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Quasi-static large deviations

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Abstract. We consider the symmetric simple exclusion with open boundaries that are in contact with particle reservoirs at different densities. The reservoir densities changes at a slower time scale with respect to the natural time scale the system reaches the stationary state.This gives rise to the quasi static hydrodynamic limit proven in (*Journal of Statistical Physics* **161** (5) (2015) 1037–1058). We study here the large deviations with respect to this limit for the particle density field and the total current. We identify explicitely the large deviation functional and prove that it satisfies a fluctuation relation.

Résumé. Nous considérons l'exclusion simple symétrique avec des frontières ouvertes en contact avec des réservoirs de particules à différentes densités. Les densités des réservoirs changent plus lentement par rapport à l'échelle de temps naturelle où le système atteint l'état stationnaire. Cela donne lieu à la limite hydrodynamique quasi-statique. Nous étudions ici les grandes déviations par rapport à cette limite pour le champ de densité de particules et le courant total. Nous identifions explicitement le fonctionnelle de grandes déviations et nous prouvons qu'il satisfait une relation de fluctuation.

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1. Introduction

In usual hydrodynamic limits it is studied the macroscopic evolution of the conserved quantities of a *large* microscopic system, under a space-time scaling such that the time scale is the typical one where the system reaches equilibrium. When the system is open and connected to thermal or particle reservoirs, the parameters of these reservoirs give the boundary conditions for the partial differential equations which describe the macroscopic evolution of the conserved quantities.

In this paper we consider the case in which the parameters of the boundary reservoirs (for example the particle density or the temperature) are not constant but change in time on a slower scale than that of relaxation at equilibrium. Rescaling the time at this slower scale give a *quasi static* macroscopic evolution for the conserved quantities: at each *macroscopic* time the profile of the conserved quantity is equal to the stationary one corresponding to the given boundary condition at that time. When these stationary profiles are of equilibrium, this limit models the thermodynamic quasi static transformations, where the Clausius equality holds, i.e. the *work* is done by the boundaries.

In [4,15] and [14], quasi static transformations are obtained from a time rescaling of the macroscopic diffusive equation.

In [9] we studied, for various stochastic particle systems whose macroscopic evolution is described by a diffusive equation, the direct hydrodynamic quasi static limit by rescaling properly space and time in the microscopic dynamics,

We study here the large deviation for one of these models, the symmetric simple exclusion process.

The system we consider is composed by $2N+1$ sites (denoted by $-N, \dots, N$) where particles move like symmetric random walks with exclusion. We add birth and death processes at the left and right boundaries that describe the interaction with reservoirs. These reservoirs have densities $\rho_-(t)$ on the left and $\rho_+(t)$ on the right, that are time dependent.

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The time scale at which these density $\rho_\pm$ change defines the *macroscopic time scale*. If the particles jump and are created and destroyed at rate N^2, the macroscopic density profile evolves following a linear diffusive equation with boundary conditions $\rho_\pm(t)$. This is the usual hydrodynamic limit and for $\rho_\pm$ constant this result is known since the '80's, [11].

In the quasi static limit particles jump (or are created and destroyed at the boundaries) with rate on order $N^{2+\alpha}$ with $\alpha > 0$. As proved in [9], at each time t, the empirical density converges, as $N \to \infty$, to the solution of the stationary heat equation with boundary conditions $\rho_\pm(t)$:

$$\bar\rho(t,y) = \frac{1}{2}\big[\rho_+(t) - \rho_-(t)\big]y + \frac{1}{2}\big[\rho_+(t) + \rho_-(t)\big], \quad y \in [-1,1], \tag{1.1}$$

and the total current through any bond will converge to

$$\bar J(t) = -\int_0^t \frac{1}{2}\big[\rho_+(s) - \rho_-(s)\big]\,ds. \tag{1.2}$$

Thus the quasi-static limit gives the evolution of macroscopic profile, completely driven by the boundary conditions.

In [9] it is also proven that the distribution of the process is close to a product Bernoulli measure with parameter $\bar\rho(t,y)$. Stronger results are proved by controlling the correlation functions with the same techniques used in [10]. First order corrections have been studied in [8].

Observe that if $\rho_+(t) = \rho_-(t)$ then the above result says that the stochastic process at any macroscopic time is close to equilibrium but the order parameter changes in time thus performing a quasi static transformation.

We study in this article *the joint* large deviations *of the density and the current* with respect to this quasi-static limit. We prove that the probability to observe a density profile $\rho(t,y)$ that satisfies the boundary conditions $\rho_\pm(t)$ and a total current $J(t)$ is asymptotically $\mathbb{P}(\rho, J) \sim e^{-N^{1+\alpha} I(J,\rho)}$ where

$$I(J,\rho) = \frac{1}{4}\int_0^T \int_{-1}^1 \frac{(J'(t) + \partial_y \rho(t,y))^2}{\rho(t,y)[1-\rho(t,y)]}\,dy\,dt. \tag{1.3}$$

See Section 4 for a precise statement. Notice that $J(t)$ is space constant: in the quasi static limit the total current must be homogeneous in space. Notice also that I does not depend on the initial configuration.

We also prove that I statisfies the fluctuation relation

$$I(J,\rho) - I(-J,\rho) = \int_0^T J'(t)\big(z_+(t) - z_-(t)\big)\,dt, \tag{1.4}$$

where $z_\pm(t) = \log \frac{\rho_\pm(t)}{1-\rho_\pm(t)}$. There are two interesting consequences of (1.4): the first one is that the difference $I(J,\rho) - I(-J,\rho)$ does not depend on ρ (as long as ρ satisfy the boundary conditions). The other is that, since $I(\bar J, \bar\rho) = 0$,

$$\begin{aligned} I(-\bar J, \bar\rho) &= \int_0^T \frac{1}{2}\big[\rho_+(t) - \rho_-(t)\big]\big(z_+(t) - z_-(t)\big)\,dt \\ &= \frac{1}{2}\int_0^T \big[\mathcal{H}\big(\rho_+(t), \rho_-(t)\big) + \mathcal{H}\big(\rho_-(t), \rho_+(t)\big)\big]\,dt, \end{aligned}$$

where $\mathcal{H}(\rho, \nu) = \rho \log \frac{\rho}{\nu} + (1-\rho)\log\frac{1-\rho}{1-\nu}$. Thus the cost to invert the Fick's law is explicetely computable in terms of the boundary conditions.

The formula (1.3) looks similar to the *fundamental formula* of the Macroscopic Fluctuation Theory [1,3], but differs in some important points. In the usual hydrodynamic scaling, i.e. for $\alpha = 0$, the probability to observe a density profile $\rho(t,y)$ (satisfying the boundary conditions $\rho(t,\pm 1) = \rho_\pm(t)$) and a current field $J(t,y)$, which must be related by the conservation law $\partial_t \rho(t,y) = -\partial_y \partial_t J(t,y)$, behaves like $e^{-N I_0(J,\rho)}$ with

$$I_0(J,\rho) = \frac{1}{4}\int_0^T \int_{-1}^1 \frac{(\partial_t J(t,y) + \partial_y \rho(t,y))^2}{\rho(t,y)[1-\rho(t,y)]}\,dy\,dt + \int_{-1}^1 \mathcal{H}\big(\rho(0,y), \nu_0(y)\big)\,dy. \tag{1.5}$$

The second term on the right hand side of (1.5) is due to the large deviations of the initial profile, if the initial distribution of the process is an inhomodeneous Bernoulli distribution with $\nu_0(y)$ as macroscopic profile of density.

Formula (1.5) appears first in [1], while a time homogeneous version was previously introduced in [6]. More precisely, in the case $\rho_\pm$ constant in time and $\alpha = 0$, the probability to observe a *total* and *time averaged* current $\frac{Q(T)}{T} = \frac{1}{T}\int_{-1}^{1} J(T,y)\,dy = 2q$ behaves like $e^{-NT I_{BD}(q)}$ with

$$I_{BD}(q,\rho_+,\rho_-) = \inf_{\rho(\cdot):\rho(\pm 1)=\rho_\pm} \frac{1}{4}\int_{-1}^{1} \frac{(q+\rho'(y))^2}{\rho(y)[1-\rho(y)]}\,dy \tag{1.6}$$

In [6] such large deviations behaviour is obtained under the assumption that the optimal density profile to obtain the *total* current q is independent of time. This is true in the symmetric simple exclusion case but not in all dynamics, and the existence of time dependent optimal profile is related to the existence of dynamical phase transitions (cf. [2,7]). Notice that if we look at the large deviation such that $\partial_t J(t,y) = 2q$ (for all t *and* y), then the minimizing density must satisfy the continuity equation and consequently it is constant in time. For these *space-homogeneous* deviations of the current field, I_{BD} gives the right rate function.

In the quasistatic case, $\alpha > 0$, the current field $J(t,y)$ has to be constant in y: we cannot have large deviations where the intensity of the current is not constant in space, see (2.10). Consequently there is no continuity equation to be satisfied between $J(t)$ and $\rho(t,y)$.

This gives a direct connection with the Bodineau–Derrida functional I_{BD}: minimising the rate function $I(J,\rho)$ defined in (1.3) over $\rho(t,y)$ in order to obtain the large deviation in the quasistatic limit of the current J we have (cf. (3.28) at the end of Section 3)

$$I(J) = \int_0^T I_{BD}\big(J'(t),\rho_+(t),\rho_-(t)\big)\,dt. \tag{1.7}$$

For simplicity we have restricted our attention to the symmetric simple exclusion, but in principle the result can be extended to other dynamics (like weakly asymmetric exclusion, zero range, KMP models etc.). Since in the symmetric simple exclusion $I_{BD}(J)$ is convex, we do not expect in the quasi static case the existence of *dynamical phase transitions* (cf. [1,2]). In the other dynamics this remain an interesting question to be investigated.

The main scheme of the proof goes along the lines of [13] and [5], but with the further feature of controlling also the deviations of the current. Using a variational characterization of I, for deviations such that $I(J,\rho) < +\infty$, it is possible to find suitable regular approximations J_ε and ρ_ε such that $I(J_\varepsilon,\rho_\varepsilon) \to I(J,\rho)$. For regular J, ρ, it is possible to construct the weakly asymmetric exclusion dynamics as described above such that the corresponding quasi-static limit is given by J and ρ and whose relative entropy with respect to the original process converges to $I(J,\rho)$. This takes care of the lower bound. For the upper bound we use suitable exponential martingales that control also the current, and a superexponential estimate adapted from the original idea in [13].

2. Simple exclusion with boundaries

We consider the exclusion process in $\{0,1\}^{\Lambda_N}$, $\Lambda_N := \{-N,\dots,N\}$ with reservoirs at the boundaries with density $\rho_\pm(t) \in [a,1-a]$ for some $a > 0$. We assume that $\rho_\pm(t) \in C^1$.

Denoting by $\eta(x) \in \{0,1\}$ the occupation number at $x \in \Lambda_N$ we define the dynamics via the generator

$$L_{N,t} = N^{2+\alpha}[L_{\text{exc}} + L_{b,t}], \quad t \geq 0, \alpha > 0 \tag{2.1}$$

where for a given $\gamma > 0$,

$$L_{\text{exc}} f(\eta) = \gamma \sum_{x=-N}^{N-1} \big(f\big(\eta^{(x,x+1)}\big) - f(\eta)\big) =: \sum_{x=-N}^{N-1} \nabla_{x,x+1} f(\eta) \tag{2.2}$$

$\eta^{(x,y)}$ is the configuration obtained from η by exchanging the occupation numbers at x and y, and

$$L_{b,t} f(\eta) = \sum_{J=\pm} \rho_J(t)^{1-\eta(JN)}\big(1-\rho_J(t)\big)^{\eta(JN)}\big[f\big(\eta^{JN}\big) - f(\eta)\big] \tag{2.3}$$

where $\eta^x(x) = 1-\eta(x)$, and $\eta^x(y) = \eta(y)$ for $x \neq y$.

We recall the quasi-static hydrodynamic limit proven in [9]:

Theorem 2.1. *For any $\alpha > 0$ and for any macroscopic time $t > 0$ the following holds. For any initial configuration η_0, for any $y \in [-1, 1]$ and for any local function φ*

$$\lim_{N\to\infty} \mathbb{E}_{\eta_0}\big(\theta_{[Ny]}\varphi(\eta_t)\big) = \langle\varphi(\eta)\rangle_{\bar\rho(y,t)} =: \hat\varphi\big(\bar\rho(y,t)\big) \tag{2.4}$$

where $[\cdot]$ denotes integer part, θ is the shift operator, $\langle\cdot\rangle_\rho$ is the expectation with respect to the product Bernoulli measure of density ρ, and

$$\bar\rho(y,t) = \frac{1}{2}\big[\rho_+(t) - \rho_-(t)\big]y + \frac{1}{2}\big[\rho_+(t) + \rho_-(t)\big], \quad y \in [-1,1] \tag{2.5}$$

is the quasi-static profile of density at time t. In particular for any $t > 0$ and any $y \in [-1, 1]$

$$\lim_{N\to\infty} \mathbb{E}_{\eta_0}\big[\eta_t\big([Ny]\big)\big] = \bar\rho(y,t) \tag{2.6}$$

In the following we will use the notation

$$\phi(\rho) = \rho(1-\rho). \tag{2.7}$$

Define the following counting processes: for $x = -N-1, \dots, N$

$$\begin{aligned} \mathsf{h}_+(t,x) &= \{\text{number of jumps } x \to x+1 \text{ up to time } t\}, \\ \mathsf{h}_-(t,x) &= \{\text{number of jumps } x+1 \to x \text{ up to time } t\}, \\ \mathsf{h}(t,x) &= \mathsf{h}_+(t,x) - \mathsf{h}_-(t,x) \end{aligned} \tag{2.8}$$

When $x = -N-1$ the corresponding $\mathsf{h}_+(t, -N-1)$ is the number of particles that enters on the left boundary, and $\mathsf{h}_+(t, N)$ is the number of particles that exit at the right boundary.

The conservation law is microscopically given by the relation

$$\eta_t(x) - \eta_0(x) = \mathsf{h}(t, x-1) - \mathsf{h}(t,x), \quad x = -N, \dots, N \tag{2.9}$$

Furthermore we have that for $x = -N, \dots, N-1$:

$$\mathsf{h}(t,x) = \gamma N^{2+\alpha} \int_0^t \big(\eta_s(x) - \eta_s(x+1)\big)\,ds + M(t),$$

where $M(t)$ is a martingale.

For $y \in [-1, 1]$, define

$$\mathsf{h}_N(t,y) = \frac{1}{N^{1+\alpha}} \mathsf{h}\big(t, [Ny]\big).$$

Notice that, for any $x, x' \in \{-N, \dots, N\}$, by (2.9) we must have $|\mathsf{h}(t,x) - \mathsf{h}(t,x')| \le 2N$, that implies

$$\big|\mathsf{h}_N(t,y) - \mathsf{h}_N\big(t,y'\big)\big| \le \frac{2}{N^\alpha}, \quad \forall y, y' \in [-1,1]. \tag{2.10}$$

It follows that

$$\begin{aligned} \frac{1}{N^{1+\alpha}} \mathbb{E}_{\eta_0}\big(\mathsf{h}(t,x)\big) &= \frac{1}{N^{1+\alpha}} \frac{1}{2N} \mathbb{E}_{\eta_0}\left(\sum_{x'=-N}^{N-1} \mathsf{h}\big(t,x'\big)\right) + O\big(N^{-\alpha}\big) \\ &= \frac{1}{N^{1+\alpha}} \frac{1}{2N} \mathbb{E}_{\eta_0}\left(\gamma N^{2+\alpha} \int_0^t \sum_{x'=-N}^{N-1} \big(\eta_s\big(x'\big) - \eta_s\big(x'+1\big)\big)\right) + O\big(N^{-\alpha}\big) \\ &= \frac{\gamma}{2} \mathbb{E}_{\eta_0}\left(\int_0^t \big(\eta_s(-N) - \eta_s(N)\big)\,ds\right) + O\big(N^{-\alpha}\big) \end{aligned}$$

Thus from (2.6)

$$\lim_{N\to\infty} \mathbb{E}_{\eta_0}\big(\mathsf{h}_N(t,y)\big) = \bar{h}(t) := -\frac{\gamma}{2}\int_0^t \big[\rho_+(s)-\rho_-(s)\big]\,ds. \tag{2.11}$$

Observe that the total current depends on γ while the quasi-static profile does not. In the sequel we will set $\gamma = 1$.

3. The rate function

We denote by $\mathcal{M} = \{\rho(t,y)\text{measurable}, t\in[0,T], y\in[-1,1], 0\le\rho(t,y)\le 1\}$. We endow $\mathcal{M}$ of the weak topology, i.e. for any continuous function $G\in\mathcal{C}([0,T]\times[-1,1])$, $\rho\mapsto\int_0^T dt\int_{-1}^1 dy\rho(t,y)G(t,y)$ is continuous on $\mathcal{M}$. Note that $\mathcal{M}$ is compact under this topology.

Let $\rho(t,y)\in\mathcal{M}$ and $J(t)\in\mathcal{D}([0,T],\mathbb{R})$, with $J(0)=0$. The rate function is defined by:

$$I(J,\rho) = \sup_{\substack{H\in\mathcal{C}^{1,2}([0,T]\times[-1,1]),\\ H(\cdot,-1)=0}} \left\{\mathcal{L}(H;J,\rho) - \int_0^T dt\int_{-1}^1 dy\big(\partial_y H(t,y)\big)^2\phi\big(\rho(t,y)\big)\right\} \tag{3.1}$$

where

$$\begin{aligned}\mathcal{L}(H;J,\rho) := H(1,T)J(T) + \int_0^T dt\bigg(&-\partial_t H(t,1)J(t) - \int_{-1}^1 \partial_{yy}H(t,y)\rho(t,y)\,dy\\ &+\big(\partial_y H(t,1)\rho_+(t) - \partial_y H(t,-1)\rho_-(t)\big)\bigg)\end{aligned} \tag{3.2}$$

Observe that if $\partial_y\rho$ and J' exist and are regular enough and $\rho(t,\pm1)=\rho_\pm(t)$, then $I(J,\rho)$ is given by

$$I(J,\rho) = \frac{1}{4}\int_0^T\int_{-1}^1 \frac{(J'(t)+\partial_y\rho(t,y))^2}{\phi(\rho(t,y))}\,dy\,dt, \tag{3.3}$$

and the maximum is reached on the function

$$\bar{H}(t,y) = \frac{1}{2}\int_{-1}^y \frac{J'(t)+\partial_y\rho(t,y')}{\phi(\rho(t,y'))}\,dy'. \tag{3.4}$$

Define

$$\mathcal{H}_+ = \text{clos}\bigg\{H(t,y)\in\mathcal{C}^{1,2}\big([0,T]\times[-1,1]\big), H(\cdot,-1)=0:\ \|H\|^2_{\rho,+} = \int_0^T dt\int_{-1}^1 dy\big(\partial_y H(t,y)\big)^2\phi\big(\rho(t,y)\big) < +\infty\bigg\}.$$

and the dual space

$$\mathcal{H}_- = \text{clos}\bigg\{H(t,y)\in\mathcal{C}^{1,2}\big([0,T]\times[-1,1]\big):\ \|H\|^2_{\rho,-} = \int_0^T dt\int_{-1}^1 dy\big(\partial_y H(t,y)\big)^2\phi\big(\rho(t,y)\big)^{-1} < +\infty\bigg\}.$$

Proposition 3.1. *If $I(J,\rho)<\infty$ then the weak derivatives $J'(t)$ and $\partial_y\rho(t,y)$ exists in $\mathcal{H}_-$ and $\rho(t,\pm1)=\rho_\pm(t)$, furthermore*

$$I(J,\rho) = \frac{1}{4}\int_0^T\int_{-1}^1 \frac{(J'(t)+\partial_y\rho(t,y))^2}{\phi(\rho(t,y))}\,dy\,dt. \tag{3.5}$$

Proof. Choose $H(t, y) = z(t)(1 + y)$ for a given smooth function $z(t)$ on $[0, T]$ in the variational formula (3.1). Then defining

$$\begin{aligned}\mathcal{Q}(J,\rho) &= \sup_{z\in\mathcal{C}^1([0,T])}\Big\{2z(T)J(T) - \int_0^T \big(2z'(t)J(t)\,dt + z(t)\big(\rho_+(t)-\rho_-(t)\big)\big) \\ &\quad - \int_0^T z(t)^2\int_{-1}^1 \phi\big(\rho(t,y)\big)\,dy\Big\} \\ &:= \sup_z\Big\{\tilde{L}(J,z) - \int_0^T dt z(t)^2 \int_{-1}^1 \phi\big(\rho(t,y)\big)\,dy\Big\}\end{aligned} \tag{3.6}$$

we have that $\mathcal{Q}(J,\rho) \le I(J,\rho) < +\infty$. It follows that

$$\big|\tilde{L}(J,z)\big|^2 \le 4\mathcal{Q}(J,\rho)\int_0^T dt z(t)^2\int_{-1}^1 \phi\big(\rho(t,y)\big)\,dy.$$

This means that $z \to \tilde{L}(J,z)$ is a bounded linear functional on the Hilbert space $\mathcal{N}_+$, where

$$\mathcal{N}_\pm = \operatorname{clos}\Big\{z \in \mathcal{C}^1\big([0,T]\big) : \|z\|^2 = \int_0^T z(t)^2\bar{\phi}(t)^{\pm 1}\,dt\Big\},$$

where $\bar{\phi}(t) = \int_{-1}^1 \phi(\rho(t,y))\,dy$. Applying Riesz representation theorem, there exists a function $g(t) \in \mathcal{N}_+$ such that

$$\tilde{L}(J,z) = \int_0^T g(t)z(t)\bar{\phi}(t)\,dt.$$

This implies the existence of the weak derivative $J'(t)$ such that $2J'(t) + (\rho_+(t) - \rho_-(t)) \in \mathcal{N}_-$. Furthermore $g(t)\bar{\phi}(t) = 2J'(t) + (\rho_+(t) - \rho_-(t))$ and

$$\mathcal{Q}(J,\rho) = \frac{1}{4}\int_0^T \big[2J'(t) + \big(\rho_+(t)-\rho_-(t)\big)\big]^2\bar{\phi}(t)^{-1}\,dt. \tag{3.7}$$

The linear functional $\mathcal{L}(H; J, \rho)$ defined in (3.2) is bounded by

$$\big|\mathcal{L}(H;J,\rho)\big|^2 \le 4I(J,\rho)\int_0^T dt\int_{-1}^1 dy\big(\partial_y H(t,y)\big)^2\phi\big(\rho(t,y)\big), \tag{3.8}$$

and it can be extended to a bounded linear functional on $\mathcal{H}_+$. By Riesz representation there exists a function $G \in \mathcal{H}_+$ such that

$$\mathcal{L}(H;J,\rho) = \int_0^T dt\int_{-1}^1 dy\partial_y H(t,y)\partial_y G(t,y)\phi\big(\rho(t,y)\big). \tag{3.9}$$

Since we have already proven that $I(J,\rho) < +\infty$ implies the existence of the weak derivative $J'(t) \in \mathcal{N}_-$, we can rewrite the variational formula (3.1) as

$$I(J,\rho) = +\infty \quad \text{if } J(t) \text{ is not differentiable.}$$

$$\begin{aligned}I(J,\rho) &= \sup_{\substack{H\in\mathcal{C}^{1,2}([0,T]\times[-1,1]),\\ H(\cdot,-1)=0}}\Big\{\int_0^T dt H(t,1)J'(t) \\ &\quad + \int_0^T dt\big(\partial_y H(t,1)\rho_+(t) - \partial_y H(t,-1)\rho_-(t)\big) \\ &\quad - \int_0^T dt\int_{-1}^1 dy\big[\partial_{yy}H(t,y)\rho(t,y) + \big(\partial_y H(t,y)\big)^2\phi\big(\rho(t,y)\big)\big]\Big\} \\ &= \sup_H\Big\{\mathcal{L}(H;J,\rho) - \int_0^T dt\int_{-1}^1 dy\big(\partial_y H(t,y)\big)^2\phi\big(\rho(t,y)\big)\Big\}.\end{aligned} \tag{3.10}$$

Notice that we can rewrite

$$\mathcal{L}(H; J, \rho) = -\int_0^T dt\left[\int_{-1}^1 \partial_{yy} H(t, y)\big(J'(t)y + \rho(t, y)\big)\, dy \right.$$
$$\left. + \partial_y H(t, 1)\big(\rho_+(t) + J'(t)\big) - \partial_y H(t, -1)\big(\rho_-(t) - J'(t)\big)\right]$$

Let us first show that $I(J, \rho) < \infty$ implies that $\rho(t, \pm 1) = \rho_\pm(t)$, in the sense that, for any continuous function $g(t)$ on $[0, T]$,

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_0^T dt \int_{-1}^{-1+\delta} dy \rho(t, y) g(t) = \int_0^T \rho_-(t) g(t)\, dt,$$
$$\lim_{\delta\to 0} \frac{1}{\delta} \int_0^T dt \int_{1-\delta}^{1} dy \rho(t, y) g(t) = \int_0^T \rho_+(t) g(t)\, dt.$$

In fact assume that these boundary conditions are not satisfied, and choose the functions $H(s, y)$ such that

$$\partial_y H(t, y) = A\left(1 - \frac{1-y}{\delta}\right) 1_{[1-\delta,1]}(y) g(t)$$

This function is not smooth, but it can smoothened up by some convolution without changing the argument. On this function we have

$$\mathcal{L}(H; J, \rho) = -\frac{A}{\delta} \int_0^T dt g(t) \int_{1-\delta}^1 dy\big(J'(t)y + \rho(t, y)\big) + A\int_0^T g(t)\big(\rho_+(t) + J'(t)\big)\, dt$$
$$= \frac{1}{2} A\delta \int_0^T g(t) J'(t)\, dt - A\int_0^T dt g(t)\left(\delta^{-1} \int_{1-\delta}^1 dy \rho(t, y) - \rho_+(t)\right)$$

while we have

$$\int_0^T dt \int_{-1}^1 dy\big(\partial_y H(t, y)\big)^2 \phi\big(\rho(t, y)\big)$$
$$= A^2 \int_0^T dt g(t)^2 \int_{1-\delta}^1 dy\left(1 - \frac{1-y}{\delta}\right)^2 \phi\big(\rho(t, y)\big)$$
$$\le C A^2 T\delta. \tag{3.11}$$

Then if the boundary condition in $y = 1$ is not satisfied, one can construct a sequence of functions H such that $I(J, \rho) = +\infty$. The other side is treated in a similar way.

Thus $I(J, \rho) < \infty$ implies that the boundary conditions must be satisfied and that there exist the weak derivative $\partial_x \rho(t, x)$ and from (3.9) we can identify $\partial_x G(t, x)\phi(\rho(t, x)) = J'(t) + \partial_x \rho(t, x)$.

Notice that, from this identification we can rewrite I as

$$I(J, \rho) = \frac{1}{4} \int_0^T \int_{-1}^1 \frac{J'(t)^2 + \partial_y \rho(t, y)^2}{\phi(\rho(t, y))}\, dy\, dt$$
$$+ \frac{1}{2} \int_0^T J'(t) \log\left(\frac{\rho_+(t)(1 - \rho_-(t))}{\rho_-(t)(1 - \rho_+(t))}\right) dt \tag{3.12}$$

that proves that J and ρ are in $\mathcal{H}_-$, since $\rho_\pm(t)$ are assumed bounded away from 0 and 1. □

Proposition 3.2. *If $I(J, \rho) < \infty$, there exists an approximation by bounded functions $J_\varepsilon(t)$ and x-smooth $\rho_\varepsilon(t, x)$ of J and ρ, such that $\rho_\varepsilon(t, \pm 1) = \rho_\pm(t)$ and*

$$\lim_{\varepsilon\to 0} I(J_\varepsilon, \rho_\varepsilon) = I(J, \rho). \tag{3.13}$$

Proof. We follow a similar argument used in [5].

We proceed in two steps. We first define an approximation $\tilde{\rho}_\varepsilon(t,x)$ bounded away from 0 and 1 such that $\phi(\tilde{\rho}_\varepsilon(t,x)) \geq C\varepsilon^2$ and $I(J,\tilde{\rho}_\varepsilon) \longrightarrow I(J,\rho)$. Then assuming that $\phi(\rho(t,x)) \geq a > 0$, we construct a smooth approximation $\rho_\varepsilon(t,x)$ such that $I(J,\rho_\varepsilon) \longrightarrow I(J,\rho)$. The conclusion follows then by a diagonal argument.

For the first step, define (recall (2.5))

$$\tilde{\rho}_\varepsilon(t,x) = (1-\varepsilon)\rho(t,x) + \varepsilon\bar{\rho}(t,x). \tag{3.14}$$

Notice that $\tilde{\rho}_\varepsilon(t,\pm 1) = \rho_\pm(t)$ and

$$\phi\big(\tilde{\rho}_\varepsilon(t,x)\big) \geq \varepsilon^2\big(\rho_-(t) \wedge \rho_+(t)\big)\big(1 - \big(\rho_-(t) \vee \rho_+(t)\big)\big).$$

$I(J,\rho)$ is convex and lower semicontinuous in ρ since it is a sup of continuous and convex functions of ρ. Consequently we have

$$I(J,\tilde{\rho}_\varepsilon) \;\leq\; (1-\varepsilon)I(J,\rho) + \varepsilon I(J,\bar{\rho}) \underset{\varepsilon\to 0}{\longrightarrow} I(J,\rho), \tag{3.15}$$

while by lower semicontinuity

$$\liminf_{\varepsilon\to 0} I(J,\tilde{\rho}_\varepsilon) \;\geq\; I(J,\rho), \tag{3.16}$$

that concludes the first approximation step.

Now assuming that $\phi(\rho(t,x)) \geq \phi(a) > 0$, i.e. $a < \rho(t,x) < 1-a$, we construct a smooth approximation $\rho_\varepsilon(t,x)$ such that

$$\rho_\varepsilon(t,x) \underset{\varepsilon\to 0}{\longrightarrow} \rho(t,y), \qquad \rho_\varepsilon(t,\pm 1) = \rho_\pm(t)$$

and such that $I(J,\rho_\varepsilon) \longrightarrow I(J,\rho)$.

Let Δ_D be the laplacian on $[-1,1]$ with Dirichlet boundary conditions, and

$$R^D_\varepsilon(x,y) = (I - \varepsilon\Delta_D)^{-1}(x,y).$$

Then define

$$\rho_\varepsilon(t,x) \;= \bar{\rho}(t,x) + \int_{-1}^{1} R^D_\varepsilon(x,y)\big(\rho(t,y) - \bar{\rho}(t,y)\big)\,dy. \tag{3.17}$$

We next prove that there exist $a_* > 0$ such that

$$\phi\big(\rho_\varepsilon(t,x)\big) \geq \phi(a_*). \tag{3.18}$$

In fact notice that $0 \leq \bar{\rho}(t,x) - \int_{-1}^{1} R^D_\varepsilon(x,y)\bar{\rho}(t,y)\,dy \leq 1$, that implies

$$\rho_\varepsilon(t,x) \geq \int_{-1}^{1} R^D_\varepsilon(x,y)\rho(t,y)\,dy \geq a\int_{-1}^{1} R^D_\varepsilon(x,y)\,dy \geq a',$$

for some positive $a' < a$.

Similarly we have

$$\begin{aligned} 1 - \rho_\varepsilon(t,x) &= 1 - \bar{\rho}(t,x) - \int_{-1}^{1} R^D_\varepsilon(x,y)(1 - \bar{\rho}(t,y)\,dy + \int_{-1}^{1} R^D_\varepsilon(x,y)\big(1 - \rho(t,y)\big)\,dy \\ &\geq \int_{-1}^{1} R^D_\varepsilon(x,y)\big(1-\rho(t,y)\big)\,dy \geq (1-a)\int_{-1}^{1} R^D_\varepsilon(x,y)\,dy \geq \big(1 - a''\big). \end{aligned}$$

Then choosing $a_* = a' \wedge a''$ we obtain (3.18).

Since ρ_ε is smooth, by (3.5), we have:

$$I(J,\rho_\varepsilon) = \frac{1}{4}\int_0^T dt \int_{-1}^{1} dy\; \big(J'(t) + \partial_y\rho_\varepsilon(t,y)\big)^2\phi\big(\rho_\varepsilon(t,y)\big)^{-1} \tag{3.19}$$

Let $R_\varepsilon^N(x,y) = (I - \varepsilon\Delta_N)^{-1}(x,y)$ the resolvent for the laplacian with Neumann boundary conditions, then we have the property that

$$\partial_x R_\varepsilon^D(x,y) = -\partial_y R_\varepsilon^N(x,y), \tag{3.20}$$

that implies

$$\partial_x \rho_\varepsilon(t,x) \;= \int_{-1}^1 R_\varepsilon^N(x,y)\partial_y\rho(t,y)\,dy := \big(R_\varepsilon^N \partial_y\rho\big)(t,x), \tag{3.21}$$

because $R_\varepsilon^N(x,y)$ is a probability kernel: $\int_{-1}^1 R_\varepsilon^N(x,y)\,dy = 1$.

Since $R_\varepsilon^N(x,y) \le 1$ and symmetric, by convexity we have

$$\begin{aligned}\big(J'(t) + \partial_y\rho_\varepsilon(t,y)\big)^2 &\le \int_{-1}^1 R_\varepsilon^N(x,y)\big(J'(t) + \partial_x\rho(t,x)\big)^2\,dx \\ &\le \int_{-1}^1 \big(J'(t) + \partial_x\rho(t,x)\big)^2\,dx \le I(J,\rho)\end{aligned} \tag{3.22}$$

Then we have

$$\frac{(J'(t) + \partial_y\rho_\varepsilon(t,y))^2}{\phi(\rho_\varepsilon(t,y))} \le \frac{1}{\phi(a_*)} I(J,\rho) \tag{3.23}$$

Then by dominated convergence we have

$$\lim_{\varepsilon\to 0} \mathcal{I}(J,\rho_\varepsilon) \;=\; \mathcal{I}(J,\rho).$$

It remains to prove the J' approximation by bounded functions. Let us define

$$J_k(t) = \int_0^t J'(s)\wedge k \vee (-k)\;\,ds \tag{3.24}$$

and let us assume that $\rho(t,x)$ is smooth in x and bounded away from 0 and 1. Then it is clear that

$$\frac{(J'(t)\wedge k\vee(-k) + \partial_y\rho(t,y))^2}{\phi(\rho(t,y))}$$

is not decreasing in k for k large enough. Then by monotone convergence $I(J_k,\rho) \to I(J,\rho)$. □

Notice that the minimum of $I(J,\rho)$ is correctly achieved for $J'(t) = \bar J'(t) = -\frac{1}{2}[\rho_+(t) - \rho_-(t)]$ and $\rho(t,y) = \bar\rho(t,y)$.

Furthermore, from (3.12), we have that for any given ρ, $I(J,\rho)$ is a quadratic function of J'. In particular we have the following Gallavotti–Cohen type of symmetry:

$$I(-J,\rho) = I(J,\rho) - \int_0^T J'(t)\log\frac{\rho_+(t)(1-\rho_-(t))}{\rho_-(t)(1-\rho_+(t))}\,dt. \tag{3.25}$$

The rate function for h_N alone is obtained, by contraction principle (cf. [16]), minimizing over all possible ρ satisfying the given boundary conditions:

$$I(J) = \inf_{\rho(t,y):\rho(t,\pm1)=\rho_\pm(t)} I(J,\rho), \tag{3.26}$$

and also satisfy the symmetry relation

$$I(-J) = I(J) - \int_0^T J'(t)\log\frac{\rho_+(t)(1-\rho_-(t))}{\rho_-(t)(1-\rho_+(t))}\,dt. \tag{3.27}$$

Since in (3.12) there is no relation between $J(t)$ and $\rho(t,y)$, it is possible to exchange the $\inf_\rho$ with the time integral and obtain

$$I(J) = \int_0^T dt \inf_{\rho(y):\rho(\pm 1)=\rho_\pm(t)} \int_{-1}^1 \frac{(J'(t)+\rho'(y))^2}{\phi(\rho(y))}\, dy, \tag{3.28}$$

that proves (1.7).

4. The large deviation theorem

Let us define the empirical density profile

$$\pi_N(t,y) = \sum_{x=-N}^{N-1} \eta_x(t) 1_{[x,x+1)}(Ny), \quad y \in [-1,1] \tag{4.1}$$

that has values on $\mathcal{M}$, so that the couple $(\mathsf{h}_N(1), \pi_N)$ has values on $\Xi = \mathcal{D}([0,T],\mathbb{R}) \times \mathcal{M}$.

Theorem 4.1. *Under the dynamics generated by* (2.1), *starting with an arbitrary configuration* η, *the couple* $(\mathsf{h}_N(1), \pi_N)$ *satisfy a large deviation principle with rate function* $I(J,\rho)$, *i.e.*

- *For any closed set* $C \subset \Xi$

$$\limsup_{N\to\infty} \frac{1}{N^{1+\alpha}} \log \mathbb{P}_\eta\big(\big(\mathsf{h}_N(1), \pi_N\big) \in C\big) \le - \inf_{(J,\rho)\in C} I(J,\rho), \tag{4.2}$$

- *For any open set* $\mathcal{O} \subset \Xi$

$$\liminf_{N\to\infty} \frac{1}{N^{1+\alpha}} \log \mathbb{P}_\eta\big(\big(\mathsf{h}_N(1), \pi_N\big) \in \mathcal{O}\big) \ge - \inf_{(J,\rho)\in \mathcal{O}} I(J,\rho). \tag{4.3}$$

5. The superexponential estimate

One of the main steps in the proof of Theorem 4.1 is a super-exponential estimate which allows the replacement of local functions by functionals of the empirical density. Define

$$V_{N,\varepsilon}(t,\eta) = \sum_{x=-N}^{N-1} G(t,x/N)\big(\eta(x)\big(1-\eta(x+1)\big) - \phi\big(\bar\eta_{N,\varepsilon}(x/N)\big)\big) \tag{5.1}$$

The local averages are defined in the bulk as

$$\bar\eta_{N,\varepsilon}(x/N) = \frac{1}{2N\varepsilon+1} \sum_{|x'-x|\le \varepsilon N} \eta\big(x'\big), \quad |x| < N(1-\varepsilon), \tag{5.2}$$

and at the boundaries as

$$\begin{aligned} \bar\eta_{N,\varepsilon}(x/N) &= \rho_+(t) \quad x \ge N(1-\varepsilon), \\ \bar\eta_{N,\varepsilon}(x/N) &= \rho_-(t) \quad x \le -N(1-\varepsilon). \end{aligned} \tag{5.3}$$

Notice that $V_{N,\varepsilon}(t,\eta)$ depends on t not only by the function $G(t,y)$ but also by the special definition (5.3).

In the following we use as reference measures the inhomogeneous product measures

$$\mu_t(\eta) = \prod_{x=-N}^{N} \bar\rho\left(\frac{x}{N}, t\right)^{\eta(x)} \left[1 - \bar\rho\left(\frac{x}{N}, t\right)\right]^{1-\eta(x)}. \tag{5.4}$$

Observe that, since $\rho_\pm(t)$ are uniformly away from 0 and 1, there is $C>0$ so that

$$\sup_t \sup_\eta \mu_t(\eta) \ge e^{-CN}. \tag{5.5}$$

Proposition 5.1. *For any $\delta > 0$ and initial configuration η*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\left(\left|\int_0^T V_{N,\varepsilon}(t,\eta_t)\,dt\right|\geq N\delta\right)=-\infty. \tag{5.6}$$

Proof. By (5.5), it is enough to prove (5.6) for the dynamics with initial distribution given by μ_0. By exponential Tchebychev inequality we have for any $a > 0$:

$$\mathbb{P}_{\mu_0}\left(\left|\int_0^T V_{N,\varepsilon}(t,\eta_t)\,dt\right|\geq N\delta\right)\leq e^{-N^{1+\alpha}\delta a}\mathbb{E}_{\mu_0}\left(e^{aN^\alpha|\int_0^T V_{N,\varepsilon}(t,\eta_t)\,dt|}\right)$$

By using that $e^{|x|}\leq e^x+e^{-x}$, all we need to prove is that there exists $K<+\infty$ such that for all $a\in\mathbb{R}$

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{E}_{\mu_0}\left(e^{N^\alpha\int_0^T aV_{N,\varepsilon}(t,\eta_t)\,dt}\right)\leq KT$$

To simplify notations, denote $\tilde V(t,\eta)=N^\alpha aV_{N,\varepsilon}(t,\eta)$. Consider the equation

$$\partial_s u(\eta,s)=L_{N,T-s}u(\eta,s)+\tilde V(T-s,\eta)u(\eta,s),\quad u(\eta,0)=1,\ 0\leq s\leq T. \tag{5.7}$$

By Feynman–Kac formula

$$u(\eta,T)=\mathbb{E}_\eta\left(e^{\int_0^T\tilde V(s,\eta_s)\,ds}\right) \tag{5.8}$$

Then

$$\begin{aligned}\frac{d}{ds}\frac{1}{2}\sum_\eta u(\eta,s)^2\mu_{T-s}(\eta)&=\frac{1}{2}\sum_\eta u(\eta,s)^2\frac{d\mu_{T-s}(\eta)}{ds}\\&\quad+\sum_\eta\left(u(\eta,s)L_{N,T-s}u(\eta,s)+\tilde V(T-s,\eta)u^2(\eta,s)\right)\mu_{T-s}(\eta)\end{aligned} \tag{5.9}$$

and, since $|\frac{d\mu_{T-s}(\eta)}{ds}|\leq CN\mu_{T-s}(\eta)$, this is bounded by

$$\leq\left(CN+\Gamma(s)\right)\sum_\eta u(\eta,s)^2\mu_{T-s}(\eta),$$

where, setting $\|f\|_s^2=\sum_\eta f(\eta)^2\mu_{T-s}(\eta)$,

$$\Gamma(s)=\sup_{f,\|f\|_s=1}\left\{\sum_\eta\tilde V(T-s,\eta)f^2(\eta)\mu_{T-s}(\eta)+\sum_\eta f(\eta)L_{N,T-s}f(\eta)\mu_{T-s}(\eta)\right\}. \tag{5.10}$$

By Gronwall inequality and (5.9) we have

$$\sum_\eta u(\eta,T)^2\mu_0(\eta)\leq e^{2\int_0^T\Gamma(s)\,ds+TCN}\sum_\eta u(\eta,0)^2\mu_T(\eta)=e^{2\int_0^T\Gamma(t)\,dt+TCN}. \tag{5.11}$$

Then using Schwarz inequality we get

$$\mathbb{E}_{\mu_0}\left(e^{\int_0^T\tilde V(\eta(t))\,dt}\right)\leq e^{\int_0^T\Gamma(t)\,dt+TCN}. \tag{5.12}$$

The Dirichlet forms associated to the generator are

$$\begin{aligned}\mathfrak{D}_{x,t,\rho}(f)&=\frac{1}{2}\sum_\eta\rho^{1-\eta(x)}(1-\rho)^{\eta(x)}\left[f\left(\eta^x\right)-f(\eta)\right]^2\mu_t(\eta),\quad x=\pm N\\\mathfrak{D}_{ex,t}(f)&=\frac{1}{2}\sum_\eta\sum_{x=-N}^{N-1}\left(\nabla_{x,x+1}f(\eta)\right)^2\mu_t(\eta)\end{aligned} \tag{5.13}$$

and we define

$$\mathfrak{D}_t(f) = \mathfrak{D}_{-N,t,\rho_-(t)}(f) + \mathfrak{D}_{N,t,\rho_+(t)}(f) + \mathfrak{D}_{ex,t}(f).$$

Observe that

$$\sum_\eta f(\eta)(L_{N,t}f)(\eta)\mu_t(\eta) = -\frac{1}{2}N^{2+\alpha}\sum_\eta\sum_{x=-N}^{N-1}\nabla_{x,x+1}f(\eta)\nabla_{x,x+1}(f\mu_t)(\eta) - N^{2+\alpha}\big(\mathfrak{D}_{N,t,\rho_+(t)}(f) + \mathfrak{D}_{-N,t,\rho_-(t)}(f)\big) \tag{5.14}$$

and

$$\begin{aligned}\sum_\eta\sum_{x=-N}^{N-1}\nabla_{x,x+1}f(\eta)\nabla_{x,x+1}(f\mu_t)(\eta) &= \sum_\eta\sum_{x=-N}^{N-1}(\nabla_{x,x+1}f)^2\mu_t(\eta)\\ &\quad+\sum_\eta\sum_{x=-N}^{N-1}f\big(\eta^{x,x+1}\big)\nabla_{x,x+1}f(\eta)\nabla_{x,x+1}\mu_t(\eta)\\ &= 2\mathfrak{D}_{ex,t}(f) + \frac{1}{N}\tilde{B}_N(t) + O\big(N^{-1}\big)\end{aligned} \tag{5.15}$$

where

$$\tilde{B}_N(t) = \sum_\eta\sum_{x=-N}^{N-1}f\big(\eta^{x,x+1}\big)\nabla_{x,x+1}f(\eta)\big(\eta(x)-\eta(x+1)\big)B\left(\frac{x}{N},t\right)\mu_t(\eta)$$

and

$$B\left(\frac{x}{N},t\right) = \frac{\bar{\rho}'(\frac{x}{N},t)}{\bar{\rho}(\frac{x}{N},t)(1-\bar{\rho}(\frac{x}{N},t))}, \qquad \|B\|_\infty \le c\|\rho'\|_\infty \tag{5.16}$$

where $c > 0$ since $\rho_\pm(t)$ are uniformly away from 0 and 1. By an elementary inequality we have:

$$\begin{aligned}\big|\tilde{B}_N(t)\big| &\le \frac{N}{2}\mathfrak{D}_{ex,t}(f) + \frac{2}{2N}\sum_\eta\sum_{x=-N}^{N-1}f\big(\eta^{x,x+1}\big)^2\big(\eta(x)-\eta(x+1)\big)^2B\left(\frac{x}{N},t\right)^2\mu_t(\eta)\\ &\le \frac{N}{2}\mathfrak{D}_{ex,t}(f) + \frac{1}{N}\sum_{x=-N}^{N-1}B\left(\frac{x}{N},t\right)^2\sum_\eta f^2(\eta)\mu_t\big(\eta^{x,x+1}\big)\end{aligned}$$

Since $\mu_t(\eta^{x,x+1}) = F(x,t)\mu_t(\eta)$ with $F(x,t)$ a uniformly bounded function of $\bar{\rho}(\frac{x}{N},t)$ and $\bar{\rho}(\frac{x+1}{N},t)$ and η, using (5.16) we get

$$\big|\tilde{B}_N(t)\big| \le \frac{N}{2}\mathfrak{D}_{ex}(f) + C \le \frac{N}{2}\mathfrak{D}_t(f) + C \tag{5.17}$$

From (5.14), (5.15) and (5.17) we get

$$\begin{aligned}\Gamma(s) &\le \sup_f\left\{N^\alpha a\sum_\eta V_{N,\varepsilon}(T-s,\eta)f^2(\eta)\mu_{T-s}(\eta) - N^{2+\alpha}\frac{1}{2}\mathfrak{D}_{T-s}(f)\right\} + N^{1+\alpha}C\\ &\le N^{1+\alpha}\sup_f\left\{\frac{a}{N}\sum_\eta V_{N,\varepsilon}(T-s,\eta)f^2(\eta)\mu_{T-s}(\eta) - N\frac{1}{2}\mathfrak{D}_{T-s}(f)\right\} + N^{1+\alpha}C\end{aligned} \tag{5.18}$$

Then we are left to prove that for any a and any $t > 0$:

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_f\left\{\frac{1}{N}\sum_\eta aV_{N,\varepsilon}(t,\eta)f^2(\eta)\mu_t(\eta) - \frac{N}{2}\mathfrak{D}_t(f)\right\} \le C' \tag{5.19}$$

This follows by proving that, for a suitable constant C

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_{\mathfrak{D}_t(f)\le CN^{-1}}\frac{1}{N}\sum_{\eta}V_{N,\varepsilon}(t,\eta)f^2(\eta)\mu_t(\eta)\le 0 \tag{5.20}$$

The rest of the proof is identical as in theorem 3.1 in [9] (see pages 1051–1052). □

With a similar argument follows also the super-exponential control of the densities at the boundaries:

Proposition 5.2. *For any $\delta>0$ we have*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\left(\int_0^T\left|\frac{1}{\varepsilon N}\sum_{x=\pm N}^{\pm N(1\mp\varepsilon)}\eta_t(x)-\rho_\pm(t)\right|\ge\delta\right)=-\infty \tag{5.21}$$

6. The exponential martingales

We use the following notations: for $x=-N,\dots,N-1$

$$\phi_-(\eta,x)=\eta(x+1)\big(1-\eta(x)\big),\qquad \phi_+(\eta,x)=\eta(x)\big(1-\eta(x+1)\big), \tag{6.1}$$

and at the boundaries

$$\begin{aligned}
&\phi_-(\eta,-N-1,t):=\big(1-\rho_-(t)\big)\eta(-N)\\
&\phi_+(\eta,-N-1,t):=\rho_-(t))\big(1-\eta(-N)\big)\\
&\phi_-(\eta,N,t):=\rho_+(t)\big(1-\eta(N)\big)\\
&\phi_+(\eta,N,t):=\big(1-\rho_+(t)\big)\eta(N).
\end{aligned} \tag{6.2}$$

Given two functions $z_\pm(t,y)$, we associate the following exponential martingales, for $x=-N-1,\dots,N$

$$\begin{aligned}
&\mathcal{E}_\pm(z_\pm,x,T)\\
&\quad=\exp\left\{\int_0^T z_\pm(t,x/N)\,d\mathsf{h}_\pm(t,x)-\int_0^T N^{2+\alpha}\big(e^{z_\pm(t,x/N)}-1\big)\phi_\pm(\eta_t,x)\,dt\right\}\\
&\quad=\exp\left\{z_\pm(T,x/N)\mathsf{h}_\pm(T,x)\right.\\
&\qquad\left.-\int_0^T\big[\partial_t z_\pm(t,x/N)\mathsf{h}_\pm(t,x)+N^{2+\alpha}\big(e^{z_\pm(t,x/N)}-1\big)\phi_\pm(\eta_t,x)\big]\,dt\right\}
\end{aligned} \tag{6.3}$$

We now choose a smooth function $H(t,y)$, $y\in[-1,1]$ such that $H(t,-1)=0$ and we set

$$z_+(t,y)=H(t,y+1/N)-H(t,y)=-z_-(t,y),\quad -1\le y\le 1-1/N \tag{6.4}$$

and at the boundaries

$$z_+(t,-1-1/N):=\frac{1}{N}\partial_y H(t,-1),\qquad z_+(t,1):=\frac{1}{N}\partial_y H(t,1). \tag{6.5}$$

The martingales defined by (6.3) are orthogonal, consequently taking the product we still have an exponential martingale equal to

$$
\begin{aligned}
&\prod_{x=-N-1}^{N}\prod_{\sigma=\pm}\mathcal{E}_\sigma(z_\sigma,x,T)\\
&\quad=\exp\Bigg\{\sum_{x=-N-1}^{N}\bigg[z_+(T,x/N)\mathsf{h}(T,x)-\int_0^T\partial_t z_+(t,x/N)\mathsf{h}(t,x)\,dt\\
&\qquad-N^{2+\alpha}\int_0^T\big(\big(e^{z_+(t,x/N)}-1\big)\phi_+(\eta_t,x)+\big(e^{-z_+(t,x/N)}-1\big)\phi_-(\eta_t,x)\big)\,dt\bigg]\Bigg\}
\end{aligned}
\tag{6.6}
$$

For large N we use Taylor approximation, for $x=-N,\dots,N-1$,

$$
\begin{aligned}
&\big(e^{z_+(t,x/N)}-1\big)\phi_+(\eta_t,x)+\big(e^{-z_+(t,x/N)}-1\big)\phi_-(\eta_t,x)\\
&\quad=-z_+(t,x/N)\big(\eta_t(x+1)-\eta_t(x)\big)\\
&\qquad+\frac{1}{2}z_+(t,x/N)^2\big(\phi_+(\eta_t,x)+\phi_-(\eta_t,x)\big)+O\big(z_+(t,x/N)^3\big)
\end{aligned}
$$

and on the boundaries

$$
\begin{aligned}
&\big(e^{z_+(t,-1-1/N)}-1\big)\phi_+(\eta_t,-N-1,t)+\big(e^{-z_+(t,-1-1/N)}-1\big)\phi_-(\eta_t,-N-1,t)\\
&\quad=-z_+(t,-1-1/N)\big(\eta_t(-N)-\rho_-(t)\big)\\
&\qquad+\frac{1}{2}z_+(t,-1-1/N)^2\big(\phi_+(\eta_t,-N-1,t)+\phi_-(\eta_t,-N-1,t)\big)+O\big(z_+(t,-1-1/N)^3\big).\\
&\big(e^{z_+(t,1)}-1\big)\phi_+(\eta_t,N)+\big(e^{-z_+(t,1)}-1\big)\phi_-(\eta_t,N)\\
&\quad=-z_+(t,1)\big(\rho_+(t)-\eta(t,N)\big)+\frac{1}{2}z_+(t,1)^2\big(\phi_+(\eta_t,N)+\phi_-(\eta_t,N)\big)+O\big(z_+(t,1)^3\big).
\end{aligned}
$$

We can rewrite the exponential martingale (6.6) as

$$
\begin{aligned}
&\prod_{x=-N-1}^{N}\prod_{\sigma=\pm}\mathcal{E}_\sigma(z_\sigma,x,T)\\
&\quad=\exp\Bigg\{\sum_{x=-N-1}^{N}\bigg[z_+(T,x/N)\mathsf{h}(T,x)-\int_0^T\partial_t z_+(t,x/N)\mathsf{h}(t,x)\,dt\bigg]\\
&\qquad+N^{2+\alpha}\int_0^T\sum_{x=-N-1}^{N}\bigg[z_+(t,x/N)\big(\eta_t(x+1)-\eta_t(x)\big)\\
&\qquad-\frac{1}{2}z_+(t,x/N)^2\big(\phi_+(\eta_t,x)+\phi_-(\eta_t,x)\big)\bigg]\,dt+O\big(N^\alpha\big)\Bigg\}
\end{aligned}
\tag{6.7}
$$

where in the above expression we set the convention $\eta_t(N+1):=\rho_+(t)$ and $\eta_t(-N-1):=\rho_-(t)$.

After a summation by parts we have

$$
\begin{aligned}
&\exp\Bigg\{\sum_{x=-N-1}^{N}\bigg[z_+(T,x/N)\mathsf{h}(T,x)-\int_0^T\partial_t z_+(t,x/N)\mathsf{h}(t,x)\,dt\bigg]\\
&\quad+N^{2+\alpha}\int_0^T\sum_{x=-N}^{N}\bigg[\bigg(z_+\bigg(t,\frac{x-1}{N}\bigg)-z_+\bigg(t,\frac{x}{N}\bigg)\bigg)\eta_t(x)
\end{aligned}
$$

$$
-\frac{1}{2}z_+(t,x/N)^2\big(\phi_+(\eta_t,x)+\phi_-(\eta_t,x)\big)\Bigg]\,dt
$$

$$
+N^{2+\alpha}\int_0^T\big(z_+(t,1)\rho_+(t)-z_+(t,-1-1/N)\rho_-(t)\big)\,dt+O\big(N^\alpha\big)\Bigg\} \tag{6.8}
$$

Using (2.9) and the special expression for z_+, we have for the logarithm of (6.8)

$$
\begin{aligned}
&\sum_{x=-N}^{N}H(T,x/N)\big[\eta_T(x)-\eta_0(x)\big]\\
&\quad+\Big[H(T,1)+\frac{1}{N}\partial_y H(t,1)\Big]\mathsf{h}(T,N)+\frac{1}{N}\partial_y H(T,-1)\mathsf{h}(T,-N-1)\\
&\quad-\int_0^T\sum_{x=-N}^{N}\partial_t H(t,x/N)\big[\eta_t(x)-\eta_0(x)\big]\,dt\\
&\quad-\int_0^T\Big(\Big[\partial_t H(t,1)+\frac{1}{N}\partial_t\partial_y H(t,1)\Big]\mathsf{h}(t,N)+\frac{1}{N}\partial_t\partial_y H(t,-1)\mathsf{h}(t,-N-1)\Big)\,dt\\
&\quad-N^\alpha\int_0^T\Bigg(\sum_{x=-N-1}^{N}\Big[\partial_{yy}H\Big(t,\frac{x}{N}\Big)\eta_t(x)+\frac{1}{2}\Big(\partial_y H\Big(t,\frac{x}{N}\Big)\Big)^2\big(\phi_+(\eta_t,x)+\phi_-(\eta_t,x)\big)\Big]\\
&\quad+N\big[\partial_y H(t,1)\rho_+(t)-\partial_y H(t,-1)\rho_-(t)\big]\Bigg)\,dt+O\big(N^\alpha\big)
\end{aligned}
$$

Since we are interested only in the terms that have order $N^{1+\alpha}$, we can forget all terms of order N in the above expression, and the exponential martingale has the form:

$$
\begin{aligned}
&\exp\Bigg[N^{1+\alpha}\Bigg\{H(T,1)\mathsf{h}_N(T,1)-\int_0^T\partial_t H(t,1)\mathsf{h}_N(t,1)\,dt\\
&\quad-\int_0^T\Bigg(\frac{1}{N}\sum_{x=-N-1}^{N}\partial_{yy}H\Big(t,\frac{x}{N}\Big)\eta_t(x)+\partial_y H(t,1)\rho_+(t)-\partial_y H(t,-1)\rho_-(t)\\
&\quad+\frac{1}{2N}\sum_{x=-N-1}^{N}\Big(\partial_y H\Big(t,\frac{x}{N}\Big)\Big)^2\big(\phi_+(\eta_t,x)+\phi_-(\eta_t,x)\big)\Bigg)\,dt+O\big(N^{-(\alpha\wedge 1)}\big)\Bigg\}\Bigg].
\end{aligned} \tag{6.9}
$$

After the superexponential estimate proved in the previous section, that also fix the densities at the boundaries, it follows the variational formula for the rate function given by (3.1).

7. The upper bound

7.1. *Exponential tightness*

The following proposition uses standard arguments and we give a proof for completeness in the Appendix:

Proposition 7.1. *There exist a sequence of compact sets K_L in $\mathcal{D}([0,T],\mathbb{R})$ such that*

$$
\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\big(\mathsf{h}_N(1)\in K_L^c\big)=-\infty. \tag{7.1}
$$

7.2. *Proof of the upper bound*

By (6.9) we can rewrite the exponential martingale defined by (6.7) as

$$M_N^H = \exp\Bigg[N^{1+\alpha}\Bigg\{H(T,1)\mathsf{h}_N(T,1) - \int_0^T \partial_t H(t,1)\mathsf{h}_N(t,1)\,dt - \int_0^T \Bigg(\int_{-1}^1 \pi_N(t,y)\partial_{yy}H(t,y)\,dy + \partial_y H(t,1)\rho_+(t) - \partial_y H(t,-1)\rho_-(t) + \frac{1}{2N}\sum_{x=-N-1}^{N}\Big(\partial_y H\Big(t,\frac{x}{N}\Big)\Big)^2\big(\phi_+(\eta_t,x)+\phi_-(\eta_t,x)\big)\Bigg)dt + O\big(N^{-(\alpha\wedge 1)}\big)\Bigg\}\Bigg], \tag{7.2}$$

where

$$\lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\eta}\big(O\big(N^{-(\alpha\wedge 1)}\big)\big)=0.$$

After applying the superexponential estimates of Section 5, we have

$$M_N^H = \exp\Bigg[N^{1+\alpha}\Bigg\{H(T,1)\mathsf{h}_N(T,1) - \int_0^T \partial_t H(t,1)\mathsf{h}_N(t,1)\,dt - \int_0^T \Bigg(\int_{-1}^1 \pi_N(t,y)\partial_{yy}H(t,y)\,dy + \partial_y H(t,1)\rho_+(t) - \partial_y H(t,-1)\rho_-(t) + \int_{-1}^1 \big(\partial_y H(t,y)\big)^2\phi\big(\bar\eta_{N,\varepsilon}(t,y)\big)\,dy\Bigg)dt + O_{\varepsilon,N} + O\big(N^{-(\alpha\wedge 1)}\big)\Bigg\}\Bigg].$$

Then for every set $A\subset\Xi$ we have

$$1=\mathbb{E}^{\mathbb{P}_\eta}\big(M_N^H\big)\ge e^{N^{1+\alpha}\sup_H\inf_{(J,\rho)\in A} I(H,J,\rho)}\mathbb{P}_\eta\big(\big(\mathsf{h}_N(1),\pi_N\big)\in A\big)$$

where

$$I(H,J,\rho)=H(T,1)J(T)-\int_0^T \partial_t H(t,1)J(t)\,dt - \int_0^T\Bigg(\int_{-1}^1 \rho(t,y)\partial_{yy}H(t,y)\,dy + \partial_y H(t,1)\rho_+(t) - \partial_y H(t,-1)\rho_-(t) + \int_{-1}^1\big(\partial_y H(t,y)\big)^2\phi\big(\rho(t,y)\big)\,dy\Bigg)dt.$$

Using the lower semicontinuity of $I(J,\rho)$ and a standard argument (see [16], lemma 11.3 or [12] lemma A2.3.3) we have for a compact set $C\subset\Xi$:

$$\limsup_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\big(\big(\mathsf{h}_N(1),\pi_N\big)\in C\big)\le -\inf_{(J,\rho)\in C} I(J,\rho).$$

The extension to closed set follows from the exponential compactness proved in (7.1), see [12] page 271.

8. The lower bound

The proof of the lower bound follows a standard argument, consequently we will only sketch it here, since all the ingredients are already proven. It is enough to prove that given $(J,\rho)\in\Xi$ such that $I(J,\rho)<\infty$, then for any open neighbor $\mathcal{O}$ of it we have

$$\liminf_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\big(\big(\mathsf{h}_N(1),\mu_N\big)\in\mathcal{O}\big)\ \ge\ -I(J,\rho). \tag{8.1}$$

By proposition 3.2, we can assume J and ρ such that $J'(t)$ exists and is bounded, ρ bounded away from 0 and 1, and $\partial_y \rho(t, y)$ exists and is bounded. Then we consider the weakly asymmetric exclusion dynamics with drift given by

$$\partial_y \bar{H}(t, y) = \frac{1}{2} \frac{J'(t) + \partial_y \rho(t, y)}{\phi(\rho(t, y))}, \tag{8.2}$$

more precisely, recalling the definition of $z(t, y)$ given by (6.4) and (6.5), the jump rate from x to $x+1$ at time t is taken to be $N^{2+\alpha} e^{z(t,x/N)}$, and from $x+1$ to x is given by $N^{2+\alpha} e^{-z(t,x/N)}$, while at the boundaries the birth rates are given by $N^{2+\alpha} e^{\frac{1}{N}\partial_y H(t,\pm 1)}$ and the death rates by $N^{2+\alpha} e^{-\frac{1}{N}\partial_y H(t,\pm 1)}$.

We call Q_N the law of this weakly asymmetric process that start with the same initial condition η. The Radon–Nykodyn derivative $\frac{dQ_N}{d\mathbb{P}_\eta}$ is given by (6.6).

The quasi static limit for this process is the following:

Proposition 8.1. *Let $\tilde{Q}_N$ the law on Ξ of $(\mathsf{h}_N(1), \pi_N)$ under Q_N, then*

$$\tilde{Q}_N \longrightarrow \delta_{(J,\rho)} \tag{8.3}$$

Proof of proposition 8.1. By (A.1) we can extend the superexponential estimates contained in Section 5 to Q_N. In fact we have that

$$\mathbb{E}^{\mathbb{P}_\eta}\left(\left(\frac{dQ_N}{d\mathbb{P}_\eta}\right)^2\right)^{1/2} \le e^{cN^{1+\alpha}}$$

and by Schwarz inequality

$$Q_N(A_{N,\varepsilon}) \le \mathbb{P}_\eta(A_{N,\varepsilon}) e^{cN^{1+\alpha}}$$

where $A_{N,\varepsilon} = \{\int_0^T V_{N,\varepsilon}(t, \eta_t)\, dt \ge N\delta\}$, and (5.6) extends immediately to Q_N. At this point the proof of the quasi-static hydrodynamic limit follows similar to the one in [9]. □

We then write

$$\mathbb{P}_\eta\big((\mathsf{h}_N(1), \pi_N) \in \mathcal{O}\big) = \mathbb{E}^{Q_N}\left(\frac{d\mathbb{P}_\eta}{dQ_N} \mathbf{1}_{(\mathsf{h}_N,\pi_N)\in\mathcal{O}}\right)$$

Since $\mathcal{O}$ contains (J, ρ), by Proposition 8.1, under Q_N the probability of the event $(\mathsf{h}_N(1), \pi_N) \in \mathcal{O}$ is close to one. By Jensen inequality

$$\frac{1}{N^{1+\alpha}} \log \mathbb{P}_\eta\big((\mathsf{h}_N(1), \pi_N) \in \mathcal{O}\big) \ge -\mathbb{E}^{Q_N}\left(\frac{1}{N^{1+\alpha}} \log \frac{dQ_N}{d\mathbb{P}_\eta}\right) = -\frac{1}{N^{1+\alpha}} H(Q_N|\mathbb{P}_\eta)$$

where $H(Q_N|\mathbb{P}_\eta)$ is the relative entropy of Q_N with respect to $\mathbb{P}_\eta$.

The lower bound is then a consequence of the following Proposition.

Proposition 8.2. *Let $H(Q_N|\mathbb{P}_\eta) = \mathbb{E}^{Q_N}(\log \frac{dQ_N}{d\mathbb{P}_\eta})$ the relative entropy of Q_N with respect to $\mathbb{P}_\eta$. Then*

$$\lim_{N\to\infty} \frac{1}{N^{1+\alpha}} H(Q_N|\mathbb{P}_\eta) = I(J, \rho), \tag{8.4}$$

The proof of proposition 8.2 is a direct consequence of (6.6) and of Proposition 8.1.

Appendix: The exponential tightness

We prove here Proposition 7.1. The arguments used here are just slight variations of the standard ones (e.g. Section 10.4 in [12]). SInce $\mathcal{M}$ is compact, we have only to control that the distribution of $\mathsf{h}_N(1, t)$ is exponentially tight. This is consequence of the following 2 propositions.

Proposition A.1.

$$\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\Big(\sup_{0\le t\le T}\big|\mathsf{h}_N(t,1)\big|\ge L\Big)=-\infty. \tag{A.1}$$

Proposition A.2. *For any* $\varepsilon>0$:

$$\lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\Big(\sup_{|t-s|\le\delta}\big|\mathsf{h}_N(t,1)-\mathsf{h}_N(s,1)\big|\ge\varepsilon\Big)=-\infty. \tag{A.2}$$

Proof of Proposition A.1. Since the difference between $\mathsf{h}_N(t,1)$ and $\bar{\mathsf{h}}_N(t):=\frac{1}{2N}\sum_x\mathsf{h}_N(t,x/N)$ is uniformly small, we have just to prove it for $\bar{\mathsf{h}}_N(t)$. For $\beta\in\mathbb{R}$, consider the exponential martingale (6.6) with $z_+(t,x)=\frac{\beta}{N}$, $x=-N,\dots,N-1$ and $z_+(t,-N-1)=0$. This is given by

$$M_t=\exp\big\{N^{1+\alpha}\big(\beta\bar{\mathsf{h}}_N(t)-A_N(\beta,t)\big)\big\}$$

$$A_N(\beta,t)=N\int_0^t\sum_{x=-N}^{N-1}\big(\big(e^{\beta/N}-1\big)\phi_+(\eta_s,x)+\big(e^{-\beta/N}-1\big)\phi_-(\eta_s,x)\big)\,ds$$

Notice that, by expanding the exponentials and using the explicit form of $\phi_\pm$, we have that $0\le A_N(\beta,t)\le CT(|\beta|+\beta^2)$ for some constant C. Then for any $\beta>0$ we have, by Doob's inequality,

$$\begin{aligned}\mathbb{P}_\eta\Big(\sup_{0\le t\le T}\big|\bar{\mathsf{h}}_N(t)\big|\ge L\Big)&\le\mathbb{P}_\eta\Big(\sup_{0\le t\le T}|\log M_t|\ge N^{1+\alpha}\big(\beta L-CT\big(\beta+\beta^2\big)\big)\Big)\\&\le\mathbb{P}_\eta\Big(\sup_{0\le t\le T}\log M_t\ge N^{1+\alpha}\big(\beta L-CT\big(\beta+\beta^2\big)\big)\Big)\\&=\mathbb{P}_\eta\Big(\sup_{0\le t\le T}M_t\ge e^{N^{1+\alpha}(\beta L-CT(\beta+\beta^2))}\Big)\\&\le e^{-N^{1+\alpha}(\beta L-CT(\beta+\beta^2))},\end{aligned} \tag{A.3}$$

that concludes the proof. □

Proof of Proposition A.2. Since

$$\Big\{\sup_{|t-s|\le\delta}\big|\bar{\mathsf{h}}_N(t)-\bar{\mathsf{h}}_N(s)\big|\ge\varepsilon\Big\}\subset\bigcup_{k=0}^{[T\delta^{-1}]}\Big\{\sup_{k\delta\le t\le(k+1)\delta}\big|\bar{\mathsf{h}}_N(t)-\bar{\mathsf{h}}_N(k\delta)\big|\ge\varepsilon/4\Big\}$$

Since

$$\begin{aligned}\log\mathbb{P}_\eta\Big(\sup_{|t-s|\le\delta}\big|\bar{\mathsf{h}}_N(t)-\bar{\mathsf{h}}_N(s)\big|\ge\varepsilon\Big)\le{}&\max_k\log\mathbb{P}_\eta\Big(\sup_{k\delta\le t\le(k+1)\delta}\big|\bar{\mathsf{h}}_N(t)-\bar{\mathsf{h}}_N(k\delta)\big|\ge\varepsilon/4\Big)\\&+\log\big(\big[T\delta^{-1}\big]\big)\end{aligned}$$

By the same estimate made in (A.3) we have

$$\log\mathbb{P}_\eta\Big(\sup_{k\delta\le t\le(k+1)\delta}\big|\bar{\mathsf{h}}_N(t)-\bar{\mathsf{h}}_N(k\delta)\big|\ge\varepsilon/4\Big)\le-N^{1+\alpha}\big(\beta\varepsilon/4-C\delta\big(\beta+\beta^2\big)\big)$$

and with a proper choice of β we get the following bound with a constant C' independent of k:

$$\frac{1}{N^{1+\alpha}}\log\mathbb{P}_\eta\Big(\sup_{k\delta\le t\le(k+1)\delta}\big|\bar{\mathsf{h}}_N(t)-\bar{\mathsf{h}}_N(k\delta)\big|\ge\varepsilon/4\Big)\le-\frac{C'\varepsilon^2}{\delta}.$$

□

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