Journal of Statistical Physics, Vol. 45, Nos. 5/6, 1986

Microscopic Selection Principle for a Diffusion-Reaction Equation

M. Bramson,[1] P. Calderoni,[2,3] A. De Masi,[2,4] P. Ferrari,[2,5] J. Lebowitz,[2] and R. H. Schonmann[2,5]

Received August 6, 1986

We consider a model of stochastically interacting particles on $\mathbb{Z}$, where each site is assumed to be empty or occupied by at most one particle. Particles jump to each empty neighboring site with rate $\gamma/2$ and also create new particles with rate $1/2$ at these sites. We show that as seen from the rightmost particle, this process has precisely one invariant distribution. The average velocity of this particle $V(\gamma)$ then satisfies $\gamma^{-1/2}V(\gamma) \to \sqrt{2}$ as $\gamma \to \infty$. This limit corresponds to that of the macroscopic density obtained by rescaling lengths by a factor $\gamma^{1/2}$ and letting $\gamma \to \infty$. This density solves the reaction-diffusion equation $u_t = \frac{1}{2}u_{xx} + u(1-u)$, and under Heaviside initial data converges to a traveling wave moving at the same rate $\sqrt{2}$.

KEY WORDS: Diffusion-reaction equation.

1. INTRODUCTION

In recent years it has become possible to derive increasingly complex hydrodynamic-type equations from microscopic dynamical models. These models are generally lattice systems of infinitely many interacting particles that evolve via some stochastic dynamics (see De Masi *et al.*,[1] Presutti,[2] Spohn,[3] and Fritz[4] for reviews of this work).

In this context, De Masi *et al.*[5,6] used a stochastic lattice-gas model (each site can be occupied or empty), evolving according to a combination

---

[1] School of Mathematics, University of Minnesota, Minneapolis, Minnesota.
[2] Department of Mathematics, Rutgers University, New Brunswick, New Jersey.
[3] Permanent Address: 2nd University of Rome "Tor Vergata," Italy.
[4] Permanent Address: University of L'Aquila, Italy.
[5] Permanent Address: University of São Paulo, Brazil.

0022-4715/86/1200-0905$05.00/0

of exchange and Glauber dynamics, to derive general diffusion-reaction (DR) equations of the form

$$\partial u(x, t)/\partial t = \tfrac{1}{2}\nabla^2 u(x, t) + f(u(x, t)) \tag{1.1}$$

Here $f(u)$ is a polynomial in u which vanishes at $u=0$ and $u=1$; $u(x, t)$ represents the macroscopic particle density at $x \in \mathbb{R}^d$ evolving on the macroscopic time scale t. The exact form of f in (1.1) is fairly arbitrary. We shall consider the simplest example of such an f, $f(u) = u(1-u)$. This is representative of the class $f(0) = f(1) = 0$, $f(u) > 0$, $f'(0) > f'(u)$, for $0 < u < 1$; then (1.1) corresponds to an equation first studied by Fisher[7] and Kolmogoroff *et al.*[9] as a model for the spread of certain genetic traits through a population. It was later studied by others in a variety of contexts.[8,10,11] An important feature of this type of DR equation is that they admit traveling front solutions: $u(x, t) = \phi_c(x - ct)$, $x \in \mathbb{R}$, where $\phi_c(y)$ satisfies the equation

$$\begin{gathered} \phi_c''(y) + c\phi_c'(y) + f(\phi_c) = 0 \\ \lim_{y \to -\infty} \phi_c(y) = 1, \qquad \lim_{y \to +\infty} \phi_c(y) = 0 \end{gathered} \tag{1.2}$$

which (for suitable f's) has solutions for all speeds $c \geq c^*$. (There are, of course, also fronts traveling in the opposite direction.) The marginal speed $c^* = [2f'(0)]^{1/2}$—which equals $\sqrt{2}$ for the case $f(u) = u(1-u)$—is singled out in the sense that all positive data $u_0(x)$ such that $u_0(x) \to 1$ as $x \to -\infty$, $u_0(x) = 0$ for $x > R$, converge as $t \to \infty$ to $\phi^*(x - m(t))$ for appropriate $m(t)$, where $\phi^* = \phi_{c^*}$ and $m(t)/t \to c^*$.

This "selection principle" was investigated from a physical point of view by Langer and co-workers.[12] The interest in this problem stems from a desire to understand pattern selection principles for physical phenomena described by complex equations, e.g., dendritic growth of a solid front into a melt. For an up-to-date review see the articles by Langer[18] and Eckman.[19]

In this note we investigate certain one-dimensional microscopic models leading to the moving front solutions of the DR equations (1.1) with $f(u) = u(1-u)$. We show that, as seen from the rightmost particle, this process has precisely one invariant distribution (Theorem 1) and that the average velocity $V(\gamma)$ of this particle satisfies $\gamma^{-1/2}V(\gamma) \to \sqrt{2}$ as $\gamma \to \infty$ (Theorem 2). The significance of this result for other pattern selection problems is not clear. We note, however, that this stochastic microscopic system has been employed independently as a model for flame front propagation by Kerstein,[13] where occupied sites represent burned regions. His work was the direct motivation for our analysis.

The outline of this paper is as follows. In Section 2 we describe the microscopic model that leads to the desired DR equations and prove their basic properties. In Section 3 we prove the main result. In Section 4 we discuss some extensions of the results.

2. THE MODEL

The microscopic models we consider are Markov processes with state space $\Omega = \{0, 1\}^{\mathbb{Z}}$. One can think of particles that can either jump to or create new particles at nearest neighbor empty sites on $\mathbb{Z}$. A particle on the site i waits an exponential random time with mean $(1+\gamma)^{-1}$. At this time it jumps to the position j if it is empty, with probability $\gamma(1+\gamma)^{-1}\, p(j-i)$, $p(-1) = p(1) = 1/2$, $p(k) = 0$ if $|k| \neq 1$, and with probability $(1+\gamma)^{-1}\, p(j-i)$, it creates a new particle at the site j if it is empty. If the site j is occupied, nothing happens.

Before we give a formal definition of this process, we introduce some notation:

Elements of Ω (configurations) will be represented by symbols such as η, ξ, ζ.

Given $\eta \in \Omega$ and $i \in \mathbb{Z}$, $\eta(i) \in \{0, 1\}$ is the projection of η at the site i.

$\{0, 1\}$ is endowed with the discrete topology and Ω with the corresponding product topology and σ-field Σ. Probability distributions on (Ω, Σ) will be represented by symbols like μ, ν.

η^j is defined by

$$\eta^j(i) = \begin{cases} \eta(i) & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\eta^{j,k}$ is defined by

$$\eta^{j,k}(i) = \begin{cases} \eta(i) & \text{if } i \notin \{j, k\} \\ \eta(j) & \text{if } i = k \\ \eta(k) & \text{if } i = j \end{cases}$$

For any cylindrical function $f\colon \Omega \to R$ the generator L of the process is defined by

$$(Lf)(\eta) = (\gamma/2) \sum_{i,j \in \mathbb{Z}} p(j-i)[f(\eta^{i,j}) - f(\eta)] + \sum_{i,j \in \mathbb{Z}} p(j-i)[f(\eta^j) - f(\eta)]\, \eta(i)$$

This generator defines a unique Markov process.[14] Let $(\xi_t^\mu, t \geq 0)$ be the process starting at time zero from a random configuration distributed according to μ. If μ is the point mass on a single configuration η, we write $\xi_t^\eta = \xi_t^\mu$.

Next we describe the relations between the macroscopic equation (1.1) and the microscopic system. Let μ^γ be a family of initial measures satisfying conditions (i)–(iii) of Definition 2.2 in Ref. 6; for example, one may think of μ^γ as a sequence of Bernoulli measures such that $\mu^\gamma(\eta(x)=1) = m((1/\sqrt{\gamma})\, x)$, where m is a smooth function $\mathbb{R} \to [0, 1]$. Under these conditions, it was proved in Ref. 6 that for any $t \geqslant 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}(\xi_t^{\mu^\gamma}(\sqrt{\gamma}\, x)) \xrightarrow[\gamma \to \infty]{} u(x, t) \tag{2.1}$$

where $u(x, t)$ is the solution of (1.1) with $f(u) = u(1-u)$ and initial data $u(x, 0) = m(x)$. (In fact, almost sure convergence for suitable functions was proved.)

We remark that it follows from the proof in Ref. 6 that one can also consider as initial measure μ^γ a point mass concentrated on the configuration η^-; $\eta^-(j) = 1$ if $j \leqslant 0$ and $\eta^-(j) = 0$ if $j > 0$. In this case (2.1) holds for any $t > 0$ and $x \in \mathbb{R}$ and the corresponding initial data $m(x)$ for $u(x, t)$ is the Heaviside profile, i.e., $m(x) = 1$ if $x \leqslant 0$ and $m(x) = 0$ if $x > 0$.

We will be interested in the case in which the initial microscopic configuration has a rightmost particle. In this case there will be at all times a rightmost particle. It will be useful then to look at the system from this first particle. To do so, we set

$$\tilde{\Omega} = \{\eta \in \Omega: \quad \eta(0) = 1, \eta(j) = 0 \text{ if } j > 0\}$$

For $\eta \in \Omega$, $i \in \mathbb{Z}$, let $\eta - i$ denote the configuration defined by $(\eta - i)(j) = \eta(i + j)$.

For μ concentrated on $\tilde{\Omega}$, define

$$X_t^\mu = \sup\{i \in \mathbb{Z}: \quad \xi_t^\mu(i) = 1\}$$

Set

$$\zeta_t^\mu = \xi_t^\mu - X_t^\mu$$

$(\zeta_t^\eta, t \geqslant 0)$ is a process with states on $\tilde{\Omega}$. A basic result about this process is the following:

Theorem 1. The family $\{(\zeta_t^\eta, t \geqslant 0), \eta \in \tilde{\Omega}\}$ has a unique invariant measure ν, which is concentrated on

$$\tilde{\Omega}_0 = \{\eta \in \tilde{\Omega}: \quad \eta(i) = 1 \text{ if } i < M, \text{ for some } M > -\infty\}$$

Proof. Existence. $(\zeta_t, t \geqslant 0)$ is a Feller process. This fact can be proven in a way analogous to Proposition (1.4) of Chapter I of Ref. 15

(p. 15). The only difference is that here the process is seen from the perspective of the first particle, but since the movement of this particle is bounded by appropriate Poisson processes, this is not a problem. (One should observe that the situation is completely different for systems in which particles can die[16]; in that case the process seen from the first particle is not in general Feller, since this first particle can make unbounded jumps in a very short time interval.) Existence follows now from well-known results, since $\tilde{\Omega}$ is compact. (See Proposition 1.8 of Chapter I of Ref. 14.)

Uniqueness. Assume that ν is invariant. Then since particles are created and not destroyed, ν must be concentrated on configurations in $\tilde{\Omega}$ with infinitely many particles. These configurations are completely specified by the random variables $R_1, R_2, \ldots$, where $R_1 = 0$,

$$R_{i+1} = \sup\{x < R_i\colon \quad \eta(x) = 1\}, \qquad i = 1, 2, \ldots$$

The basic point is that ν is concentrated on $\tilde{\Omega}_0$, i.e., that with ν-probability one, $R_i - R_{i+1} > 1$ for only a finite number of indices i. Uniqueness then follows, since due to the countability of $\tilde{\Omega}_0$, ζ_t is equivalent to an irreducible Markov chain and hence cannot have more than one invariant probability measure. Now R_n jumps one unit (up or down) when either the first or the nth particle in ξ_t^η jumps. It jumps up at least one unit when a particle is created to the right of the nth and the left of the first particle in ξ_t^η and it does not decrease when a particle is created to the right of the first particle. So, with respect to (ξ_t^γ), chosen from the stationary measure ν,

$$\begin{aligned}
&\left(\frac{d}{dt} ER_n\right)_{t=0} \\
&\quad \geqslant -\frac{\gamma}{2} - \frac{\gamma}{2}\nu\{R_n - R_{n+1} > 1\} \\
&\quad\quad + \frac{\gamma}{2}\nu\{R_1 - R_2 > 1\} + \frac{\gamma}{2}\nu\{R_{n-1} - R_n > 1\} \\
&\quad\quad + \sum_{i=1}^{n-1} \nu\{R_i - R_{i+1} > 1\}
\end{aligned} \tag{2.2}$$

Since ν is invariant, the left-hand side must be zero. Therefore, for any n,

$$\sum_{i=1}^{n-1} \nu\{R_i - R_{i+1} > 1\} < \gamma < \infty$$

By the Borel–Cantelli Lemma, $\nu(\tilde{\Omega}_0) = 1$, as desired. ∎

The process ξ_t^γ represents a microscopic propagating front. Indeed, the position of the first particle X_t^γ is a process with stationary increments and, as seen from this particle, the system is in a stationary state. There are various different (but equivalent) ways to define the microscopic velocity of propagation for this system. For instance:

1. Since $(X_t^\nu, t \geq 0)$ has stationary increments,

$$t^{-1} E X_t^\nu = V(\gamma)$$

where $V(\gamma)$ is a constant—the velocity of the first particle.

2. For μ concentrated on $\tilde{\Omega}_0$, the random variables

$$\begin{aligned} Y_t^\mu = {} & \text{number of particles created in the process} \\ & (\xi_t^\mu) \text{ from time 0 to } t \end{aligned}$$

are well defined and finite. Clearly $(Y_t^\nu, t \geq 0)$ has stationary increments and therefore

$$t^{-1} E Y_t^\nu = \tilde{V}(\gamma)$$

where $\tilde{V}(\gamma)$ is a constant—the rate of creation of particles.

It is easy to see that $V(\gamma) = \tilde{V}(\gamma)$. Indeed, let

$$H_t = \sum_{i \leqslant X_t^\nu} [1 - \xi_t^\nu(i)]$$

and $H_t^K = \min(H, K_t)$. Notice that H_t increases by one if the first particle jumps to the right, and decreases by one if the first particle jumps to the left or there is a creation to the left of the first particle. Then, with respect to the invariant measure ν:

$$\begin{aligned} \frac{d}{dt} E H_t^K \bigg|_{t=0} = {} & \frac{\gamma}{2} \nu\{H_t < K\} - \frac{\gamma}{2} \nu\{\zeta(-1) = 0, H_t < K\} \\ & - \sum_{i<0} \nu\{\zeta(i)\,(1 - \zeta(i+1)),\ H \leqslant K\}. \end{aligned}$$

The left-hand side above is zero by the invariance of ν. Since $\gamma\{H_t < K\} \uparrow 1$, by Dominate Convergence Theorem, the right-hand side converges as $K \to \infty$ to $V(\gamma) - V(\gamma)$. Note that

$$\begin{aligned} \frac{dEY_t^\mu}{dt} = {} & \frac{1}{2} \sum_{i \in \mathbb{Z}} P(\xi_t^\mu(i) = 1, \xi_t^\mu(i+1) = 0) \\ & + \frac{1}{2} \sum_{i \in \mathbb{Z}} P(\xi_t^\mu(i) = 1, \xi_t^\mu(i-1) = 0) \\ = {} & E\left\{ \sum_{i \in \mathbb{Z}} \xi_t^\mu(i)[1 - \xi_t^\mu(i+1)] \right\} - \frac{1}{2} \end{aligned} \tag{2.3}$$

One can consider the velocity of the ith particle or of the ith hole, etc., and get an equivalent definition for the velocity. It is also possible to define the velocity in terms of the "middle position" M_t specified by

$$\sum_{i > M_t^\mu} \xi_t^\mu(i) = \sum_{i \leqslant M_t^\mu} [1 - \xi_t^\mu(i)]$$

Here $M_t^\mu = M_0^\mu + Y_t^\mu$, as one can verify by induction. The movement of this middle point is therefore governed by the creation of particles, even if $\mu \neq \nu$. (See Ref. 12 for some simulation results.) Our main result is the following:

Theorem 2. $\lim_{\gamma \to \infty} \gamma^{-1/2} V(\gamma) = \sqrt{2}$.

This corresponds to $\lim_{t \to \infty} t^{-1} m(t) = \sqrt{2}$ for the solution $u(x, t)$ of (1.1) and (2.1) under Heaviside initial data.

Note that this is the limit one obtains for the asymptotic velocity of the rightmost particle of one-dimensional branching Brownian motion.[11,17] What Theorem 2 asserts, then, is that the differences in behavior for the two processes, including the saturation exhibited by our lattice model, do not give rise to different asymptotic velocities for the rightmost particles.

We prove Theorem 2 in the next section. The asymptotics of $V(\gamma)$ for γ small are considerably easier to derive, which we shall do now. Clearly

$$N(t) - M_1(t) \leqslant X_t^\nu \leqslant N(t) + M_2(t)$$

for appropriate Poisson processes $N(t)$, $M_1(t)$, and $M_2(t)$, where $N(t)$ has rate 1/2 and corresponds to creation of particles and $M_i(t)$, $i = 1, 2$, have rates $\gamma/2$ and correspond to jumps. So

$$1/2 - \gamma/2 \leqslant V(\gamma) \leqslant 1/2 + \gamma/2 \tag{2.4}$$

and $V(\gamma) \to 1/2$ as $\gamma \to 0$.

It is moreover true that

$$(\gamma/2)[V(\gamma) - \tfrac{1}{2}] \to 1 \qquad \text{as} \quad \gamma \to 0 \tag{2.5}$$

i.e., $V(\gamma) = 1/2 + \gamma/2 + o(\gamma)$. To see this, use the notation introduced in the proof of the uniqueness of ν. We have

$$\begin{aligned} V(\gamma) &= 1/2 - \tfrac{1}{2}\gamma\nu\{R_1 - R_2 > 1\} + \gamma/2 \\ &= 1/2 + \tfrac{1}{2}\gamma\nu\{R_1 - R_2 = 1\} \end{aligned} \tag{2.6}$$

But from (2.2)

$$\begin{aligned} 0 = \frac{d}{dt} ER_2 &\geqslant -\frac{\gamma}{2} - \frac{\gamma}{2}\nu\{R_2 - R_3 > 1\} + (\gamma + 1)\,\nu\{R_1 - R_2 > 1\} \\ &\geqslant -\gamma + (\gamma + 1)\,\nu\{R_1 - R_2 > 1\} \end{aligned}$$

Hence

$$\nu\{R_1 - R_2 > 1\} \leqslant \gamma/(\gamma+1)$$

and therefore

$$\nu\{R_1 - R_2 = 1\} \geqslant 1/(\gamma+1)$$

From (2.4) and (2.6)

$$\frac{1}{2}+\frac{\gamma}{2(\gamma+1)} \leqslant V(\gamma) \leqslant \frac{1}{2}+\frac{\gamma}{2}$$

from which (2.5) follows.

3. PROOF OF THEOREM 2

We break up the proof of Theorem 2 into two parts, corresponding to finding a lower bound and an upper bound on $\gamma^{-1/2}V(\gamma)$. We use the definitions 1 and 2 of $V(\gamma)$ in the different parts. Miraculously, both bounds converge to $\sqrt{2}$ as $\gamma \to \infty$.

Part 1:

$$\liminf_{\gamma \to \infty} \gamma^{-1/2} V(\gamma) \geqslant \sqrt{2}$$

Proof. We use the abbreviations $\bar{\xi}_t = \xi_t^{\eta_-}$, $\bar{X}_t = X_t^{\eta_-}$, $\bar{Y}_t = Y_t^{\eta_-}$.

Lemma 1. For any $\eta \in \tilde{\Omega}_0$, Y_t^η is stochastically greater than $\bar{Y}_t$. In particular, $EY_t^\eta \geqslant E\bar{Y}_t$.

Proof. We start by defining a partial order on $\tilde{\Omega}_0$. For $\eta \in \tilde{\Omega}_0$ recall the definition of R_i, $i = 1, 2,...$, which we now denote by $R_i(\eta)$. Write $\eta' \geqslant \eta$ if $R_i(\eta') - R_{i+1}(\eta') \geqslant R_i(\eta) - R_{i+1}(\eta)$ for $i = 1, 2,...$.

The basic argument involves a coupling, which can be most easily described in the following informal way: suppose that the particles that define η and η_- are white. The particles created in $\bar{\xi}_t$ after time 0 will be blue, whereas the particles created in ξ_t^η will be either blue or red; the coupling will be such that to each blue particle in $\bar{\xi}_t$ there will correspond a blue particle in ξ_t^η. Then $\bar{Y}_t$ = number of blue particles at time $t \leqslant$ number of blue and red particles at time $t = Y_t^\eta$.

To construct this coupling, we do the following. Up to the time of the first attempt at creation T_1 we couple the ith particle of ξ_t^η (counted from right to left) with the ith particle of $\bar{\xi}_t$: they attempt to jump or create a particle to the right or left together. Then the order $\xi_t^\eta \geqslant \bar{\xi}_t$ is maintained up to T_1. At T_1 there are three possibilities:

(a) No particle is created in either system. We can then proceed as before until the time T_2 of the second attempt at creation.

(b) A particle is created in both systems. In this case, color both the new particles blue. We renumber the particles (without distinguishing between white and blue) and continue with the same rule as before up to time T_2.

(c) A particle is created in ξ_t^η, but not in ξ_t. The particle created in the first system is then colored red. The red particles will not be considered in the enumeration of the particles and will not be coupled to any other.

The same rules are employed at the other times when either a white or blue particle attempts to reproduce. Red particles, on the other hand, produce red particles. Moreover, for red particles, the following extra rules are applied (red particles can be thought of as second-class particles):

1. If a red particle tries to create a particle in the position of a white, blue, or red particle, nothing happens.

2. If a white or blue particle in ξ_t^η tries to create a particle in the position of a red particle and creation also occurs in the other system, then this red particle becomes blue and will be coupled with a particle in the other system as in (b).

3. If a red particle tries to jump to the position of a white or blue particle, nothing happens.

4. If a white or blue particle tries to jump to the position of a red particle, they exchange positions.

Let $\bar{\xi}_t^\eta$ be the process defined by the white and blue particles in the system started from η. Proceeding by induction as above, then for all t

$$\xi_t \leqslant \bar{\xi}_t^\eta$$

Therefore, whenever a blue particle is created in ξ_t, a corresponding one is created in $\bar{\xi}_t^\eta$. ∎

From the definition of the velocity $V(\gamma)$ in terms of the rate of creation (see Section 2.2) and from Lemma 1 we have that for any $t > 0$

$$\gamma^{-1/2} V(\gamma) = \gamma^{-1/2} t^{-1} E(Y_t^\gamma) \geqslant \gamma^{-1/2} t^{-1} E(\bar{Y}_t) \tag{3.1}$$

Furthermore, by (2.3) we obtain that

$$E(\bar{Y}_t) = \int_0^t ds \left(E\left\{ \sum_{j \in Z} \xi_s(j)[1 - \xi_s(j+1)] \right\} - 1/2 \right) \tag{3.2}$$

Let $a_1, a_2 \in \mathbb{R}$, $a_1 > a_2$. By (2.1) and Lemma (3.5) in Ref. 2 it follows that for all $s > 0$

$$\lim_{\gamma \to \infty} \gamma^{-1/2} \sum_{j=[\gamma^{-1/2}a_2]}^{[\gamma^{-1/2}a_1]} E(\bar{\xi}_s(j)[1-\bar{\xi}_s(j+1)])$$

$$= \int_{a_2}^{a_1} dx\, u(x, s)[1-u(x, s)] \tag{3.3}$$

where $u(x, s)$ is the solution of (1.1) with initial data $u(x, 0) = 1$, $x \leqslant 0$; $u(x, 0) = 0$, $x > 0$.

As mentioned in the introduction, it is known (see Ref. 10, theorem (K.P.P.) p. 34, and Ref. 11) that as $t \to \infty$, $u(x + m(t), t)$ converges uniformly in x to $\phi^*(x)$, the traveling front solution satisfying (1.2) with $f(\phi_c) = \phi_c(1-\phi_c)$ and $c = \sqrt{2}$, where $m(t)$ is the median of u $[u(t, m) = 1/2]$.

Hence, for any positive b, $b < +\infty$,

$$\lim_{s \to \infty} \int_{-b+m(s)}^{b+m(s)} dx\, u(x, s)[1-u(x, s)]$$

$$= \int_{-b}^{b} dx\, \phi^*(x)[1-\phi^*(x)] \tag{3.4}$$

Taking supremum over b and using (1.2) finishes the proof.

Part 2:

$$\limsup_{\gamma \to \infty} \gamma^{-1/2} V(\gamma) \leqslant \sqrt{2}$$

Proof. The basic strategy is to again compare the process starting at v with the process starting at η_-. Instead of Y_t^γ, we now use X_t^γ to compute the velocity $V(\gamma)$.

The first step is to realize that

$$E(X_t^\gamma) \leqslant E(\bar{X}_t) \tag{3.5}$$

This follows by coupling ξ_t^γ and $\bar{\xi}_t$ so that whenever a particle of ξ_t^γ and of $\bar{\xi}_t$ occupy the same site, they attempt to jump and to reproduce at the same random times. It is easy to see that for this coupling $\xi_t^\gamma \subset \bar{\xi}_t$; (3.5) therefore holds. Now, for any y,

$$\gamma^{-1/2} V(\gamma) = \gamma^{-1/2} t^{-1} E(X_t^\gamma) \leqslant \gamma^{-1/2} t^{-1} E(\bar{X}_t)$$

$$\leqslant y + E(\gamma^{-1/2} t^{-1} \bar{X}_t; \bar{X}_t \geqslant \gamma^{1/2} t y) \tag{3.6}$$

But $|\bar{X}_t|$ is bounded by an appropriate Poisson random variable with mean $(\gamma+1)\,t$. Therefore,

$$E(\bar{X}_t^2) \leqslant (\gamma+1)\,t + (\gamma+1)^2\,t^2 \tag{3.7}$$

Using the Schwarz inequality, it follows from (3.7) that

$$\begin{aligned} &E(\gamma^{-1/2}t^{-1}\bar{X}_t;\, \bar{X}_t \geqslant \gamma^{1/2}ty) \\ &\quad \leqslant \gamma^{-1/2}t^{-1}[E(\bar{X}_t^2)]^{1/2}[P(\bar{X}_t \geqslant \gamma^{1/2}ty)]^{1/2} \\ &\quad \leqslant [\gamma^{-1}(1+\gamma)^2 + (1+\gamma^{-1})\,t^{-1}]^{1/2}[P(\bar{X}_t \geqslant \gamma^{1/2}ty)]^{1/2} \end{aligned} \tag{3.8}$$

Part 2 will thus follow from (3.6) and (3.8), if we show that for any $y > \sqrt{2}$, there exist ρ, $c > 0$, and $\gamma_0 = \gamma_0(y)$ such that for any $\gamma > \gamma_0$

$$P(\{\bar{X}_t \geqslant \gamma^{1/2}ty\}) \leqslant C\gamma t^2 e^{-\rho t} \tag{3.9}$$

The proof of (3.9) is long, so we shall break it into several steps. The basic idea is to apply the exponential Chebyshev inequality to a system of branching random walks ζ_t which dominates $\bar{\zeta}_t$. We begin by constructing ζ_t.

Step 1. Construction of ζ_t.

Particles are assumed to reproduce and to jump to neighboring sites.

As in $\bar{\xi}_t$, particles give birth to new particles at each nearest neighbor site independently at rate 1/2; here, however, more than one particle is allowed per site.

Particles are assumed to move according to the following scheme. For each site at which there is more than one particle, select one of them, e.g., choose the first one to arrive at a given site. Associate with these particles a stirring substructure.[15] That is, at each occupied site i, the corresponding particle jumps to $i+1$ (resp. $i-1$) at rate $\gamma/2$, at which time the particle at $i+1$ (resp. $i-1$), if present, is required to jump to i. Particles at occupied pairs of sites thus exchange positions at rate γ. When there is more than one particle at a given site, the particles that have not been selected jump to neighboring sites at rate $\gamma/2$.

Denote by $\zeta_t(i)$ the number of particles at site i for this process under $\zeta_0 = \eta_-$. It is not difficult to check that

$$\bar{\xi}_t(i) \leqslant \zeta_t(i) \qquad \text{for} \quad i \in Z \tag{3.10}$$

Step 2. Structure of ζ_t.

We shall denote by $(\zeta_t^n,\ t \geqslant 0)$ the progeny in $(\zeta_t,\ t \geqslant 0)$ of the particle starting at time zero at the site $-n$. Also denote by $z_n(t)$ the position of the members of ζ_t^n that are furthest to the right, and set

$$z(t) = \max_{n \geqslant 0}(z_n(t)) \tag{3.11}$$

From (3.10) it follows that for any $y > 0$,

$$\begin{aligned} P(\bar{X}_t \geqslant \gamma^{1/2} yt) &\leqslant P(z(t) \geqslant \gamma^{1/2} yt) \\ &\leqslant \sum_{n=0}^{\infty} P(z_n(t) \geqslant \gamma^{1/2} yt) \end{aligned} \tag{3.12}$$

In order to obtain bounds on $z_n(t)$, we need to introduce some notation for the branches of the process ζ_t^n. Let $J(\sigma)$ denote the branch associated with the sequence $\sigma = (\sigma_1, \sigma_2,...)$ of 0's and 1's in the following way. At time zero start following the particle originally at the site $-n$. When this particle first reproduces, we continue to follow it if $\sigma_1 = 0$, whereas we follow its offspring if $\sigma_1 = 1$; let T_1^σ denote this reproduction time. Continue to follow this particle until time T_2^σ, its next reproduction time. If $\sigma_2 = 0$, continue to follow the parent, whereas if $\sigma_2 = 1$, follow its offspring. Proceeding in this manner, one can inductively define the branch $J(\sigma)$ and the reproduction times T_k^σ, $k = 1, 2,...$. For convenience, let $T_0^\sigma = 0$. Also, set

$$L_t^\sigma = \max\{k\colon T_k^\sigma \leqslant t\}$$

and define $x^\sigma(t)$ as the position of the branch $J(\sigma)$ at time t. Then

$$\Delta_i = x^\sigma(T_i^\sigma) - x^\sigma(T_i^\sigma - 0)$$

is the jump that $J(\sigma)$ undergoes at time T_i^σ. ($\Delta_i = 0$ if $\sigma_i = 0$, and $\Delta_i = \pm 1$ otherwise.)

Step 3. Behavior along branches. It is easy to check that

$$\begin{aligned} x^\sigma(t) = x^\sigma(T_1^\sigma - 0) &+ [x^\sigma(T_2^\sigma - 0) - x^\sigma(T_1^\sigma) + \Delta_1] + \cdots \\ &+ [x^\sigma(t) - x^\sigma(T_L) + \Delta_L] \end{aligned}$$

where $L = L_t^\sigma$. Consequently,

$$x^\sigma(s) - \sum_{i=1}^{L_s^\sigma} \Delta_i + n, \qquad 0 \leqslant s \leqslant t \tag{3.13}$$

is a random walk $W(s)$ with rate γ and $W(0) = 0$. To control the total effect up to time t of reproduction, i.e., $\sum_{i=1}^{L_t^\sigma} \Delta_i$, we introduce the event

$$A(M_n, t) = \{\forall \sigma,\ L_t^\sigma \leqslant M_n t - 1\}$$

$A(M_n, t)$ is the event that by time t no branch of ζ_t^n has more than $M_n t - 1$ births.

We now prove that for M_n large enough,

$$P((A(M_n, t))^c) \leqslant e^{-M_n t/2} \tag{3.14}$$

To see this, first note that

$$(A(M_n, t))^c = \bigcup_\sigma \{T^\sigma_{[M_n t]} \leqslant t\} \tag{3.15}$$

and write

$$T^\sigma_{[M_n t]} = \sum_{i=1}^{[M_n t]} (T_i^\sigma - T_{i-1}^\sigma)$$

Now for all σ and i, $\tau_i^\sigma = T_i^\sigma - T_{i-1}^\sigma$ are independent exponential random times of parameter 1. We therefore obtain for all $\beta > 0$ the following exponential Chebyshev inequality:

$$\begin{aligned} P(T^\sigma_{[M_n t]} \leqslant t) &\leqslant [\exp(\beta t)][E(\exp(-\beta \tau_i^\sigma))]^{[M_n t]} \\ &= [\exp(\beta t)](1+\beta)^{-[M_n t]} \end{aligned}$$

Now, to check whether $A(M_n, t)$ occurs, one has only to follow the $2^{M_n t}$ branch segments $(\sigma_1, \sigma_2, \ldots, \sigma_{M_n t})$. Therefore, from (3.15)

$$P((A(M_n, t))^c) \leqslant 2^{M_n t} e^{\beta t}(1+\beta)^{-[M_n t]}$$

For $\beta = 2$ and M_n large enough, one obtains

$$P((A(M_n, t))^c) \leqslant e^{-M_n t/2} \tag{3.16}$$

On the other hand, it follows from the definition of $A(M_n, t)$ that for any σ

$$\begin{aligned} &A(M_n, t) \cap \{x^\sigma(t) \geqslant \gamma^{1/2} y t\} \\ &\quad \subset \left\{ x^\sigma(t) - \sum_{i=1}^{L_t^\sigma} \Delta_i + n \geqslant \gamma^{1/2} y t - M_n t + n \right\} \end{aligned} \tag{3.17}$$

One therefore obtains from (3.13) and the exponential Chebyshev inequality for the random walk that for any $\theta > 0$

$$\begin{aligned} &P(A(M_n, t) \cap \{x^\sigma(t) \geqslant \gamma^{1/2} y t\}) \\ &\quad \leqslant P(W(t) \geqslant \gamma^{1/2} y t - M_n t + n) \\ &\quad \leqslant \exp[t\gamma(\cosh\theta - 1) - \gamma^{1/2}\theta t(y - M_n \gamma^{-1/2}) - n\theta] \end{aligned} \tag{3.18}$$

Step 4. Conclusion. We are finally in a position to prove (3.10). Set $M_n = M$ for $n < \gamma t^2$ and $M_n = 2 \log n$ for $n \geqslant \gamma t^2$. Then, by (3.12), (3.14), and (3.18), we obtain

$$\begin{aligned} P(\bar{X}_t \geqslant \gamma^{1/2} y t) &\leqslant \sum_{n=0}^{\infty} \mathbb{E}\left(\sum_{\sigma} \mathbb{1}_{\{x^\sigma(s) \geqslant \gamma^{1/2} y t\}}, A(M_n, t)\right) \\ &\quad + P((A(M_n, t))^c) \\ &\leqslant \sum_{n=0}^{\infty} E(|\zeta_t^n|) \exp[-n\theta + \theta t M_n + t\gamma(\cosh\theta - 1 - \gamma^{1/2} y \theta)] \\ &\quad + \gamma t^2 e^{-Mt/2} + \sum_{n=[\gamma t^2]+1}^{\infty} n^{-t} \end{aligned} \tag{3.19}$$

Set $\theta = \gamma^{-1/2} y$ and expand $\cosh\theta$ up to the third power. We obtain that the first term in the last inequality of (3.19) is bounded by

$$\begin{aligned} &\exp\{t[-y^2/2 + \gamma^{-1/2} M y + O(y^4/\gamma)]\} \\ &\quad \times (\exp t) \sum_{n=0}^{[\gamma t^2]} \exp(-ny/\sqrt{\gamma}) \\ &\quad + \exp[t(-y^2/2 + O(y^4/\gamma))] \sum_{n=[\gamma t^2]+1}^{\infty} \exp(-ny/2\sqrt{\gamma}) \\ &\leqslant 2\gamma^{1/2} y^{-1} \exp\{-t[y^2/2 - 1 - \gamma^{-1/2} M y + O(y^4/\gamma)]\} \end{aligned} \tag{3.20}$$

For any fixed $y > \sqrt{2}$ and M, we can find γ_0, depending on y and M, such that for any $\gamma \geqslant \gamma_0$

$$y^2/2 - 1 - \gamma^{-1/2} M y + O(y^4/\gamma) > 0$$

Thus, (3.9) follows from (3.19) and (3.20). ∎

4. EXTENSIONS

It is natural to consider the more general class of systems on Z for which a particle attempts to jump from i to j according to some probability $p(i, j) = p(0, j-i)$ with $p(0, k) = p(0, -k)$; and a particle at i attempts to create another at j with probability $q(i, j) = q(0, j-i)$. One can extend Theorems 1 and 2 to the case where $p(\cdot, \cdot)$ is as before and $q(0, 1) + q(0, -1) = 1$. The proofs are then essentially the same. We can also extend part of these results when $p(\cdot, \cdot)$ and $q(\cdot, \cdot)$ are irreducible and the interactions have finite range, i.e., there exists an $L < \infty$ such that $p(0, k) = q(0, k) = 0$ if $|k| > L$. In this case:

1. Theorem 1 is still true. The proof is essentially the same as before.
2. The corresponding DR equation is

$$\frac{\partial u(x,t)}{\partial t}=\frac{D}{2}\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t)[1-u(x,t)]$$

where

$$D=\sum_{i\in Z} i^2 p(0,i)$$

This result is not stated in Ref. 6, but follows using the same techniques. Note that traveling fronts exist for velocities larger than or equal to $c_D^*=(2D)^{1/2}$.

3. The upper bound

$$\limsup_{\gamma\to\infty} \gamma^{-1/2} V(\gamma)\leqslant c_D^*$$

can be obtained in the same way as before.

ACKNOWLEDGMENTS

We would like to thank A. Kerstein for bringing this question to our attention and for valuable discussions. We also thank the Institute for Mathematics and its Applications, University of Minnesota, which provided the facilities for joining the upper and lower bounds. P. C., A. DeM., P. F., and R. H. S. thank the Rutgers Mathematics Department for their kind hospitality.

This work was partially supported by CNPq (Brazil), CNR (Italy), FAPESP (Brazil), and the NSF (grants DMS-831080 and DMR81-14726).

REFERENCES

1. A. De Masi, N. Ianiro, A. Pellegrinotti, and E. Presutti, A survey of the hydrodynamical behavior of many particle systems, in *Studies in Statistical Mechanics*, Vol. II, J. L. Lebowitz and E. W. Montroll, eds. (North-Holland, Amsterdam, 1984).
2. E. Presutti, Collective Phenomena in Stochastic Particle Systems, Proceedings, BiBOS Conference—Bielefeld.
3. H. Spohn, Equilibrium fluctuation for some stochastic particle systems, in *Statistical Physics and Dynamical Systems*, J. Fritz, A. Jaffe, and D. Szàsz, eds. (Birkhäuser, Boston, 1985).
4. J. Fritz, The Euler equation for the stochastic dynamics of a one-dimensional continuous spin system, Preprint (1986).

5. A. De Masi, P. Ferrari, and J. Lebowitz, Rigorous derivation of reaction-diffusion equation with fluctuations, *Phys. Rev. Lett.* **35**:19 (1985).
6. A. De Masi, P. Ferrari, and J. Lebowitz, Reaction-diffusion equations for interacting particle systems, *J. Stat. Phys* **44**:589 (1986).
7. R. A. Fisher, The advance of advantageous genes, *Ann. Eugenics* **7**:355–369 (1937).
8. D. G. Aronson and H. F. Weinberger, Non linear diffusion in population genetics, combustion and nerve propagation, in *Partial Differential Equations and Related Topics*, J. Goldstein, ed. (Lecture Notes in Mathematics, No. 446, Springer, New York).
9. A. Kolmogorov, I. Petrovskii, and N. Piscounov, Etudes de l'équations de la diffusion avec croissance de la quantité de matière et son application a un problème biologique, *Bull. Univ. Mosc. Ser. Int. A* **1**(6):1–25.
10. M. Bramson, *Convergence of Solutions of the Kolmogorov Equation to Travelling Waves*, Memoirs American Mathematical Society 285.
11. H. P. McKean, Application of Brownian Motion to the Equation of Kolmogorov–Petrovskii–Piscounov, *Commun. Pure Appl. Math.* **XXVIII**:323–331.
12. E. Ben Jacob, H. Brand, G. Dee, L. Kramer, and I. S. Langer, Pattern propagation in nonlinear dissipative systems, *Physica* **14D**:348 (1985).
13. A. R. Kerstein, Computational study of propagating fronts in a lattice-gas model, *J. Stat. Phys.*, this issue, preceding paper.
14. L. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, 1985).
15. D. Griffeath, *Additive and Cancellative Interacting Particle Systems* (Springer Lecture Notes in Mathematics, 724).
16. R. Durrett, Oriented percolation in two dimensions, *Ann. Prob.* **12**:999 (1984).
17. M. Bramson, Maximal displacement of branching Brownian motion, *Commun. Pure Appl. Math.* **31**:531–581.
18. J. S. Langer, in Proceedings of the 1986 Les Houche summer school.
19. J. P. Eckman, in Proceedings of the 1986 Les Houche summer school.