A}(s)$ and $\hat{B}(s)$ such that

$$\hat{A}(s)D(s) + \hat{B}(s)N(s) = 0 \tag{9.7}$$

Obviously $\hat{A}(s) = -N(s)$ and $\hat{B}(s) = D(s)$ are such solutions. Then for any polynomial $Q(s)$,

$$A(s) = \bar{A}(s)F(s) + Q(s)\hat{A}(s) \qquad B(s) = \bar{B}(s)F(s) + Q(s)\hat{B}(s) \tag{9.8}$$

are general solutions of (9.4). This can easily be verified by substituting (9.8) into (9.4) and using (9.5) and (9.7).

EXAMPLE 9.1 Given $D(s) = s^2 - 1$, $N(s) = s - 2$, and $F(s) = s^3 + 4s^2 + 6s + 4$, then

$$A(s) = \tfrac{1}{3}(s^3 + 4s^2 + 6s + 4) + Q(s)(-s + 2)$$
$$B(s) = -\tfrac{1}{3}(s + 2)(s^3 + 4s^2 + 6s + 4) + Q(s)(s^2 - 1) \tag{9.9}$$

for any polynomial $Q(s)$, are solutions of (9.4).

Although the classical method can yield general solutions, the solutions are not necessarily convenient to use in design. For example, we may be interested in solving $A(s)$ and $B(s)$ with least degrees to meet (9.4). For the polynomials in Example 9.1, after some manipulation, we find that if $Q(s) = (s^2 + 6s + 15)/3$, then (9.9) reduces to

$$A(s) = s + 34/3 \qquad B(s) = (-22s - 23)/3 \tag{9.10}$$

They are the least-degree solutions of Example 9.1. In this chapter, instead of solving the compensator equation directly as shown, we will change it into solving a set of linear algebraic equations as in Section 7.3. By so doing, we can bypass some polynomial theorems.

9.2 Unity-Feedback Configuration—Pole Placement

Consider the unity-feedback system shown in Fig. 9.1(b). The plant transfer function $\hat{g}(s)$ is assumed to be strictly proper and of degree n. The problem is to design a proper compensator $C(s)$ of least possible degree m so that the resulting overall system has any set of $n + m$ desired poles. Because all transfer functions are required to have real coefficients, complex conjugate poles must be assigned in pairs. This will be a standing assumption throughout this chapter.

Let $\hat{g}(s) = N(s)/D(s)$ and $C(s) = B(s)/A(s)$. Then the overall transfer function from r to y in Fig. 9.1(b) is

$$\hat{g}_o(s) = \frac{pC(s)\hat{g}(s)}{1 + C(s)\hat{g}(s)} = \frac{p\dfrac{B(s)}{A(s)}\dfrac{N(s)}{D(s)}}{1 + \dfrac{B(s)}{A(s)}\dfrac{N(s)}{D(s)}}$$
$$= \frac{pB(s)N(s)}{A(s)D(s) + B(s)N(s)} \tag{9.11}$$

In pole assignment, we are interested in assigning all poles of $\hat{g}_o(s)$ or, equivalently, all roots of $A(s)D(s) + B(s)N(s)$. In this design, nothing is said about the zeros of $\hat{g}_o(s)$. As we can see from (9.11), the design not only has no effect on the plant zeros (roots of $N(s)$) but also introduces new zeros (roots of $B(s)$) into the overall transfer function. On the other hand, the poles of the plant and compensator are shifted from $D(s)$ and $A(s)$ to the roots of $A(s)D(s) + B(s)N(s)$. Thus *feedback can shift poles but has no effect on zeros.*

Given a set of desired poles, we can readily form a polynomial $F(s)$ that has the desired poles as its roots. Then the pole-placement problem becomes one of solving the polynomial equation

$$A(s)D(s) + B(s)N(s) = F(s) \tag{9.12}$$

Instead of solving (9.12) directly, we will transform it into solving a set of linear algebraic equations. Let $\deg N(s) < \deg D(s) = n$ and $\deg B(s) \le \deg A(s) = m$. Then $F(s)$ in (9.12) has degree at most $n + m$. Let us write

$$D(s) = D_0 + D_1 s + D_2 s^2 + \cdots + D_n s^n \qquad D_n \neq 0$$
$$N(s) = N_0 + N_1 s + N_2 s^2 + \cdots + N_n s^n$$
$$A(s) = A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m$$
$$B(s) = B_0 + B_1 s + B_2 s^2 + \cdots + B_m s^m$$
$$F(s) = F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m}$$

where all coefficients are real constants, not necessarily nonzero. Substituting these into (9.12) and matching the coefficients of like powers of s, we obtain

$$A_0D_0 + B_0N_0 = F_0$$

$$A_0D_1 + B_0N_1 + A_1D_0 + B_1N_0 = F_1$$

$$\vdots$$

$$A_mD_n + B_mN_n = F_{n+m}$$

There are a total of $(n + m + 1)$ equations. They can be arranged in matrix form as

$$[A_0\ B_0\ A_1\ B_1\ \cdots\ A_m\ B_m]\mathbf{S}_m = [F_0\ F_1\ F_2\ \cdots\ F_{n+m}] \tag{9.13}$$

with

$$\mathbf{S}_m := \begin{bmatrix} D_0 & D_1 & \cdots & D_n & 0 & \cdots & 0 \\ N_0 & N_1 & \cdots & N_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & D_0 & \cdots & D_{n-1} & D_n & \cdots & 0 \\ 0 & N_0 & \cdots & N_{n-1} & N_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & D_0 & \cdots & D_n \\ 0 & 0 & \cdots & 0 & N_0 & \cdots & N_n \end{bmatrix} \tag{9.14}$$

If we take the transpose of (9.13), then it becomes the standard form studied in Theorems 3.1 and 3.2. We use the form in (9.13) because it can be extended directly to the matrix case. The matrix $\mathbf{S}_m$ has $2(m + 1)$ rows and $(n + m + 1)$ columns and is formed from the coefficients of $D(s)$ and $N(s)$. The first two rows are simply the coefficients of $D(s)$ and $N(s)$ arranged in ascending powers of s. The next two rows are the first two rows shifted to the right by one position. We repeat the process until we have $(m + 1)$ sets of coefficients. The left-hand-side row vector of (9.13) consists of the coefficients of the compensator $C(s)$ to be solved. If $C(s)$ has degree m, then the row vector has $2(m + 1)$ entries. The right-hand-side row vector of (9.13) consists of the coefficients of $F(s)$. Now solving the compensator equation in (9.12) becomes solving the linear algebraic equation in (9.13).

Applying Corollary 3.2, we conclude that (9.13) has a solution for any $F(s)$ if and only if $\mathbf{S}_m$ has full column rank. A necessary condition for $\mathbf{S}_m$ to have full column rank is that $\mathbf{S}_m$ is square or has more rows than columns, that is,

$$2(m + 1) \geq n + m + 1 \quad \text{or} \quad m \geq n - 1$$

If $m < n - 1$, then $\mathbf{S}_m$ does not have full column rank and solutions may exist for some $F(s)$, but not for every $F(s)$. Thus if the degree of the compensator is less than $n - 1$, it is not possible to achieve arbitrary pole placement.

If $m = n - 1$, $\mathbf{S}_{n-1}$ becomes a square matrix of order $2n$. It is the transpose of the Sylvester resultant in (7.28) with $n = 4$. As discussed in Section 7.3, $\mathbf{S}_{n-1}$ is nonsingular if and only if $D(s)$ and $N(s)$ are coprime. Thus if $D(s)$ and $N(s)$ are coprime, then $\mathbf{S}_{n-1}$ has rank $2n$ (full column rank). Now if m increases by 1, the number of columns increases by 1 but the the

number of rows increases by 2. Because $D_n \neq 0$, the new D row is linearly independent of its preceding rows. Thus the $2(n + 1) \times (2n + 1)$ matrix $\mathbf{S}_n$ has rank $(2n + 1)$ (full column rank). Repeating the argument, we conclude that if $D(s)$ and $N(s)$ are coprime and if $m \geq n - 1$, then the matrix $\mathbf{S}_m$ in (9.14) has full column rank.

Theorem 9.2

Consider the unity-feedback system shown in Fig. 9.1(b). The plant is described by a strictly proper transfer function $\hat{g}(s) = N(s)/D(s)$ with $N(s)$ and $D(s)$ coprime and deg $N(s) <$ deg $D(s) = n$. Let $m \geq n - 1$. Then for any polynomial $F(s)$ of degree $(n + m)$, there exists a proper compensator $C(s) = B(s)/A(s)$ of degree m such that the overall transfer function equals

$$\hat{g}_o(s) = \frac{pN(s)B(s)}{A(s)D(s) + B(s)N(s)} = \frac{pN(s)B(s)}{F(s)}$$

Furthermore, the compensator can be obtained by solving the linear algebraic equation in (9.13).

As discussed earlier, the matrix $\mathbf{S}_m$ has full column rank for $m \geq n - 1$; therefore, for any $(n + m)$ desired poles or, equivalently, for any $F(s)$ of degree $(n + m)$, solutions exist in (9.13). Next we show that $B(s)/A(s)$ is proper or $A_m \neq 0$. If $N(s)/D(s)$ is strictly proper, then $N_n = 0$ and the last equation of (9.13) reduces to

$$A_mD_n + B_mN_n = D_nA_m = F_{n+m}$$

Because $F(s)$ has degree $(n + m)$, we have $F_{n+m} \neq 0$ and, consequently, $A_m \neq 0$. This establishes the theorem. If $m = n - 1$, the compensator is unique; if $m > n - 1$, compensators are not unique and free parameters can be used to achieve other design objectives, as we will discuss later.

9.2.1 Regulation and Tracking

Pole placement can be used to achieve the regulation and tracking discussed in Section 8.3. In the regulator problem, we have $r = 0$ and the problem is to design a compensator $C(s)$ so that the response excited by any nonzero initial state will die out at a desired rate. For this problem, if all poles of $\hat{g}_o(s)$ are selected to have negative real parts, then for any gain p, in particular $p = 1$ (no feedforward gain is needed), the overall system will achieve regulation.

We discuss next the tracking problem. Let the reference signal be a step function with magnitude a. Then $\hat{r}(s) = a/s$ and the output $\hat{y}(s)$ equals

$$\hat{y}(s) = \hat{g}_o(s)\hat{r}(s) = \hat{g}_o(s)\frac{a}{s}$$

If $\hat{g}_o(s)$ is BIBO stable, the output will approach the constant $\hat{g}_o(0)a$ (Theorem 5.2). This can also be obtained by employing the final-value theorem of the Laplace transform as

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s) = \hat{g}_o(0)a$$

Thus in order to track asymptotically any step reference input, $\hat{g}_o(s)$ must be BIBO stable and $\hat{g}_o(0) = 1$. The transfer function from r to y in Fig. 9.1(b) is $\hat{g}_o(s) = pN(s)B(s)/F(s)$. Thus we have

$$\hat{g}_o(0) = p\frac{N(0)B(0)}{F(0)} = p\frac{B_0N_0}{F_0}$$

which implies

$$p = \frac{F_0}{B_0N_0} \qquad (9.15)$$

Thus in order to track any step reference input, we require $B_0 \neq 0$ and $N_0 \neq 0$. The constant B_0 is a coefficient of the compensator and can be designed to be nonzero. The coefficient N_0 is the constant term of the plant numerator. Thus if the plant transfer function has one or more zeros at $s = 0$, then $N_0 = 0$ and the plant cannot be designed to track any step reference input. This is consistent with the discussion in Section 8.3.

If the reference signal is a ramp function or $r(t) = at$, for $t \geq 0$, then using a similar argument, we can show that the overall transfer function $\hat{g}_o(s)$ must be BIBO stable and has the properties $\hat{g}_o(0) = 1$ and $\hat{g}'_o(0) = 0$ (Problems 9.13 and 9.14). This is summarized in the following.

- Regulation$\Leftrightarrow$ $\hat{g}_o(s)$ BIBO stable.
- Tracking step reference input$\Leftrightarrow$ $\hat{g}_o(s)$ BIBO stable and $\hat{g}_o(0) = 1$.
- Tracking ramp reference input$\Leftrightarrow$ $\hat{g}_o(s)$ BIBO stable, $\hat{g}_o(0) = 1$, and $\hat{g}'_o(0) = 0$.

Example 9.2 Given a plant with transfer function $\hat{g}(s) = (s-2)/(s^2-1)$, find a proper compensator $C(s)$ and a gain p in the unity-feedback configuration in Fig. 9.1(b) so that the output y will track asymptotically any step reference input.

The plant transfer function has degree $n = 2$. Thus if we choose $m = 1$, all three poles of the overall system can be assigned arbitrarily. Let the three poles be selected as $-2, -1 \pm j1$; they spread evenly in the sector shown in Fig. 8.3(a). Then we have

$$F(s) = (s+2)(s+1+j1)(s+1-j1) = (s+2)(s^2+2s+2) = s^3+4s^2+6s+4$$

We use the coefficients of $D(s) = -1 + 0\cdot s + 1\cdot s^2$ and $N(s) = -2 + 1\cdot s + 0\cdot s^2$ to form (9.13) as

$$[A_0\ B_0\ A_1\ B_1]\begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} = [4\quad 6\quad 4\quad 1]$$

Its solution is

$$A_1 = 1 \qquad A_0 = 34/3 \qquad B_1 = -22/3 \qquad B_0 = -23/3$$

This solution can easily be obtained using the MATLAB function / (slash), which denotes matrix right division. Thus we have[2]

2. This is the solution obtained in (9.10). This process of solving the polynomial equation in (9.13) is considerably simpler than the procedure discussed in Section 9.1.1.

$$A(s) = s + 34/3 \qquad B(s) = (-22/3)s - 23/3 = (-22s-23)/3$$

and the compensator

$$C(s) = \frac{B(s)}{A(s)} = \frac{-(23+22s)/3}{34/3+s} = \frac{-22s-23}{3s+34} \qquad (9.16)$$

will place the three poles of the overall system at -2 and $-1 \pm j1$. If the system is designed to achieve regulation, we set $p = 1$ (no feedforward gain is needed) and the design is completed. To design tracking, we check whether or not $N_0 \neq 0$. This is the case; thus we can find a p so that the overall system will track asymptotically any step reference input. We use (9.15) to compute p:

$$p = \frac{F_0}{B_0N_0} = \frac{4}{(-23/3)(-2)} = \frac{6}{23} \qquad (9.17)$$

Thus the overall transfer function from r to y is

$$\hat{g}_o(s) = \frac{6}{23}\frac{[-(22s+23)/3](s-2)}{(s^3+4s^3+6s+4)} = \frac{-2(22s+23)(s-2)}{23(s^3+4s^2+6s+4)} \qquad (9.18)$$

Because $\hat{g}_o(s)$ is BIBO stable and $\hat{g}_o(0) = 1$, the overall system will track any step reference input.

9.2.2 Robust Tracking and Disturbance Rejection

Consider the design problem in Example 9.2. Suppose after the design is completed, the plant transfer function $\hat{g}(s)$ changes, due to load variations, to $\bar{\hat{g}}(s) = (s-2.1)/(s^2-0.95)$. Then the overall transfer function becomes

$$\bar{\hat{g}}_o(s) = \frac{pC(s)\bar{\hat{g}}(s)}{1+C(s)\bar{\hat{g}}(s)} = \frac{6}{23}\frac{\dfrac{-22s-23}{3s+34}\dfrac{s-2.1}{s-0.95}}{1+\dfrac{-22s-23}{3s+34}\dfrac{s-2.1}{s-0.95}}$$
$$= \frac{-6(22s+23)(s-2.1)}{23(3s^3+12s^2+20.35s+16)} \qquad (9.19)$$

This $\bar{\hat{g}}_o(s)$ is still BIBO stable, but $\bar{\hat{g}}_o(0) = (6\cdot 23\cdot 2.1)/(23\cdot 16) = 0.7875 \neq 1$. If the reference input is a unit step function, the output will approach 0.7875 as $t \to \infty$. There is a tracking error of over 20%. Thus the overall system will no longer track any step reference input after the plant parameter variations, and the design is said to be nonrobust.

In this subsection, we discuss a design that can achieve robust tracking and disturbance rejection. Consider the system shown in Fig. 9.2, in which a disturbance enters at the plant input as shown. The problem is to design an overall system so that the plant output y will track asymptotically a class of reference signal r even with the presence of the disturbance and with plant parameter variations. This is called *robust tracking and disturbance rejection*.

Before proceeding, we discuss the nature of the reference signal $r(t)$ and the disturbance $w(t)$. If both $r(t)$ and $w(t)$ approach zero as $t \to \infty$, then the design is automatically

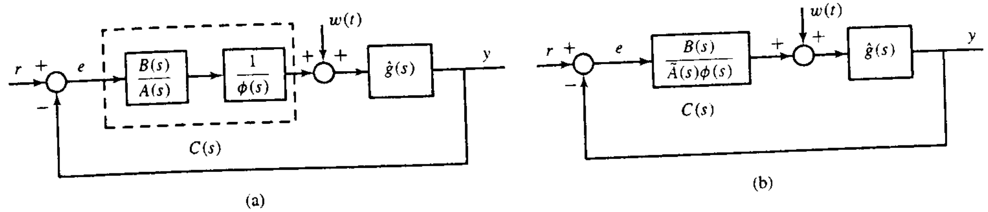


Figure 9.2 Robust tracking and disturbance rejection.

achieved if the overall system in Fig. 9.2 is designed to be BIBO stable. To exclude this trivial case, we assume that $r(t)$ and $w(t)$ do not approach zero as $t \to \infty$. If we have no knowledge whatsoever about the nature of $r(t)$ and $w(t)$, it is not possible to achieve asymptotic tracking and disturbance rejection. Therefore we need some information of $r(t)$ and $w(t)$ before carrying out the design. We assume that the Laplace transforms of $r(t)$ and $w(t)$ are given by

$$\hat{r}(s) = \mathcal{L}[r(t)] = \frac{N_r(s)}{D_r(s)} \qquad \hat{w}(s) = \mathcal{L}[w(t)] = \frac{N_w(s)}{D_w(s)} \tag{9.20}$$

where $D_r(s)$ and $D_w(s)$ are known polynomials; however, $N_r(s)$ and $N_w(s)$ are unknown to us. For example, if $r(t)$ is a step function with unknown magnitude a, then $\hat{r}(s) = a/s$. Suppose the disturbance is $w(t) = b + c\sin(\omega_o t + d)$; it consists of a constant biasing with unknown magnitude b and a sinusoid with known frequency ω_o but unknown amplitude c and phase d. Then we have $\hat{w}(s) = N_w(s)/s(s^2+\omega_o^2)$. Let $\phi(s)$ be the least common denominator of the unstable poles of $\hat{r}(s)$ and $\hat{w}(s)$. The stable poles are excluded because they have no effect on y as $t \to \infty$. Thus all roots of $\phi(s)$ have zero or positive real parts. For the examples just discussed, we have $\phi(s) = s(s^2 + \omega_o^2)$.

Theorem 9.3

Consider the unity-feedback system shown in Fig. 9.2(a) with a strictly proper plant transfer function $\hat{g}(s) = N(s)/D(s)$. It is assumed that $D(s)$ and $N(s)$ are coprime. The reference signal $r(t)$ and disturbance $w(t)$ are modeled as $\hat{r}(s) = N_r(s)/D_r(s)$ and $\hat{w}(s) = N_w(s)/D_w(s)$. Let $\phi(s)$ be the least common denominator of the unstable poles of $\hat{r}(s)$ and $\hat{w}(s)$. If no root of $\phi(s)$ is a zero of $\hat{g}(s)$, then there exists a proper compensator such that the overall system will track $r(t)$ and reject $w(t)$, both asymptotically and robustly.

Proof: If no root of $\phi(s)$ is a zero of $\hat{g}(s) = N(s)/D(s)$, then $D(s)\phi(s)$ and $N(s)$ are coprime. Thus there exists a proper compensator $B(s)/A(s)$ such that the polynomial $F(s)$ in

$$A(s)D(s)\phi(s) + B(s)N(s) = F(s)$$

has any desired roots, in particular, has all roots lying inside the sector shown in Fig. 8.3(a). We claim that the compensator

$$C(s) = \frac{B(s)}{A(s)\phi(s)}$$

as shown in Fig. 9.2(a) will achieve the design. Let us compute the transfer function from w to y:

$$\hat{g}_{yw}(s) = \frac{N(s)/D(s)}{1 + (B(s)/A(s)\phi(s))(N(s)/D(s))} = \frac{N(s)A(s)\phi(s)}{A(s)D(s)\phi(s) + B(s)N(s)} = \frac{N(s)A(s)\phi(s)}{F(s)}$$

Thus the output excited by $w(t)$ equals

$$\hat{y}_w(s) = \hat{g}_{yw}(s)\hat{w}(s) = \frac{N(s)A(s)\phi(s)}{F(s)}\frac{N_w(s)}{D_w(s)} \tag{9.21}$$

Because all unstable roots of $D_w(s)$ are canceled by $\phi(s)$, all poles of $\hat{y}_w(s)$ have negative real parts. Thus we have $y_w(t) \to 0$ as $t \to \infty$. In other words, the response excited by $w(t)$ is asymptotically suppressed at the output.

Next we compute the output $\hat{y}_r(s)$ excited by $\hat{r}(s)$:

$$\hat{y}_r(s) = \hat{g}_{yr}(s)\hat{r}(s) = \frac{B(s)N(s)}{A(s)D(s)\phi(s) + B(s)N(s)}\hat{r}(s)$$

Thus we have

$$\hat{e}(s) := \hat{r}(s) - \hat{y}_r(s) = (1 - \hat{g}_{yr}(s))\hat{r}(s) = \frac{A(s)D(s)\phi(s)}{F(s)}\frac{N_r(s)}{D_r(s)} \tag{9.22}$$

Again all unstable roots of $D_r(s)$ are canceled by $\phi(s)$ in (9.22). Thus we conclude $r(t) - y_r(t) \to 0$ as $t \to \infty$. Because of linearity, we have $y(t) = y_w(t) + y_r(t)$ and $r(t) - y(t) \to 0$ as $t \to \infty$. This shows asymptotic tracking and disturbance rejection. From (9.21) and (9.22), we see that even if the parameters of $D(s)$, $N(s)$, $A(s)$, and $B(s)$ change, as long as the overall system remains BIBO stable and the unstable roots of $D_r(s)$ and $D_w(s)$ are canceled by $\phi(s)$, the system still achieve tracking and rejection. Thus the design is robust. Q.E.D.

This robust design consists of two steps. First find a model $1/\phi(s)$ of the reference signal and disturbance and then carry out pole-placement design. Inserting the model inside the loop is referred to as the *internal model principle*. If the model $1/\phi(s)$ is not located in the forward path from w to y and from r to e, then $\phi(s)$ will appear in the numerators of $\hat{g}_{yw}(s)$ and $\hat{g}_{er}(s)$ (see Problem 9.7) and cancel the unstable poles of $\hat{w}(s)$ and $\hat{r}(s)$, as shown in (9.21) and (9.22). Thus the design is achieved by unstable pole–zero cancellations of $\phi(s)$. It is important to mention that there are no unstable pole–zero cancellations in the pole-placement design and the resulting unity-feedback system is totally stable, which will be defined in Section 9.3. Thus the internal model principle can be used in practical design.

In classical control system design, if a plant transfer function or a compensator transfer function is of type 1 (has one pole at $s = 0$), and the unity-feedback system is designed to be

BIBO stable, then the overall system will track asymptotically and robustly any step reference input. This is a special case of the internal model principle.

EXAMPLE 9.3 Consider the plant in Example 9.2 or $\hat{g}(s) = (s-2)/(s^2-1)$. Design a unity-feedback system with a set of desired poles to track robustly any step reference input.

First we introduce the internal model $\phi(s) = 1/s$. Then $B(s)/A(s)$ in Fig. 9.2(a) can be solved from

$$A(s)D(s)\phi(s) + B(s)N(s) = F(s)$$

Because $\bar{D}(s) := D(s)\phi(s)$ has degree 3, we may select $A(s)$ and $B(s)$ to have degree 2. Then $F(s)$ has degree 5. If we select five desired poles as -2, $-2 \pm j1$, and $-1 \pm j2$, then we have

$$\begin{aligned} F(s) &= (s+2)(s^2+4s+5)(s^2+2s+5) \\ &= s^5 + 8s^4 + 30s^3 + 66s^2 + 85s + 50 \end{aligned}$$

Using the coefficients of $\bar{D}(s) = (s^2-1)s = 0 - s + 0\cdot s^2 + s^3$ and $N(s) = -2 + s + 0\cdot s^2 + 0\cdot s^3$, we form

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2]\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 & 0 \end{bmatrix} = [50\ 85\ 66\ 30\ 8\ 1]$$

Its solution is $[127.3\ \ -25\ \ 0\ \ -118.7\ \ 1\ \ -96.3]$. Thus we have

$$\frac{B(s)}{A(s)} = \frac{-96.3s^2 - 118.7s - 25}{s^2 + 127.3}$$

and the compensator is

$$C(s) = \frac{B(s)}{A(s)\phi(s)} = \frac{-96.3s^2 - 118.7s - 25}{(s^2+127.3)s}$$

Using this compensator of degree 3, the unity-feedback system in Fig. 9.2(a) will track robustly any step reference input and has the set of desired poles.

9.2.3 Embedding Internal Models

The design in the preceding subsection was achieved by first introducing an internal model $1/\phi(s)$ and then designing a proper $B(s)/A(s)$. Thus the compensator $B(s)/A(s)\phi(s)$ is always strictly proper. In this subsection, we discuss a method of designing a biproper compensator whose denominator will include the internal model as a factor as shown in Fig. 9.2(b). By so doing, the degree of compensators can be reduced.

Consider

$$A(s)D(s) + B(s)N(s) = F(s)$$

If deg $D(s) = n$ and if deg $A(s) = n-1$, then the solution $A(s)$ and $B(s)$ is unique. If we increase the degree of $A(s)$ by one, then solutions are not unique, and there is one free parameter we can select. Using the free parameter, we may be able to include an internal model in the compensator, as the next example illustrates.

EXAMPLE 9.4 Consider again the design problem in Example 9.2. The degree of $D(s)$ is 2. If $A(s)$ has degree 1, then the solution is unique. Let us select $A(s)$ to have degree 2. Then $F(s)$ must have degree 4 and can be selected as

$$F(s) = (s^2+4s+5)(s^2+2s+5) = s^4 + 6s^3 + 18s^2 + 30s + 25$$

We form

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2]\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 \end{bmatrix} = [25\ 30\ 18\ 6\ 1] \qquad (9.23)$$

In order for the proper compensator

$$C(s) = \frac{B_0 + B_1 s + B_2 s^2}{A_0 + A_1 s + A_2 s^2}$$

to have $1/s$ as a factor, we require $A_0 = 0$. There are five equations and six unknowns in (9.23). Thus one of the unknowns can be arbitrarily assigned. Let us select $A_0 = 0$. This is equivalent to deleting the first row of the 6×5 matrix in (9.23). The remaining 5×5 matrix is nonsingular, and the remaining five unknowns can be solved uniquely. The solution is

$$[A_0\ \ B_0\ \ A_1\ \ B_1\ \ A_2\ \ B_2] = [0\ \ -12.5\ \ 34.8\ \ -38.7\ \ 1\ \ -28.8]$$

Thus the compensator is

$$C(s) = \frac{B(s)}{A(s)} = \frac{-28.8s^2 - 38.7s - 12.5}{s^2 + 34.8s}$$

This biproper compensator can achieve robust tracking. This compensator has degree 2, one less than the one obtained in Example 9.3. Thus this is a better design.

In the preceding example, we mentioned that one of the unknowns in (9.23) can be arbitrarily assigned. This does not mean that any one of them can be arbitrarily assigned. For example, if we assign $A_2 = 0$ or, equivalently, delete the fifth row of the 6×5 matrix in (9.23), then the remaining square matrix is singular and no solution may exist. In Example 9.4, if we select $A_0 = 0$ and if the remaining equation in (9.23) does not have a solution, then we must increase the degree of the compensator and repeat the design. Another way to carry out the design is to find the general solution of (9.23). Using Corollary 3.2, we can express the general solution as

$$[A_0\ B_0\ A_1\ B_1\ A_2\ B_2] = [1\ \ -13\ \ 34.3\ \ -38.7\ \ 1\ \ -28.3] + \alpha[2\ \ -1\ \ -1\ \ 0\ \ 0\ \ 1]$$

with one free parameter α. If we select $\alpha = -0.5$, then $A_0 = 0$ and we will obtain the same compensator.

We give one more example and discuss a different method of embedding $\phi(s)$ in the compensator.

EXAMPLE 9.5 Consider the unity-feedback system in Fig. 9.2(b) with $\hat{g}(s) = 1/s$. Design a proper compensator $C(s) = B(s)/A(s)$ so that the system will track asymptotically any step reference input and reject disturbance $w(t) = a\sin(2t + \theta)$ with unknown a and θ.

In order to achieve the design, the polynomial $A(s)$ must contain the disturbance model $(s^2 + 4)$. Note that the reference model s is not needed because the plant already contains the factor. Consider

$$A(s)D(s) + B(s)N(s) = F(s)$$

For this equation, we have $\deg D(s) = n = 1$. Thus if $m = n - 1 = 0$, then the solution is unique and we have no freedom in assigning $A(s)$. If $m = 2$, then we have two free parameters that can be used to assign $A(s)$. Let

$$A(s) = \bar{A}_0(s^2 + 4) \qquad B(s) = B_0 + B_1 s + B_2 s^2$$

Define

$$\bar{D}(s) = D(s)(s^2 + 4) = \bar{D}_0 + \bar{D}_1 s + \bar{D}_2 s^2 + \bar{D}_3 s^3 = 0 + 4s + 0\cdot s^2 + s^3$$

We write $A(s)D(s) + B(s)N(s) = F(s)$ as

$$\bar{A}_0\bar{D}(s) + B(s)N(s) = F(s)$$

Equating its coefficients, we obtain

$$[\bar{A}_0\ B_0\ B_1\ B_2]\begin{bmatrix} \bar{D}_0 & \bar{D}_1 & \bar{D}_2 & \bar{D}_3 \\ N_0 & N_1 & 0 & 0 \\ 0 & N_0 & N_1 & 0 \\ 0 & 0 & N_0 & N_1 \end{bmatrix} = [F_0\ F_1\ F_2\ F_3]$$

For this example, if we select

$$F(s) = (s+2)(s^2 + 2s + 2) = s^3 + 4s^2 + 6s + 4$$

then the equation becomes

$$[\bar{A}_0\ B_0\ B_1\ B_2]\begin{bmatrix} 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = [4\ 6\ 4\ 1]$$

Its solution is [1 4 2 4]. Thus the compensator is

$$C(s) = \frac{B(s)}{A(s)} = \frac{4s^2 + 2s + 4}{1 \times (s^2 + 4)} = \frac{4s^2 + 2s + 4}{s^2 + 4}$$

This biproper compensator will place the poles of the unity-feedback system in the assigned positions, track any step reference input, and reject the disturbance $a\sin(2t + \theta)$, both asymptotically and robustly.

9.3 Implementable Transfer Functions

Consider again the design problem posed in Fig. 8.1 with a given plant transfer function $\hat{g}(s)$. Now the problem is the following: given a desired overall transfer function $\hat{g}_o(s)$, find a feedback configuration and compensators so that the transfer function from r to y equals $\hat{g}_o(s)$. This is called the *model matching* problem. This problem is clearly different from the pole-placement problem. In pole placement, we specify only poles; its design will introduce some zeros over which we have no control. In model matching, we specify not only poles but also zeros. Thus model matching can be considered as pole-and-zero placement and should yield a better design.

Given a proper plant transfer function $\hat{g}(s)$, we claim that $\hat{g}_o(s) = 1$ is the best possible overall system we can design. Indeed, if $\hat{g}_o(s) = 1$, then $y(t) = r(t)$ for $t \geq 0$ and for any $r(t)$. Thus the overall system can track immediately (not asymptotically) any reference input no matter how erratic $r(t)$ is. Note that although $y(t) = r(t)$, the power levels at the reference input and plant output may be different. The reference signal may be provided by turning a knob by hand; the plant output may be the angular position of an antenna with weight over several tons.

Although $\hat{g}_o(s) = 1$ is the best overall system, we may not be able to match it for a given plant. The reason is that in matching or implementation, there are some physical constraints that every overall system should meet. These constraints are listed in the following:

1. All compensators used have proper rational transfer functions.
2. The configuration selected has *no plant leakage* in the sense that all forward paths from r to y pass through the plant.
3. The closed-loop transfer function of every possible input–output pair is proper and BIBO stable.

Every compensator with a proper rational transfer function can be implemented using the op-amp circuit elements shown in Fig. 2.6. If a compensator has an improper transfer function, then its implementation requires the use of pure differentiators, which are not standard op-amp circuit elements. Thus compensators used in practice are often required to have proper transfer functions. The second constraint requires that all power passes through the plant and no compensator be introduced in parallel with the plant. All configurations in Fig. 9.1 meet this constraint. In practice, noise and disturbance may exist in every component. For example, noise may be generated in using potentiometers because of brush jumps and wire irregularity. The load of an antenna may change because of gusting or air turbulence. These will be modeled as exogenous inputs entering the input and output terminals of every block as shown in Fig. 9.3.

Clearly we cannot disregard the effects of these exogenous inputs on the system. Although the plant output is the signal we want to control, we should be concerned with all variables inside the system. For example, suppose the closed-loop transfer function from r to u is not BIBO stable; then any r will excite an unbounded u and the system will either saturate or burn out. If the closed-loop transfer function from n_1 to u is improper, and if n_1 contains high-frequency noise, then the noise will be greatly amplified at u and the amplified noise will drive the system crazy. Thus the closed-loop transfer function of every possible input–output pair of the overall system should be proper and BIBO stable. An overall system is said to be *well posed* if the closed-loop transfer function of every possible input–output pair is proper; it is *totally stable* if the closed-loop transfer function of every possible input–output pair is BIBO stable.

Total stability can readily be met in design. If the overall transfer function from r to y is BIBO stable and if there is no unstable pole–zero cancellation in the system, then the overall system is totally stable. For example, consider the system shown in Fig. 9.3(a). The overall transfer function from r to y is

$$\hat{g}_{yr}(s) = \frac{1}{s+1}$$

which is BIBO stable. However, the system is not totally stable because it involves an unstable pole–zero cancellation of $(s-2)$. The closed-loop transfer function from n_2 to y is $s/(s-2)(s+1)$, which is not BIBO stable. Thus the output will grow unbounded if noise n_2, even very small, enters the system. Thus we require BIBO stability not only of $\hat{g}_o(s)$ but also of every possible closed-loop transfer function. Note that whether or not $\hat{g}(s)$ and $C(s)$ are BIBO stable is immaterial.

The condition for the unity-feedback configuration in Fig. 9.3 to be well posed is $C(\infty)\hat{g}(\infty) \neq -1$ (Problem 9.9). This can readily be established by using Mason's formula. See Reference [7, pp. 200–201]. For example, for the unity-feedback system in Fig. 9.3(b), we have $C(\infty)\hat{g}(\infty) = (-1/2) \times 2 = -1$. Thus the system is not well posed. Indeed, the closed-loop transfer function from r to y is

$$\hat{g}_o(s) = \frac{(-s+2)(2s+2)}{s+3}$$

which is improper. The condition for the two-parameter configuration in Fig. 9.1(d) to be well posed is $\hat{g}(\infty)C_2(\infty) \neq -1$. In the unity-feedback and two-parameter configurations, if $\hat{g}(s)$ is strictly proper or $\hat{g}(\infty) = 0$, then $\hat{g}(\infty)C(\infty) = 0 \neq -1$ for any proper $C(s)$ and the overall systems will automatically be well posed. In conclusion, total stability and well-posedness can easily be met in design. Nevertheless, they do impose some restrictions on $\hat{g}_o(s)$.

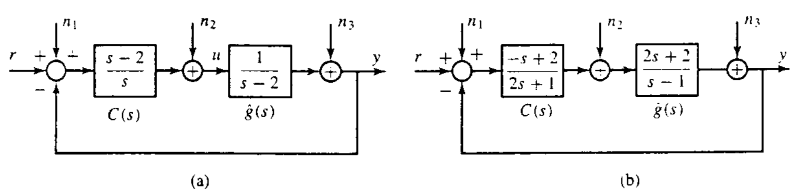


Figure 9.3 Feedback systems.

> ***Definition 9.1*** *Given a plant with proper transfer function $\hat{g}(s)$, an overall transfer function $\hat{g}_o(s)$ is said to be* implementable *if there exists a no-plant-leakage configuration and proper compensators so that the transfer function from r to y in Fig. 8.1 equals $\hat{g}_o(s)$ and the overall system is well posed and totally stable.*

If an overall transfer function $\hat{g}_o(s)$ is not implementable, then no matter what configuration is used to implement it, the design will violate at least one of the aforementioned constraints. Therefore, in model matching, the selected $\hat{g}_o(s)$ must be implementable; otherwise, it is not possible to implement it in practice.

Theorem 9.4

Consider a plant with proper transfer function $\hat{g}(s)$. Then $\hat{g}_o(s)$ is implementable if and only if $\hat{g}_o(s)$ and

$$\hat{t}(s) := \frac{\hat{g}_o(s)}{\hat{g}(s)} \tag{9.24}$$

are proper and BIBO stable.

Corollary 9.4

Consider a plant with proper transfer function $\hat{g}(s) = N(s)/D(s)$. Then $\hat{g}_o(s) = E(s)/F(s)$ is implementable if and only if

1. All roots of $F(s)$ have negative real parts ($F(s)$ is Hurwitz).
2. Deg $F(s)$ − deg $E(s) \geq$ deg $D(s)$ − deg $N(s)$ (pole-zero excess inequality).
3. All zeros of $N(s)$ with zero or positive real parts are retained in $E(s)$ (retainment of nonminimum-phase zeros).

We first develop Corollary 9.4 from Theorem 9.4. If $\hat{g}_o(s) = E(s)/F(s)$ is BIBO stable, then all roots of $F(s)$ have negative real parts. This is condition (1). We write (9.24) as

$$\hat{t}(s) = \frac{\hat{g}_o(s)}{\hat{g}(s)} = \frac{E(s)D(s)}{F(s)N(s)}$$

The condition for $\hat{t}(s)$ to be proper is

$$\deg F(s) + \deg N(s) \geq \deg E(s) + \deg D(s)$$

which implies (2). In order for $\hat{t}(s)$ to be BIBO stable, all roots of $N(s)$ with zero or positive real parts must be canceled by the roots of $E(s)$. Thus $E(s)$ must contain the nonminimum-phase zeros of $N(s)$. This is condition (3). Thus Corollary 9.4 follows directly Theorem 9.4.

Now we show the necessity of Theorem 9.4. For any configuration that has no plant leakage, if the closed-loop transfer function from r to y is $\hat{g}_o(s)$, then we have

$$\hat{y}(s) = \hat{g}_o(s)\hat{r}(s) = \hat{g}(s)\hat{u}(s)$$

which implies

$$\hat{u}(s) = \frac{\hat{g}_o(s)}{\hat{g}(s)}\hat{r}(s) = \hat{t}(s)\hat{r}(s)$$

Thus the closed-loop transfer function from r to u is $\hat{t}(s)$. Total stability requires every closed-loop transfer function to be BIBO stable. Thus $\hat{g}_o(s)$ and $\hat{t}(s)$ must be BIBO stable. Well-posedness requires every closed-loop transfer function to be proper. Thus $\hat{g}_o(s)$ and $\hat{t}(s)$ must be proper. This establishes the necessity of the theorem. The sufficiency of the theorem will be established constructively in the next subsection. Note that if $\hat{g}(s)$ and $\hat{t}(s)$ are proper, then $\hat{g}_o(s) = \hat{g}(s)\hat{t}(s)$ is proper. Thus the condition for $\hat{g}_o(s)$ to be proper can be dropped from Theorem 9.4.

In pole placement, the design will always introduce some zeros over which we have no control. In model matching, other than retaining nonminimum-phase zeros and meeting the pole–zero excess inequality, we have complete freedom in selecting poles and zeros: any pole inside the open left-half s-plane and any zero in the entire s-plane. Thus model matching can be considered as pole-and-zero placement and should yield a better overall system than pole-placement design.

Given a plant transfer function $\hat{g}(s)$, how to select an implementable model $\hat{g}_o(s)$ is not a simple problem. For a discussion of this problem, see Reference [7, Chapter 9].

9.3.1 Model Matching—Two-Parameter Configuration

This section discusses the implementation of $\hat{g}_o(s) = \hat{g}(s)\hat{t}(s)$. Clearly, if $C(s) = \hat{t}(s)$ in Fig. 9.1(a), then the open-loop configuration has $\hat{g}_o(s)$ as its transfer function. This implementation may involve unstable pole–zero cancellations and, consequently, may not be totally stable. Even if it is totally stable, the configuration can be very sensitive to plant parameter variations. Therefore the open-loop configuration should not be used. The unity-feedback configuration in Fig. 9.1(b) can be used to achieve every pole placement; however it cannot be used to achieve every model matching, as the next example shows.

EXAMPLE 9.6 Consider a plant with transfer function $\hat{g}(s) = (s-2)/(s^2-1)$. We can readily show that

$$\hat{g}_o(s) = \frac{-(s-2)}{s^2+2s+2} \tag{9.25}$$

is implementable. Because $\hat{g}_o(0) = 1$, the plant output will track asymptotically any step reference input. Suppose we use the unity-feedback configuration with $p = 1$ to implement $\hat{g}_o(s)$. Then from

$$\hat{g}_o(s) = \frac{C(s)\hat{g}(s)}{1+C(s)\hat{g}(s)}$$

we can compute the compensator as

$$C(s) = \frac{\hat{g}_o(s)}{\hat{g}(s)[1-\hat{g}_o(s)]} = \frac{-(s^2-1)}{s(s+3)}$$

This compensator is proper. However, the tandem connection of $C(s)$ and $\hat{g}(s)$ involves the pole–zero cancellation of $(s^2-1) = (s+1)(s-1)$. The cancellation of the stable pole $s+1$ will not cause any serious problem in the overall system. However, the cancellation of the unstable pole $s-1$ will make the overall system not totally stable. Thus the implementation is not acceptable.

Model matching in general involves some pole–zero cancellations. The same situation arises in state-feedback state-estimator design; all eigenvalues of the estimator are not controllable from the reference input and are canceled in the overall transfer function. However, because we have complete freedom in selecting the eigenvalues of the estimator, if we select them properly, the cancellation will not cause any problem in design. In using the unity-feedback configuration in model matching, as we saw in the preceding example, the canceled poles are dictated by the plant transfer function. Thus, if a plant transfer function has poles with positive real parts, the cancellation will involve unstable poles. Therefore the unity-feedback configuration, in general, cannot be used in model matching.

The open-loop and the unity-feedback configurations in Figs. 9.1(a) and 9.1(b) have one degree of freedom and cannot be used to achieve every model matching. The configurations in Figs. 9.1(c) and 9.1(d) both have two degrees of freedom. In using either configuration, we have complete freedom in assigning canceled poles; therefore both can be used to achieve every model matching. Because the two-parameter configuration in Fig. 9.1(d) seems to be more natural and more suitable for practical implementation, we discuss only that configuration here. For model matching using the configuration in Fig. 9.1(c), see Reference [6].

Consider the two-parameter configuration in Fig. 9.1(d). Let

$$C_1(s) = \frac{L(s)}{A_1(s)} \qquad C_2(s) = \frac{M(s)}{A_2(s)}$$

where $L(s)$, $M(s)$, $A_1(s)$, and $A_2(s)$ are polynomials. We call $C_1(s)$ the *feedforward compensator* and $C_2(s)$ the *feedback compensator*. In general, $A_1(s)$ and $A_2(s)$ need not be the same. It turns out that even if they are chosen to be the same, the configuration can still be used to achieve any model matching. Furthermore, a simple design procedure can be developed. Therefore we assume $A_1(s) = A_2(s) = A(s)$ and the compensators become

$$C_1(s) = \frac{L(s)}{A(s)} \qquad C_2(s) = \frac{M(s)}{A(s)} \tag{9.26}$$

The transfer function from r to y in Fig. 9.1(d) then becomes

$$\hat{g}_o(s) = C_1(s)\frac{\hat{g}(s)}{1+\hat{g}(s)C_2(s)} = \frac{L(s)}{A(s)}\frac{\dfrac{N(s)}{D(s)}}{1+\dfrac{N(s)}{D(s)}\dfrac{M(s)}{A(s)}}$$

$$= \frac{L(s)N(s)}{A(s)D(s)+M(s)N(s)} \tag{9.27}$$

Thus in model matching, we search for proper $L(s)/A(s)$ and $M(s)/A(s)$ to meet

$$\hat{g}_o(s) = \frac{E(s)}{F(s)} = \frac{L(s)N(s)}{A(s)D(s)+M(s)N(s)} \tag{9.28}$$

Note that the two-parameter configuration has no plant leakage. If the plant transfer function $\hat{g}(s)$ is strictly proper as assumed and if $C_2(s) = M(s)/A(s)$ is proper, then the overall system is automatically well posed. The question of total stability will be discussed in the next subsection.

Problem *Given $\hat{g}(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are coprime and deg $N(s) < \deg D(s) = n$, and given an implementable $\hat{g}_o(s) = E(s)/F(s)$, find proper $L(s)/A(s)$ and $M(s)/A(s)$ to meet (9.28).*

Procedure 9.1

1. Compute

$$\frac{\hat{g}_o(s)}{N(s)} = \frac{E(s)}{F(s)N(s)} =: \frac{\bar{E}(s)}{\bar{F}(s)} \tag{9.29}$$

where $\bar{E}(s)$ and $\bar{F}(s)$ are coprime. Since $E(s)$ and $F(s)$ are implicitly assumed to be coprime, common factors may exist only between $E(s)$ and $N(s)$. Cancel all common factors between them and denote the rest as $\bar{E}(s)$ and $\bar{F}(s)$. Note that if $E(s) = N(s)$, then $\bar{F}(s) = F(s)$ and $\bar{E}(s) = 1$. Using (9.29), we rewrite (9.28) as

$$\hat{g}_o(s) = \frac{\bar{E}(s)N(s)}{\bar{F}(s)} = \frac{L(s)N(s)}{A(s)D(s) + M(s)N(s)} \tag{9.30}$$

From this equation, we may be tempted to set $L(s) = \bar{E}(s)$ and solve for $A(s)$ and $M(s)$ from $\bar{F}(s) = A(s)D(s) + M(s)N(s)$. However, no proper $C_2(s) = M(s)/A(s)$ may exist in the equation. See Problem 9.1. Thus we need some additional manipulation.

2. Introduce an arbitrary Hurwitz polynomial $\hat{F}(s)$ such that the degree of $\bar{F}(s)\hat{F}(s)$ is $2n - 1$ or higher. In other words, if deg $\bar{F}(s) = p$, then deg $\hat{F}(s) \geq 2n - 1 - p$. Because the polynomial $\hat{F}(s)$ will be canceled in the design, its roots should be chosen to lie inside the sector shown in Fig. 8.3(a).

3. Rewrite (9.30) as

$$\hat{g}_o(s) = \frac{\bar{E}(s)\hat{F}(s)N(s)}{\bar{F}(s)\hat{F}(s)} = \frac{L(s)N(s)}{A(s)D(s) + M(s)N(s)} \tag{9.31}$$

Now we set

$$L(s) = \bar{E}(s)\hat{F}(s) \tag{9.32}$$

and solve $A(s)$ and $M(s)$ from

$$A(s)D(s) + M(s)N(s) = \bar{F}(s)\hat{F}(s) \tag{9.33}$$

If we write

$$A(s) = A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m$$
$$M(s) = M_0 + B_1 s + M_2 s^2 + \cdots + M_m s^m$$
$$\bar{F}(s)\hat{F}(s) = F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m}$$

with $m \geq n - 1$, then $A(s)$ and $M(s)$ can be obtained by solving

$$[A_0\ M_0\ A_1\ M_1\ \cdots\ A_m\ M_m]\mathbf{S}_m = [F_0\ F_1\ F_2\ \cdots\ F_{n+m}] \tag{9.34}$$

with

$$\mathbf{S}_m := \begin{bmatrix} D_0 & D_1 & \cdots & D_n & 0 & \cdots & 0 \\ N_0 & N_1 & \cdots & N_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & D_0 & \cdots & D_{n-1} & D_n & \cdots & 0 \\ 0 & N_0 & \cdots & N_{n-1} & N_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & D_0 & \cdots & D_n \\ 0 & 0 & \cdots & 0 & N_0 & \cdots & N_n \end{bmatrix}$$

The computed compensators $L(s)/A(s)$ and $M(s)/A(s)$ are proper.

We justify the procedure. By introducing $\hat{F}(s)$, the degree of $\bar{F}(s)\hat{F}(s)$ is $2n - 1$ or higher and, following Theorem 9.2, solutions $A(s)$ and $M(s)$ with deg $M(s) \leq$ deg $A(s) = m$ and $m \geq n - 1$ exist in (9.34) for any $\bar{F}(s)\hat{F}(s)$. Thus the compensator $M(s)/A(s)$ is proper. Note that if we do not introduce $\hat{F}(s)$, proper compensator $M(s)/A(s)$ may not exist in (9.34).

Next we show deg $L(s) \leq$ deg $A(s)$. Applying the pole–zero excess inequality to (9.31) and using (9.32), we have

$$\deg(\bar{F}(s)\hat{F}(s)) - \deg N(s) - \deg L(s) \geq \deg D(s) - \deg N(s)$$

which implies

$$\deg L(s) \leq \deg(\bar{F}(s)\hat{F}(s)) - \deg D(s) = \deg A(s)$$

Thus the compensator $L(s)/A(s)$ is proper.

Example 9.7 Consider the model matching problem studied in Example 9.6. That is, given $\hat{g}(s) = (s - 2)/(s^2 - 1)$, match $\hat{g}_o(s) = -(s - 2)/(s^2 + 2s + 2)$. We implement it in the two-parameter configuration shown in Fig. 9.1(d). First we compute

$$\frac{\hat{g}_o(s)}{N(s)} = \frac{-(s-2)}{(s^2 + 2s + 2)(s - 2)} = \frac{-1}{s^2 + 2s + 2} =: \frac{\bar{E}(s)}{\bar{F}(s)}$$

Because the degree of $\bar{F}(s)$ is 2, we select arbitrarily $\hat{F}(s) = s + 4$ so that the degree of $\bar{F}(s)\hat{F}(s)$ is $3 = 2n - 1$. Thus we have

$$L(s) = \bar{E}(s)\hat{F}(s) = -(s + 4) \tag{9.35}$$

and $A(s)$ and $M(s)$ can be solved from

$$A(s)D(s) + M(s)N(s) = \bar{F}(s)\hat{F}(s) = (s^2 + 2s + 2)(s + 4)$$
$$= s^3 + 6s^2 + 10s + 8$$

or

$$[A_0\ M_0\ A_1\ M_1]\begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} = [8\ \ 10\ \ 6\ \ 1]$$

The solution is $A_0 = 18$, $A_1 = 1$, $M_0 = -13$, and $M_1 = -12$. Thus we have $A(s) = 18 + s$ and $M(s) = -13 - 12s$ and the compensators are

$$C_1(s) = \frac{L(s)}{A(s)} = \frac{-(s+4)}{s+18} \qquad C_2(s) = \frac{M(s)}{A(s)} = \frac{-(12s+13)}{s+18}$$

This completes the design. Note that, because $\hat{g}_o(0) = 1$, the output of the feedback system will track any step reference input.

Example 9.8 Given $\hat{g}(s) = (s-2)/(s^2-1)$, match

$$\hat{g}_o(s) = \frac{-(s-2)(4s+2)}{(s^2+2s+2)(s+2)} = \frac{-4s^2+6s+4}{s^3+4s^2+6s+4}$$

This $\hat{g}_o(s)$ is BIBO stable and has the property $\hat{g}_o(0) = 1$ and $\hat{g}'_o(s) = 0$; thus the overall system will track asymptotically not only any step reference input but also any ramp input. See Problems 9.13 and 9.14. This $\hat{g}_o(s)$ meets all three conditions in Corollary 9.4; thus it is implementable. We use the two-parameter configuration. First we compute

$$\frac{\hat{g}_o(s)}{N(s)} = \frac{-(s-2)(4s+2)}{(s^2+2s+2)(s+2)(s-2)} = \frac{-(4s+2)}{s^3+4s^2+6s+4} =: \frac{\bar{E}(s)}{\bar{F}(s)}$$

Because the degree of $\bar{F}(s)$ is 3, which equals $2n - 1 = 3$, there is no need to introduce $\hat{F}(s)$ and we set $\hat{F}(s) = 1$. Thus we have

$$L(s) = \hat{F}(s)\bar{E}(s) = -(4s+2)$$

and $A(s)$ and $M(s)$ can be solved from

$$[A_0\ M_0\ A_1\ M_1]\begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} = [4\ \ 6\ \ 4\ \ 1]$$

as $A_0 = 1$, $A_1 = 34/3$, $M_0 = -23/3$, and $M_1 = -22/3$. Thus the compensators are

$$C_1(s) = \frac{-(4s+2)}{s+34/3} \qquad C_2(s) = \frac{-(22s+23)}{3s+34}$$

This completes the design. Note that this design does not involve any pole–zero cancellation because $\hat{F}(s) = 1$.

9.3.2 Implementation of Two-Parameter Compensators

Given a plant with transfer function $\hat{g}(s)$ and an implementable model $\hat{g}_o(s)$, we can implement the model in the two-parameter configuration shown in Fig. 9.1(d) and redrawn in Fig. 9.4(a). The compensators $C_1(s) = L(s)/A(s)$ and $C_2(s) = M(s)/A(s)$ can be obtained by using Procedure 9.1. To complete the design, the compensators must be built or implemented. This is discussed in this subsection.

Consider the configuration in Fig. 9.4(a). The denominator $A(s)$ of $C_1(s)$ is obtained by solving the compensator equation in (9.33) and may or may not be a Hurwitz polynomial. See Problem 9.12. If it is not a Hurwitz polynomial and if we implement $C_1(s)$ as shown in Fig. 9.4(a), then the output of $C_1(s)$ will grow without bound and the overall system is not totally stable. Therefore, in general, we should not implement the two compensators as shown in Fig. 9.4(a). If we move $C_2(s)$ outside the loop as shown in Fig. 9.4(b), then the design will involve the cancellation of $M(s)$. Because $M(s)$ is also obtained by solving (9.33), we have no direct control of $M(s)$. Thus the design is in general not acceptable. If we move $C_1(s)$ inside the loop, then the configuration becomes the one shown in Fig. 9.4(c). We see that the connection involves the pole–zero cancellation of $L(s) = \hat{F}(s)\bar{E}(s)$. We have freedom in selecting $\hat{F}(s)$. The polynomial $\bar{E}(s)$ is part of $E(s)$, which, other than the nonminimum-phase zeros of $N(s)$, we can also select. The nonminimum-phase zeros, however, are completely canceled in $\bar{E}(s)$. Thus $L(s)$ can be Hurwitz[3] and the implementation in Fig. 9.4(c) can be totally stable and is

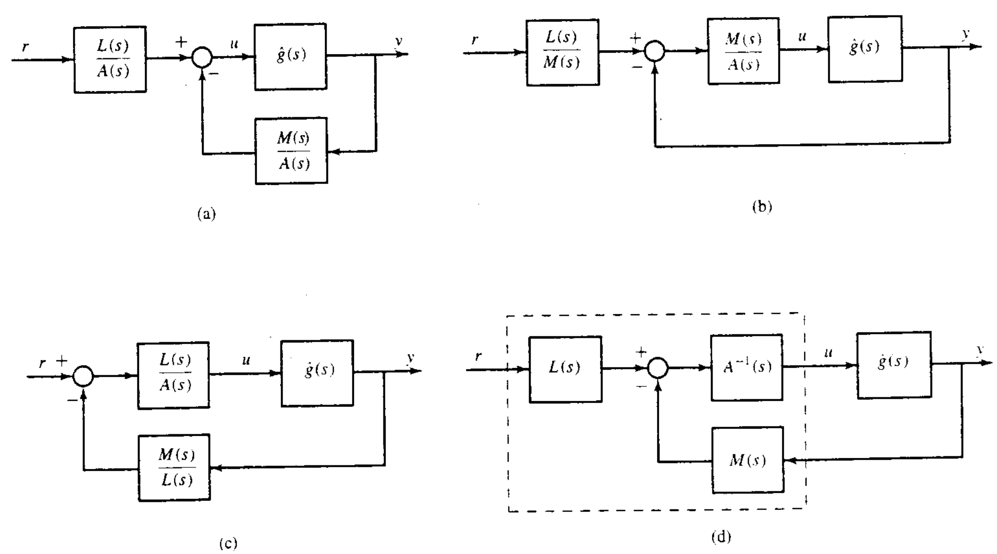


Figure 9.4 Two-degrees-of-freedom configurations.

3. This may not be true in the multivariable case.

acceptable. However, because the two compensators $L(s)/A(s)$ and $M(s)/L(s)$ have different denominators, their implementations require a total of $2m$ integrators. We discuss next a better implementation that requires only m integrators and involves only the cancellation of $\hat{F}(s)$.

Consider

$$\hat{u}(s) = C_1(s)\hat{r}(s) - C_2(s)\hat{y}(s) = \frac{L(s)}{A(s)}\hat{r}(s) - \frac{M(s)}{A(s)}\hat{y}(s)$$
$$= A^{-1}(s)[L(s) \quad -M(s)]\begin{bmatrix}\hat{r}(s)\\ \hat{y}(s)\end{bmatrix}$$

This can be plotted as shown in Fig. 9.4(d). Thus we can consider the two compensators as a single compensator with two inputs and one output with transfer matrix

$$\mathbf{C}(s) = [C_1(s) \quad -C_2(s)] = A^{-1}(s)[L(s) \quad -M(s)] \tag{9.36}$$

If we find a minimal realization of (9.36), then its dimension is m and the two compensators can be implemented using only m integrators. As we can see from (9.31), the design involves only the cancellation of $\hat{F}(s)$. Thus the implementation in Fig. 9.4(d) is superior to the one in Fig. 9.4(c). We see that the four configurations in Fig. 9.4 all have two degrees of freedom and are mathematically equivalent. However, they can be different in actual implementation.

EXAMPLE 9.9 Implement the compensators in Example 9.8 using an op-amp circuit. We write

$$\hat{u}(s) = C_1(s)\hat{r}(s) - C_2(s)\hat{y}(s) = \begin{bmatrix}\frac{-(4s+2)}{s+11.33} & \frac{7.33s+7.67}{s+11.33}\end{bmatrix}\begin{bmatrix}\hat{r}(s)\\ \hat{y}(s)\end{bmatrix}$$
$$= \left([-4 \quad 7.33] + \frac{1}{s+11.33}[43.33 \quad -75.38]\right)\begin{bmatrix}\hat{r}(s)\\ \hat{y}(s)\end{bmatrix}$$

Its state-space realization is, using the formula in Problem 4.10,

$$\dot{x} = -11.33x + [43.33 \quad -75.38]\begin{bmatrix}r\\ y\end{bmatrix}$$
$$u = x + [-4 \quad 7.33]\begin{bmatrix}r\\ y\end{bmatrix}$$

(See Problem 4.14.) This one-dimensional state equation can be realized as shown in Fig. 9.5. This completes the implementation of the compensators.

9.4 Multivariable Unity-Feedback Systems

This section extends the pole placement discussed in Section 9.2 to the multivariable case. Consider the unity-feedback system shown in Fig. 9.6. The plant has p inputs and q outputs and is described by a $q \times p$ strictly proper rational matrix $\hat{\mathbf{G}}(s)$. The compensator $\mathbf{C}(s)$ to be designed must have q inputs and p outputs in order for the connection to be possible. Thus $\mathbf{C}(s)$ is required to be a $p \times q$ proper rational matrix. The matrix $\mathbf{P}$ is a $q \times q$ constant gain matrix. For the time being, we assume $\mathbf{P} = \mathbf{I}_q$. Let the transfer matrix from $\mathbf{r}$ to $\mathbf{y}$ be denoted by $\hat{\mathbf{G}}_o(s)$, a $q \times q$ matrix. Then we have

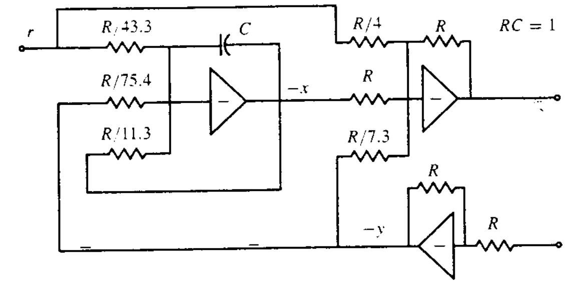


Figure 9.5 Op-amp circuit implementation.

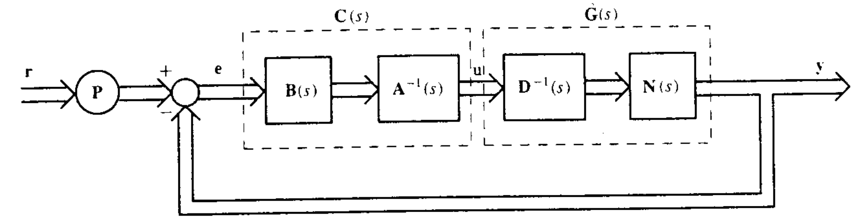


Figure 9.6 Multivariable unity feedback system with $\mathbf{P} = \mathbf{I}_q$.

$$\hat{\mathbf{G}}_o(s) = [\mathbf{I}_q + \hat{\mathbf{G}}(s)\mathbf{C}(s)]^{-1}\hat{\mathbf{G}}(s)\mathbf{C}(s)$$
$$= \hat{\mathbf{G}}(s)\mathbf{C}(s)[\mathbf{I}_q + \hat{\mathbf{G}}(s)\mathbf{C}(s)]^{-1}$$
$$= \hat{\mathbf{G}}(s)[\mathbf{I}_p + \mathbf{C}(s)\hat{\mathbf{G}}(s)]^{-1}\mathbf{C}(s) \tag{9.37}$$

The first equality is obtained from $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\mathbf{C}(s)[\hat{\mathbf{r}}(s) - \hat{\mathbf{y}}(s)]$; the second one from $\hat{\mathbf{e}}(s) = \hat{\mathbf{r}}(s) - \hat{\mathbf{G}}(s)\mathbf{C}(s)\hat{\mathbf{e}}(s)$; and the third one from $\hat{\mathbf{u}}(s) = \mathbf{C}(s)[\hat{\mathbf{r}}(s) - \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)]$. They can also be verified directly. For example, pre- and postmultiplying by $[\mathbf{I}_q + \hat{\mathbf{G}}(s)\mathbf{C}(s)]$ in the first two equations yield

$$\hat{\mathbf{G}}(s)\mathbf{C}(s)[\mathbf{I}_q + \hat{\mathbf{G}}(s)\mathbf{C}(s)] = [\mathbf{I}_q + \hat{\mathbf{G}}(s)\mathbf{C}(s)]\hat{\mathbf{G}}(s)\mathbf{C}(s)$$

which is an identity. This establishes the second equality. The third equality can similarly be established.

Let $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$ be a right coprime fraction and let $\mathbf{C}(s) = \mathbf{A}^{-1}(s)\mathbf{B}(s)$ be a left fraction to be designed. Then (9.37) implies

$$\hat{\mathbf{G}}_o(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)[\mathbf{I} + \mathbf{A}^{-1}(s)\mathbf{B}(s)\mathbf{N}(s)\mathbf{D}^{-1}(s)]^{-1}\mathbf{A}^{-1}(s)\mathbf{B}(s)$$
$$= \mathbf{N}(s)\mathbf{D}^{-1}(s)\left\{\mathbf{A}^{-1}(s)[\mathbf{A}(s)\mathbf{D}(s) + \mathbf{B}(s)\mathbf{N}(s)]\mathbf{D}^{-1}(s)\right\}^{-1}\mathbf{A}^{-1}(s)\mathbf{B}(s)$$
$$= \mathbf{N}(s))[\mathbf{A}(s)\mathbf{D}(s) + \mathbf{B}(s)\mathbf{N}(s)]^{-1}\mathbf{B}(s)$$
$$= \mathbf{N}(s)\mathbf{F}^{-1}(s)\mathbf{B}(s) \tag{9.38}$$

where

$$\mathbf{A}(s)\mathbf{D}(s) + \mathbf{B}(s)\mathbf{N}(s) = \mathbf{F}(s) \qquad (9.39)$$

It is a polynomial matrix equation. Thus the design problem can be stated as follows: given $p \times p$ $\mathbf{D}(s)$ and $q \times p$ $\mathbf{N}(s)$ and an arbitrary $p \times p$ $\mathbf{F}(s)$, find $p \times p$ $\mathbf{A}(s)$ and $p \times q$ $\mathbf{B}(s)$ to meet (9.39). This is the matrix version of the polynomial compensator equation in (9.12).

Theorem 9.M1

Given polynomial matrices $\mathbf{D}(s)$ and $\mathbf{N}(s)$, polynomial matrix solutions $\mathbf{A}(s)$ and $\mathbf{B}(s)$ exist in (9.39) for any polynomial matrix $\mathbf{F}(s)$ if and only if $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are right coprime.

Suppose $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are not right coprime, then there exists a nonunimodular polynomial matrix $\mathbf{R}(s)$ such that $\mathbf{D}(s) = \hat{\mathbf{D}}(s)\mathbf{R}(s)$ and $\mathbf{N}(s) = \hat{\mathbf{N}}(s)\mathbf{R}(s)$. Then $\mathbf{F}(s)$ in (9.39) must be of the form $\hat{\mathbf{F}}(s)\mathbf{R}(s)$, for some polynomial matrix $\hat{\mathbf{F}}(s)$. Thus if $\mathbf{F}(s)$ cannot be expressed in such a form, no solutions exist in (9.39). This shows the necessity of the theorem. If $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are right coprime, there exist polynomial matrices $\bar{\mathbf{A}}(s)$ and $\bar{\mathbf{B}}(s)$ such that

$$\bar{\mathbf{A}}(s)\mathbf{D}(s) + \bar{\mathbf{B}}(s)\mathbf{N}(s) = \mathbf{I}$$

The polynomial matrices $\bar{\mathbf{A}}(s)$ and $\bar{\mathbf{B}}(s)$ can be obtained by a sequence of elementary operations. See Reference [6, pp. 587–595]. Thus $\mathbf{A}(s) = \mathbf{F}(s)\bar{\mathbf{A}}(s)$ and $\mathbf{B}(s) = \mathbf{F}(s)\bar{\mathbf{B}}(s)$ are solutions of (9.39) for any $\mathbf{F}(s)$. This establishes Theorem 9.M1. As in the scalar case, it is possible to develop general solutions for (9.39). However, the general solutions are not convenient to use in our design. Thus they will not be discussed.

Next we will change solving (9.39) into solving a set of linear algebraic equations. Consider $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$, where $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are right coprime and $\mathbf{D}(s)$ is column reduced. Let μ_i be the degree of the ith column of $\mathbf{D}(s)$. Then we have, as discussed in Section 7.8.2,

$$\deg \hat{\mathbf{G}}(s) = \deg\det \mathbf{D}(s) = \mu_1 + \mu_2 + \cdots + \mu_p =: n \qquad (9.40)$$

Let $\mu := \max(\mu_1, \mu_2, \ldots, \mu_p)$. Then we can express $\mathbf{D}(s)$ and $\mathbf{N}(s)$ as

$$\mathbf{D}(s) = \mathbf{D}_0 + \mathbf{D}_1 s + \mathbf{D}_2 s^2 + \cdots + \mathbf{D}_\mu s^\mu \qquad \mathbf{D}_\mu \neq \mathbf{0}$$
$$\mathbf{N}(s) = \mathbf{N}_0 + \mathbf{N}_1 s + \mathbf{N}_2 s^2 + \cdots + \mathbf{N}_\mu s^\mu$$

Note that $\mathbf{D}_\mu$ is singular unless $\mu_1 = \mu_2 = \cdots = \mu_p$. Note also that $\mathbf{N}_\mu = \mathbf{0}$, following the strict properness assumption of $\hat{\mathbf{G}}(s)$. We also express $\mathbf{A}(s)$, $\mathbf{B}(s)$, and $\mathbf{F}(s)$ as

$$\mathbf{A}(s) = \mathbf{A}_0 + \mathbf{A}_1 s + \mathbf{A}_2 s^2 + \cdots + \mathbf{A}_m s^m$$
$$\mathbf{B}(s) = \mathbf{B}_0 + \mathbf{B}_1 s + \mathbf{B}_2 s^2 + \cdots + \mathbf{B}_m s^m$$
$$\mathbf{F}(s) = \mathbf{F}_0 + \mathbf{F}_1 s + \mathbf{F}_2 s^2 + \cdots + \mathbf{F}_{\mu+m} s^{\mu+m}$$

Substituting these into (9.39) and matching the coefficients of like powers of s, we obtain

$$[\mathbf{A}_0\ \mathbf{B}_0\ \mathbf{A}_1\ \mathbf{B}_1\ \cdots\ \mathbf{A}_m\ \mathbf{B}_m]\mathbf{S}_m = [\mathbf{F}_0\ \mathbf{F}_1\ \cdots\ \mathbf{F}_{\mu+m}] =: \bar{\mathbf{F}} \qquad (9.41)$$

where

$$\mathbf{S}_m := \left[\begin{array}{cccccccc}
\mathbf{D}_0 & \mathbf{D}_1 & \cdots & \mathbf{D}_\mu & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{N}_0 & \mathbf{N}_1 & \cdots & \mathbf{N}_\mu & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{D}_0 & \cdots & \mathbf{D}_{\mu-1} & \mathbf{D}_\mu & \mathbf{0} & \cdots & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{N}_0 & \cdots & \mathbf{N}_{\mu-1} & \mathbf{N}_\mu & \mathbf{0} & \cdots & \mathbf{0} \\
\hline
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\hline
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \cdots & \mathbf{D}_\mu \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 & \cdots & \mathbf{N}_\mu
\end{array}\right] \qquad (9.42)$$

The matrix $\mathbf{S}_m$ has $m + 1$ block rows; each block row consists of p D-rows and q N-rows. Thus $\mathbf{S}_m$ has $(m + 1)(p + q)$ number of rows. Let us search linearly independent rows of $\mathbf{S}_m$ in order from top to bottom. It turns out that if $\mathbf{N}(s)\mathbf{D}^{-1}(s)$ is proper, then all D-rows are linearly independent of their previous rows. An N-row can be linearly independent of its previous rows. However, if an N-row becomes linearly dependent, then, because of the structure of $\mathbf{S}_m$, the same N-rows in subsequent N-block rows will be linearly dependent. Let ν_i be the number of linearly independent ith N-rows and let

$$\nu := \max\{\nu_1, \nu_2, \ldots, \nu_q\}$$

It is called the *row index* of $\hat{\mathbf{G}}(s)$. Then all q N-rows in the last N-block row of $\mathbf{S}_\nu$ are linearly dependent on their previous rows. Thus $\mathbf{S}_{\nu-1}$ contains all linearly independent N-rows and its total number equals, as discussed in Section 7.8.2, the degree of $\hat{\mathbf{G}}(s)$, that is,

$$\nu_1 + \nu_2 + \cdots + \nu_q = n \qquad (9.43)$$

Because all D-rows are linearly independent and there are a total of $p\nu$ D-rows in $\mathbf{S}_{\nu-1}$, we conclude that $\mathbf{S}_{\nu-1}$ has $n + p\nu$ independent rows or rank $n + p\nu$.

Let us consider

$$\mathbf{S}_0 = \begin{bmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \cdots & \mathbf{D}_{\mu-1} & \mathbf{D}_\mu \\ \mathbf{N}_0 & \mathbf{N}_1 & \cdots & \mathbf{N}_{\mu-1} & \mathbf{N}_\mu \end{bmatrix}$$

It has $p(\mu + 1)$ number of columns; however, it has at least a total of $\sum_{i=1}^{p}(\mu - \mu_i)$ zero columns. In the matrix $\mathbf{S}_1$, some new zero columns will appear in the rightmost block column. However, some zero columns in $\mathbf{S}_0$ will not be zero columns in $\mathbf{S}_1$. Thus the number of zero columns in $\mathbf{S}_1$ remains as

$$\alpha := \sum_{i=1}^{p}(\mu - \mu_i) = p\mu - (\mu_1 + \mu_2 + \cdots + \mu_p) = p\mu - n \qquad (9.44)$$

In fact, this is the number of zero columns in $\mathbf{S}_m$, $m = 2, 3, \ldots$. Let $\tilde{\mathbf{S}}_{\mu-1}$ be the matrix $\mathbf{S}_{\mu-1}$ after deleting these zero columns. Because the number of columns in $\mathbf{S}_m$ is $p(\mu + 1 + m)$, the number of columns in $\tilde{\mathbf{S}}_{\nu-1}$ is

$$\beta := p(\mu + 1 + \nu - 1) - (p\mu - n) = p\nu + n \tag{9.45}$$

The rank of $\tilde{\mathbf{S}}_{\mu-1}$ clearly equals the rank of $\mathbf{S}_{\mu-1}$ or $p\nu + n$. Thus $\tilde{\mathbf{S}}_{\mu-1}$ has full column rank. Now if m increases by 1, the rank and the number of the columns of $\tilde{\mathbf{S}}_\mu$ both increase by p (because the p new D-rows are all linearly independent of their previous rows); thus $\tilde{\mathbf{S}}_\mu$ still has full column rank. Proceeding forward, we conclude that $\tilde{\mathbf{S}}_m$, for $m \geq \mu - 1$, has full column rank.

Let us define

$$\mathbf{H}_c(s) := \text{diag}(s^{\mu_1}, s^{\mu_2}, \ldots, s^{\mu_p}) \tag{9.46}$$

and

$$\mathbf{H}_r(s) = \text{diag}(s^{m_1}, s^{m_2}, \ldots, s^{m_p}) \tag{9.47}$$

Then we have the following matrix version of Theorem 9.2.

Theorem 9.M2

Consider the unity-feedback system shown in Fig. 9.6 with $\mathbf{P} = \mathbf{I}_q$. The plant is described by a $q \times p$ strictly proper rational matrix $\hat{\mathbf{G}}(s)$. Let $\hat{\mathbf{G}}(s)$ be factored as $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$, where $\mathbf{D}(s)$ and $\mathbf{N}(s)$ are right coprime and $\mathbf{D}(s)$ is column reduced with column degrees μ_i, $i = 1, 2, \ldots, p$. Let ν be the row index of $\hat{\mathbf{G}}(s)$ and let $m_i \geq \nu - 1$ for $i = 1, 2, \ldots, p$. Then for any $p \times p$ polynominal matrix $\mathbf{F}(s)$, such that

$$\lim_{s \to \infty} \mathbf{H}_r^{-1}(s)\mathbf{F}(s)\mathbf{H}_c^{-1}(s) = \mathbf{F}_h \tag{9.48}$$

is a nonsingular constant matrix, there exists a $p \times q$ proper compensator $\mathbf{A}^{-1}(s)\mathbf{B}(s)$, where $\mathbf{A}(s)$ is row reduced with row degrees m_i, such that the transfer matrix from $\mathbf{r}$ to $\mathbf{y}$ equals

$$\hat{\mathbf{G}}_o(s) = \mathbf{N}(s)\mathbf{F}^{-1}(s)\mathbf{B}(s)$$

Furthermore, the compensator can be obtained by solving sets of linear algebraic equations in (9.41).

Proof: Let $m = \max(m_1, m_2, \ldots, m_p)$. Consider the constant matrix

$$\bar{\mathbf{F}} := [\mathbf{F}_0 \;\; \mathbf{F}_1 \;\; \mathbf{F}_2 \;\; \cdots \;\; \mathbf{F}_{m+\mu}]$$

It is formed from the coefficient matrices of $\mathbf{F}(s)$ and has order $p \times (m + \mu + 1)$. Clearly $\mathbf{F}(s)$ has column degrees at most $m + \mu_i$. Thus $\bar{\mathbf{F}}$ has at least α number of zero columns, where α is given in (9.44). Furthermore, the positions of these zero columns coincide with those of $\mathbf{S}_m$. Let $\tilde{\mathbf{F}}$ be the constant matrix $\bar{\mathbf{F}}$ after deleting these zero columns. Now consider

$$[\mathbf{A}_0 \; \mathbf{B}_0 \; \mathbf{A}_1 \; \mathbf{B}_1 \; \cdots \; \mathbf{A}_m \; \mathbf{B}_m]\tilde{\mathbf{S}}_m = \tilde{\mathbf{F}} \tag{9.49}$$

It is obtained from (9.41) by deleting α number of zero columns in $\mathbf{S}_m$ and the corresponding zero columns in $\bar{\mathbf{F}}$. Now because $\tilde{\mathbf{S}}_m$ has full column rank if $m \geq \nu - 1$, we conclude that for any $\mathbf{F}(s)$ of column degrees at most $m + \mu_i$, solutions $\mathbf{A}_i$ and $\mathbf{B}_i$ exist in (9.49). Or, equivalently, polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ of row degree m or less exist in (9.49). Note that generally $\tilde{\mathbf{S}}_m$ has more rows than columns; therefore solutions of (9.49) are not unique.

Next we show that $\mathbf{A}^{-1}(s)\mathbf{B}(s)$ is proper. Note that $\mathbf{D}_\mu$ is, in general, singular and the method of proving Theorem 9.2 cannot be used here. Using $\mathbf{H}_r(s)$ and $\mathbf{H}_c(s)$, we write, as in (7.80),

$$\mathbf{D}(s) = [\mathbf{D}_{hc} + \mathbf{D}_{lc}(s)\mathbf{H}_c^{-1}(s)]\mathbf{H}_c(s)$$

$$\mathbf{N}(s) = [\mathbf{N}_{hc} + \mathbf{N}_{lc}(s)\mathbf{H}_c^{-1}(s)]\mathbf{H}_c(s)$$

$$\mathbf{A}(s) = \mathbf{H}_r(s)[\mathbf{A}_{hr} + \mathbf{H}_r^{-1}(s)\mathbf{A}_{lr}(s)]$$

$$\mathbf{B}(s) = \mathbf{H}_r(s)[\mathbf{B}_{hr} + \mathbf{H}_r^{-1}(s)\mathbf{B}_{lr}(s)]$$

$$\mathbf{F}(s) = \mathbf{H}_r(s)[\mathbf{F}_h + \mathbf{H}_r^{-1}(s)\mathbf{F}_l(s)\mathbf{H}_c^{-1}(s)]\mathbf{H}_c(s)$$

where $\mathbf{D}_{lc}(s)\mathbf{H}_c^{-1}(s)$, $\mathbf{N}_{lc}(s)\mathbf{H}_c^{-1}(s)$, $\mathbf{H}_r^{-1}(s)\mathbf{A}_{lr}(s)$, $\mathbf{H}_r^{-1}(s)\mathbf{B}_{lr}(s)$, and $\mathbf{H}_r^{-1}(s)\mathbf{F}_l(s)\mathbf{H}_c^{-1}(s)$ are all strictly proper rational functions. Substituting the above into (9.39) yields, at $s = \infty$,

$$\mathbf{A}_{hr}\mathbf{D}_{hc} + \mathbf{B}_{hr}\mathbf{N}_{hc} = \mathbf{F}_h$$

which reduces to, because $\mathbf{N}_{hc} = \mathbf{0}$ following strict properness of $\hat{\mathbf{G}}(s)$,

$$\mathbf{A}_{hr}\mathbf{D}_{hc} = \mathbf{F}_h$$

Because $\mathbf{D}(s)$ is column reduced, $\mathbf{D}_{hc}$ is nonsingular. The constant matrix $\mathbf{F}_h$ is nonsingular by assumption; thus $\mathbf{A}_{hr} = \mathbf{F}_h\mathbf{D}_{hc}^{-1}$ is nonsingular and $\mathbf{A}(s)$ is row reduced. Therefore $\mathbf{A}^{-1}(s)\mathbf{B}(s)$ is proper (Corollary 7.8). This establishes the theorem. Q.E.D.

A polynomial matrix $\mathbf{F}(s)$ meeting (9.48) is said to be *row–column reduced* with row degrees m_i and column degrees μ_i. If $m_1 = m_2 = \cdots = m_p = m$, then the row–column reducedness is the same as column reducedness with column degrees $m + \mu_i$. In application, we can select $\mathbf{F}(s)$ to be diagonal or triangular with polynomials with desired roots as its diagonal entries. Then $\mathbf{F}^{-1}(s)$ and, consequently, $\hat{\mathbf{G}}_o(s)$ have the desired roots as their poles.

Consider again $\mathbf{S}_{\nu-1}$. It is of order $(p + q)\nu \times (\mu + \nu)p$. It has $\alpha = p\mu - n$ number of zero columns. Thus the matrix $\tilde{\mathbf{S}}_{\nu-1}$ is of order $(p + q)\nu \times [(\mu + \nu)p - (p\mu - n)]$ or $(p + q)\nu \times (\nu p + n)$. The matrix $\tilde{\mathbf{S}}_{\nu-1}$ contains $p\nu$ linearly independent D-rows but contains only $\nu_1 + \cdots + \nu_q = n$ linearly independent N-rows. Thus $\tilde{\mathbf{S}}_{\nu-1}$ contains

$$\gamma := (p + q)\nu - p\nu - n = q\nu - n$$

linearly dependent N-rows. Let $\check{\mathbf{S}}_{\nu-1}$ be the matrix $\tilde{\mathbf{S}}_{\nu-1}$ after deleting these linearly dependent N-rows. Then the matrix $\check{\mathbf{S}}_{\nu-1}$ is of order

$$[(p + q)\nu - (q\nu - n)] \times (\nu p + n) = (\nu p + n) \times (\nu p + n)$$

Thus $\check{\mathbf{S}}_{\nu-1}$ is square and nonsingular.

Consider (9.49) with $m = \nu - 1$:

$$\mathbf{K}\tilde{\mathbf{S}}_{\nu-1} := [\mathbf{A}_0\ \mathbf{B}_0\ \mathbf{A}_1\ \mathbf{B}_1\ \cdots\ \mathbf{A}_{\nu-1}\ \mathbf{B}_{\nu-1}]\tilde{\mathbf{S}}_{\nu-1} = \tilde{\mathbf{F}}$$

It actually consists of the following p sets of linear algebraic equations

$$\mathbf{k}_i\tilde{\mathbf{S}}_{\nu-1} = \bar{\mathbf{f}}_i \qquad i = 1, 2, \ldots, p \tag{9.50}$$

where $\mathbf{k}_i$ and $\bar{\mathbf{f}}_i$ are the ith row of $\mathbf{K}$ and $\tilde{\mathbf{F}}$, respectively. Because $\tilde{\mathbf{S}}_{\nu-1}$ has full column rank, for any $\bar{\mathbf{f}}_i$, solutions $\mathbf{k}_i$ exist in (9.50). Because $\tilde{\mathbf{S}}_{\nu-1}$ has more γ rows than columns, the general solution of (9.50) contains γ free parameters (Corollary 3.2). If m in $\mathbf{S}_m$ increases by 1 from $\nu - 1$ to ν, then the number of rows of $\tilde{\mathbf{S}}_\nu$ increases by $(p + q)$ but the rank of $\tilde{\mathbf{S}}_\nu$ increases only by p. In this case, the number of free parameters will increase from γ to $\gamma + q$. Thus in the MIMO case, we have a great deal of freedom in carrying out the design.

We discuss a special case of (9.50). The matrix $\tilde{\mathbf{S}}_{\nu-1}$ has γ linearly dependent N-rows. If we delete these linearly dependent N-rows from $\tilde{\mathbf{S}}_{\nu-1}$ and assign the corresponding columns in $\mathbf{B}_i$ as zero, then (9.50) becomes

$$[\mathbf{A}_0\ \ \bar{\mathbf{B}}_0\ \cdots\ \mathbf{A}_{\nu-1}\ \ \bar{\mathbf{B}}_{\nu-1}]\check{\mathbf{S}}_{\nu-1} = \tilde{\mathbf{F}}$$

where $\check{\mathbf{S}}_{\nu-1}$ is, as discussed earlier, square and nonsingular. Thus the solution is unique. This is illustrated in the next example.

Example 9.10 Consider a plant with the strictly proper transfer matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} 1/s^2 & 1/s \\ 0 & 1/s \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}^{-1} =: \mathbf{N}(s)\mathbf{D}^{-1}(s) \tag{9.51}$$

The fraction is right coprime and $\mathbf{D}(s)$ is column reduced with column degrees $\mu_1 = 2$ and $\mu_2 = 1$. We write

$$\mathbf{D}(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}s^2$$

and

$$\mathbf{N}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}s^2$$

We use MATLAB to compute the row index. The QR decomposition discussed in Section 7.3.1 can reveal linearly independent columns, from left to right, of a matrix. Here we need linearly independent rows, from top to bottom, of $\mathbf{S}_m$; therefore we will apply QR decomposition to the transpose of $\mathbf{S}_m$. We type

```
d1=[0 0 0 0 1 0];d2=[0 0 0 1 0 0];
n1=[1 1 0 0 0 0];n2=[0 1 0 0 0 0];
s1=[d1 0 0;d2 0 0;n1 0 0;n2 0 0;...
   0 0 d1;0 0 d2;0 0 n1;0 0 n2];
[q,r]=qr(s1')
```


which yields, as in Example 7.7,

$$r = \begin{bmatrix} d1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d2 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & n1 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix `q` is not needed and is not typed. In the matrix `r`, we use x, di, and ni to denote nonzero entries. The nonzero diagonal entries of `r` yield the linearly independent columns of $\mathbf{S}'_1$ or, equivalently, linearly independent rows of $\mathbf{S}_1$. We see that there are two linearly independent $N1$-rows and one linearly independent $N2$-row. The degree of $\hat{\mathbf{G}}(s)$ is 3 and we have found three linearly independent N-rows. Therefore there is no need to search further and we have $\nu_1 = 2$ and $\nu_2 = 1$. Thus the row index is $\nu = 2$. We select $m_1 = m_2 = m = \nu - 1 = 1$. Thus for any column-reduced $\mathbf{F}(s)$ of column degrees $m + \mu_1 = 3$ and $m + \mu_2 = 2$, we can find a proper compensator such that the resulting unity-feedback system has $\mathbf{F}(s)$ as its denominator matrix. Let us choose arbitrarily

$$\mathbf{F}(s) = \begin{bmatrix} (s^2 + 4s + 5)(s + 3) & 0 \\ 0 & s^2 + 2s + 5 \end{bmatrix} = \begin{bmatrix} 15 + 17s + 7s^2 + s^3 & 0 \\ 0 & 5 + 2s + s^2 \end{bmatrix} \tag{9.52}$$

and form (9.41) with $m = \nu - 1 = 1$:

$$[\mathbf{A}_0\ \mathbf{B}_0\ \mathbf{A}_1\ \mathbf{B}_1] \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} 15 & 0 & 17 & 0 & 7 & 0 & 1 & 0 \\ 0 & 5 & 0 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} = \bar{\mathbf{F}} \tag{9.53}$$

The α in (9.44) is 1 for this problem. Thus $\mathbf{S}_1$ and $\bar{\mathbf{F}}$ both have one zero column as we can see in (9.53). After deleting the zero column, the remaining $\tilde{\mathbf{S}}_1$ has full column rank and, for any $\bar{\mathbf{F}}$, solutions exist in (9.53). The matrix $\tilde{\mathbf{S}}_1$ has order 8×7 and solutions are not unique.

In searching the row index, we knew that the last row of $\bar{\mathbf{S}}_1$ is a linearly dependent row. If we delete the row, then we must assign the second column of $\mathbf{B}_1$ as $\mathbf{0}$ and the solution will be unique. We type

```
d1=[0 0 0 0 0 1];d2=[0 0 0 1 0 0];
n1=[1 1 0 0 0 0];n2=[0 1 0 0 0 0];
d1t=[0 0 0 0 1];d2t=[0 0 0 1 0];n1t=[0 1 0 0 0];
s1t=[d1 0;d2 0;n1 0;n2 0;0 0 d1t;0 0 d2t;0 0 n1t];
f1t=[15 0 17 0 7 1];
f1t/s1t
```

which yields [7 −17 15 −15 1 0 17]. Computing once again for the second row of $\bar{\mathbf{F}}$, we can finally obtain

$$[\mathbf{A}_0\ \mathbf{B}_0\ \mathbf{A}_1\ \mathbf{B}_1] = \begin{bmatrix} 7 & -17 & 15 & -15 & 1 & 0 & 17 & 0 \\ 0 & 2 & 0 & 5 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Note that MATLAB yields the first 7 columns; the last column $\mathbf{0}$ is assigned by us (due to deleting the last row of $\bar{\mathbf{S}}_1$). Thus we have

$$\mathbf{A}(s) = \begin{bmatrix} 7+s & -17 \\ 0 & 2+s \end{bmatrix} \qquad \mathbf{B}(s) = \begin{bmatrix} 15+17s & -15 \\ 0 & 5 \end{bmatrix}$$

and the proper compensator

$$\mathbf{C}(s) = \begin{bmatrix} s+7 & -17 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 17s+15 & -15 \\ 0 & 5 \end{bmatrix}$$

will yield the overall transfer matrix

$$\hat{\mathbf{G}}_o(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (s^2+4s+5)(s+3) & 0 \\ 0 & s^2+2s+5 \end{bmatrix}^{-1} \times \begin{bmatrix} 17s+15 & -15 \\ 0 & 5 \end{bmatrix} \tag{9.54}$$

This transfer matrix has the desired poles. This completes the design.

The design in Theorem 9.M2 is carried out by using a right coprime fraction of $\hat{\mathbf{G}}(s)$. We state next the result by using a left coprime fraction of $\hat{\mathbf{G}}(s)$.

Corollary 9.M2

Consider the unity-feedback system shown in Fig. 9.7. The plant is described by a $q \times p$ strictly proper rational matrix $\hat{\mathbf{G}}(s)$. Let $\hat{\mathbf{G}}(s)$ be factored as $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$, where $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ are left coprime and $\bar{\mathbf{D}}(s)$ is row reduced with row degrees v_i, $i = 1, 2, \ldots, q$. Let μ be the column index of $\hat{\mathbf{G}}(s)$ and let $m_i \geq \mu - 1$. Then for any $q \times q$ row-column reduced polynominal matrix $\bar{\mathbf{F}}(s)$ such that

$$\lim_{s\to\infty} \operatorname{diag}(s^{-v_1}, s^{-v_2}, \ldots, s^{-v_q})\bar{\mathbf{F}}(s)\operatorname{diag}(s^{-m_1}, s^{-m_2}, \ldots, s^{-m_q}) = \bar{\mathbf{F}}_h$$

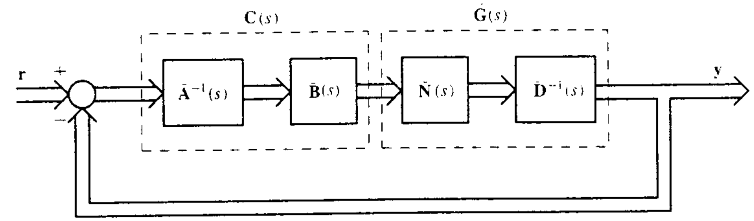


Figure 9.7 Unity feedback with $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$.

is a nonsingular constant matrix, there exists a $p \times q$ proper compensator $\mathbf{C}(s) = \bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)$, where $\bar{\mathbf{A}}(s)$ is column reduced with column degrees m_i, to meet

$$\bar{\mathbf{D}}(s)\bar{\mathbf{A}}(s) + \bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s) = \bar{\mathbf{F}}(s) \tag{9.55}$$

and the transfer matrix from $\mathbf{r}$ to $\mathbf{y}$ equals

$$\hat{\mathbf{G}}_o(s) = \mathbf{I} - \bar{\mathbf{A}}(s)\bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{D}}(s) \tag{9.56}$$

Substituting $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}\bar{\mathbf{N}}(s)$ and $\mathbf{C}(s) = \bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)$ into the first equation in (9.37) yields

$$\begin{aligned}\hat{\mathbf{G}}_o(s) &= [\mathbf{I} + \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)]^{-1}\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s) \\ &= \bar{\mathbf{A}}(s)[\bar{\mathbf{D}}(s)\bar{\mathbf{A}}(s) + \bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)]^{-1}\bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)\end{aligned}$$

which becomes, after substituting (9.55),

$$\begin{aligned}\hat{\mathbf{G}}_o(s) &= \bar{\mathbf{A}}(s)\bar{\mathbf{F}}^{-1}(s)[\bar{\mathbf{F}}(s) - \bar{\mathbf{D}}(s)\bar{\mathbf{A}}(s)]\bar{\mathbf{A}}^{-1}(s) \\ &= \mathbf{I} - \bar{\mathbf{A}}(s)\bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{D}}(s)\end{aligned}$$

This establishes the transfer matrix from $\mathbf{r}$ to $\mathbf{y}$ in the theorem. The design in Corollary 9.M2 hinges on solving (9.55). Note that the transpose of (9.55) becomes (9.39); left coprime and row reducedness become right coprime and column reducedness. Thus the linear algebraic equation in (9.41) can be used to solve the transpose of (9.55). We can also solve (9.55) directly by forming

$$\mathbf{T}_m \begin{bmatrix} \bar{\mathbf{B}}_0 \\ \bar{\mathbf{A}}_0 \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{A}}_1 \\ \vdots \\ \bar{\mathbf{B}}_m \\ \bar{\mathbf{A}}_m \end{bmatrix} = \left[\begin{array}{cc:cc:c:cc} \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 & \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \bar{\mathbf{D}}_n & \bar{\mathbf{N}}_n & \bar{\mathbf{D}}_{n-1} & \bar{\mathbf{N}}_{n-1} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{D}}_n & \bar{\mathbf{N}}_n & \cdots & \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \bar{\mathbf{D}}_n & \bar{\mathbf{N}}_n \end{array}\right]$$

$$\times \begin{bmatrix} \bar{\mathbf{B}}_0 \\ \bar{\mathbf{A}}_0 \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{A}}_1 \\ \vdots \\ \bar{\mathbf{B}}_m \\ \bar{\mathbf{A}}_m \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{F}}_0 \\ \bar{\mathbf{F}}_1 \\ \bar{\mathbf{F}}_2 \\ \vdots \\ \bar{\mathbf{F}}_{n+m} \end{bmatrix} \tag{9.57}$$

We search linearly independent columns of $\mathbf{T}_m$ in order from left to right. Let μ be the column index of $\hat{\mathbf{G}}(s)$ or the least integer such that $\mathbf{T}_{\mu-1}$ contains n linearly independent $\bar{N}$-columns. Then the compensator can be solved from (9.57) with $m = \mu - 1$.

9.4.1 Regulation and Tracking

As in the SISO case, pole placement can be used to achieve regulation and tracking in multivariable systems. In the regulator problem, we have $\mathbf{r} \equiv \mathbf{0}$ and if all poles of the overall system are assigned to have negative real parts, then the responses caused by any nonzero initial state will decay to zero. Furthermore, the rate of decaying can be controlled by the locations of the poles; the larger the negative real parts, the faster the decay.

Next we discuss tracking of any step reference input. In this design, we generally require a feedforward constant gain matrix $\mathbf{P}$. Suppose the compensator in Fig. 9.6 has been designed by using Theorem 9.M2. Then the $q \times q$ transfer matrix from $\mathbf{r}$ to $\mathbf{y}$ is given by

$$\hat{\mathbf{G}}_o(s) = \mathbf{N}(s)\mathbf{F}^{-1}(s)\mathbf{B}(s)\mathbf{P} \tag{9.58}$$

If $\hat{\mathbf{G}}_o(s)$ is BIBO stable, then the steady-state response excited by $\mathbf{r}(t) = \mathbf{d}$, for $t \ge 0$, or $\hat{\mathbf{r}}(s) = \mathbf{d}s^{-1}$, where $\mathbf{d}$ is an arbitrary $q \times 1$ constant vector, can be computed as, using the final-value theorem of the Laplace transform,

$$\lim_{t\to\infty} \mathbf{y}(t) = \lim_{s\to 0} s\hat{\mathbf{G}}_o(s)\mathbf{d}s^{-1} = \hat{\mathbf{G}}_o(0)\mathbf{d}$$

Thus we conclude that in order for $\mathbf{y}(t)$ to track asymptotically any step reference input, we need, in addition to BIBO stability,

$$\hat{\mathbf{G}}_o(0) = \mathbf{N}(0)\mathbf{F}^{-1}(0)\mathbf{B}(0)\mathbf{P} = \mathbf{I}_q \tag{9.59}$$

Before discussing the conditions for meeting (9.59), we need the concept of transmission zeros.

Transmission zeros Consider a $q \times p$ proper rational matrix $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$, where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime. A number λ, real or complex, is called a *transmission zero* of $\hat{\mathbf{G}}(s)$ if the rank of $\mathbf{N}(\lambda)$ is smaller than $\min(p, q)$.

Example 9.11 Consider the 3×2 proper rational matrix

$$\hat{\mathbf{G}}_1(s) = \begin{bmatrix} \frac{s}{s+2} & 0 \\ 0 & \frac{s+1}{s^2} \\ \frac{s+1}{s+2} & \frac{1}{s} \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s+1 \\ s+1 & s \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s^2 \end{bmatrix}^{-1}$$

This $\mathbf{N}(s)$ has rank 2 at every s; thus $\hat{\mathbf{G}}_1(s)$ has no transmission zero. Consider the 2×2 proper rational matrix

$$\hat{\mathbf{G}}_2(s) = \begin{bmatrix} \frac{s}{s+2} & 0 \\ 0 & \frac{s+2}{s} \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s \end{bmatrix}^{-1}$$

This $\mathbf{N}(s)$ has rank 1 at $s = 0$ and $s = -2$. Thus $\hat{\mathbf{G}}_2(s)$ has two transmission zeros at 0 and -2. Note that $\hat{\mathbf{G}}(s)$ has poles also at 0 and -2.

From this example, we see that a transmission zero cannot be defined directly from $\hat{\mathbf{G}}(s)$; it must be defined from its coprime fraction. Either a right coprime or left coprime fraction can be used and each yields the same set of transmission zeros. Note that if $\hat{\mathbf{G}}(s)$ is square and if $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$, where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime and $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ are left coprime, then the transmission zeros of $\hat{\mathbf{G}}(s)$ are the roots of det $\mathbf{N}(s)$ or the roots of det $\bar{\mathbf{N}}(s)$. Transmission zeros can also be defined from a minimal realization of $\hat{\mathbf{G}}(s)$. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be any n-dimensional minimal realization of a $q \times p$ proper rational matrix $\hat{\mathbf{G}}(s)$. Then the transmission zeros are those λ such that

$$\text{rank} \begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & \mathbf{D} \end{bmatrix} < n + \min(p, q)$$

This is used in the MATLAB function `tzero` to compute transmission zeros. For a more detailed discussion of transmission zeros, see Reference [6, pp. 623–635].

Now we are ready to discuss the conditions for achieving tracking or for meeting (9.59). Note that $\mathbf{N}(s)$, $\mathbf{F}(s)$, and $\mathbf{B}(s)$ are $q \times p$, $p \times p$, and $p \times q$. Because $\mathbf{I}_q$ has rank q, a necessary condition for (9.59) to hold is that the $q \times p$ matrix $\mathbf{N}(0)$ has rank q. Necessary conditions for $\rho(\mathbf{N}(0)) = q$ are $p \ge q$ and $s = 0$ is not a transmission zero of $\hat{\mathbf{G}}(s)$. Thus we conclude that in order for the unity-feedback configuration in Fig. 9.6 to achieve asymptotic tracking, the plant must have the following two properties:

- The plant has the same or a greater number of inputs than outputs.
- The plant transfer function has no transmission zero at $s = 0$.

Under these conditions, $\mathbf{N}(0)$ has rank q. Because we have freedom in selecting $\mathbf{F}(s)$, we can select it such that $\mathbf{B}(0)$ has rank q and the $q \times q$ constant matrix $\mathbf{N}(0)\mathbf{F}^{-1}(0)\mathbf{B}(0)$ is nonsingular. Under these conditions, the constant gain matrix $\mathbf{P}$ can be computed as

$$\mathbf{P} = [\mathbf{N}(0)\mathbf{F}^{-1}(0)\mathbf{B}(0)]^{-1} \tag{9.60}$$

Then we have $\hat{\mathbf{G}}_o(0) = \mathbf{I}_q$, and the unity-feedback system in Fig. 9.6 with $\mathbf{P}$ in (9.60) will track asymptotically any step reference input.

9.4.2 Robust Tracking and Disturbance Rejection

As in the SISO case, the asymptotic tracking design in the preceding section is not robust. In this section, we discuss a different design. To simplify the discussion, we study only plants with

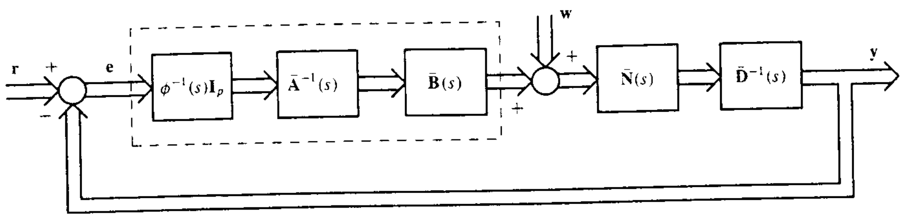


Figure 9.8 Robust tracking and disturbance rejection.

an equal number of input terminals and output terminals or $p = q$. Consider the unity-feedback system shown in Fig. 9.8. The plant is described by a $p \times p$ strictly proper transfer matrix factored in left coprime fraction as $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$. It is assumed that $p \times 1$ disturbance $\mathbf{w}(t)$ enters the plant input as shown. The problem is to design a compensator so that the output $\mathbf{y}(t)$ will track asymptotically a class of $p \times 1$ reference signals $\mathbf{r}(t)$ even with the presence of the disturbance $\mathbf{w}(t)$ and with plant parameter variations. This is called *robust tracking and disturbance rejection*.

As in the SISO case, we need some information on $\mathbf{r}(t)$ and $\mathbf{w}(t)$ before carrying out the design. We assume that the Laplace transforms of $\mathbf{r}(t)$ and $\mathbf{w}(t)$ are given by

$$\hat{\mathbf{r}}(s) = \mathcal{L}[\mathbf{r}(t)] = \mathbf{N}_r(s)D_r^{-1}(s) \qquad \hat{\mathbf{w}}(s) = \mathcal{L}[\mathbf{w}(t)] = \mathbf{N}_w(s)D_w^{-1}(s) \tag{9.61}$$

where $D_r(s)$ and $D_w(s)$ are known polynomials; however, $\mathbf{N}_r(s)$ and $\mathbf{N}_w(s)$ are unknown to us. Let $\phi(s)$ be the least common denominator of the unstable poles of $\hat{\mathbf{r}}(s)$ and $\hat{\mathbf{w}}(s)$. The stable poles are excluded because they have no effect on $\mathbf{y}(t)$ as $t \to \infty$. We introduce the internal model $\phi^{-1}(s)\mathbf{I}_p$ as shown in Fig. 9.8. If $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ are left coprime and if no root of $\phi(s)$ is a transmission zero of $\hat{\mathbf{G}}(s)$ or, equivalently, $\phi(s)$ and $\det \bar{\mathbf{N}}(s)$ are coprime, then it can be shown that $\bar{\mathbf{D}}(s)\phi(s)$ and $\bar{\mathbf{N}}(s)$ are left coprime. See Reference [6, p. 443]. Then Corollary 9.M2 implies that there exists a proper compensator $\mathbf{C}(s) = \bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)$ such that

$$\phi(s)\bar{\mathbf{D}}(s)\bar{\mathbf{A}}(s) + \bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s) = \bar{\mathbf{F}}(s) \tag{9.62}$$

for any $\bar{\mathbf{F}}(s)$ meeting the condition in Corollary 9.M2. Clearly $\bar{\mathbf{F}}(s)$ can be chosen to be diagaonl with the roots of its diagonal entries lying inside the sector shown in Fig. 8.3(a). The unity-feedback system in Fig. 9.8 so designed will track asymptotically and robustly the reference signal $\mathbf{r}(t)$ and reject the disturbance $\mathbf{w}(t)$. This is stated as a theorem.

Theorem 9.M3

Consider the unity-feedback system shown in Fig. 9.8 where the plant has a $p \times p$ strictly proper transfer matrix $\hat{\mathbf{G}}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$. It is assumed that $\bar{\mathbf{D}}(s)$ and $\bar{\mathbf{N}}(s)$ are left coprime and $\bar{\mathbf{D}}(s)$ is row reduced with row degrees v_i, $i = 1, 2, \ldots, p$. The reference signal $\mathbf{r}(t)$ and disturbance $\mathbf{w}(t)$ are modeled as $\hat{\mathbf{r}}(s) = \mathbf{N}_r(s)D_r^{-1}(s)$ and $\hat{\mathbf{w}}(s) = \mathbf{N}_w(s)D_w^{-1}(s)$. Let $\phi(s)$ be the least common denominator of the unstable poles of $\hat{\mathbf{r}}(s)$ and $\hat{\mathbf{w}}(s)$. If no root of $\phi(s)$ is a transmission zero of $\hat{\mathbf{G}}(s)$, then there exists a proper compensator $\mathbf{C}(s) = \bar{\mathbf{B}}(s)(\bar{\mathbf{A}}(s)\phi(s))^{-1}$ such that the overall system will robustly and asymptotically track the reference signal $\mathbf{r}(t)$ and reject the disturbance $\mathbf{w}(t)$.

Proof: First we show that the system will reject the disturbance at the output. Let us assume $\mathbf{r} = 0$ and compute the output $\hat{\mathbf{y}}_w(s)$ excited by $\hat{\mathbf{w}}(s)$. Clearly we have

$$\hat{\mathbf{y}}_w(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)[\hat{\mathbf{w}}(s) - \bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)\phi^{-1}(s)\hat{\mathbf{y}}_w(s)]$$

which implies

$$\begin{aligned}\hat{\mathbf{y}}_w(s) &= [\mathbf{I} + \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)\phi^{-1}(s)]^{-1}\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)\hat{\mathbf{w}}(s)\\ &= \left[\bar{\mathbf{D}}^{-1}(s)[\bar{\mathbf{D}}(s)\phi(s)\bar{\mathbf{A}}(s) + \bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)]\bar{\mathbf{A}}^{-1}(s)\phi^{-1}(s)\right]^{-1}\\ &\quad \times \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)\hat{\mathbf{w}}(s)\\ &= \phi(s)\bar{\mathbf{A}}(s)[\bar{\mathbf{D}}(s)\phi(s)\bar{\mathbf{A}}(s) + \bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)]^{-1}\bar{\mathbf{N}}(s)\hat{\mathbf{w}}(s)\end{aligned}$$

Thus we have, using (9.61) and (9.62),

$$\hat{\mathbf{y}}_w(s) = \bar{\mathbf{A}}(s)\bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{N}}(s)\phi(s)\mathbf{N}_w(s)D_w^{-1}(s) \tag{9.63}$$

Because all unstable roots of $D_w(s)$ are canceled by $\phi(s)$, all poles of $\hat{\mathbf{y}}_w(s)$ have negative real parts. Thus we have $\mathbf{y}_w(t) \to 0$ as $t \to \infty$ and the response excited by $\mathbf{w}(t)$ is suppressed asymptotically at the output.

Next we compute the error $\hat{\mathbf{e}}_r(s)$ excited by the reference signal $\hat{\mathbf{r}}(s)$:

$$\hat{\mathbf{e}}_r(s) = \hat{\mathbf{r}}(s) - \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)\phi^{-1}(s)\hat{\mathbf{e}}_r(s)$$

which implies

$$\begin{aligned}\hat{\mathbf{e}}_r(s) &= [\mathbf{I} + \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)\bar{\mathbf{A}}^{-1}(s)\phi^{-1}(s)]^{-1}\hat{\mathbf{r}}(s)\\ &= \phi(s)\bar{\mathbf{A}}(s)[\bar{\mathbf{D}}(s)\phi(s)\bar{\mathbf{A}}(s) + \bar{\mathbf{N}}(s)\bar{\mathbf{B}}(s)]^{-1}\bar{\mathbf{D}}(s)\hat{\mathbf{r}}(s)\\ &= \bar{\mathbf{A}}(s)\bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{D}}(s)\phi(s)\mathbf{N}_r(s)D_r^{-1}(s)\end{aligned} \tag{9.64}$$

Because all unstable roots of $D_r(s)$ are canceled by $\phi(s)$, the error vector $\hat{\mathbf{e}}_r(s)$ has only stable poles. Thus its time response approaches zero as $t \to \infty$. Consequently, the output $\mathbf{y}(t)$ will track asymptotically the reference signal $\mathbf{r}(t)$. The tracking and disturbance rejection are accomplished by inserting the internal model $\phi^{-1}(s)\mathbf{I}_p$. If there is no perturbation in the internal model, the tracking property holds for any plant and compensator parameter perturbations, even large ones, as long as the unity-feedback system remains BIBO stable. Thus the design is robust. This establishes the theorem. Q.E.D.

In the robust design, because of the internal model, $\phi(s)$ becomes zeros of every nonzero entry of the transfer matrices from $\mathbf{w}$ to $\mathbf{y}$ and from $\mathbf{r}$ to $\mathbf{e}$. Such zeros are called *blocking zeros*. These blocking zeros cancel all unstable poles of $\hat{\mathbf{w}}(s)$ and $\hat{\mathbf{r}}(s)$; thus the responses due to these unstable poles are completely blocked at the output. It is clear that every blocking zero is a transmission zero. The converse, however, is not true. To conclude this section, we mention that if we use a right coprime fraction for $\hat{\mathbf{G}}(s)$, insert an internal model and stabilize it, we can show only disturbance rejection. Because of the noncommutative property of matrices, we are not able to establish robust tracking. However, it is believed that the system still achieves robust tracking. The design discussed in Section 9.2.3 can also be extended to the multivariable case; the design, however, will be more complex and will not be discussed.

9.5 Multivariable Model Matching—Two-Parameter Configuration

In this section, we extend the SISO model matching to the multivariable case. We study only plants with square and nonsingular strictly proper transfer matrices. As in the SISO case, given a plant transfer matrix $\hat{\mathbf{G}}(s)$, a model $\hat{\mathbf{G}}_o(s)$ is said to be *implementable* if there exists a no-plant-leakage configuration and proper compensators so that the resulting system has the overall transfer matrix $\hat{\mathbf{G}}_o(s)$ and is totally stable and well posed. The next theorem extends Theorem 9.4 to the matrix case.

Theorem 9.M4

Consider a plant with $p \times p$ strictly proper transfer matrix $\hat{\mathbf{G}}(s)$. Then a $p \times p$ transfer matrix $\hat{\mathbf{G}}_o(s)$ is implementable if and only if $\hat{\mathbf{G}}_o(s)$ and

$$\hat{\mathbf{T}}(s) := \hat{\mathbf{G}}^{-1}(s)\hat{\mathbf{G}}_o(s) \tag{9.65}$$

are proper and BIBO stable.[4]

For any no-plant-leakage configuration, the closed-loop transfer matrix from **r** to **u** is $\hat{\mathbf{T}}(s)$. Thus well posedness and total stability require $\hat{\mathbf{G}}_o(s)$ and $\hat{\mathbf{T}}(s)$ to be proper and BIBO stable. This shows the necessity of Theorem 9.M4. Let us write (9.65) as

$$\hat{\mathbf{G}}_o(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{T}}(s) \tag{9.66}$$

Then (9.66) can be implemented in the open-loop configuration in Fig. 9.1(a) with $\mathbf{C}(s) = \hat{\mathbf{T}}(s)$. This design, however, is not acceptable either because it is not totally stable or because it is very sensitive to plant parameter variations. If we implement it in the unity-feedback configuration, we have no freedom in assigning canceled poles. Thus the configuration may not be acceptable. In the unity-feedback configuration, we have

$$\hat{\mathbf{u}}(s) = \mathbf{C}(s)[\hat{\mathbf{r}}(s) - \hat{\mathbf{y}}(s)]$$

Now we extend it to

$$\hat{\mathbf{u}}(s) = \mathbf{C}_1(s)\hat{\mathbf{r}}(s) - \mathbf{C}_2(s)\hat{\mathbf{y}}(s) \tag{9.67}$$

This is a two-degrees-of-freedom configuration. As in the SISO case, we may select $\mathbf{C}_1(s)$ and $\mathbf{C}_2(s)$ to have the same denominator matrix as

$$\mathbf{C}_1(s) = \mathbf{A}^{-1}(s)\mathbf{L}(s) \qquad \mathbf{C}_2(s) = \mathbf{A}^{-1}(s)\mathbf{M}(s) \tag{9.68}$$

Then the two-parameter configuration can be plotted as shown in Fig. 9.9. From the figure, we have

$$\hat{\mathbf{u}}(s) = \mathbf{A}^{-1}(s)[\mathbf{L}(s)\hat{\mathbf{r}}(s) - \mathbf{M}(s)\mathbf{N}(s)\mathbf{D}^{-1}(s)\hat{\mathbf{u}}(s)]$$

which implies

4. If $\hat{\mathbf{G}}(s)$ is not square, then $\hat{\mathbf{G}}_o(s)$ is implementable if and only if $\hat{\mathbf{G}}_o(s)$ is proper, is BIBO stable, and can be expressed as $\hat{\mathbf{G}}_o(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{T}}(s)$, where $\hat{\mathbf{T}}(s)$ is proper and BIBO stable. See Reference [6, pp. 517–523].

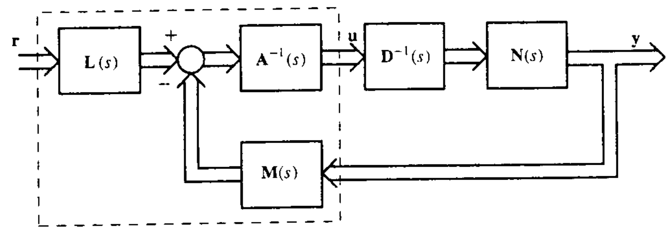


Figure 9.9 Two-parameter configuration.

$$\hat{\mathbf{u}}(s) = [\mathbf{I} + \mathbf{A}^{-1}(s)\mathbf{M}(s)\mathbf{N}(s)\mathbf{D}^{-1}(s)]^{-1}\mathbf{A}^{-1}(s)\mathbf{L}(s)\hat{\mathbf{r}}(s)$$
$$= \mathbf{D}(s)[\mathbf{A}(s)\mathbf{D}(s) + \mathbf{M}(s)\mathbf{N}(s)]^{-1}\mathbf{L}(s)\hat{\mathbf{r}}(s)$$

Thus we have

$$\hat{\mathbf{y}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\hat{\mathbf{u}}(s) = \mathbf{N}(s)[\mathbf{A}(s)\mathbf{D}(s) + \mathbf{M}(s)\mathbf{N}(s)]^{-1}\mathbf{L}(s)\hat{\mathbf{r}}(s)$$

and the transfer matrix from **r** to **y** is

$$\hat{\mathbf{G}}_o(s) = \mathbf{N}(s)[\mathbf{A}(s)\mathbf{D}(s) + \mathbf{M}(s)\mathbf{N}(s)]^{-1}\mathbf{L}(s) \tag{9.69}$$

Thus model matching becomes the problem of solving $\mathbf{A}(s)$, $\mathbf{M}(s)$ and $\mathbf{L}(s)$ in (9.69).

Problem *Given a $p \times p$ strictly proper rational matrix $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$, where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are right coprime and $\mathbf{D}(s)$ is column reduced with column degrees μ_i, $i = 1, 2, \ldots, p$, and given any implementable $\hat{\mathbf{G}}_o(s)$, find proper compensators $\mathbf{A}^{-1}(s)\mathbf{L}(s)$ and $\mathbf{A}^{-1}(s)\mathbf{M}(s)$ in Fig. 9.9 to implement $\hat{\mathbf{G}}_o(s)$.*

Procedure 9.M1

1. Compute

$$\mathbf{N}^{-1}(s)\hat{\mathbf{G}}_o(s) = \bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{E}}(s) \tag{9.70}$$

where $\bar{\mathbf{F}}(s)$ and $\bar{\mathbf{E}}(s)$ are left coprime and $\bar{\mathbf{F}}(s)$ is row reduced.

2. Compute the row index ν of $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$. This can be achieved by using QR decomposition.
3. Select

$$\hat{\mathbf{F}}(s) = \text{diag}(\alpha_1(s), \alpha_2(s), \ldots, \alpha_p(s)) \tag{9.71}$$

where $\alpha_i(s)$ are arbitrary Hurwitz polynomials, such that $\hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s)$ is row–column reduced with column degrees μ_i and row degrees m_i with

$$m_i \geq \nu - 1 \tag{9.72}$$

for $i = 1, 2, \ldots, p$.

4. Set

$$\mathbf{L}(s) = \hat{\mathbf{F}}(s)\bar{\mathbf{E}}(s) \tag{9.73}$$

and solve $\mathbf{A}(s)$ and $\mathbf{M}(s)$ from

$$\mathbf{A}(s)\mathbf{D}(s) + \mathbf{M}(s)\mathbf{N}(s) = \hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s) =: \mathbf{F}(s) \tag{9.74}$$

Then proper compensators $\mathbf{A}^{-1}(s)\mathbf{L}(s)$ and $\mathbf{A}^{-1}(s)\mathbf{M}(s)$ can be obtained to achieve the model matching.

This procedure reduces to Procedure 9.1 if $\hat{\mathbf{G}}(s)$ is scalar. We first justify the procedure. Substituting (9.73) and (9.74) into (9.69) yields

$$\hat{\mathbf{G}}_o(s) = \mathbf{N}(s)[\hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s)]^{-1}\hat{\mathbf{F}}(s)\bar{\mathbf{E}}(s) = \mathbf{N}(s)\bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{E}}(s)$$

This is (9.70). Thus the compensators implement $\hat{\mathbf{G}}_o(s)$. Define

$$\mathbf{H}_c(s) = \operatorname{diag}(s^{\mu_1}, s^{\mu_2}, \ldots, s^{\mu_p}) \qquad \mathbf{H}_r(s) = \operatorname{diag}(s^{m_1}, s^{m_2}, \ldots, s^{m_p})$$

By assumption, the matrix

$$\lim_{s\to\infty} \mathbf{H}_r^{-1}(s)\mathbf{F}(s)\mathbf{H}_c^{-1}(s) =: \mathbf{F}_h$$

is a nonsingular constant matrix. Thus solutions $\mathbf{A}(s)$, having row degrees m_i and being row reduced, and $\mathbf{M}(s)$, having row degrees m_i or less, exist in (9.74) (Theorem 9.M2). Thus $\mathbf{A}^{-1}(s)\mathbf{M}(s)$ is proper. To show that $\mathbf{A}^{-1}(s)\mathbf{L}(s)$ is proper, we consider

$$\begin{aligned}
\hat{\mathbf{T}}(s) &= \hat{\mathbf{G}}^{-1}(s)\hat{\mathbf{G}}_o(s) = \mathbf{D}(s)\mathbf{N}^{-1}(s)\hat{\mathbf{G}}_o(s)\\
&= \mathbf{D}(s)\bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{E}}(s) = \mathbf{D}(s)[\hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s)]^{-1}\hat{\mathbf{F}}(s)\bar{\mathbf{E}}(s)\\
&= \mathbf{D}(s)[\mathbf{A}(s)\mathbf{D}(s) + \mathbf{M}(s)\mathbf{N}(s)]^{-1}\mathbf{L}(s)\\
&= \mathbf{D}(s)\left\{\mathbf{A}(s)[\mathbf{I} + \mathbf{A}^{-1}(s)\mathbf{M}(s)\mathbf{N}(s)\mathbf{D}^{-1}(s)]\mathbf{D}(s)\right\}^{-1}\mathbf{L}(s)\\
&= [\mathbf{I} + \mathbf{A}^{-1}(s)\mathbf{M}(s)\hat{\mathbf{G}}(s)]^{-1}\mathbf{A}^{-1}(s)\mathbf{L}(s)
\end{aligned}$$

which implies, because $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$ is strictly proper and $\mathbf{A}^{-1}(s)\mathbf{M}(s)$ is proper,

$$\lim_{s\to\infty}\hat{\mathbf{T}}(s) = \lim_{s\to\infty}\mathbf{A}^{-1}(s)\mathbf{L}(s)$$

Because $\mathbf{T}(\infty)$ is finite by assumption, we conclude that $\mathbf{A}^{-1}(s)\mathbf{L}(s)$ is proper. If $\hat{\mathbf{G}}(s)$ is strictly proper and if all compensators are proper, then the two-parameter configuration is automatically well posed. The design involves the pole–zero cancellation of $\hat{\mathbf{F}}(s)$, which we can select. If $\hat{\mathbf{F}}(s)$ is selected as diagonal with Hurwitz polynomials as its entries, then pole–zero cancellations involve only stable poles and the system is totally stable. This completes the justification of the design procedure.

Example 9.12 Consider a plant with transfer matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} 1/s^2 & 1/s \\ 0 & 1/s \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}^{-1} \tag{9.75}$$

It has column degrees $\mu_1 = 2$ and $\mu_2 = 1$. Let us select a model as

$$\hat{\mathbf{G}}_o(s) = \begin{bmatrix} \dfrac{2}{s^2+2s+2} & 0 \\ 0 & \dfrac{2}{s^2+2s+2} \end{bmatrix} \tag{9.76}$$

which is proper and BIBO stable. To check whether or not $\hat{\mathbf{G}}_o(s)$ is implementable, we compute

$$\hat{\mathbf{T}}(s) := \hat{\mathbf{G}}^{-1}(s)\hat{\mathbf{G}}_o(s) = \begin{bmatrix} s^2 & -s^2 \\ 0 & s \end{bmatrix}\hat{\mathbf{G}}_o(s) = \begin{bmatrix} \dfrac{2s^2}{s^2+2s+2} & \dfrac{-2s^2}{s^2+2s+2} \\ 0 & \dfrac{2s}{s^2+2s+2} \end{bmatrix}$$

which is proper and BIBO stable. Thus $\hat{\mathbf{G}}_o(s)$ is implementable. We compute

$$\begin{aligned}
\mathbf{N}^{-1}(s)\hat{\mathbf{G}}_o(s) &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{2}{s^2+2s+2} & 0 \\ 0 & \dfrac{2}{s^2+2s+2} \end{bmatrix}\\
&= \begin{bmatrix} \dfrac{2}{s^2+2s+2} & \dfrac{-2}{s^2+2s+2} \\ 0 & \dfrac{2}{s^2+2s+2} \end{bmatrix}\\
&= \begin{bmatrix} s^2+2s+2 & 0 \\ 0 & s^2+2s+2 \end{bmatrix}^{-1}\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}\\
&=: \bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{E}}(s)
\end{aligned}$$

For this example, the degree of $\mathbf{N}^{-1}(s)\hat{\mathbf{G}}_o(s)$ can easily be computed as 4. The determinant of $\bar{\mathbf{F}}(s)$ has degree 4; thus the pair $\bar{\mathbf{F}}(s)$ and $\bar{\mathbf{E}}(s)$ are left coprime. It is clear that $\bar{\mathbf{F}}(s)$ is row reduced with row degrees $r_1 = r_2 = 2$. The row index of $\hat{\mathbf{G}}(s)$ was computed in Example 9.10 as $\nu = 2$. Let us select

$$\hat{\mathbf{F}}(s) = \operatorname{diag}((s+2), 1)$$

Then we have

$$\begin{aligned}
\hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s) &= \begin{bmatrix} (s^2+2s+2)(s+2) & 0 \\ 0 & s^2+2s+2 \end{bmatrix}\\
&= \begin{bmatrix} 4+6s+4s^2+s^3 & 0 \\ 0 & 2+2s+s^2 \end{bmatrix}
\end{aligned} \tag{9.77}$$

It is row–column reduced with row degrees $\{m_1 = m_2 = 1 = \nu - 1\}$ and column degrees $\{\mu_1 = 2, \mu_2 = 1\}$. Note that without introducing $\hat{\mathbf{F}}(s)$, proper compensators may not exist. We set

$$\begin{aligned}
\mathbf{L}(s) = \hat{\mathbf{F}}(s)\bar{\mathbf{E}}(s) &= \begin{bmatrix} s+2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}\\
&= \begin{bmatrix} 2(s+2) & -2(s+2) \\ 0 & 2 \end{bmatrix}
\end{aligned} \tag{9.78}$$

and solve $\mathbf{A}(s)$ and $\mathbf{M}(s)$ from

$$\mathbf{A}(s)\mathbf{D}(s) + \mathbf{M}(s)\mathbf{N}(s) = \hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s) =: \mathbf{F}(s) \tag{9.79}$$

From the coefficient matrices of $\mathbf{D}(s)$ and $\mathbf{N}(s)$ and the coefficient matrices of (9.77), we can readily form

$$[\mathbf{A}_0\ \mathbf{M}_0\ \mathbf{A}_1\ \mathbf{M}_1]\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 6 & 0 & 4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{9.80}$$

As discussed in Example 9.10, if we delete the last column of $\mathbf{S}_1$, then the remaining $\tilde{\mathbf{S}}_1$ has full column rank and for any $\mathbf{F}(s)$, after deleting the last zero column, solutions exist in (9.80). Now if we delete the last N-row of $\tilde{\mathbf{S}}_1$, which is linearly dependent on its previous row, the set of solutions is unique and can be obtained, using MATLAB, as

$$[\mathbf{A}_0\ \mathbf{M}_0\ \mathbf{A}_1\ \mathbf{M}_1] = \begin{bmatrix} 4 & -6 & 4 & -4 & 1 & 0 & 6 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{9.81}$$

Note that the last zero column is by assignment because of the deletion of the last N-row in $\tilde{\mathbf{S}}_1$. Thus we have

$$\mathbf{A}(s) = \begin{bmatrix} 4+s & -6 \\ 0 & 2+s \end{bmatrix} \qquad \mathbf{M}(s) = \begin{bmatrix} 4+6s & -4 \\ 0 & 2 \end{bmatrix} \tag{9.82}$$

The two compensators $\mathbf{A}^{-1}(s)\mathbf{M}(s)$ and $\mathbf{A}^{-1}(s)\mathbf{L}(s)$ are clearly proper. This completes the design. As a check, we compute

$$\hat{\mathbf{G}}_o(s) = \mathbf{N}(s)\mathbf{F}^{-1}(s)\mathbf{L}(s) = \begin{bmatrix} \dfrac{2(s+2)}{s^3+4s^2+6s+4} & 0 \\ 0 & \dfrac{2}{s^2+2s+2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{s^2+2s+2} & 0 \\ 0 & \dfrac{2}{s^2+2s+2} \end{bmatrix}$$

This is the desired model. Note that the design involves the cancellation of $(s+2)$, which we can select. Thus the design is satisfactory.

Let us discuss a special case of model matching. Given $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$, let us select $\hat{\mathbf{T}}(s) = \mathbf{D}(s)\mathbf{D}_f^{-1}(s)$, where $\mathbf{D}_f(s)$ has the same column degrees and the same column-degree coefficient matrix as $\mathbf{D}(s)$. Then $\hat{\mathbf{T}}(s)$ is proper and $\hat{\mathbf{G}}_o(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{T}}(s) = \mathbf{N}(s)\mathbf{D}_f^{-1}(s)$. This is the feedback transfer matrix discussed in (8.72). Thus the state feedback design discussed in Chapter 8 can also be carried out by using Procedure 9.M1.

9.5.1 Decoupling

Consider a $p \times p$ strictly proper rational matrix $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$. We have assumed that $\hat{\mathbf{G}}(s)$ is nonsingular. Thus $\hat{\mathbf{G}}^{-1}(s) = \mathbf{D}(s)\mathbf{N}^{-1}(s)$ is well defined; however, it is in general improper. Let us select $\hat{\mathbf{T}}(s)$ as

$$\hat{\mathbf{T}}(s) = \hat{\mathbf{G}}^{-1}(s)\text{diag}(d_1^{-1}(s), d_2^{-1}(s), \ldots, d_p^{-1}(s)) \tag{9.83}$$

where $d_i(s)$ are Hurwitz polynomials of least degrees to make $\hat{\mathbf{T}}(s)$ proper. Define

$$\boldsymbol{\Sigma}(s) = \text{diag}(d_1(s), d_2(s), \ldots, d_p(s)) \tag{9.84}$$

Then we can write $\hat{\mathbf{T}}(s)$ as

$$\hat{\mathbf{T}}(s) = \mathbf{D}(s)\mathbf{N}^{-1}(s)\boldsymbol{\Sigma}^{-1}(s) = \mathbf{D}(s)[\boldsymbol{\Sigma}(s)\mathbf{N}(s)]^{-1} \tag{9.85}$$

If all transmission zeros of $\hat{\mathbf{G}}(s)$ or, equivalently, all roots of det $\mathbf{N}(s)$ have negative real parts, then $\hat{\mathbf{T}}(s)$ is proper and BIBO stable. Thus the overall transfer matrix

$$\hat{\mathbf{G}}_o(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{T}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\mathbf{D}(s)[\boldsymbol{\Sigma}(s)\mathbf{N}(s)]^{-1}$$
$$= \mathbf{N}(s)[\boldsymbol{\Sigma}(s)\mathbf{N}(s)]^{-1} = \boldsymbol{\Sigma}^{-1}(s) \tag{9.86}$$

is implementable. This overall transfer matrix is a diagonal matrix and is said to be *decoupled*. This is the design in Example 9.12.

If $\hat{\mathbf{G}}(s)$ has nonminimum-phase transmission zeros or transmission zeros with zero or positive real parts, then the preceding design cannot be employed. However, with some modification, it is still possible to design a decoupled overall system. Consider again $\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$. We factor $\mathbf{N}(s)$ as

$$\mathbf{N}(s) = \mathbf{N}_1(s)\mathbf{N}_2(s)$$

with

$$\mathbf{N}_1(s) = \text{diag}(\beta_{11}(s), \beta_{12}(s), \ldots, \beta_{1p}(s))$$

where $\beta_{1i}(s)$ is the greatest common divisor of the ith row of $\mathbf{N}(s)$. Let us compute $\mathbf{N}_2^{-1}(s)$, and let $\beta_{2i}(s)$ be the least common denominator of the *unstable* poles of the ith column of $\mathbf{N}_2^{-1}(s)$. Define

$$\mathbf{N}_{2d} := \text{diag}(\beta_{21}(s), \beta_{22}(s), \ldots, \beta_{2p}(s))$$

Then the matrix

$$\bar{\mathbf{N}}_2(s) := \mathbf{N}_2^{-1}(s)\mathbf{N}_{2d}(s)$$

has no unstable poles. Now we select $\hat{\mathbf{T}}(s)$ as

$$\hat{\mathbf{T}}(s) = \mathbf{D}(s)\bar{\mathbf{N}}_2(s)\mathbf{\Sigma}^{-1}(s) \tag{9.87}$$

with

$$\mathbf{\Sigma}(s) = \text{diag}(d_1(s), d_2(s), \ldots, d_p(s))$$

where $d_i(s)$ are Hurwitz polynomials of least degrees to make $\hat{\mathbf{T}}(s)$ proper. Because $\bar{\mathbf{N}}_2(s)$ has only stable poles, and $d_i(s)$ are Hurwitz, $\hat{\mathbf{T}}(s)$ is BIBO stable. Consider

$$\begin{aligned}\hat{\mathbf{G}}_o(s) &= \hat{\mathbf{G}}(s)\hat{\mathbf{T}}(s) = \mathbf{N}_1(s)\mathbf{N}_2(s)\mathbf{D}^{-1}(s)\mathbf{D}(s)\bar{\mathbf{N}}_2(s)\mathbf{\Sigma}^{-1}(s)\\ &= \mathbf{N}_1(s)\mathbf{N}_{2d}(s)\mathbf{\Sigma}^{-1}(s)\\ &= \text{diag}\left(\frac{\beta_1(s)}{d_1(s)}, \frac{\beta_2(s)}{d_2(s)}, \ldots, \frac{\beta_p(s)}{d_p(s)}\right)\end{aligned} \tag{9.88}$$

where $\beta_i(s) = \beta_{1i}(s)\beta_{2i}(s)$. It is proper because both $\mathbf{T}(s)$ and $\hat{\mathbf{G}}(s)$ are proper. It is clearly BIBO stable. Thus $\hat{\mathbf{G}}_o(s)$ is implementable and is a decoupled system.

EXAMPLE 9.13 Consider

$$\hat{\mathbf{G}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) = \begin{bmatrix} s & 1 \\ s-1 & s-1 \end{bmatrix}\begin{bmatrix} s^3+1 & 1 \\ 0 & s^2 \end{bmatrix}^{-1}$$

We compute $\det \mathbf{N}(s) = (s-1)(s-1) = (s-1)^2$. The plant has two nonminimum-phase transmission zeros. We factor $\mathbf{N}(s)$ as

$$\mathbf{N}(s) = \mathbf{N}_1(s)\mathbf{N}_2(s) = \begin{bmatrix} 1 & 0 \\ 0 & s-1 \end{bmatrix}\begin{bmatrix} s & 1 \\ 1 & 1 \end{bmatrix}$$

with $\mathbf{N}_1(s) = \text{diag}(1, (s-1))$, and compute

$$\mathbf{N}_2^{-1}(s) = \frac{1}{(s-1)}\begin{bmatrix} 1 & -1 \\ -1 & s \end{bmatrix}$$

If we select

$$\mathbf{N}_{2d} = \text{diag}((s-1), (s-1)) \tag{9.89}$$

then the rational matrix

$$\bar{\mathbf{N}}_2(s) = \mathbf{N}_2^{-1}(s)\mathbf{N}_{2d}(s) = \begin{bmatrix} 1 & -1 \\ -1 & s \end{bmatrix}$$

has no unstable poles. We compute

$$\begin{aligned}\mathbf{D}(s)\bar{\mathbf{N}}_2(s) &= \begin{bmatrix} s^3+1 & 1 \\ 0 & s^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & s \end{bmatrix}\\ &= \begin{bmatrix} s^3 & -s^3+s-1 \\ -s^2 & s^3+1 \end{bmatrix}\end{aligned}$$

If we choose

$$\mathbf{\Sigma}(s) = \text{diag}((s^2+2s+2)(s+2), (s^2+2s+2)(s+2))$$

then

$$\hat{\mathbf{T}}(s) = \mathbf{D}(s)\bar{\mathbf{N}}_2(s)\mathbf{\Sigma}^{-1}(s)$$

is proper. Thus the overall transfer matrix

$$\hat{\mathbf{G}}_o(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{T}}(s) = \text{diag}\left(\frac{s-1}{(s^2+2s+2)(s+2)}, \frac{(s-1)^2}{(s^2+2s+2)(s+2)}\right)$$

is implementable. This is a decoupled system. This decoupled system will not track any step reference input. Thus we modify it as

$$\hat{\mathbf{G}}_o(s) = \text{diag}\left(\frac{-4(s-1)}{(s^2+2s+2)(s+2)}, \frac{4(s-1)^2}{(s^2+2s+2)(s+2)}\right) \tag{9.90}$$

which has $\hat{\mathbf{G}}_o(0) = \mathbf{I}$ and will track any step reference input.

Next we implement (9.90) in the two-parameter configuration. We follow Procedure 9.M1. To save space, we define $d(s) := (s^2+2s+2)(s+2)$. First we compute

$$\begin{aligned}\mathbf{N}^{-1}(s)\hat{\mathbf{G}}_o(s) &= \begin{bmatrix} s & 1 \\ s-1 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{-4(s-1)}{d(s)} & 0 \\ 0 & \dfrac{4(s+1)^2}{d(s)} \end{bmatrix}\\ &= \frac{1}{(s-1)^2}\begin{bmatrix} s-1 & -1 \\ 1-s & s \end{bmatrix}\begin{bmatrix} \dfrac{-4(s-1)}{d(s)} & 0 \\ 0 & \dfrac{4(s-1)^2}{d(s)} \end{bmatrix}\\ &= \begin{bmatrix} \dfrac{-4}{d(s)} & \dfrac{-4}{d(s)} \\ \dfrac{4}{d(s)} & \dfrac{4s}{d(s)} \end{bmatrix} = \begin{bmatrix} d(s) & 0 \\ 0 & d(s) \end{bmatrix}^{-1}\begin{bmatrix} -4 & -4 \\ 4 & 4s \end{bmatrix}\\ &=: \bar{\mathbf{F}}^{-1}(s)\bar{\mathbf{E}}(s)\end{aligned}$$

It is a left coprime fraction.

From the plant transfer matrix, we have

$$\mathbf{D}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}s + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}s^3$$

and

$$\mathbf{N}(s) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}s + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}s^3$$

We use QR decomposition to compute the row index of $\hat{\mathbf{G}}(s)$. We type

```
d1=[1 1 0 0 0 0 1 0];d2=[0 0 0 0 0 1 0 0];
n1=[0 1 1 0 0 0 0 0];n2=[-1 -1 1 1 0 0 0 0];
```

```
s2=[d1 0 0 0 0;d2 0 0 0 0;n1 0 0 0 0;n2 0 0 0 0;...
   0 0 d1 0 0;0 0 d2 0 0;0 0 n1 0 0;0 0 n2 0 0;...
   0 0 0 0 d1;0 0 0 0 d2;0 0 0 0 n1;0 0 0 0 n2];
[q,r]=qr(s2')
```

From the matrix r (not shown), we can see that there are three linearly independent $N1$-rows and two linearly independent $N2$-rows. Thus we have $\nu_1 = 3$ and $\nu_2 = 2$ and the row index equals $\nu = 3$. If we select

$$\hat{\mathbf{F}}(s) = \text{diag}\left((s+3)^2,\ (s+3)\right)$$

then $\hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s)$ is row–column reduced with row degrees $\{2, 2\}$ and column degrees $\{3, 2\}$. We set

$$\mathbf{L}(s) = \hat{\mathbf{F}}(s)\bar{\mathbf{E}}(s) = \begin{bmatrix} -4(s+3)^2 & -4(s+3)^2 \\ 4(s+3) & 4s(s+3) \end{bmatrix} \tag{9.91}$$

and solve $\mathbf{A}(s)$ and $\mathbf{M}(s)$ from

$$\mathbf{A}(s)\mathbf{D}(s) + \mathbf{M}(s)\mathbf{N}(s) = \hat{\mathbf{F}}(s)\bar{\mathbf{F}}(s) := \mathbf{F}(s)$$

Using MATLAB, they can be solved as

$$\mathbf{A}(s) = \begin{bmatrix} s^2 + 10s + 329 & 100 \\ -46 & s^2 + 7s + 6 \end{bmatrix} \tag{9.92}$$

and

$$\mathbf{M}(s) = \begin{bmatrix} -290s^2 - 114s - 36 & 189s + 293 \\ 46s^2 + 34s + 12 & -34s - 46 \end{bmatrix} \tag{9.93}$$

The compensators $\mathbf{A}^{-1}(s)\mathbf{M}(s)$ and $\mathbf{A}^{-1}(s)\mathbf{L}(s)$ are clearly proper. This completes the design.

The model matching discussed can be modified in several ways. For example, if stable roots of det $\mathbf{N}_2(s)$ are not inside the sector shown in Fig. 8.3(a), they can be included in β_{2i}. Then they will be retained in $\hat{\mathbf{G}}_o(s)$ and will not be canceled in the design. Instead of decoupling the plant for each pair of input and output, we may decouple it for a group of inputs and a group of outputs. In this case, the resulting overall transfer matrix is a block-diagonal matrix. These modifications are straightforward and will not be discussed.

9.6 Concluding Remarks

In this chapter, we used coprime fractions to carry out designs to achieve pole placement or model matching. For pole placement, the unity-feedback configuration shown in Fig. 9.1(a), a one-degree-of-freedom configuration, can be used. If a plant has degree n, then any pole placement can be achieved by using a compensator of degree $n-1$ or larger. If the degree of a compensator is larger than the minimum required, the extra degrees can be used to achieve robust tracking, disturbance rejection, or other design objectives.

Model matching generally involves pole–zero cancellations. One-degree-of-freedom configurations cannot be used here because we have no freedom in selecting canceled poles. Any two-degree-of-freedom configuration can be used because we have freedom in selecting canceled poles. This text discusses only the two-parameter configuration shown in Fig. 9.4.

All designs in this chapter are achieved by solving sets of linear algebraic equations. The basic idea and procedure are the same for both the SISO and MIMO cases. All discussion in this chapter can be applied, without any modification, to discrete-time systems; the only difference is that desired poles must be chosen to lie inside the region shown in Fig. 8.3(b) instead of in Fig. 8.3(a).

This chapter studies only strictly proper $\hat{\mathbf{G}}(s)$. If $\hat{\mathbf{G}}(s)$ is proper, the basic idea and procedure are still applicable, but the degree of compensators must be increased to ensure properness of compensators and well posedness of overall systems. See Reference [6]. The model matching in Section 9.5 can also be extended to nonsquare plant transfer matrices. See also Reference [6].

The controller-estimator design in Chapter 8 can be carried out using polynomial fractions. See References [6, pp. 506–514; 7, pp. 461–465]. Conversely, because of the equivalence of controllable and observable state equations and coprime fractions, we should be able to use state equations to carry out all designs in this chapter. The state-space design, however, will be more complex and less intuitively transparent, as we may conclude from comparing the designs in Sections 8.3.1 and 9.2.2.

The state-space approach first appeared in the 1960s, and by the 1980s the concepts of controllability and observability and controller-estimator design were integrated into most undergraduate control texts. The polynomial fraction approach was developed in the 1970s; its underlying concept of coprimeness, however, is an ancient one. Even though the concepts and design procedures of the coprime fraction approach are as simple as, if not simpler than, the state-space approach, the approach appears to be less widely known. It is hoped that this chapter has demonstrated its simplicity and usefulness and will help in its dissemination.

PROBLEMS

9.1 Consider

$$A(s)D(s) + B(s)N(s) = s^2 + 2s + 2$$

where $D(s)$ and $N(s)$ are given in Example 9.1. Do solutions $A(s)$ and $B(s)$ exist in the equation? Can you find solutions with deg $B(s) \le$ deg $A(s)$ in the equation?

9.2 Given a plant with transfer function $\hat{g}(s) = (s-1)/(s^2-4)$, find a compensator in the unity-feedback configuration so that the overall system has desired poles at -2 and $-1 \pm j1$. Also find a feedforward gain so that the resulting system will track any step reference input.

9.3 Suppose the plant transfer function in Problem 9.2 changes to $\bar{\hat{g}}(s) = (s-0.9)/(s^2-4.1)$ after the design is completed. Can the overall system still track asymptotically any step reference input? If not, give two different designs, one with a compensator of degree 3 and another with degree 2, that will track asymptotically and robustly any step reference input. Do you need additional desired poles? If yes, place them at -3.

9.4 Repeat Problem 9.2 for a plant with transfer function $\hat{g}(s) = (s-1)/s(s-2)$. Do you need a feedforward gain to achieve tracking of any step reference input? Give your reason.

9.5 Suppose the plant transfer function in Problem 9.4 changes to $\hat{\bar{g}}(s) = (s-0.9)/s(s-2.1)$ after the design is completed. Can the overall system still track any step reference input? Is the design robust?

9.6 Consider a plant with transfer function $\hat{g}(s) = 1/(s-1)$. Suppose a disturbance of form $w(t) = a\sin(2t+\theta)$, with unknown amplitude a and phase θ, enters the plant as shown in Fig. 9.2. Design a biproper compensator of degree 3 in the feedback system so that the output will track asymptotically any step reference input and reject the disturbance. Place the desired poles at $-1 \pm j2$ and $-2 \pm j1$.

9.7 Consider the unity feedback system shown in Fig. 9.10. The plant transfer function is $\hat{g}(s) = 2/s(s+1)$. Can the output track robustly any step reference input? Can the output reject any step disturbance $w(t) = a$? Why?

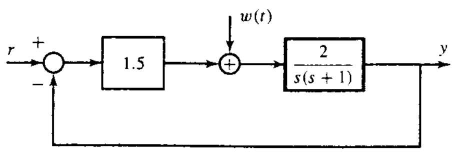


Figure 9.10

9.8 Consider the unity-feedback system shown in Fig. 9.11(a). Is the transfer function from r to y BIBO stable? Is the system totally stable? If not, find an input–output pair whose closed-loop transfer function is not BIBO stable.

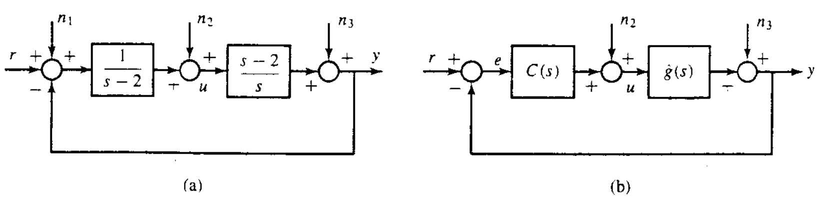


Figure 9.11

9.9 Consider the unity-feedback system shown in Fig. 9.11(b). (1) Show that the closed-loop transfer function of every possible input–output pair contains the factor $(1+C(s)\hat{g}(s))^{-1}$. (2) Show that $(1+C(s)\hat{g}(s))^{-1}$ is proper if and only if

$$1 + C(\infty)\hat{g}(\infty) \neq 0$$

(3) Show that if $C(s)$ and $\hat{g}(s)$ are proper, and if $C(\infty)\hat{g}(\infty) \neq -1$, then the system is well posed.

9.10 Given $\hat{g}(s) = (s^2-1)/(s^3+a_1s^2+a_2s+a_3)$, which of the following $\hat{g}_o(s)$ are implementable for any a_i and b_i.

$$\frac{s-1}{(s+1)^2} \qquad \frac{s+1}{(s+2)(s+3)} \qquad \frac{s^2-1}{(s-2)^3}$$

$$\frac{(s^2-1)}{(s+2)^2} \qquad \frac{(s-1)(b_0s+b_1)}{(s+2)^2(s^2+2s+2)} \qquad \frac{1}{1}$$

9.11 Given $\hat{g}(s) = (s-1)/s(s-2)$, implement the model $\hat{g}_o(s) = -2(s-1)/(s^2+2s+2)$ in the open-loop and the unity-feedback configurations. Are they totally stable? Can the implementations be used in practice?

9.12 Implement Problem 9.11 in the two-parameter configuration. Select the poles to be canceled at $s = -3$. Is $A(s)$ a Hurwitz polynomial? Can you implement the two compensators as shown in Fig. 9.4(a)? Implement the two compensators in Fig. 9.4(d) and draw its op-amp circuit.

9.13 Given a BIBO stable $\hat{g}_o(s)$, show that the steady-state response $y_{ss}(t) := \lim_{t\to\infty} y(t)$ excited by the ramp reference input $r(t) = at$ for $t \geq 0$ is given by

$$y_{ss}(t) = \hat{g}_o(0)at + \hat{g}'_o(0)a$$

Thus if the output is to track asymptotically the ramp reference input we require $\hat{g}_o(0) = 1$ and $\hat{g}'_o(0) = 0$.

9.14 Given a BIBO stable

$$\hat{g}_o(s) = \frac{b_0 + b_1s + \cdots + b_ms^m}{a_0 + a_1s + \cdots + a_ns^n}$$

with $n \geq m$, show that $\hat{g}_o(0) = 1$ and $\hat{g}'_o(0) = 0$ if and only if $a_0 = b_0$ and $a_1 = b_1$.

9.15 Given a plant with transfer function $\hat{g}(s) = (s+3)(s-2)/(s^3+2s-1)$. (1) find conditions on b_1, b_0, and a so that

$$\hat{g}_o(s) = \frac{b_1s + b_0}{s^2 + 2s + a}$$

is implementable; and (2) determine if

$$\hat{g}_o(s) = \frac{(s-2)(b_1s+b_0)}{(s+2)(s^2+2s+2)}$$

is implementable. Find conditions on b_1 and b_0 so that the overall transfer function will track any ramp reference input.

9.16 Consider a plant with transfer matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{s+1}{s(s-1)} \\[2mm] \dfrac{1}{s^2-1} \end{bmatrix}$$

Find a compensator in the unity-feedback configuration so that the poles of the overall system are located at $-2, -1 \pm j$ and the rest at $s = -3$. Can you find a feedforward gain so that the overall system will track asymptotically any step reference input?

9.17 Repeat Problem 9.16 for a plant with transfer matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s+1}{s(s-1)} & \frac{1}{s^2-1} \end{bmatrix}$$

9.18 Repeat Problem 9.16 for a plant with transfer matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s-2}{s^2-1} & \frac{1}{s-1} \\ \frac{1}{s} & \frac{2}{s-1} \end{bmatrix}$$

9.19 Given a plant with the transfer matrix in Problem 9.18, is

$$\hat{\mathbf{G}}_o(s) = \begin{bmatrix} \frac{4(s^2-4s+1)}{(s^2+2s+2)(s+2)} & 0 \\ 0 & \frac{4(s^2-4s+1)}{(s^2+2s+2)(s+2)} \end{bmatrix}$$

implementable? If yes, implement it.

9.20 Diagonalize a plant with transfer matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} 1 & s \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & s^2+1 \\ s & 0 \end{bmatrix}^{-1}$$

Set the denominator of each diagonal entry of the resulting overall system as s^2+2s+2.

References

We list only the references used in preparing this edition. The list does not indicate original sources of the results presented. For more extensive lists of references, see References [2, 6, 13, 15].

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Answers to Selected Problems

CHAPTER 2

2.1 (a) Linear. (b) and (c) Nonlinear. In (b), shift the operating point to $(0, y_o)$; then $(u, \bar{y})$ is linear, where $\bar{y} = y - y_o$.

2.3 Linear, time varying, causal.

2.5 No, yes, no for $\mathbf{x}(0) \neq \mathbf{0}$. Yes, yes, yes for $\mathbf{x}(0) = \mathbf{0}$. The reason is that the superposition property must also apply to the initial states.

2.9 $y(t) = 0$ for $t < 0$ and $t > 4$.

$$y(t) = \begin{cases} 0.5t^2 & \text{for } 0 \le t < 1 \\ -1.5t^2 + 4t - 2 & \text{for } 1 \le t \le 2 \\ -y(4-t) & \text{for } 2 < t \le 4 \end{cases}$$

2.10 $\hat{g}(s) = 1/(s+3)$, $g(t) = e^{-3t}$ for $t \ge 0$.

2.12

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}^{-1} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

2.15 (a) Define $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then $\dot{x}_1 = x_2$, $\dot{x}_2 = -(g/l)\sin x_1 - (u/ml)\cos x_1$. If θ is small, then

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ -1/mg \end{bmatrix}u$$

It is a linear state equation.

(b) Define $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$, and $x_4 = \dot{\theta}_2$. Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(g/l_1)\sin x_1 + (m_2 g/m_1 l)\cos x_3 \sin(x_3 - x_1) \\ &\quad + (1/m_1 l)\sin x_3 \sin(x_3 - x_1)\cdot u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -(g/l_2)\sin x_3 + (1/m_2 l_2)(\cos x_3)u \end{aligned}$$

This is a set of nonlinear equations. If $\theta_i \approx 0$ and $\theta_i\theta_j \approx 0$, then it can be linearized as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -g(m_1+m_2)/m_1l_1 & 0 & m_2g/m_1l_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -g/l_2 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2l_2 \end{bmatrix}u$$

2.16 From

$$m\ddot{h} = f_1 - f_2 = k_1\theta - k_2 u$$
$$I\ddot{\theta} + b\dot{\theta} = (l_1 + l_2)f_2 - l_1 f_1$$

we can readily obtain a state equation. Assuming $I = 0$ and taking their Laplace transforms can yield the transfer function.

2.18 $\hat{g}_1(s) = \hat{y}_1(s)/\hat{u}(s) = 1/(A_1R_1s + 1)$, $\hat{g}_2(s) = \hat{y}(s)/\hat{y}_1(s) = 1/(A_2R_2s + 1)$. Yes. $\hat{y}(s)/\hat{u}(s) = \hat{g}_1(s)\hat{g}_2(s)$.

2.20

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1/C_1 \\ 0 & 0 & 1/C_2 \\ -1/L_1 & -1/L_1 & -(R_1+R_2)/L_1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & -1/C_1 \\ 0 & 0 \\ 1/L_1 & R_1/L_1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = [-1 \ \ -1 \ \ -R_1]\mathbf{x} + [1 \ \ R_1]\mathbf{u}$$

$$\hat{g}_1(s) = \frac{\hat{y}(s)}{\hat{u}_1(s)} = \frac{s^2 + (R_2/L_1)s}{s^2 + \left(\dfrac{R_1+R_2}{L_1}\right)s + \left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)\dfrac{1}{L_1}}$$

$$\hat{g}_2(s) = \frac{\hat{y}(s)}{\hat{u}_2(s)} = \frac{(R_1s + (1/C_1))(s + (R_2/L_1))}{s^2 + \left(\dfrac{R_1+R_2}{L_1}\right)s + \left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)\dfrac{1}{L_1}}$$

$$\hat{y}(s) = \hat{g}_1(s)\hat{u}_1(s) + \hat{g}_2(s)\hat{u}_2(s)$$

CHAPTER 3

3.1 $[\frac{1}{3} \ \frac{8}{3}]'$, $[-2 \ \ 1.5]'$.

3.3

$$\mathbf{q}_1 = \frac{1}{3.74}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{q}_2 = \frac{1}{1.732}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.5 $\rho(\mathbf{A}_1) = 2$, nullity$(\mathbf{A}_1)$=1; 3, 0; 3, 1.

3.7 $\mathbf{x} = [1 \ \ 1]$ is a solution. Unique. No solution if $y = [1 \ 1 \ 1]'$.

3.9 $\alpha_1 = 4/11, \alpha_2 = 16/11$. The solution

$$\mathbf{x} = \begin{bmatrix} \dfrac{4}{11} & \dfrac{-8}{11} & \dfrac{-4}{11} & \dfrac{-16}{11} \end{bmatrix}'$$

has the smallest 2-norm.

3.12

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

3.13

$$\mathbf{Q}_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \hat{\mathbf{A}}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{Q}_4 = \begin{bmatrix} 5 & 4 & 0 \\ 0 & 20 & 1 \\ 0 & -25 & 0 \end{bmatrix} \qquad \hat{\mathbf{A}}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3.18

$$\Delta_1(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2) \qquad \psi_1(\lambda) = \Delta_1(\lambda)$$
$$\Delta_2(\lambda) = (\lambda - \lambda_1)^4 \qquad \psi_2(\lambda) = (\lambda - \lambda_1)^3$$
$$\Delta_3(\lambda) = (\lambda - \lambda_1)^4 \qquad \psi_3(\lambda) = (\lambda - \lambda_1)^2$$
$$\Delta_4(\lambda) = (\lambda - \lambda_1)^4 \qquad \psi_4(\lambda) = (\lambda - \lambda_1)$$

3.21

$$\mathbf{A}^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^{103} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

3.22

$$e^{\mathbf{A}_4t} = \begin{bmatrix} 1 & 4t + 2.5t^2 & 3t + 2t^2 \\ 0 & 1 + 20t & 16t \\ 0 & -25t & 1 - 20t \end{bmatrix}$$

3.24

$$\mathbf{B}=\begin{bmatrix}\ln\lambda_1 & 0 & 0\\ 0 & \ln\lambda_2 & 0\\ 0 & 0 & \ln\lambda_3\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}\ln\lambda & 1/\lambda & 0\\ 0 & \ln\lambda & 0\\ 0 & 0 & \ln\lambda\end{bmatrix}$$

3.32 Eigenvalues: 0, 0. No solution for $\mathbf{C}_1$. For any m_1, $[m_1 \;\; 3-m_1]'$ is a solution for $\mathbf{C}_2$.

3.34 $\sqrt{6}$, 1.4.7, 1.7.

CHAPTER 4

4.2 $y(t)=5e^{-t}\sin t \quad$ for $t\geq 0$.

4.3 For $T=\pi$,

$$\mathbf{x}[k+1]=\begin{bmatrix}-0.0432 & 0\\ 0 & -0.0432\end{bmatrix}\mathbf{x}[k]+\begin{bmatrix}1.5648\\ -1.0432\end{bmatrix}u[k]$$

$$y[k]=[2\;\;3]\mathbf{x}[k]$$

4.5 MATLAB yields $|y|_{max}=0.55$, $|x_1|_{max}=0.5$, $|x_2|_{max}=1.05$, and $|x_3|_{max}=0.52$ for unit step input. Define $\bar{x}_1=x_1$, $\bar{x}_2=0.5x_2$, and $\bar{x}_3=x_3$. Then

$$\dot{\bar{\mathbf{x}}}=\begin{bmatrix}-2 & 0 & 0\\ 0.5 & 0 & 0.5\\ 0 & -4 & -2\end{bmatrix}\bar{\mathbf{x}}+\begin{bmatrix}1\\0\\1\end{bmatrix}u \qquad y=[1\;\;-2\;\;0]\bar{\mathbf{x}}$$

The largest permissible a is $10/0.55=18.2$.

4.8 They are not equivalent but are zero-state equivalent.

4.11 Using (4.34), we have

$$\dot{\mathbf{x}}=\begin{bmatrix}-3 & 0 & -2 & 0\\ 0 & -3 & 0 & -2\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}u$$

$$\mathbf{y}=\begin{bmatrix}2 & 2 & 4 & -3\\ -3 & -2 & -6 & -2\end{bmatrix}\mathbf{x}+\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}u$$

4.13

$$\dot{\mathbf{x}}=\begin{bmatrix}-3 & 1 & 0 & 0\\ -2 & 0 & 0 & 0\\ 0 & 0 & -3 & 1\\ 0 & 0 & -2 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}2 & 2\\ 4 & -3\\ -3 & -2\\ -6 & -2\end{bmatrix}u$$

$$\mathbf{y}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}u$$

Both have dimension 4.

4.16

$$\mathbf{X}(t)=\begin{bmatrix}1 & \int_0^t e^{0.5\tau^2}d\tau\\ 0 & e^{0.5t^2}\end{bmatrix}$$

$$\mathbf{\Phi}(t,t_0)=\begin{bmatrix}1 & -e^{0.5t^2}\int_{t_0}^t e^{0.5\tau^2}d\tau\\ 0 & e^{0.5(t^2-t_0^2)}\end{bmatrix}$$

$$\mathbf{X}(t)=\begin{bmatrix}e^{-t} & e^{t}\\ 0 & 2e^{-t}\end{bmatrix}$$

$$\mathbf{\Phi}(t,t_0)=\begin{bmatrix}e^{-(t-t_0)} & 0.5(e^{t}e^{t_0}-e^{-t}e^{t_0}\\ 0 & e^{-(t-t_0)}\end{bmatrix}$$

4.20

$$\mathbf{\Phi}(t,t_0)=\begin{bmatrix}e^{\cos t-\cos t_0} & 0\\ 0 & e^{-\sin t+\sin t_0}\end{bmatrix}$$

4.23 Let $\bar{\mathbf{x}}(t)=\mathbf{P}(t)\mathbf{x}(t)$ with

$$\mathbf{P}(t)=\begin{bmatrix}e^{-\cos t} & 0\\ 0 & e^{\sin t}\end{bmatrix}$$

Then $\dot{\bar{\mathbf{x}}}(t)=\mathbf{0}\cdot\bar{\mathbf{x}}=\mathbf{0}$.

4.25

$$\dot{\mathbf{x}}=\mathbf{0}\cdot\mathbf{x}+\begin{bmatrix}t^2e^{-\lambda t}\\ -2te^{-\lambda t}\\ e^{-\lambda t}\end{bmatrix}u \quad y=[e^{\lambda t}\;\;te^{\lambda t}\;\;t^2e^{\lambda t}]\mathbf{x}$$

$$\dot{\mathbf{x}}=\begin{bmatrix}3\lambda & -3\lambda^2 & \lambda^3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\0\\0\end{bmatrix}u \quad y=[0\;\;0\;\;2]\mathbf{x}$$

CHAPTER 5

5.1 The bounded input $u(t)=\sin t$ excites $y(t)=0.5t\sin t$, which is not bounded. Thus the system is not BIBO stable.

5.3 No. Yes.

5.6 If $u(t)=3$, then $y(t)\to -6$. If $u(t)=\sin 2t$, then $y(t)\to 1.26\sin(2t+1.25)$.

5.8 Yes.

5.10 Not asymptotically stable. Its minimal polynomial can be found from its Jordan form as $\psi(\lambda) = \lambda(\lambda+1)$. $\lambda = 0$ is a simple eigenvalue, thus the equation is marginally stable.

5.13 Not asymptotically stable. Its minimal polynomial can be found from its Jordan form as $\psi(\lambda) = (\lambda-1)^2(\lambda+1)$. $\lambda = 1$ is a repeated eigenvalue, thus the equation is not marginally stable.

5.15 If $\mathbf{N} = \mathbf{I}$, then

$$\mathbf{M} = \begin{bmatrix} 2.2 & 1.6 \\ 1.6 & 4.8 \end{bmatrix}$$

It is positive definite; thus all eigenvalues have magnitudes less than 1.

5.17 Because $\mathbf{x}'\mathbf{M}\mathbf{x} = \mathbf{x}'[0.5(\mathbf{M}+\mathbf{M}')]\mathbf{x}$, the only way to check positive definiteness of nonsymmetric $\mathbf{M}$ is to check positive definiteness of symmetric $0.5(\mathbf{M}+\mathbf{M}')$.

5.20 Both are BIBO stable.

5.22 BIBO stable, marginally stable, not asymptotically stable. $P(t)$ is not a Lyapunov transformation.

CHAPTER 6

6.2 Controllable, observable.

6.5

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u \qquad y = [0 \ \ -1]\mathbf{x} + 2u$$

Not controllable, not observable.

6.7 $\mu_i = 1$ for all i and $\mu = 1$.

6.9 $y = 2u$.

6.14 Controllable, not observable.

6.17 Using x_1 and x_2 yields

$$\dot{\mathbf{x}} = \begin{bmatrix} -2/11 & 0 \\ 3/22 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -2/11 \\ 3/22 \end{bmatrix}u \qquad y = [-1 \ \ -1]\mathbf{x}$$

This two-dimensional equation is not controllable but is observable.
Using x_1, x_2, and x_3 yields

$$\dot{\mathbf{x}} = \begin{bmatrix} -2/11 & 0 & 0 \\ 3/22 & 0 & 0 \\ 1/22 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -2/11 \\ 3/22 \\ 1/22 \end{bmatrix}u \qquad y = [0 \ 0 \ 1]\mathbf{x}$$

This three-dimensional equation is not controllable and not observable.

6.19 For $T = \pi$, not controllable and not observable.

6.21 Not controllable at any t. Observable at every t.

CHAPTER 7

7.1

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u \qquad y = [0 \ 1 \ -1]\mathbf{x}$$

Not observable.

7.3

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 & 2 & a_1 \\ 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & a_4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u \qquad y = [0 \ 1 \ -1 \ c_4]\mathbf{x}$$

For any a_i and c_4, it is an uncontrollable and unobservable realization.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u \qquad y = [0 \ 1]\mathbf{x}$$

A controllable and observable realization.

7.5

Solve the monic null vector of

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} -N_0 \\ D_0 \\ -N_1 \\ D_1 \end{bmatrix} = \mathbf{0}$$

as $[-0.5 \ 0.5 \ 0 \ 1]'$. Thus $0.5/(0.5+s) = 1/(2s+1)$.

7.10

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad y = [1 \ 0]\mathbf{x}$$

7.13

$$\Delta_1(s) = s(s+1)(s+3) \qquad \deg = 3$$
$$\Delta_2(s) = (s+1)^3(s+2)^2 \qquad \deg = 5$$

7.17 Its right coprime fraction can be computed from any left fraction as

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} 2.5 & s+0.5 \\ 2.5 & s+2.5 \end{bmatrix}\begin{bmatrix} 0.5s & s^2+0.5s \\ s-0.5 & -0.5 \end{bmatrix}^{-1}$$

or, by interchanging their two columns,

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} s+0.5 & 2.5 \\ s+2.5 & 2.5 \end{bmatrix} \begin{bmatrix} s^2+0.5s & 0.5s \\ -0.5 & s-0.5 \end{bmatrix}^{-1}$$

Using the latter, we can obtain

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.5 & -0.25 & -0.25 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & -0.5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0.5 & 2.5 \\ 1 & 2.5 & 2.5 \end{bmatrix}\mathbf{x}$$

CHAPTER 8

8.1 $\mathbf{k} = [4 \;\; 1]$.

8.3 For

$$\mathbf{F} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \bar{\mathbf{k}} = [1 \;\; 1]$$

we have

$$\mathbf{T} = \begin{bmatrix} 0 & 1/13 \\ 1 & 9/13 \end{bmatrix} \qquad \mathbf{k} = \bar{\mathbf{k}}\mathbf{T}^{-1} = [4 \;\; 1]$$

8.5 Yes, yes, yes.

8.7 $u = pr - \mathbf{kx}$ with $\mathbf{k} = [15 \;\; 47 \;\; -8]$ and $p = 0.5$.

8.9 $u[k] = pr[k] - \mathbf{kx}[k]$ with $\mathbf{k} = [1 \;\; 5 \;\; 2]$ and $p = 0.5$. The overall transfer function is

$$\hat{g}_f(z) = \frac{0.5(2z^2 - 8z + 8)}{z^3}$$

and the output excited by $r[k] = a$ is $y[0] = 0$, $y[1] = a$, $y[2] = -3a$, and $y[k] = a$ for $k \geq 3$.

8.11 Two-dimensional estimator:

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0.6282 \\ -0.3105 \end{bmatrix}u + \begin{bmatrix} 1 \\ 0 \end{bmatrix}y$$

$$\hat{\mathbf{x}} = \begin{bmatrix} -12 & -27.5 \\ 19 & 32 \end{bmatrix}\mathbf{z}$$

One-dimesnional estimator:

$$\dot{z} = -3z + (13/21)u + y$$

$$\hat{\mathbf{x}} = \begin{bmatrix} -4 & 21 \\ 5 & -21 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

8.13 Select

$$\mathbf{F} = \begin{bmatrix} -4 & 3 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 0 & 0 & -5 & 4 \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

If $\bar{\mathbf{K}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then $\mathbf{K} = \begin{bmatrix} 62.5 & 147 & 20 & 515.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

If $\bar{\mathbf{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ then $\mathbf{K} = \begin{bmatrix} -606.2 & -168 & -14.2 & -2 \\ 371.1 & 119.2 & 14.9 & 2.2 \end{bmatrix}$

CHAPTER 9

9.2 $C(s) = (10s + 20)/(s - 6)$, $p = -0.2$.

9.4 $C(s) = (22s - 4)/(s - 16)$, $p = 1$. There is no need for a feed forward gain because $\hat{g}(s)$ contains $1/s$.

9.6 $C(s) = (7s^3 + 14s^2 + 34s + 25)/(s(s^2 + 4))$.

9.8 Yes, no. The transfer function from r to u is $\hat{g}_{ur}(s) = s/(s+1)(s-2)$, which is not BIBO stable.

9.10 Yes, no, no, no, yes, no (row-wise).

9.12 $C_1(s) = -2(s+3)/(s-21)$, $C_2(s) = (28s-6)/(s-21)$. $A(s) = s - 21$ is not Hurwitz. Its implementation in Fig. 9.4(a) will not be totally stable. A minimal realization of $[C_1(s) \;\; -C_2(s)]$ is

$$\dot{x} = 21x + [-48 \;\; -582]\begin{bmatrix} r \\ y \end{bmatrix} \qquad y = x + [-2 \;\; -28]\begin{bmatrix} r \\ y \end{bmatrix}$$

from which an op-amp circuit can be drawn.

9.15 (1) $a > 0$ and $b_0 = -2b_1$. (2) Yes, $b_0 = -2$, $b_1 = -4$.

9.16 The 1×2 compensator

$$\mathbf{C}(s) = \frac{1}{s+3.5}[3.5s + 12 \quad -2]$$

will place the closed-loop poles at -2, $-1 \pm j$, -3. No.

9.18 If we select

$$\mathbf{F}(s) = \text{diag}((s+2)(s^2+2s+2)(s+3),\ (s^2+2s+2))$$

then we cannot find a feedforward gain to achieve tracking. If

$$\mathbf{F}(s) = \begin{bmatrix} (s+2)(s^2+2s+2)(s+3) & 0 \\ 1 & s^2+2s+2 \end{bmatrix}$$

then the compensator

$$\mathbf{C}(s) = \mathbf{A}^{-1}(s)\mathbf{B}(s) = \begin{bmatrix} s-4.7 & -53.7 \\ -3.3 & s-4.3 \end{bmatrix}^{-1} \begin{bmatrix} -30.3s-29.7 & 4.2s-12 \\ -0.7s-0.3 & 4s-1 \end{bmatrix}$$

and the feedforward gain matrix

$$\mathbf{P} = \begin{bmatrix} 0.92 & 0 \\ -4.28 & 1 \end{bmatrix}$$

will achieve the design.

9.20 The diagonal transfer matrix

$$\hat{\mathbf{G}}_o(s) = \begin{bmatrix} \dfrac{-2(s-1)}{s^2+2s+2} & 0 \\ 0 & \dfrac{-2(s-1)}{s^2+2s+2} \end{bmatrix}$$

is implementable. The proper compensators $\mathbf{A}^{-1}(s)\mathbf{L}(s)$ and $\mathbf{A}^{-1}(s)\mathbf{M}(s)$ with

$$\mathbf{A}(s) = \begin{bmatrix} s+5 & -14 \\ 0 & s+4 \end{bmatrix} \qquad \mathbf{L}(s) = \begin{bmatrix} -2(s+3) & 2(s+3) \\ 2 & -2s \end{bmatrix}$$

$$\mathbf{M}(s) = \begin{bmatrix} -6 & 13s+1 \\ 2 & -2s \end{bmatrix}$$

in the two-parameter configuration will achieve the design.

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