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Stability of invariant manifolds in one and two dimensions*

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Abstract
We consider the gradient flow associated with a nonlocal free energy functional and extend to such a case results obtained for the Allen–Cahn equation on 'slow motions on invariant manifolds'. The manifolds in question are time-invariant one-dimensional curves in an L^2 space which connect the two ground states (interpreted as the pure phases of the system) to the first excited state (interpreted as a diffuse interface). Local and structural stability of the manifolds are proved and applications to the characterization of optimal tunnelling are discussed.

Mathematics Subject Classification: 82C05

1. Introduction

In this paper we consider the evolution equation for $u(x, t)$, $|x| \leqslant L/2$, $t \geqslant 0$

$$u_t = f_L(u), \tag{1.1}$$

with u_t the t-derivative of u and the 'velocity field' $f_L(u)$ given by

$$f_L(u) = J^{\text{neum}} * u - \frac{1}{\beta}\operatorname{arctanh}(u), \qquad \beta > 1, \tag{1.2}$$

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where $J^{\text{neum}} * u(x) = \int_{-L/2}^{L/2} J^{\text{neum}}(x, y)u(y)\,dy$,

$$J^{\text{neum}}(x, y) := J(x, y) + J(x, R_L(y)) + J(x, R_{-L}(y)),$$

with $R_L(y) = \frac{L}{2} + (\frac{L}{2} - y)$, $R_{-L}(y) = -\frac{L}{2} - (y - \frac{L}{2})$, namely $R_{\pm L}(y)$ are the reflections of y around $\pm L$. We suppose that $J(x, y)$, $x, y \in \mathbb{R}$, is a smooth, symmetric translational invariant (i.e. $J(x, y) = J(0, |y - x|)$) probability kernel supported in $|y - x| \leqslant 1$; we also need that $J(0, x)$ is not increasing in $x \geqslant 0$, i.e. $J'(0, x) \leqslant 0$ for $x \in (0, 1)$. Later in the paper, see section 7, we will extend our analysis to the case $d = 2$.

We are here mainly interested in the analysis of the stationary points and invariant sets of (1.1) as well as their stability. The function m is stationary if it solves the 'nonlocal mean field equation'

$$m = \tanh\{\beta J^{\text{neum}} * m\}. \tag{1.3}$$

As we will see, some of the motivations of our work are related to the fact that (1.1) is the gradient flow equation $u_t = -\delta F_L(u)/\delta u$ of the 'free energy' functional

$$F_L(m) := \int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_\beta(m)\,dx + \frac{1}{4}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} J^{\text{neum}}(x, y)[m(x) - m(y)]^2\,dx\,dy, \tag{1.4}$$

where $\beta > 1$ and, denoting by $m_\beta \in (0, 1)$ the number that satisfies the mean field equation

$$m_\beta = \tanh\{\beta m_\beta\}, \tag{1.5}$$

we let

$$\phi_\beta(m) := \tilde\phi_\beta(m) - \tilde\phi_\beta(m_\beta), \qquad \tilde\phi_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta}S(m),$$

$$S(m) := -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}.$$

The terminology comes from statistical mechanics where $F_L(m)$ is the large deviation rate function as $\gamma \to 0$ of an Ising system at temperature $T = 1/k\beta$, k the Boltzmann constant, and with Kac potential J_γ, $J_\gamma(x, y) = \gamma J(\gamma x, \gamma y)$, $\gamma > 0$ the scaling parameter; m represents a magnetization density profile and $F_L(m)$ its free energy; thus the parameter β in (1.2) has the meaning of an inverse temperature, $\beta = 1/(kT)$, T being the absolute temperature and k the Boltzmann constant. We will not pursue here the connection with the Ising model and take $F_L(m)$ and (1.1) as primitive notions of a theory which lives at a mesoscopic level intermediate between microscopics (statistical mechanics) and macroscopics (thermodynamics and continuum theory).

Since the stationary points of (1.1) are critical points of $F_L(m)$, our analysis of stationarity is intimately related to the study of variational problems for F_L. Indeed the minimizers of $F_L(m)$ are the two spatially homogeneous solutions of (1.3) $m^{(\pm)}(x) \equiv \pm m_\beta$; in fact $F_L(m^{(\pm)}) = 0$ and $F_L(m) > 0$ for any other m as follows from (1.4) recalling that J is a smooth probability kernel. The choice $\beta > 1$ is responsible for the nonuniqueness of minimizers, see (1.5), which, in the mesoscopic theory, is interpreted as a phase transition, each minimizer being a 'pure phase'. Our model has a plus phase, $m^{(+)}$, and a minus phase, $m^{(-)}$. At $\beta \leqslant 1$ instead the minimizer is unique and given by $m^{(0)} \equiv 0$; $m^{(0)}$ remains a critical point also at $\beta > 1$, but it is no longer a minimizer (or a pure phase). If L is large enough there are also space dependent critical points, in particular $\hat m_L$, which is an antisymmetric increasing solution of (1.3) the existence and properties of which have been studied in [4, 7, 8, 11]. There are several seemingly different reasons to study $\hat m_L$ that we outline below.

- *Energy.* For L large enough $\hat m_L$ is the first excited state of F_L: in [2] it is in fact proved that there is $\epsilon > 0$ so that

  $$\text{if } m = \tanh\{\beta J * m\} \text{ and } F_L(m) < F_L(\hat m_L) + \epsilon \text{ then } m \in \{m^{(+)}, m^{(-)}, \hat m_L\}. \tag{1.6}$$

- *Interfaces.* There is a stationary solution $\bar{m}(x)$ of the equation (1.1) extended to the whole $\mathbb{R}$,

$$\bar{m}(x) = \tanh\{\beta(J * \bar{m})(x)\}, \quad x \in \mathbb{R}, \qquad \text{such that } \lim_{x\to\pm\infty} \bar{m}(x) = \pm m_\beta. \tag{1.7}$$

  Thus $\bar{m}$ (called an instanton) interpolates between the two equilibrium states $m^{(\pm)}$ at $\pm\infty$ and it has the interpretation of the interface connecting the two coexisting phases. In [4, 11] it was proved that

$$\lim_{L\to\pm\infty} \sup_{|x|\leqslant L/2} |\bar{m}(x) - \hat{m}_L(x)| = 0. \tag{1.8}$$

  Thus $\hat{m}_L$ has the meaning of the interface at finite volumes.
- *The Wulff problem.* According to thermodynamics, the minimizers of the free energy are the equilibrium states. If the total magnetization is a conserved quantity, then equilibrium at 0 magnetization is described by minimizing $F_L(m)$ with the constraint $\int m = 0$, Wulff problem. In [20] it is proved that for L large enough the infimum of F_L with the 0 magnetization constraint is attained at a unique point, $\hat{m}_L$.
- *Tunnelling.* While the instanton $\bar{m}$ and its finite volume version $\hat{m}_L$ describe the 'optimal' spatial pattern to connect $m^{(\pm)}$, $\hat{m}_L$ is also the saddle point of optimal orbits connecting $m^{(\pm)}$ in time, as proved in [1].

In this paper we will investigate the latter issue establishing first the existence of a dynamical connection between $\hat{m}_L$ and $m^{(\pm)}$. We will prove that there are two one-dimensional, invariant manifolds, $\mathcal{W}_\pm = \{v^{(\pm)}(x,s) : |x| \leqslant L/2, s \in \mathbb{R}\}$, such that

$$\lim_{s\to-\infty} \|v^{(\pm)}(\cdot,s) - \hat{m}_L\|_\infty = 0, \qquad \lim_{s\to\infty} \|v^{(\pm)}(\cdot,s) - m^{(\pm)}\|_\infty = 0, \tag{1.9}$$

and moreover, denoting by $S_t(m)$ the semigroup generated by (1.1) (see section 2.1), for any $t \geqslant 0$ and any $s \in \mathbb{R}$,

$$S_t(v^{(\pm)}(\cdot,s)) = v^{(\pm)}(\cdot, t+s). \tag{1.10}$$

The existence of such manifolds has been much studied in the context of the Allen–Cahn equation $u_t = u_{xx} + u - u^3$, starting from the works [6, 17]. It has also been studied [2–4] for the nonlocal 'Glauber' evolution equation

$$u_t = -u + \tanh\{\beta J^{\text{neum}} * u\}, \tag{1.11}$$

which is similar to (1.1) and with the same critical points. But unfortunately the orbits of (1.11) and (1.1) do not coincide: therefore a new analysis is needed, and this is what we carry out in the present paper. Our main motivation for doing it is related to the tunnelling problems mentioned above. A characterization of the optimal orbit connecting $m^{(\pm)}$ requires a proof of existence of the two manifolds $v^{(\pm)}$ and its local stability, we refer to [1] for a discussion on this point. We also mention that the question also appears in tunnelling in $d = 2$ dimensions. Indeed a key estimate in the proof in [3] (namely that any optimizing orbit from $m^{(-)}$ to $m^{(+)}$ in $d = 2$ dimensions is planar and follows first $v^{(-)}$ with time reversed and then $v^{(+)}$) is based on the analysis we carry out here. The present paper deals only with the $d = 1$ case, but the extension to $d = 2$ is immediate using the spectral analysis in [3], as discussed in section 7.

The plan of the paper is the following. In section 2.1 we recall some properties of solutions to (1.1), such as comparison principle (proposition 2.1) and contractivity (proposition 2.2). See also appendix A for other useful statements and for the proofs of propositions 2.2 and 2.3. In section 2.2 we list some known properties of $\bar{m}$, such as its asymptotic decay (inequalities (2.5)). We also define the instantons manifold and $\hat{m}_L$. In section 2.3 we give some properties of the spectrum of the linearization Ω_m (defined in (2.7)) of (1.11), see (2.8), (2.9) and (2.10).

Similar results (theorem 2.4) for the operator $\mathcal{L}_m$ in (2.11), obtained by linearizing (1.1), are proved in appendix B. In section 2.4 we state the first of the two main results of the present paper (theorem 2.5), i.e. the existence of the invariant manifolds $\mathcal{W}_\pm$ for which (1.9) and (1.10) hold. The proof of theorem 2.5 is given in section 4; preliminarily, in section 3 we study manifolds which approximate $\mathcal{W}_\pm$. Stability of the invariant manifolds is discussed in section 2.5; proofs are given in section 5, precisely in theorems 5.5, 5.7 and 5.9, which are the other main results of the paper. In section 6 we prove a lower bound for $F_L(m) - F_L(\hat{m}_L)$, for m in the basin of attraction of $\hat{m}_L$; this result is needed in the applications to tunnelling. In section 7 we discuss some extensions of the above results. In appendix C we study some regularizing properties of the evolution, needed in the proofs of section 5.

Sections 6 and 7 are an addendum where we prove for future reference some important properties which are a consequence of the theory developed in the previous sections. In section 6 we study the energy landscape $F_L(m)$ on the manifold $m \in \Sigma$, where Σ is the set of all m which are attracted by $\hat{m}_L$, so that $F_L(m) \geqslant F_L(\hat{m}_L)$ on Σ. In theorem 6.1 we prove a quadratic bound: for any $m \in \Sigma$, $F_L(m) - F_L(\hat{m}_L) \geqslant c_\Sigma \|m - \hat{m}_L\|_2^2$, where c_Σ is a positive constant and $\|\cdot\|_2$ is the L^2 norm in $[-L/2, L/2]$. In section 7 we sketch the extension to $d = 2$ of our results on the existence and stability of the invariant manifolds.

2. Definitions and results

2.1. Existence of dynamics

The velocity field $f_L(m)$ is Lipschitz when restricted to sets of the form $\{\|m\|_\infty \leqslant b\}$, $b < 1$. Then, by classical arguments, the Cauchy problem for (1.1) with initial datum $m \in L^\infty((-L/2, L/2); (-1, 1))$ has a unique, local (in time) solution, denoted by $S_t(m)$. See appendix A for details on this and the other statements in the present section.

To derive global existence we use *a priori* bounds, namely that if $\|m\|_\infty < 1$, then there is $b < 1$ such that for all t, $\|S_t(m)\|_\infty < b$, a statement which follows from the comparison theorem. Recall that a smooth function $v(x, t)$ is a sub-solution (respectively super-solution) of (1.1) if

$$v_t \leqslant f_L(v) \qquad \text{(respectively } v_t \geqslant f_L(v)\text{)}. \tag{2.1}$$

Proposition 2.1. *Let $m \in L^\infty((-L/2, L/2); (-1, 1))$. If v is a sub-solution (respectively, super-solution) of (1.1) and $v(\cdot, 0) \leqslant m$ (respectively, $v(\cdot, 0) \geqslant m$) then*

$$v(\cdot, t) \leqslant S_t(m) \qquad \text{(respectively } v(\cdot, t) \geqslant S_t(m)\text{)}. \tag{2.2}$$

In this way we will prove that $S_t(m)$ is well defined for all $m \in L^\infty((-L/2, L/2); (-1, 1))$ and all $t > 0$. Moreover

Proposition 2.2. *If m and $\tilde{m}$ are both in $L^\infty((-L/2, L/2); (-1, 1))$, then for any $t > 0$*

$$\|S_t(m) - S_t(\tilde{m})\|_\infty \leqslant \mathrm{e}^t \|m - \tilde{m}\|_\infty. \tag{2.3}$$

By continuity S_t extends to $L^\infty((-L/2, L/2); [-1, 1])$.

Proposition 2.3. *Let $\|m\|_\infty \leqslant 1$. For any $t > 0$, $\|S_t(m)\|_\infty < 1$, $S_t(m)$ solves (1.1) for $t > 0$, and $S_t(m) \to m$ in L^∞ as $t \to 0$.*

Using sub and super solutions, existence, uniqueness and regularity theorems extend to the case of a bounded, smooth external force as considered in section 5, see (5.1).

2.2. Instantons manifold and the finite volume instanton

In [13, 14] it is proved that there exists a unique solution $\bar{m}$ of (1.7), called an instanton, which is a strictly increasing $\mathcal{C}^\infty$ function, such that $\bar{m}(0) = 0$ and satisfying $\bar{m}(x) = -\bar{m}(-x)$ for any $x \in \mathbb{R}$. Moreover, with $\alpha > 0$ such that

$$\beta(1 - m_\beta^2) \int J(0, z)\mathrm{e}^{\alpha z}\, \mathrm{d}z = 1, \tag{2.4}$$

there are $a > 0$, $\alpha_0 > \alpha$, and $c > 0$ so that, for all $x \geqslant 0$,

$$|\bar{m}(x) - (m_\beta - a\mathrm{e}^{-\alpha x})| + |\bar{m}'(x) - \alpha a\mathrm{e}^{-\alpha x}| + |\bar{m}''(x) + \alpha^2 a\mathrm{e}^{-\alpha x}| \leqslant c\mathrm{e}^{-\alpha_0 x}, \tag{2.5}$$

where $\bar{m}'$ and $\bar{m}''$ are, respectively, the first and second derivatives of $\bar{m}$.

Given any $\xi \in \mathbb{R}$ we denote by

$$\bar{m}_\xi(x) = \bar{m}(x - \xi), \qquad x \in \mathbb{R},$$

the instanton with centre ξ and $\{\bar{m}_\xi : \xi \in \mathbb{R}\}$ the instantons manifold. Any solution of (1.7) which is definitively strictly negative (respectively, positive) as $x \to -\infty$ (respectively, as $x \to \infty$) is an element of the instantons manifold.

In [4, 7, 8, 11] it was proved that a finite volume analogue of the instanton does exist. If L is large enough there is in fact a C^∞ solution $\hat{m}_L$ of (1.3), called the finite volume instanton, which is antisymmetric and strictly increasing; moreover $\hat{m}_L$ converges to $\bar{m}$ as $L \to \infty$ in the sense of (1.8). For finite L, however, there is no analogue of the instantons manifold, but we will prove that there are invariant manifolds starting from $\hat{m}_L$ where the motion is 'extremely slow' as L becomes large. Uniqueness of finite volume instantons is proved (in the above quoted references) in the following sense. Given $\epsilon > 0$ and $r \in (0, 1)$ let

$$B_{\epsilon,r} := \left\{ m \in L^\infty((-L/2, L/2); (-1, 1)) : \inf_{|\xi| \leqslant rL/2} \|m - \bar{m}_\xi \mathbf{1}_{|x| \leqslant L}\|_\infty \leqslant \epsilon \right\}. \tag{2.6}$$

Then for any $r \in (0, 1)$ and $\epsilon > 0$ small enough, there is $L_{\epsilon,r}$ such that $\hat{m}_L$ is the only solution of (1.3) in $B_{\epsilon,r}$ for any $L \geqslant L_{\epsilon,r}$.

2.3. Spectral properties of linearized operators

The content of this section is based on [7, 8, 10]; in appendix B we fill in the missing details.

Let $m \in L^\infty((-L/2, L/2); (-1, 1))$ and define

$$\Omega_m u := -u + p_m J * u, \qquad p_m(x) := \beta[1 - m(x)^2]. \tag{2.7}$$

If m is a stationary solution of (1.11) then Ω_m is obtained by linearizing (1.11) around m. Ω_m is self-adjoint in $L^2((-L/2, L/2), p_m^{-1} dx)$. We will denote by $\langle \cdot, \cdot \rangle$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the scalar product, the L^2 and the L^∞ norm in $L^2((-L/2, L/2), dx)$ and add a subscript p_m when the scalar product is with weight p_m^{-1}.

In [7, 8, 10] it is proved that there are positive constants $\hat{c}_\pm$, ω and $\epsilon(L)$ so that for any L large enough if $\|m - \hat{m}_L\|_\infty \leqslant \epsilon(L)$, then Ω_m has a maximal eigenvalue λ (dependence on m is not made explicit)

$$\hat{c}_- \mathrm{e}^{-2\alpha L} \leqslant \lambda \leqslant \hat{c}_+ \mathrm{e}^{-2\alpha L}, \tag{2.8}$$

with eigenfunction e smooth and strictly positive. To the left of λ the spectrum (regarding Ω_m as an operator on $L^2((-L/2, L/2), p_m^{-1} dx)$) has a gap, namely

$$\langle u, \Omega_m u \rangle_{p_m} \leqslant -\omega \langle u, u \rangle_{p_m}, \qquad \text{for } u \text{ such that } \langle u, e \rangle_{p_m} = 0. \tag{2.9}$$

The semigroup $\mathrm{e}^{\Omega_m t}$ has a spectral gap also in L^∞, in fact it has been proved in [7, 8, 10] that there are positive constants λ', ω' and c' so that

$$\|\mathrm{e}^{\Omega_m t} u\|_\infty \leqslant c' \mathrm{e}^{\lambda' t} \|u\|_\infty, \qquad \|\mathrm{e}^{\Omega_m t} u\|_\infty \leqslant c' \mathrm{e}^{-\omega' t} \|u\|_\infty, \ \langle u, e \rangle_{p_m} = 0. \tag{2.10}$$

We prove in appendix B that the L^2 analysis extends to the operator $\mathcal{L}_m = p_m^{-1}\Omega_m$, explicitly

$$\mathcal{L}_m u := -\frac{1}{p_m}u + J^{\text{neum}} * u. \tag{2.11}$$

The operator $\mathcal{L}_m$ is obtained by linearizing (1.1) around m. Referring to appendix B for details of the proof (based on the previous analysis on Ω_m) we have the following result.

Theorem 2.4. *There are positive constants $c_\pm$ and ω such that the following holds. Given any $\delta > 0$ there exists $\epsilon(L,\delta)$ so that for L large enough if $\|m-\hat{m}_L\|_2 \leqslant \epsilon(L,\delta)$ and $\|m\|_\infty < 1-\delta$, then the L^2 norm of the operator $\mathcal{L}_m$ is bounded by $1+\beta^{-1}\text{arctanh}''((1+m_\beta)/2)$, and*

(i) $\mathcal{L}_m$ has a positive eigenvalue λ_m satisfying

$$c_- \mathrm{e}^{-2\alpha L} \leqslant \lambda_m \leqslant c_+ \mathrm{e}^{-2\alpha L}, \tag{2.12}$$

with eigenfunction e_m which is smooth and strictly positive;

(ii) $\langle u, \mathcal{L}_m u\rangle \leqslant -\omega \langle u, u\rangle, \qquad u \in L^2((-L/2, L/2); [-1,1]), \qquad \langle u, e_m\rangle = 0;$ (2.13)

(iii) let $\frac{\delta e_m}{\delta m}$ be the linear operator such that $\left.\frac{\mathrm{d}e_{m(s)}}{\mathrm{d}s}\right|_{s=0} = \frac{\delta e_m}{\delta m}\left.\frac{\mathrm{d}m(s)}{\mathrm{d}s}\right|_{s=0}$ for any smooth curve $m(s)$, $m(0) = m$. Then there is $c_1 > 0$ so that

$$\left\| \frac{\delta e_m}{\delta m} \right\|_2 \leqslant c_1. \tag{2.14}$$

Notice that for L large enough, $\lambda_m - \omega < 0$ and there is a constant $c > 0$ so that

$$\|\mathcal{L}_m^{-1} u\|_2 \leqslant c \qquad \text{for any } u \text{ such that } \langle u, e_m\rangle = 0. \tag{2.15}$$

2.4. Invariant manifolds: existence

The first of the two main results in this paper is the following theorem which extends to the present case results proved in [4] for the evolution (1.11).

Theorem 2.5. *There is $L_0 > 0$ so that the following holds. For any $L > L_0$, there are two distinct one-dimensional manifolds*

$$\mathcal{W}_\pm = \{v^\pm(x,s) : |x| \leqslant L/2, s \in \mathbb{R}\},$$

for which (1.9)–(1.10) hold.

The proof is reported in section 4.

2.5. Invariant manifolds: stability

There is an L^2 neighbourhood of $\mathcal{W}_+$ which is attracted by $m^{(+)}$ (for $\mathcal{W}_-$ the analogous statements hold). As discussed at the beginning of section 5 the statement is almost evident due to the continuity of S_t and the stability of $m^{(+)}$. Less trivial is the property that for any $\epsilon > 0$ there is $\delta > 0$ so that if $\|m - \hat{m}_L\|_2 < \delta$ and $\lim_{t\to\infty} \|S_t(m) - m^{(+)}\|_2 = 0$, then

$$\inf_{s\in\mathbb{R}} \|S_t(m) - v^{(+)}(s)\|_2 < \epsilon \qquad \forall t > 0. \tag{2.16}$$

A stronger statement is actually proved in theorems 5.5, 5.7 and 5.9 (the other main results in this paper) where also sufficiently small external forces are added to (1.1). In section 6 we prove that the initial condition can be controlled in terms of the energy, as we will prove a lower bound for $F_L(m) - F_L(\hat{m}_L)$, $m \in \Sigma := \{u : \lim_{t\to\infty} \|S_t(u) - \hat{m}_L\|_2 = 0\}$, proportional to $\|m - \hat{m}_L\|_2^2$.

2.6. Tunnelling

Let

$$\mathcal{U}_\tau[m^{(-)}, m^{(+)}] := \Big\{u \in C^\infty((-L/2, L/2) \times (0, \tau); (-1, 1)) : \lim_{t\to 0^+} u(\cdot, t) = m^{(-)},$$
$$\lim_{t\to\tau^-} u(\cdot, t) = m^{(+)}\Big\}. \tag{2.17}$$

the set of tunnelling events in a time τ;

$$I_\tau(u) := \frac{1}{4}\int_0^\tau \int_{-L}^{L} K(x, t)^2\, \mathrm{d}x\, \mathrm{d}t, \qquad K = u_t - f_L(u) \tag{2.18}$$

the 'cost' of an orbit u and

$$P_{[m^{(-)},m^{(+)}]} := \inf_{\tau>0} \inf_{u\in\mathcal{U}_\tau[m^{(-)},m^{(+)}]} I_\tau(u) \tag{2.19}$$

the tunnelling penalty. Then in [1] it is proved that

Theorem 2.6. *For any L large enough,*

$$P_{[m^{(-)},m^{(+)}]} = F_L(\hat{m}_L). \tag{2.20}$$

The proof of theorem 2.6 in [1] shows that any minimizing sequence in (2.19) passes arbitrarily close to $\hat{m}_L$. The validity of theorem 2.6 extends to $d = 2$ dimensions for the model defined in the square $Q_L = [-L/2, L/2]^d$. In [3] it is proved that the penalty $P_{[m^{(-)},m^{(+)}]}$ is again given by the $d = 1$ energy $F_L(\hat{m}_L)$ multiplied by the factor $2L$. Indeed the function $\hat{m}_L^e$ on Q_L which only depends on the x-coordinate and as a function of x is equal to $\hat{m}_L$ is stationary and again any minimizing sequence passes arbitrarily close to $\hat{m}_L^e$. In [3] using our theorems 5.5, 5.7 and 5.9 (whose validity extends to $d = 2$, as briefly discussed in section 7), it is shown that any minimizing sequence is an orbit which follows backward in time $v^{(-)}$ from $m^{(-)}$ to $\hat{m}_L^e$ and then forward in time $v^{(+)}$ from $\hat{m}_L^e$ to $m^{(+)}$ (the analogous statement holds in $d = 1$ as well).

While theorem 2.6 answers the first question about tunnelling, namely the minimal cost for the tunnelling to occur, the above result specifies also the way the tunnelling occurs. While it is well established that a minimizing sequence can be obtained by following the reversed flow on the invariant manifolds, see [15], our statements in theorems 5.5, 5.7 and 5.9 complete the picture by saying that 'this is in fact the only possible way', as any other pattern would lead to a larger penalty.

3. Approximants of the invariant manifolds

In this section we study manifolds which approximate the invariant manifolds $\mathcal{W}_\pm$ of theorem 2.4. In agreement with section 2.2 we denote by $\hat{m}_L$ the finite volume instanton and write $\hat{\lambda}$ and $\hat{e}$ for the maximal eigenvalue and corresponding eigenfunction of

$$\hat{\mathcal{L}} := \mathcal{L}_{\hat{m}_L}, \tag{3.1}$$

recalling from theorem 2.4 that $\hat{\lambda} > 0$ and $\hat{e} := e_{\hat{m}_L}$ is strictly positive and smooth, and we normalize $\hat{e}$ so that $\langle \hat{e}, \hat{e}\rangle = 1$. Finally we denote by $\hat{\omega}$ the L^2 spectral gap of $\hat{\mathcal{L}}$ and by ω the spectral gap of $\mathcal{L}_m$ when m is in a suitable neighbourhood of $\hat{m}_L$, see (2.13). Recall that the remaining part of the spectrum is strictly negative.

The proof of theorem 2.5 is based on constructing portions of the invariant manifolds $\mathcal{W}_\pm$ in a small neighbourhood of $\hat{m}_L$ and then prove that such manifolds can be continued reaching, respectively, $m^{(+)}$ and $m^{(-)}$. By the symmetry under change of sign it suffices to consider the former case to which we restrict ourselves in the following. The proof is in spirit close to the one in [4] relative to the evolution $u_t = -u + \tanh\{\beta J^{\text{neum}} * u\}$, but the absence in our case of an L^∞ spectral gap for the linearized evolution around $\hat{m}_L$ complicates the proofs.

3.1. a, t-coordinates of m

In a small neighbourhood of $\hat{m}_L$, the manifold $\mathcal{W}_+$ is to a linear order approximation given by

$$\mathcal{M}_+ = \{v_a = \hat{m}_L + a\hat{e}, a > 0\} \tag{3.2}$$

(we will only consider small positive values of a). However if nonlinear effects are taken into account,

$$v_a(t) = S_t(v_a) \tag{3.3}$$

leaves $\mathcal{M}_+$ as soon as t is positive, while, in contrast, the solution of the linearized evolution around $\hat{m}_L$ is $v_{e^{\hat{\lambda}t}a} \in \mathcal{M}_+$ (for t not too large). We will prove that in a suitably small neighbourhood of $\hat{m}_L$, the orbits $v_a(\cdot)$ are close to $\mathcal{M}_+$ and to the linearized evolution on $\mathcal{M}_+$ converging, as $a \to 0$ to a limit manifold which in the end will be identified with $\mathcal{W}_+$.

We thus have two natural ways to parametrize points in a small neighbourhood of $\mathcal{M}_+$, by using orthogonal projections either onto $\mathcal{M}_+$ or onto $v_a(\cdot)$.

Definition 3.1. *The a-coordinate of m is*

$$a(m) = \langle m - \hat{m}_L, \hat{e}\rangle. \tag{3.4}$$

We denote by $a(t; b)$ the a-coordinate of $v_b(t)$, so that we can write

$$v_b(t) = v_{a(t;b)} + \psi(t; b), \qquad \langle\psi(t; b), \hat{e}\rangle = 0. \tag{3.5}$$

Given $b > 0$, the t-coordinate of m relative to the orbit $\{v_b(\cdot)\}$ is a nonnegative number $\tau_{m;b}$ such that

$$\langle m - v_b(\tau_{m;b}), e_{v_b(\tau_{m;b})}\rangle = 0. \tag{3.6}$$

We will prove later that the t-coordinate relative to the orbit $\{v_b(\cdot)\}$, $b > 0$, is well defined provided m is sufficiently close to $\mathcal{M}_+$ and b is sufficiently smaller than the a-coordinate of m (see proposition 3.8(ii)). We will also establish relations between a- and t-coordinates.

We start with some *a priori* estimates on the orbit $S_t(\hat{m}_L + u_0)$. We suppose u_0 to be a small, smooth function and study the orbit up to times t which grow proportionally to $|\log(\|u_0\|_2)|$. We linearize around $\hat{m}_L$ and control the nonlinear terms using smallness and smoothness of the initial datum. We prove that the orbit follows $\mathcal{M}_+$ with orthogonal deviations which are much smaller than the displacement along $\mathcal{M}_+$.

Lemma 3.2. *There exists a constant $c_0' > 0$ so that the following holds. Let $u_0 \in \mathcal{C}^1([-L/2, L/2])$ be such that*

$$\|u_0\|_2^{2/3} < \frac{\hat{\lambda}}{c_0'}, \qquad \|u_0\|_\infty \leqslant \frac{1 + m_\beta}{2}. \tag{3.7}$$

Define

$$\sigma = \sigma_{u_0} := \frac{1}{\hat{\lambda}} \log\left(\frac{\hat{\lambda}}{c_0'\|u_0\|_2^{2/3}}\right) > 0, \tag{3.8}$$

$$u(t) := S_t(\hat{m}_L + u_0) - \hat{m}_L.$$

Then

$$\|u(t) - \mathrm{e}^{\hat{\mathcal{L}}t}u_0\|_2 < \frac{c_0'}{\hat{\lambda}}(\mathrm{e}^{\hat{\lambda}t}\|u_0\|_2)^{5/3} \qquad \forall t \in [0,\sigma], \tag{3.9}$$

$$\|u(t)\|_2 \leqslant 2\mathrm{e}^{\hat{\lambda}t}\|u_0\|_2 \qquad \forall t \in [0,\sigma]. \tag{3.10}$$

Proof. We will prove the lemma with $c_0' = 2^6 CC_2$, $C = C_{t=0}$ as in (C.13), $C_2 = (2\beta)^{-1}\,\mathrm{arctanh}''(x)$, $x = (m_\beta + 1)/2$, see (C.15).

Since $S_t(\hat{m}_L) = \hat{m}_L$ and $f_L(\hat{m}_L) = 0$ we have

$$u_t = \hat{\mathcal{L}}u + [f_L(\hat{m}_L + u) - f_L(\hat{m}_L) - \hat{\mathcal{L}}u], \tag{3.11}$$

which implies

$$u = \mathrm{e}^{\hat{\mathcal{L}}t}u_0 + \int_0^t \mathrm{d}s\, \mathrm{e}^{\hat{\mathcal{L}}(t-s)}[f_L(\hat{m}_L + u) - f_L(\hat{m}_L) - \hat{\mathcal{L}}u]. \tag{3.12}$$

Then by (C.15), (C.13), recalling that $\hat{\lambda}$ is the maximal eigenvalue of $\hat{\mathcal{L}}$, and using Jensen's inequality (with respect to the measure $\mathrm{e}^{\hat{\lambda}(t-s)}\,\mathrm{d}s/\hat{\lambda}^{-1}$)

$$\|u(t) - \mathrm{e}^{\hat{\mathcal{L}}t}u_0\|_2^2 \leqslant CC_2 \int_0^t \mathrm{e}^{\hat{\lambda}(t-s)}\|u(s)\|_2^{10/3}\,\mathrm{d}s. \tag{3.13}$$

We prove (3.9) by contradiction. We suppose, without loss of generality, that $\|u_0\|_2 > 0$. Let $T \leqslant \sigma$ be the first time when the inequality (3.9) becomes an equality. Note that for any s such that (3.9) holds

$$\|u(s)\|_2 \leqslant \mathrm{e}^{\hat{\lambda}s}\|u_0\|_2 + \frac{c_0'}{\hat{\lambda}}(\mathrm{e}^{\hat{\lambda}s}\|u_0\|_2)^{5/3}. \tag{3.14}$$

Define

$$\rho_t := [\mathrm{e}^{\hat{\lambda}t}\|u_0\|_2]^{2/3}.$$

We use (3.14) in (3.13) with $t = T$. We estimate the integral on the right-hand side of (3.13) as follows

$$\begin{aligned}
&\int_0^T \mathrm{e}^{\hat{\lambda}(T-s)}\left[\mathrm{e}^{\hat{\lambda}s}\|u_0\|_2 + \frac{c_0'}{\hat{\lambda}}(\mathrm{e}^{\hat{\lambda}s}\|u_0\|_2)^{5/3}\right]^{10/3}\mathrm{d}s \\
&\qquad \leqslant \int_0^T \mathrm{e}^{\hat{\lambda}(T-s)+(10/3)\hat{\lambda}s}\|u_0\|_2^{10/3}\left[1 + \frac{c_0'}{\hat{\lambda}}\rho_s\right]^{10/3}\mathrm{d}s \\
&\qquad \leqslant \left(1 + \frac{c_0'\rho_\sigma}{\hat{\lambda}}\right)^{10/3}\|u_0\|_2^{10/3}\mathrm{e}^{\hat{\lambda}T}\int_0^T \mathrm{e}^{\frac{7}{3}\hat{\lambda}s}\,\mathrm{d}s < \frac{6}{7\hat{\lambda}}(\mathrm{e}^{\hat{\lambda}T}\|u_0\|_2)^{10/3}.
\end{aligned} \tag{3.15}$$

In the last inequality we have used that $c_0'\rho_\sigma < \hat{\lambda}$, which follows from (3.7). Observing that $\left(2CC_2\frac{3}{7\hat{\lambda}}\right)^{1/2} < \frac{c_0'}{\hat{\lambda}}$ for L sufficiently large (see (2.12)), from (3.15) and (3.13) we get

$$\|u(T) - \mathrm{e}^{\hat{\mathcal{L}}T}u_0\|_2 < \frac{c_0'}{\hat{\lambda}}(\mathrm{e}^{\hat{\lambda}T}\|u_0\|_2)^{5/3}. \tag{3.16}$$

We have thus reached a contradiction and (3.9) is proved.

By (3.9) and (3.14) it follows that

$$\|u(t)\|_2 \leqslant \left(1 + \frac{c_0'}{\hat{\lambda}}\rho_\sigma\right)\mathrm{e}^{\hat{\lambda}t}\|u_0\|_2 \leqslant 2\mathrm{e}^{\hat{\lambda}t}\|u_0\|_2 \tag{3.17}$$

for all $t \leqslant \sigma$, and (3.10) is proved. □

In the next lemmas we study the orbits $S_t(\hat{m}_L + b\hat{e}) = v_b(t)$ with $b > 0$ small, first for short times and then, by an iterative procedure, for longer times.

Lemma 3.3. *Let c_0' be as in lemma 3.2, and define*

$$\sigma_b := \frac{1}{\hat\lambda}\log\left(\frac{\hat\lambda}{c_0' b^{2/3}}\right), \qquad b \in \left(0, \left(\frac{\hat\lambda}{c_0'}\right)^{3/2}\right). \tag{3.18}$$

Let $\epsilon_0 \in \left(0, \frac{1-m_\beta}{4\|\hat e\|_\infty}\right)$. *Then*

$$\|v_b(t) - v_{e^{\hat\lambda t}b}\|_2 \leqslant \frac{c_0'}{\hat\lambda}(e^{\hat\lambda t}b)^{5/3} \qquad \forall b \in (0,\epsilon_0), \quad \forall t \in [0,\sigma_b]. \tag{3.19}$$

Moreover

$$\|\psi(t;b)\|_2 \leqslant \frac{c_0'}{\hat\lambda}(e^{\hat\lambda t}b)^{5/3}, \quad |a(t;b) - e^{\hat\lambda t}b| \leqslant \frac{c_0'}{\hat\lambda}(e^{\hat\lambda t}b)^{5/3} \qquad \forall t \in [0,\sigma_b]. \tag{3.20}$$

Proof. We have $\|v_b - \hat m_L\|_\infty \leqslant \|\hat e\|_\infty b < (1-m_\beta)/2$. Then (3.19) follows from lemma 3.2 observing that

$$e^{\hat{\mathcal{L}} t} b\hat e = e^{\hat\lambda t} b\hat e.$$

By (3.5) we have $\|\psi(t;b)\|_2^2 = \inf_{a\in\mathbb{R}} \|v_b(t) - v_a\|_2^2 \leqslant \|v_b(t) - v_{e^{\hat\lambda t}b}\|_2^2 \leqslant \left\{\frac{c_0'}{\hat\lambda}[e^{\hat\lambda t}b]^{5/3}\right\}^2$, where the last inequality follows from (3.19). Then the first inequality in (3.20) follows. We write

$$v_b(t) = v_{e^{\hat\lambda t}b} + \phi, \qquad \phi := v_b(t) - v_{e^{\hat\lambda t}b}$$

so that $a(t;b) = e^{\hat\lambda t}b + \langle \phi, \hat e\rangle$ and the second inequality in (3.20) then follows from (3.19). □

Lemma 3.4. *Let* $\epsilon_0 \in \left(0, \frac{1-m_\beta}{4\|\hat e\|_\infty}\right)$. *There exists* $\epsilon_1 \in \left(0, \min\left(\epsilon_0, \frac{\hat\lambda}{8c_0'}\right)\right)$ *so that the following holds. If at time* $t^* > 0$ *equality (3.5) holds with* $b \in (0,\epsilon_1)$ *and*

$$a(t^*;b) \leqslant \epsilon_1, \qquad \|\psi(t^*;b)\|_2 \leqslant \frac{4c_0'}{\hat\lambda} a(t^*;b)^{5/3}, \tag{3.21}$$

then (3.5) holds at time $t^* + \sigma_{a(t^*;b)}$ *and*

$$\|\psi(t^* + \sigma_{a(t^*;b)}; b)\|_2 \leqslant \frac{4c_0'}{\hat\lambda} a(t^* + \sigma_{a(t^*;b)}; b)^{5/3}. \tag{3.22}$$

Proof. Set for simplicity $a_* := a(t^*;b)$. We observe that from (3.5), (3.2) and (2.13),

$$\|e^{\hat{\mathcal{L}} s}(v_b(t^*) - \hat m_L) - e^{\hat\lambda s} a_* \hat e\|_2 \leqslant e^{-\hat\omega s}\|\psi(t^*;b)\|_2, \qquad s \geqslant 0.$$

Then (3.9) and (3.19) imply that for $s \in (0, \sigma_{a_*}]$,

$$\begin{aligned}
\|v_b(t^*+s) - v_{e^{\hat\lambda s}a_*}\|_2 &= \|S_s(v_b(t^*)) - \hat m_L - e^{\hat\lambda s}a_*\hat e\|_2 \\
&\leqslant \|u(s) - e^{\hat{\mathcal{L}} s}(v_b(t^*) - \hat m_L)\|_2 + \|e^{\hat{\mathcal{L}} s}(v_b(t^*) - \hat m_L) - e^{\hat\lambda s}a_*\hat e\|_2 \\
&\leqslant e^{-\hat\omega s}\|\psi(t^*;b)\|_2 + \frac{c_0'}{\hat\lambda}(e^{\hat\lambda s}[a_* + \|\psi(t^*;b)\|_2])^{5/3} \\
&\leqslant e^{-\hat\omega s}\frac{4c_0'}{\hat\lambda}a_*^{5/3} + \frac{c_0'}{\hat\lambda}\left(e^{\hat\lambda s}\left[a_* + \frac{4c_0'}{\hat\lambda}a_*^{5/3}\right]\right)^{5/3} \\
&\leqslant \frac{c_0'}{\hat\lambda}(e^{\hat\lambda s}a_*)^{5/3}\left(4e^{-\hat\omega s}e^{-\frac{5}{3}\hat\lambda s} + \left[1 + \frac{4c_0'}{\hat\lambda}\epsilon_1^{2/3}\right]^{5/3}\right).
\end{aligned} \tag{3.23}$$

Recalling the choice of a_*, the definition (3.18) of σ_{a_*} and by taking $\epsilon_1 > 0$ sufficiently small, from (3.23) we get

$$\|v_b(t^* + \sigma_{a_*}) - v_{\mathrm{e}^{\hat\lambda\sigma_{a*}}a_*}\|_2 \leqslant \frac{2c_0'}{\hat\lambda}(\mathrm{e}^{\hat\lambda\sigma_{a*}}a_*)^{5/3}. \tag{3.24}$$

From the second inequality in (3.20) we get that

$$|a(t^* + \sigma_{a_*}; b) - \mathrm{e}^{\hat\lambda\sigma_{a*}}a_*| \leqslant \frac{c_0'}{\hat\lambda}(\mathrm{e}^{\hat\lambda\sigma_{a*}}a_*)^{\frac{5}{3}}.$$

Recalling again the definition of σ_{a_*}, for ϵ_1 small enough, we get

$$a(t^* + \sigma_{a_*}; b) \geqslant \mathrm{e}^{\hat\lambda\sigma_{a*}}a_*\left(1 - \frac{c_0'}{\hat\lambda}\left(\frac{\hat\lambda}{c_0'}\epsilon_1^{1/3}\right)^{2/3}\right) \geqslant \mathrm{e}^{\hat\lambda\sigma_{a*}}a_* 2^{-3/5}.$$

Thus

$$\mathrm{e}^{\hat\lambda\sigma_{a*}}a_* \leqslant a(t^* + \sigma_{a_*}; b)2^{3/5}. \tag{3.25}$$

Then from (3.24) and (3.25) we get

$$\|v_b(t^* + \sigma_{a_*}) - v_{\mathrm{e}^{\hat\lambda\sigma_{a*}}a_*}\|_2 \leqslant 2\frac{c_0'}{\hat\lambda}\{\mathrm{e}^{\hat\lambda\sigma_{a*}}a_*\}^{5/3} \leqslant 4\frac{c_0'}{\hat\lambda}a(t^* + \sigma_{a_*}; b)^{5/3}.$$

Since $\|\psi(t^* + \sigma_{a_*}; b)\|_2 = \inf_{a'\in\mathbb{R}}\|v_b(t^* + \sigma_{a_*}) - v_{a'}\|_2$, (3.22) follows. □

Remark 3.5. Note that the constant $4c_0'/\hat\lambda$ in conclusion (3.22) is the same as in (3.21).

We set
$\|\hat{\mathcal{L}}\|$ the (L^2) norm of $\hat{\mathcal{L}}$,
c_0' is as in lemma 3.2,
C_2 defined as in (C.15), with a replaced by $1 - \sigma$, $\sigma > 0$ small enough,
$C = C_{t=0}$ as in (C.13), where m is of the form $\hat m_L + b\hat e$,
$\tilde c_2 := \|\hat{\mathcal{L}}\|\frac{4c_0'}{\hat\lambda} + 2C_2C$,
c_1 the constant appearing in (2.14), for m of the form $\hat m_L + b\hat e$,
ϵ_1 as in lemma 3.4 and also such that

$$4\hat\lambda c_1\epsilon_1^{1/3} < 1, \qquad \frac{4c_0' + \tilde c_2}{\hat\lambda}\epsilon_1^{2/3} < \frac{1}{4}, \tag{3.26}$$

$T(\epsilon_1; b)$ the time for which $a(T(\epsilon_1; b); b) = \epsilon_1$, for a given $b \in (0, \epsilon_1)$.

In the next proposition we prove that if b is small enough then (3.22) holds for all times $t \in (0, T(\epsilon_1; b))$, the function $a(t; b)$ is strictly positive and increasing and $f_L(v_b(t))$ is essentially parallel to $\hat e$.

Proposition 3.6. *There exists $\epsilon_2 \in (0, \epsilon_1]$ so that for all $b \in (0, \epsilon_2)$ and all $t \in (0, T(\epsilon_1; b))$:*

$$\|v_b(t) - v_{a(t;b)}\|_2 \leqslant \frac{4c_0'}{\hat\lambda}a(t; b)^{5/3}, \tag{3.27}$$

$t \to a(t; b)$ is strictly positive, differentiable and strictly increasing with respect to t,

$$\left|\frac{\mathrm{d}a(t; b)}{\mathrm{d}t} - \hat\lambda\, a(t; b)\right| \leqslant \tilde c_2\, a(t; b)^{5/3}, \tag{3.28}$$

and

$$\|f_L(v_b(t)) - \hat\lambda\, a(t; b)\hat e\|_2 \leqslant \tilde c_2 a(t; b)^{5/3}, \tag{3.29}$$

$$|\langle f_L(v_b(t)), e_{v_b(t)}\rangle - \hat\lambda\, a(t; b)| \leqslant 2\tilde c_2 a(t; b)^{5/3} \tag{3.30}$$

$$|\langle f_L(v_b(t)), e_{v_b(t)}\rangle| \geqslant \frac{\hat\lambda}{2}a(t; b) \geqslant \frac{\|f_L(v_b(t))\|_2}{4}. \tag{3.31}$$

Finally for all $t_1 < t_2 \leqslant T(\epsilon_1; b)$ *we have*

$$\mathrm{e}^{(t_2-t_1)(\hat\lambda-\tilde c_2 a(t_2;b)^{2/3})} \leqslant \frac{a(t_2;b)}{a(t_1;b)} \leqslant \mathrm{e}^{(t_2-t_1)(\hat\lambda+\tilde c_2 a(t_2;b)^{2/3})}, \tag{3.32}$$

$$\|v_b(t_2)-v_b(t_1)\|_2 \leqslant a(t_2;b)(1-\mathrm{e}^{-(t_2-t_1)(\hat\lambda+\tilde c_2 a(t_2;b)^{2/3})}) + 2\frac{4c_0'}{\hat\lambda}a(t_2;b)^{5/3}. \tag{3.33}$$

Proof. We fix $b \in (0,\epsilon_2)$ with $\epsilon_2 \in (0,\epsilon_1)$ small enough. We shorthand $a = a(t;b)$ with $t < T(\epsilon_1;b)$. Inequality (3.27) follows by an iterative procedure using lemmas 3.3 and 3.4.

Let us prove (3.29). Recalling the definition of $R(t)$ in (C.14), we have

$$f_L(v_b(t)) = \hat{\mathcal{L}}(v_b(t)-\hat m_L) + R(t) = \hat{\mathcal{L}}(v_b(t)-v_a+v_a-\hat m_L) + R(t),$$

so that

$$f_L(v_b(t)) - \hat\lambda a\hat e = \hat{\mathcal{L}}(v_b(t)-v_a) + R(t). \tag{3.34}$$

From (3.27) it follows that $\|\hat{\mathcal{L}}(v_b(t)-v_a)\|_2 \leqslant \|\hat{\mathcal{L}}\|\frac{4c_0'}{\hat\lambda}a^{5/3}$. From (C.15) and (C.13), we have $\|R(t)\|_2 \leqslant C_2C\|v_b(t)-\hat m_L\|_2^{5/3}$. Moreover

$$\|v_b(t)-\hat m_L\|_2 \leqslant \|v_b(t)-v_a\|_2 + a \leqslant \frac{4c_0'}{\hat\lambda}a^{5/3} + a \leqslant a\left(1+\frac{4c_0'}{\hat\lambda}\epsilon_1^{2/3}\right).$$

Since by (3.26), $\frac{4c_0'}{\hat\lambda}\epsilon_1^{2/3} \leqslant 1$, collecting together the above estimates we have

$$\|f_L(v_b(t)) - \hat\lambda a\hat e\|_2 \leqslant \|\hat{\mathcal{L}}\|\left\{\frac{4c_0'}{\hat\lambda}a^{5/3}\right\} + 2C_2Ca^{5/3} \leqslant \tilde c_2 a^{5/3} \tag{3.35}$$

and (3.29) is proved.

We now show (3.28). From (3.29) it follows that

$$|\langle f_L(v_b(t)),\hat e\rangle - \hat\lambda\, a(t;b)| \leqslant \tilde c_2 a(t;b)^{5/3}. \tag{3.36}$$

Then (3.28) follows from (3.4) and (3.36), and the fact that $a(t;b)$ is positive because $a(0,b) = b > 0$ and $a(t;b)$ is a continuous function of t. Observe that from (3.28) and (3.26) it follows that

$$\frac{\mathrm{d}a(t;b)}{\mathrm{d}t} \geqslant \hat\lambda a(t;b)\left[1-\frac{\tilde c_2}{\hat\lambda}\epsilon_1^{2/3}\right] \geqslant \frac{\hat\lambda}{2}a(t;b) > 0, \qquad \text{for all } t \leqslant T(\epsilon_1;b).$$

Let us now prove (3.30) and (3.31). We first observe that by the definition of $T(\epsilon_1;b)$, the fact that $b < \epsilon_2 < \epsilon_1$ and ϵ_1 satisfies (3.26), we get that

$$\frac{4c_0'}{\hat\lambda}a(t;b)^{2/3} \leqslant \frac{4c_0'}{\hat\lambda}\epsilon_1^{2/3} < \frac{1}{4}, \quad \frac{\tilde c_2}{\hat\lambda}a(t;b)^{2/3} \leqslant \frac{\tilde c_2}{\hat\lambda}\epsilon_1^{2/3} < \frac{1}{4}, \qquad \forall t \leqslant T(\epsilon_1;b). \tag{3.37}$$

Thus from (3.36) and (3.37) we get that

$$|\langle f_L(v_b(t)),\hat e\rangle| \geqslant \hat\lambda a(t;b) - \tilde c_2 a(t;b)^{5/3} \geqslant \frac{\hat\lambda}{2}a(t;b). \tag{3.38}$$

By (2.14) and (3.27)

$$\|e_{v_{a(t;b)}} - \hat e\|_2 \leqslant c_1 a(t;b), \qquad \|e_{v_b(t)} - e_{v_{a(t;b)}}\|_2 \leqslant c_1\frac{4c_0'}{\hat\lambda}a(t;b)^{5/3}. \tag{3.39}$$

We then have, by using (3.37),

$$\|e_{v_b(t)} - \hat e\|_2 \leqslant 2c_1 a(t;b). \tag{3.40}$$

By (3.29) $\|f_L(v_b(t))\|_2 \leqslant \hat{\lambda} a(t;b)\big(1 + \frac{\tilde{c}_2}{\hat{\lambda}}\epsilon_1^{2/3}\big)$ so that from (3.26) we get

$$\|f_L(v_b(t))\|_2 \leqslant 2\hat{\lambda} a(t;b). \tag{3.41}$$

From (3.36), (3.41), (3.40), (3.26) we get

$$|\langle f_L(v_b(t)), e_{v_b(t)}\rangle - \hat{\lambda}\, a(t;b)| \leqslant \tilde{c}_2 a(t;b)^{5/3} + |\langle f_L(v_b(t)), e_{v_b(t)} - \hat{e}\rangle|$$
$$\leqslant \tilde{c}_2 a(t;b)^{5/3} + 2\hat{\lambda} a(t;b)(2c_1 a(t;b)) \leqslant 2\tilde{c}_2 a(t;b)^{5/3},$$

and (3.30) is proved.

Finally, from (3.30) using (3.26) we get

$$|\langle f_L(v_b(t)), e_{v_b(t)}\rangle| \geqslant \hat{\lambda} a(t;b)\left[1 - \frac{2\tilde{c}_2}{\hat{\lambda}} a(t;b)^{2/3}\right] \geqslant \frac{\|f_L(v_b(t)\|_2}{4},$$

and this concludes the proof of (3.31).

It remains to prove (3.32) and (3.33). From (3.28), by using that $a(\cdot;b)$ is increasing, we get

$$a(t_1;b) + \int_{t_1}^{t_2} a(s;b)[\hat{\lambda} - \tilde{c}_2 a(t_2;b)]\,\mathrm{d}s \leqslant a(t_2;b) \leqslant a(t_1;b) + \int_{t_1}^{t_2} a(s;b)[\hat{\lambda} + \tilde{c}_2 a(t_2;b)]\,\mathrm{d}s$$

that gives (3.32).

Recalling that $v_b(t_i) = v_{a(t_i;b)} + \psi(t_i, b)$ for $i = 1, 2$, from (3.27) we get

$$\|v_b(t_2) - v_b(t_1)\|_2 \leqslant [a(t_2,b) - a(t_1,b)] + \|\psi(t_2,b)\|_2 + \|\psi(t_1,b)\|_2. \tag{3.42}$$

By using (3.32) and the monotonicity of a we get (3.33). □

3.2. *The function $\theta(a_1;b)$*

We will next describe functions in terms of their t-coordinates relative to orbits $v_b(\cdot)$, see definition 3.1. To this end it is convenient to give the following definition.

Definition 3.7. *We define the function $\theta(a_1;b)$ as the inverse of $t \to a(t;b)$:*

$$a(\theta(a_1;b);b) = a_1 \qquad \forall a_1 \in [b, \epsilon_1]. \tag{3.43}$$

Namely, $\theta(a_1;b)$ is the time when the orbit $v_b(\cdot)$ has its a-coordinate equal to a_1. The notion is well defined, since by proposition 3.6, $a(\cdot;b)$ is strictly increasing and $a(0;b) = b$.

For later applications we establish conditions for the existence of t-coordinates for functions m not necessarily of the form $v_a(t)$ (to which we may restrict in the proof of existence of $\mathcal{W}_+$). We will need m close to $\mathcal{M}_+$, the condition will involve the quantity $\|m - v_{a(m)}\|_2$ (see (3.44) below), $a(m)$ the a-coordinate of m defined in (3.4).

Proposition 3.8. *Let ϵ_1 be as in lemma 3.4 and $\epsilon_2 \in (0, \epsilon_1)$ be as in proposition 3.6. Then there exists $\epsilon_3 \in (0, \epsilon_2)$ such that, if we suppose*

$$\|m - v_{a(m)}\|_2 \leqslant \epsilon_3\, a(m), \quad a(m) \leqslant \epsilon_2, \qquad b \in \big(0, \tfrac{1}{2}a(m)\big) \tag{3.44}$$

and define

$$S := \frac{16}{\hat{\lambda} a(m)}\left\{\|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat{\lambda}} a(m)^{5/3}\right\}, \tag{3.45}$$

then

(i) $\theta(a(m); b) + S < T(\epsilon_1; b)$ *and for all* $t \in [\theta(a(m); b) - S, \theta(a(m); b) + S]$,

$$\frac{1}{2}a(m) < a(\theta(a(m); b) - S; b) \leqslant a(t; b) \leqslant a(\theta(a(m); b) + S; b) < 2a(m). \tag{3.46}$$

(ii) m *has a unique* t*-coordinate* $\tau_{m,b}$ *relative to* $\{v_b(\cdot)\}$ *in* $[\theta(a(m); b) - S, \theta(a(m); b) + S]$.

(iii) $\|m - v_b(t)\|_2 \leqslant 81\|m - v_{a(m)}\|_2 + 360\frac{c_0'}{\hat{\lambda}}a(m)^{5/3}$ *for all* $t \in [\theta(a(m); b) - S, \theta(a(m); b) + S]$.

(iv)
$$\|v_b(\theta(a(m); b)) - v_b(\tau_{m,b})\|_2 \leqslant 80\|m - v_{a(m)}\|_2 + 576\frac{4c_0'}{\hat{\lambda}}a(m)^{5/3}, \tag{3.47}$$

$$|a(m) - a(\tau_{m,b})| \leqslant 80\|m - v_{a(m)}\|_2 + 80\frac{4c_0'}{\hat{\lambda}}a(m)^{5/3}. \tag{3.48}$$

(v) If in particular m *has the form* $m = v_{a_1}(t^*)$ *then we take* $S = \frac{64c_0'}{\hat{\lambda}^2}a(m)^{2/3}$ *and*

$$\|m - v_b(\tau_{m,b})\|_2 \leqslant 360\frac{c_0'}{\hat{\lambda}}a(m)^{5/3}. \tag{3.49}$$

Proof. Set for simplicity $\theta = \theta(a(m); b)$ and recall that

$$a(\theta; b) = a(\theta(a(m); b); b) = a(m) < \epsilon_2 < \epsilon_1 \tag{3.50}$$

Since by (3.26)

$$\frac{\tilde{c}_2}{\hat{\lambda}}\epsilon_1^{2/3} < \frac{1}{4}, \tag{3.51}$$

we have that necessarily $T := T(\epsilon_1; b) > \theta$.

We assume by contradiction that $T \leqslant \theta + S$; then from (3.28), (3.50) and (3.51) we get

$$\epsilon_1 = a(T; b) \leqslant a(\theta) + \int_\theta^T \hat{\lambda}a(t; b)\left[1 + \frac{\tilde{c}_2}{\hat{\lambda}}\epsilon_1^{2/3}\right] \leqslant \epsilon_2 + \frac{5}{4}\hat{\lambda}\epsilon_1 S.$$

By taking ϵ_2 small enough the right-hand side of the above inequality is less than ϵ_1, which gives the desired contradiction.

From (3.33), (3.51) for ϵ_3 and ϵ_2 small enough we get that

$$\frac{a(\theta; b)}{a(\theta - S; b)} \leqslant e^{\frac{5}{4}\hat{\lambda}S} \leqslant 2, \qquad \frac{a(\theta + S; b)}{a(\theta; b)} \leqslant e^{\frac{5}{4}\hat{\lambda}S} \leqslant 2,$$

which implies (3.46).

Let us prove (iii) and (v). For $t \in [\theta - S, \theta + S]$ write

$$m - v_b(t) = [m - v_{a(m)}] + [v_{a(m)} - v_b(\theta)] + [v_b(\theta) - v_b(t)]. \tag{3.52}$$

From (3.27) (recall that $v_{a(m)} = v_{a(\theta;b)}$) we get

$$\|v_{a(m)} - v_b(\theta)\|_2 \leqslant \frac{4c_0'}{\hat{\lambda}}a(m)^{5/3}. \tag{3.53}$$

Recalling the monotonicity of $a(\cdot; b)$, we get

$$\|v_b(\theta) - v_b(t)\|_2 \leqslant a(\theta + S; b)\left\{(1 - e^{-5\hat{\lambda}S/4}) + 2\frac{4c_0'}{\hat{\lambda}}a(\theta + S; b)^{2/3}\right\}, \tag{3.54}$$

where the last inequality is a consequence of (3.33) (applied with $t_1 = t$ and $t_2 = \theta$), since

$$e^{(t_2 - t_1)(\hat{\lambda} + \tilde{c}_2 a(\theta; b)^{2/3})} \leqslant e^{S(\hat{\lambda} + \tilde{c}_2 a(\theta; b)^{2/3})},$$

and $\hat{\lambda} + \tilde{c}_2 a(\theta; b)^{2/3} \leqslant 5\hat{\lambda}/4$ from (3.51).

Hence, from (3.52), (3.53), (3.54) and (3.46) we obtain

$$\begin{aligned}\|m - v_b(t)\|_2 &\leqslant \|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat\lambda}\, a(m)^{5/3} \\ &\quad + a(\theta + S; b)\left\{(1 - \mathrm{e}^{-5\hat\lambda S/4}) + 2\frac{4c_0'}{\hat\lambda} a(\theta + S; b)^{2/3}\right\} \\ &\leqslant \|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat\lambda} a(m)^{5/3} + 2a(m)\left\{\frac{5\hat\lambda S}{2} + \frac{8c_0'}{\hat\lambda}[2a(m)]^{2/3}\right\}. \end{aligned} \tag{3.55}$$

Therefore, recalling the definition (3.46) of S we get

$$\|m - v_b(t)\|_2 \leqslant 81\|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat\lambda} a(m)^{5/3}[1 + 80 + 4\cdot 2^{2/3}].$$

Observe that for ϵ_2 small enough the above inequality implies the following inequality that we are going to use later.

$$\sup_{|t-\theta|\leqslant S} \|m - v_b(t)\|_2 \leqslant \frac{1}{8c_1}, \qquad c_1 \text{ defined in (2.14).} \tag{3.56}$$

Notice that if m has the form $m = v_{a_1}(t^*)$ then by letting $S = \frac{64c_0'}{\hat\lambda^2} a(m)^{2/3}$ in (3.55), for ϵ_2 small enough, we get (3.49).

We now show (ii). For $t \in [\theta - S, \theta + S]$ we define $\xi(t) := \langle m - v_b(t), e_{v_b(t)}\rangle$. We have to show that ξ vanishes at only one point of $[\theta - S, \theta + S]$.

We compute

$$\xi'(t) = -\langle f(v_b(t)), e_{v_b(t)}\rangle + \langle m - v_b(t), \frac{\delta e_v}{\delta v} f(v_b(t))\rangle, \tag{3.57}$$

where $\delta e_v/\delta v$ is the functional derivative at $v_b(t)$. We claim that $\xi'(t) < 0$.

From (3.30) and (3.51) it follows

$$\langle f(v_b(t)), e_{v_b(t)}\rangle \geqslant \hat\lambda a(t; b)\left[1 - 2\frac{\tilde c_2}{\hat\lambda}\epsilon_1^{2/3}\right] \geqslant \frac{1}{2}\,\hat\lambda a(t; b), \tag{3.58}$$

while from (2.14), (3.56), (3.29) and (3.51) we obtain

$$\left|\langle m - v_b(t), \frac{\delta e_v}{\delta v} f(v_b(t))\rangle\right| \leqslant c_1\|m - v_b(t)\|_2 \|f(v_b(t))\|_2 \leqslant c_1 \frac{1}{8c_1} 2\hat\lambda a(t; b). \tag{3.59}$$

Therefore, from (3.57), (3.58) and (3.59), we deduce

$$\xi'(t) \leqslant -\frac{1}{2}\hat\lambda a(t; b) + \frac{1}{4}\hat\lambda a(t; b) \leqslant -\frac{\hat\lambda}{4} a(t; b) < 0, \tag{3.60}$$

and the claim is proved. As a consequence, there exists at most one t-coordinate of m in $[\theta - S, \theta + S]$. To conclude the proof of (ii) it remains to show that ξ changes sign in $[\theta - S, \theta + S]$. We will show that if $\xi(\theta) > 0$ then $\xi(\theta + S) < 0$, and if $\xi(\theta) < 0$ then $\xi(\theta - S) > 0$.

We write $\psi = [m - v_{a(m)}] - [v_b(\theta) - v_{a(\theta,b)}]$, where we recall once more that $v_{a(m)} = v_{a(\theta,b)}$. From (3.27) and (3.53) we get

$$|\xi(\theta)| = |\langle \psi, e_{v_b(\theta)}\rangle| \leqslant \|m - v_{a(m)}\|_2 + \frac{4c_0'}{\hat\lambda} a(m)^{5/3} \tag{3.61}$$

Assume that $\xi(\theta) > 0$; from (3.60) and the fact (proved in proposition 3.6) that $a(\cdot; b)$ is strictly increasing, we get

$$\xi(\theta + S) \leqslant \xi(\theta) - \frac{\hat\lambda}{4}\int_\theta^{\theta+S} a(t; b)\mathrm{d}t < \xi(\theta) - \frac{\hat\lambda}{4} a(m) S \tag{3.62}$$

The definition (3.45) of S and (3.61) imply

$$\frac{\hat{\lambda}}{4}a(m)S = 4\|m - v_{a(m)}\|_2 + 4\frac{4c_0'}{\hat{\lambda}}a(m)^{5/3} \geqslant 4\xi(\theta). \tag{3.63}$$

Inequality (3.63) together with (3.62) implies that $\xi(\theta + S) < 0$.

If instead $\xi(\theta) < 0$ we have from (3.60) and (3.46)

$$\xi(\theta - S) \geqslant \xi(\theta) + \frac{\hat{\lambda}}{4}\int_{\theta-S}^{\theta} a(t; b)\,\mathrm{d}t \geqslant \xi(\theta) + \frac{\hat{\lambda}}{4}a(\theta - S)S \geqslant \xi(\theta) + \frac{\hat{\lambda}}{8}a(m)S$$

The definition (3.45) of S and (3.61) implies that $\frac{\hat{\lambda}}{8}a(m)S > -2\xi(\theta)$ that, in turn, implies $\xi(\theta - S) > 0$.

Thus $\xi(t)$ must change sign in $[\theta - S, \theta + S]$, and the existence of $\tau_{m;b}$ in $[\theta - S, \theta + S]$ follows.

It remains to prove (iv). From (3.33), (3.51), (3.45) and (3.46) we get

$$\begin{aligned}\|v_b(\theta) - v_b(\tau_{m;b})\|_2 &\leqslant a(\theta + S)[1 - \mathrm{e}^{2S\hat{\lambda}\frac{5}{4}}] + 2\frac{4c_0'}{\hat{\lambda}}a(\theta + S)^{5/3} \\ &\leqslant 2a(m)2S\hat{\lambda}\frac{5}{2} + 2\frac{4c_0'}{\hat{\lambda}}[2a(m)]^{5/3} \\ &\leqslant 80\|m - v_{a(m)}\|_2 + (80 + 64)\frac{4c_0'}{\hat{\lambda}}a(m)^{5/3},\end{aligned}$$

which gives (3.47).

From (3.28), (3.45) and (3.46) we get

$$\begin{aligned}|a(m) - a(\tau_{m;b})| &\leqslant \frac{5}{4}\hat{\lambda}\left|\int_{\theta-S}^{\theta+S} a(t; b)\mathrm{d}t\right| \leqslant \frac{5}{4}\hat{\lambda}a(\theta + S; b)2S \\ &\leqslant 80\|m - v_{a(m)}\|_2 + 80\frac{4c_0'}{\hat{\lambda}}a(m)^{5/3}.\end{aligned}$$

Thus (3.48) is proved and the proof of the proposition is concluded. □

Remark 3.9. From (ii) and (iii) of proposition 3.8 it follows in particular that

$$\|m - v_b(\tau_{m,b})\|_2 \leqslant 81\|m - v_{a(m)}\|_2 + 360\frac{c_0'}{\hat{\lambda}}a(m)^{5/3}. \tag{3.64}$$

Let us suppose that $m, b > 0$ and $t^* > 0$ are such that $S_t(m)$ satisfies the assumptions of proposition 3.8 for any $t \in [0, t^*]$, so that the t-coordinate $\tau_{S_t(m),b}$ of $S_t(m)$ relative to $\{v_b(\cdot)\}$ is well defined for all $t \in [0, t^*]$.

We set

$$\tau = \tau(t) = \tau_{S_t(m),b},$$

$$\mathcal{L}_{b,\tau} = \mathcal{L}_{v_b(\tau)}.$$

Moreover, we denote derivative with respect to t either by a superscript dot or by a subscript t.

Our purpose is to study the evolution of $\tau(t)$ and of $\mathtt{u}(t)$ which is the component of $S_t(m)$ orthogonal to $e_{v_b(\tau(t))}$, namely

$$S_t(m) = v_b(\tau(t)) + \mathtt{u}(t), \tag{3.65}$$

$$\langle \mathtt{u}(t), e_{v_b(\tau(t))}\rangle = 0. \tag{3.66}$$

We have

$$\mathrm{u}_t = f_L(S_t(m)) - \dot{\tau} f_L(v_b(\tau)) = (1-\dot{\tau}) f_L(v_b) + \mathcal{L}_{b,\tau}\mathrm{u} + R(S_t(m)) \qquad (3.67)$$

and

$$R(S_t(m)) := \beta^{-1}[\operatorname{arctanh} S_t(m) - \operatorname{arctanh} v_b(\tau) - \operatorname{arctanh}'(v_b(\tau))\mathrm{u}]. \qquad (3.68)$$

By differentiating (3.66) and using (3.67) we get

$$\begin{aligned} 0 &= \langle \mathrm{u}_t, e_{v_b}\rangle + \dot{\tau}\left\langle \mathrm{u}, \frac{\delta e_{v_b}}{\delta v_b} f_L(v_b)\right\rangle \\ &= (1-\dot{\tau})\langle f_L(v_b), e_{v_b}\rangle + \langle R(S_t(m)), e_{v_b}\rangle + \dot{\tau}\left\langle \mathrm{u}, \frac{\delta e_{v_b}}{\delta v_b} f_L(v_b)\right\rangle, \end{aligned}$$

so that

$$\dot{\tau} = \frac{\langle f_L(v_b), e_{v_b}\rangle + \langle R(S_t(m)), e_{v_b}\rangle}{\langle f_L(v_b), e_{v_b}\rangle - \left\langle \mathrm{u}, \frac{\delta e_{v_b}}{\delta v_b} f_L(v_b)\right\rangle},$$

provided the denominator is nonzero. In such a case we can also write

$$\dot{\tau} - 1 = A(S_t(m); b), \qquad (3.69)$$

where the 'force field' $A(m; b)$ is defined as

$$A(m; b) := \frac{\langle R(m), e_{v_b(\tau)}\rangle + \left\langle \mathrm{u}, \frac{\delta e_{v_b(\tau)}}{\delta v_b(\tau)} f_L(v_b(\tau))\right\rangle}{\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle - \left\langle \mathrm{u}, \frac{\delta e_{v_b(\tau)}}{\delta v_b(\tau)} f_L(v_b(\tau))\right\rangle}, \qquad (3.70)$$

where $v_b(\tau) = v_b(\tau_{m;b})$, $\tau_{m;b}$ the t-coordinate of m. The definition applies only if m has a t-coordinate and the denominator in (3.70) is nonzero.

By taking the scalar product with u of both sides of (3.67) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{u}, \mathrm{u}\rangle = B(S_t(m); b), \qquad (3.71)$$

where

$$B(m; b) := -A(m; b)\langle \mathrm{u}, f_L(v_b(\tau))\rangle + \langle \mathrm{u}, \mathcal{L}_{b,\tau}\mathrm{u}\rangle + \langle \mathrm{u}, R(m)\rangle, \qquad (3.72)$$

with the same meaning of symbols and the same restrictions on m.

The analysis of the system of equations (3.69)–(3.71) is based on the following bounds on the 'forces' $A(m; b)$ and $B(m; b)$.

We recall that

ϵ_3 is given by proposition 3.8,
ω is a bound for the spectral gap uniform in a neighbourhood of $\hat{m}_L$, see theorem 2.4,
c_1 is defined in (2.14),
$\|e_{v_b(\tau)}\|_\infty$, see theorem 2.4,
C_2 is defined at the beginning of the proof of lemma 3.2.

Proposition 3.10. *There exists* $\epsilon_4 \in (0, \epsilon_3)$ *so that if we suppose*

$$\|m - v_{a(m)}\|_2 \leqslant \epsilon_4 a(m), \qquad a(m) \leqslant \epsilon_4, \qquad b > 0 \text{ small enough},$$

and define $\mathrm{u} := m - v_b(\tau)$, $v_b(\tau) = v_b(\tau_{m,b})$ *(recall proposition 3.8), then*

(i) $\|\mathrm{u}\|_2 \leqslant \min\left\{\frac{1}{8c_1}, \frac{\omega}{80C_2\tilde{c}_+}\right\}$, *where* $\tilde{c}_+ \geqslant \|e_{v_b(\tau)}\|_\infty$,

(ii) $A(m; b)$ *and* $B(m; b)$ *are well defined and*

$$|A(m;b)| \leqslant 2\left\{4c_1\|\mathrm{u}\|_2 + \frac{C_2\|e_{v_b(\tau)}\|_\infty\|\mathrm{u}\|_2^2}{|\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle|}\right\}. \tag{3.73}$$

(iii) If in addition

$$\mathrm{u} = u_1 + u_2, \qquad |\mathrm{supp}(u_2)| \leqslant \zeta, \qquad \|u_1\|_\infty \leqslant \frac{\omega}{10C_2}, \tag{3.74}$$

then

$$B(m;b) \leqslant -\frac{\omega}{2}\|\mathrm{u}\|_2^2 + 8C_2\zeta. \tag{3.75}$$

Proof. By taking ϵ_4 small enough we get (i) from (3.64). By (2.14) and (3.31), shorthanding $v = v_b(\tau)$,

$$\begin{aligned}|\langle f_L(v), e_v\rangle - \langle \mathrm{u}, \frac{\delta e_v}{\delta v} f_L(v)\rangle| &\geqslant |\langle f_L(v), e_v\rangle| \left|1 - c_1\|\mathrm{u}\|_2 \frac{\|f_L(v)\|_2}{|\langle f_L(v), e_v\rangle|}\right| \\ &\geqslant |\langle f_L(v), e_v\rangle||1 - 4c_1\|\mathrm{u}\|_2|\end{aligned} \tag{3.76}$$

and by (i),

$$\left|\langle f_L(v), e_v\rangle - \left\langle \mathrm{u}, \frac{\delta e_v}{\delta v} f_L(v)\right\rangle\right| \geqslant \frac{1}{2}|\langle f_L(v), e_v\rangle|. \tag{3.77}$$

Then (3.73) follows from (C.15) and (3.31).

Let us now show (3.75). By (2.13),

$$B(m;b) \leqslant |A(m;b)\langle \mathrm{u}, f_L(v)\rangle| - \omega\|\mathrm{u}\|_2^2 + |\langle \mathrm{u}, R\rangle| \tag{3.78}$$

and by (3.73) and (3.31) we have

$$|A(m;b)\langle \mathrm{u}, f_L(v)\rangle| \leqslant 8c_1\|\mathrm{u}\|_2^2\|f_L(v)\|_2 + 8C_2\|\mathrm{u}\|_2^3\|e_v\|_\infty. \tag{3.79}$$

By (3.29) and (3.37), $\|f_L(v)\|_2 \leqslant 2\hat{\lambda}\epsilon_4$ and supposing ϵ_4 small enough, $8c_1\|f_L(v)\|_2 \leqslant \omega/10$.

By (i), $8C_2\|\mathrm{u}\|_2\|e_v\|_\infty \leqslant \omega/10$ so that

$$|A(m;b)\langle \mathrm{u}, f_L(v)\rangle| \leqslant \frac{\omega}{5}\|\mathrm{u}\|_2^2. \tag{3.80}$$

Finally from (3.74) and (C.15) we get

$$|\langle \mathrm{u}, R\rangle| \leqslant C_2\|\mathrm{u}\|_2^2\|u_1\|_\infty + 8C_2\zeta \leqslant \frac{\omega}{10}\|\mathrm{u}\|_2^2 + 8C_2\zeta. \tag{3.81}$$

Inserting the estimates (3.80) and (3.81) in (3.78) we get (3.75). □

4. Construction of the invariant manifolds

In this section we prove theorem 2.5 by constructing the invariant manifolds $\mathcal{W}_\pm$ as a suitable limit of the manifolds $\mathcal{M}_\pm$.

4.1. Existence

Given $\epsilon_4 < 1$ as in proposition 3.10, we fix $a_0 \in (0, \epsilon_4)$ and such that

$$\frac{4c_0' + \tilde{c}_2}{\hat{\lambda}} a_0^{2/3} < \epsilon_4 \tag{4.1}$$

(the ratio $(4c_0' + \tilde{c}_2)/\hat{\lambda}$ appears in (3.26)).

We call $\theta(a_0; b)$, $b \in (0, a_0)$, the time when the a-coordinate of $v_b(t)$ is equal to a_0.

Theorem 4.1. *There exists a family of functions* $v^{(+)}(s) \in L^\infty((-L/2, L/2); (-m_\beta, m_\beta))$, $s \leqslant 0$, *such that*

$$\lim_{b\to 0} \sup_{0\geqslant s'\geqslant s} \|v_b(\theta(a_0; b) + s') - v^{(+)}(s')\|_2 = 0 \qquad \forall s \leqslant 0. \tag{4.2}$$

Proof. We will prove that $\{v_b(\theta(a_0; b) + s), b \in (0, a_0)\}$ is a Cauchy sequence for any $s < 0$, namely for any $\zeta > 0$ there is $a_1 \in (0, a_0)$ so that for any $b \in (0, a_1)$

$$\|v_b(\theta(a_0; b) + s) - v_{a_1}(\theta(a_0; a_1) + s)\|_2 < \zeta. \tag{4.3}$$

By the continuity of S_t, (4.2) then follows, where

$$v^{(+)}(s) := \lim_{b\to 0^+} v_b(\theta(a_0; b) + s),$$

and the limit is taken in L^2.

Step 1. For any $s < 0$ and $\delta > 0$ there is $a_1 \in (0, a_0)$ so that for any $b < a_1$ and any t such that $a(t; a_1) \leqslant \epsilon_4$, the pair $m = v_{a_1}(t)$ and $v_b(\cdot)$ satisfy the assumptions in proposition 3.10.

Since $m = v_{a_1}(t)$ and t is such that $a(m) \leqslant \epsilon_4$, by (3.27) we find $\|m - v_{a(m)}\|_2 \leqslant (4c_0'/\hat{\lambda})a(m)^{5/3}$. By the second inequality in (3.28) and since $a(m) \leqslant \epsilon_4 \leqslant \epsilon_2$, we then have $\|m - v_{a(m)}\|_2 \leqslant \epsilon_4 a(m)$.

To verify (3.74) we set $u_2 = 0$. Since the derivative with respect to x of $\mathtt{u} = v_{a_1}(t) - v_b(\tau(t))$ is bounded, by lemma C.3 $\|\mathtt{u}\|_\infty \leqslant c\|\mathtt{u}\|_2^{2/3}$. In order to bound $\|\mathtt{u}\|_2$, we use (3.49) and obtain

$$\|v_{a_1}(t) - v_b(\tau(t))\|_2 \leqslant \frac{360c_0'}{\hat{\lambda}} a(v_{a_1}(t))^{5/3}, \qquad a_1 < \epsilon_4, \tag{4.4}$$

and the inequality in (3.74) follows, using also (i) of proposition 3.10.

Step 2. For any $s < 0$ and $\delta > 0$ there is $a_1 \in (0, a_0)$ so that if we indicate by $\theta(a_0; a_1)$ the time when $a(t; a_1) = a_0$, then the t-coordinate of $v_{a_1}(t)$ relative to $\{v_b(\cdot)\}$, denoted by $\tau(t)$, is well defined for $t \leqslant \theta(a_0; a_1)$ and

$$\|v_b(\tau(\theta(a_0; a_1) + s)) - v_{a_1}(\theta(a_0; a_1) + s)\|_2 \leqslant \delta. \tag{4.5}$$

By (4.4), $\|\mathtt{u}(0)\|_2 \leqslant ca_1^{5/3}$, c a suitable constant. By (3.71) and (3.75),

$$\frac{\mathrm{d}\|\mathtt{u}\|_2^2}{\mathrm{d}t} \leqslant -\frac{\omega}{4}\|\mathtt{u}\|_2^2,$$

so that (4.5) holds with $\delta = ca_1^{5/3}$.

Let us conclude the proof of the theorem, by proving (4.3). We then fix an $s < 0$ and we shorthand $m = v_{a_1}(\theta(a_0; a_1) + s)$ and $m^* = v_b(\tau(\theta(a_0; a_1) + s))$. Since $S_{|s|}(m) = v_{a_1}(\theta(a_0; a_1))$ then $a(S_{|s|}(m)) = a_0$. For any $t \geqslant 0$

$$\|S_t(m) - S_t(m^*)\|_2 \leqslant \mathrm{e}^{ct}\|m - m^*\|_2, \qquad c = 1 + \frac{1}{\beta} \max_{|x|\leqslant(1+m_\beta)/2} \operatorname{arctanh}'(x). \tag{4.6}$$

By using (4.6) for $t = |s|$ and (4.5) we get $\|S_{|s|}(m^*) - v_{a_1}(\theta(a_0; a_1))\|_2 \leqslant e^{c|s|}\delta$. This implies that there is a constant c' so that $|a(S_{|s|}(m^*)) - a_0| \leqslant c' e^{c|s|}\delta$.

For a in the interval $[a_0 - c' e^{c|s|}\delta, a_0 + c' e^{c|s|}\delta]$, $\mathrm{d}a/\mathrm{d}t \geqslant (\hat\lambda/2)a \geqslant (\hat\lambda/4)a_0$, see (3.28). Letting t^* be such that $(\hat\lambda a_0/4)t^* = c' e^{c|s|}\delta$, there is $|t| \leqslant t^*$ so that $a(S_{|s|+t}(m^*)) = a_0$. Thus, letting $c'' = (4/a_0\hat\lambda)c'$, we have

$$\theta(a_0, b) = \tau(\theta(a_0; a_1) + s) + |s| + t, \qquad |t| \leqslant c'' e^{c|s|}\delta, \tag{4.7}$$

hence, using (3.28) and (4.5) to derive the second inequality below,

$$\begin{aligned} &\|v_b(\theta(a_0; b) + s)) - v_{a_1}(\theta(a_0; a_1) + s))\|_2 \\ &\quad \leqslant \|v_b(\tau(\theta(a_0; a_1) + s) + t) - v_b(\tau(\theta(a_0; a_1) + s))\|_2 \\ &\qquad + \|v_b(\tau(\theta(a_0; a_1) + s)) - v_{a_1}(\theta(a_0; a_1) + s))\|_2 \\ &\quad \leqslant 2\hat\lambda a_0 t + \delta \leqslant (8c' e^{c|s|} + 1)\delta, \end{aligned}$$

which proves (4.3) if δ is small enough. □

We extend the definition of $v^{(+)}(t)$ to all $t \in \mathbb{R}$, by setting

$$v^{(+)}(t) := S_t(v^{(+)}(0)), \qquad t > 0. \tag{4.8}$$

4.2. *Time invariance and asymptotic limit*

We now prove (1.10) and (1.9) for $v^{(+)}$.

Theorem 4.2. *The family* $\{v^{(+)}(s) : s \in \mathbb{R}\}$ *is time-invariant, i.e.*

$$S_t(v^{(+)}(s)) = v^{(+)}(t + s), \qquad s \in \mathbb{R}, \quad t \geqslant 0, \tag{4.9}$$

the a-coordinate $a(v^{(+)}(t))$ *of* $v^{(+)}(t)$ *is increasing and satisfies*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} a(v^{(+)}(t)) - \hat\lambda a(v^{(+)}(t)) \right| \leqslant \tilde c_2 a(v^{(+)}(t))^{5/3}. \tag{4.10}$$

Furthermore

$$\lim_{t\to-\infty} \|v^{(+)}(t) - \hat m_L\|_2 = 0. \tag{4.11}$$

Proof. If $s \geqslant 0$ equality (4.9) follows directly from (4.8) Let then $s < 0$ and suppose first $s + t \leqslant 0$. By (4.2),

$$S_t(v^{(+)}(s)) = S_t\left(\lim_{b\to 0} v_b(\theta(a_0; b) + s)\right) = \lim_{b\to 0} v_b(\theta(a_0; b) + s + t) = v^{(+)}(s + t)$$

If $s + t > 0$, we write $S_t(v^{(+)}(s)) = S_{t-|s|}(S_{|s|}(v^{(+)}(s)))$ and since we have already proved that $S_{|s|}(v^{(+)}(s)) = v^{(+)}(0)$, (4.9) is proved.

Let us prove (4.10). By (3.4) and since $v^{(+)}(t)$ satisfies the equations of motion, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} a(v^{(+)}(t)) = \langle f_L(v^{(+)}(t)), \hat e\rangle, \tag{4.12}$$

it follows

$$\frac{\mathrm{d}}{\mathrm{d}t} a(v^{(+)}(t)) = \lim_{b\to 0} \langle f_L(v_b(\theta(a_0; b) + t)), \hat e\rangle, \tag{4.13}$$

which extends the validity of (3.28) to $a(v^{(+)}(t))$, namely (4.10) holds. Observe that from (4.10) it follows that

$$\lim_{t\to-\infty} a(v^{(+)}(t)) = 0. \tag{4.14}$$

It remains to show (4.11). Let $s < 0$ and $a_s^* := a(v^{(+)}(s))$. Since from theorem 4.1 we know that $a(v^{(+)}(0)) = a_0$, from the fact that a_s^* is an increasing function, we get that $a_s^* < a_0$.

We also call $\tilde{a}_s := a(v_b(\theta(a_0; b) + s))$ for $s < 0$. Notice that $\tilde{a}_0 = a_0$ and, since $\tilde{a}_s$ is strictly increasing, $\tilde{a}_s < a_0$ for all $s < 0$.

From (3.42) and (3.27) we get

$$\begin{aligned}\|v_b(\theta(a_0; b) + s) - v_b(\theta(a_s^*; b))\|_2 &\leqslant |\tilde{a}_s - a_s^*| + \frac{4c_0'}{\hat{\lambda}}(\tilde{a}_s)^{5/3} + \frac{4c_0'}{\hat{\lambda}}(a_s^*)^{5/3} \\ &\leqslant \tilde{a}_s[1 + \frac{4c_0'}{\hat{\lambda}}\tilde{a}_s^{2/3}] + a_s^*[1 + \frac{4c_0'}{\hat{\lambda}}(a_s^*)^{2/3}].\end{aligned}$$

From (3.28) it follows that $\tilde{a}_s \leqslant \mathrm{e}^{-\hat{\lambda}|s|}a_0\big[1 + \frac{\tilde{c}_2}{\hat{\lambda}}a_0^{2/3}\big]$. From (4.1) it follows that

$$1 + \frac{4c_0'}{\hat{\lambda}}(a_s^*)^{2/3} \leqslant 1 + \frac{4c_0'}{\hat{\lambda}}a_0^{2/3} < 2, \tag{4.15}$$

and analogously from the term containing $\tilde{a}_s$. Thus we finally get

$$\|v_b(\theta(a_0; b) + s) - v_b(\theta(a_s^*; b))\|_2 \leqslant 2\mathrm{e}^{-\hat{\lambda}|s|}a_0 + 2a_s^*. \tag{4.16}$$

From (3.27) and (4.15) we get that

$$\|v_b(\theta(a_s^*; b)) - \hat{m}_L\|_2 \leqslant a_s^* + \frac{4c_0'}{\hat{\lambda}}(a_s^*)^{5/3} \leqslant 2a_s^*. \tag{4.17}$$

Hence

$$\|v^{(+)}(s) - \hat{m}_L\|_2 = \lim_{b\to 0}\|v_b(\theta(a_0; b) + s) - \hat{m}_L\|_2 \leqslant 2\mathrm{e}^{-\hat{\lambda}|s|}a_0 + 4a_s^*.$$

Thus (4.11) follows from (4.14). □

Theorem 4.3. *The function $v^{(+)}(t)$ verifies*

$$\lim_{t\to\infty}\|v^{(+)}(t) - m^{(+)}\|_2 = 0, \tag{4.18}$$

so that the manifold $\mathcal{W}_+$ can be identified with $\{v^{(+)}(t)\}_{t\in\mathbb{R}}$.

Proof. Since $s \to F_L(v^{(+)}(s))$ is strictly decreasing and F_L is continuous in L^2, $F_L(v^{(+)}(s)) < F_L(\hat{m}_L)$. For any m, $S_t(m)$ converges by subsequences as $t \to \infty$ and any limit point is stationary. Thus $v^{(+)}(t)$ converges by subsequences and any limit point m^* is a stationary solution of (1.1). By (1.6), $m^* \in \{m^{(+)}, \hat{m}_L, m^{(-)}\}$. On the other hand $m^* \neq \hat{m}_L$ because its free energy is strictly smaller. Since $v_b \geqslant \hat{m}_L$, $v_b(t) \geqslant \hat{m}_L$ hence $v^{(+)}(t) \geqslant \hat{m}_L$. Thus $m^* \geqslant \hat{m}_L$ excluding the possibility that $m^* = m^{(-)}$. □

5. Stability of the invariant manifolds

The manifolds $\mathcal{W}_\pm$ are asymptotically stable: referring for instance to $\mathcal{W}_+$, since $m^{(+)}$ is stable, its basin of attraction $\mathcal{B}^+$ is an open set (in the L^2 topology) and each element $v^{(+)}(\cdot) \in \mathcal{W}_+$, being in $\mathcal{B}^+$, has a neighbourhood which is attracted by $m^{(+)}$; in this sense, therefore, $\mathcal{W}_+$ is asymptotically stable. The property is indeed a consequence of the existence of $\mathcal{W}_+$ and of the stability of $m^{(+)}$.

We discuss here a different stability property of $\mathcal{W}_+$, namely, that given any neighbourhood I of $\mathcal{W}_+$ there is a neighbourhood U of $\hat{m}_L$ so that for any $m \in U \cap \mathcal{B}^+$, $S_t(m)$ converges to $m^{(+)}$ being at all times in I and 'following closely' the orbit $v^{(+)}(\cdot)$. For applications to tunnelling and the characterization of its optimal orbits we generalize the context by adding a time dependent force $K(x,t)$ and thus considering the evolution equation

$$m_t = f_L(m) + K \tag{5.1}$$

supposing $K(x,t)$ smooth and that

$$\|K\|^2 = \int_0^\infty k(t)^2\,\mathrm{d}t < \infty, \qquad k(t)^2 := \int_{-\frac{L}{2}}^{\frac{L}{2}} K(x,t)^2\,\mathrm{d}x. \tag{5.2}$$

We will denote by $S_t^K(m)$ the orbit which solves (5.1) starting from $m \in L^2((-L/2, L/2); (-1,1))$ and with K satisfying (5.2). As noticed in section 2.1, $|S_t^K(m)| < 1$ for all t.

The presence of the additional force may have the effect of stabilizing $\hat{m}_L$ and we need assumptions to avoid such a case which can be dropped if $K = 0$, as in theorem 5.9. When $K \neq 0$ we thus split our results into two theorems. Theorem 5.5 below describes the initial part of the orbit which stays close to $\hat{m}_L$. Theorem 5.7 describes what happens when the orbit leaves such a neighbourhood.

Before stating and proving the above theorems we give some definitions and preliminary lemmas.

Recalling definition 3.1, we write any m as

$$m = \hat{m}_L + a(m)\hat{e} + \phi, \qquad \langle \phi, \hat{e}\rangle = 0.$$

Definition 5.1. *For any $a^* > 0$ and m such that $|a(m)| < a^*$ we define*

$$t_{a^*}(m) := \inf\{t \geqslant 0 : |a(S_t^K(m))| = a^*\}, \tag{5.3}$$

so that $t_{a^*}(m) = \infty$ if $|a(S_t^K(m))| < a^*$ for all t.

Definition 5.2. *We define $\phi(t)$ as*

$$S_t^K(m) = \hat{m}_L + a(t)\hat{e} + \phi(t), \qquad \langle \phi(t), \hat{e}\rangle = 0, \quad a(t) := a(S_t^K(m)). \tag{5.4}$$

In the next lemma we give a first estimate of $\phi(t)$ for m_0 such that $a(m_0) < \epsilon_0$.

Lemma 5.3. *Let $\epsilon_0 \in \left(0, \frac{1-m_\beta}{4\|\hat{e}\|_\infty}\right)$ be as in lemma 3.3. There exists a constant $c_M > 0$ such that if $a(m_0) \in (0, \epsilon_0)$ then*

$$\|\phi(t)\|_2^2 \leqslant \mathrm{e}^{8c_M t}\left(\|\phi(0)\|_2^2 + \frac{2\sqrt{L}}{\sqrt{c_M}}\|K\| + 8c_M\int_0^t \mathrm{e}^{-8c_M s}a(s)^2\,\mathrm{d}s\right) \qquad \forall t \in [0, t_{\epsilon_0}(m_0)]. \tag{5.5}$$

Proof. We set $m(t) := S_t^K(m_0) = m(\cdot, t)$ and define

$$\mathcal{C}_t := \left\{x \in [-L/2, L/2] : |m(x,t)| \geqslant \frac{m_\beta + 1}{2}\right\} \qquad \forall t \in [0, t_{\epsilon_0}(m_0)].$$

We claim that there is $c > 0$ so that for all $t \leqslant t_{\epsilon_0}(m_0)$

$$\phi(t)\,[\operatorname{arctanh} m(t) - \operatorname{arctanh} \hat{m}_L] \geqslant \phi(t)\operatorname{arctanh}'(\hat{m}_L)[m(t) - \hat{m}_L] - c|\phi(t)|(m(t) - \hat{m}_L)^2 \quad \text{in } \mathcal{C}_t. \tag{5.6}$$

We prove the claim assuming $m \geqslant (1+m_\beta)/2$, the case $m \leqslant -(1+m_\beta)/2$ is analogous and omitted. Write for simplicity ϕ in place of $\phi(t)$. Since $\phi = m - \hat{m}_L - a(t)\hat{e}$ and $m - \hat{m}_L > (1-m_\beta)/2$, by using that $\|a(t)\hat{e}\|_\infty \leqslant \epsilon_0\|\hat{e}\|_\infty \leqslant (1-m_\beta)/4$ (recall $t \leqslant t_{\epsilon_0}(m_0)$) we get that $\phi > 0$ so that ϕ drops from (5.6).

If $\hat{m}_L \geqslant 0$, then $[\text{arctanh}\, m - \text{arctanh}\, \hat{m}_L] \geqslant \text{arctanh}'(\hat{m}_L)[m - \hat{m}_L]$ because $x \to \text{arctanh}' x$ is increasing when $x > 0$. If instead $\hat{m}_L < 0$, $[m - \hat{m}_L] \geqslant (1 + m_\beta)/2$ and the right-hand side of (5.6) becomes negative when $c(1 + m_\beta)/2 > 1/(1 - m_\beta^2)$. Therefore (5.6) is proved.

Denoting by $\mathcal{C}_t^c = [-L/2, L/2] \backslash \mathcal{C}_t$, from (5.6) and (C.15) of lemma C.3 (proved with $K = 0$ but also true for a bounded K, possibly modifying the constant C_2 in (C.15) by replacing $\text{arctanh}''(\mathtt{a})$ with $\text{arctanh}''(1 - \sigma)$ for a suitable $\sigma > 0$ small enough) we get

$$\langle \phi, \text{arctanh}\, m - \text{arctanh}\, \hat{m}_L \rangle \geqslant \int_{\mathcal{C}_t^c} \phi [\text{arctanh}\, m - \text{arctanh}\, \hat{m}_L]\, \mathrm{d}x$$

$$+ \int_{\mathcal{C}_t} \phi\, \text{arctanh}'(\hat{m}_L)[m - \hat{m}_L]\, \mathrm{d}x - c \int |\phi| (m - \hat{m}_L)^2\, \mathrm{d}x$$

$$\geqslant \int \phi\, \text{arctanh}'(\hat{m}_L)[m - \hat{m}_L]\, \mathrm{d}x - [C_2 + c] \int |\phi| (m - \hat{m}_L)^2\, \mathrm{d}x.$$

By using that $f_L(\hat{m}_L) = 0$ and $\langle \phi, \hat{e} \rangle = 0$, from (5.1) we get

$$\langle \phi, \phi_t \rangle \leqslant \langle \phi, J^{\text{neum}} * [m - \hat{m}_L] \rangle - \langle \phi, \text{arctanh}'(\hat{m}_L)[m - \hat{m}_L] \rangle + \langle \phi, K \rangle$$
$$+ [C_2 + c] \langle |\phi| (m - \hat{m}_L)^2 \rangle.$$

Recalling the definition of $\hat{\mathcal{L}}$ and using again that $\langle \phi, \hat{e} \rangle = 0$, we then get that there is $c_M > 0$ so that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, \phi \rangle \leqslant \langle \phi, \hat{\mathcal{L}} \phi \rangle + \langle |\phi|, c_M [|\phi|^2 + (a(t)\hat{e})^2] \rangle + |\langle \phi, K \rangle|$$
$$\leqslant -\hat{\omega} \langle \phi, \phi \rangle + \langle |\phi|, c_M [|\phi|^2 + (a(t)\hat{e})^2] \rangle + |\langle \phi, K \rangle| \qquad \forall t \in (0, t_{\epsilon_0}(m_0)), \tag{5.7}$$

where the last inequality follows from (2.13). Since $|\phi| < 4$ (recall that $|m| < 1$, $|\hat{m}_L| < 1$, the assumption $a(m_0) \in (0, m_0)$ and the choice of ϵ_0), from (5.7) we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, \phi \rangle \leqslant -\hat{\omega} \langle \phi, \phi \rangle + 4 c_M (\langle \phi, \phi \rangle + a(t)^2) + |\langle \phi, K \rangle|.$$

By the Cauchy–Schwartz inequality we have

$$|\langle \phi(t), K \rangle| \leqslant 4 \sqrt{L}\, k(t), \tag{5.8}$$

and estimate

$$\langle \phi(t), \phi(t) \rangle \leqslant \mathrm{e}^{8 c_M t} \langle \phi(0), \phi(0) \rangle + 8 \int_0^t \mathrm{e}^{8 c_M (t-s)} \{ c_M a(s)^2 + k(s) \}.$$

By the Cauchy–Schwartz inequality we then have

$$8 \int_0^t \mathrm{e}^{-8 c_M s} k(s)\, \mathrm{d}s \leqslant \frac{2}{\sqrt{c_M}} \|K\|,$$

and (5.5) follows. □

To have an estimate better than (5.5), we must improve the trivial bound $|\phi| < 4$ used in the proof of lemma 5.3. The idea is to bound the L^∞ norm of ϕ in terms of its L^2 norm. This requires some regularity properties of ϕ which will follow from regularizing properties of the evolution (see appendix C, lemmas C.1, C.2 and C.3). There is, however, a very first time interval when the evolution has not yet produced the desired regularity, and in such an interval we cannot do better than (5.5).

We will use the parameters listed below:

$\delta > 0$ is the parameter which specifies the initial datum and the strength of the external force:

$$\|m - \hat{m}_L\|_2 \leqslant \delta, \quad a(0)^2 + \|\phi(0)\|_2^2 \leqslant \delta^2; \qquad \|K\| \leqslant \delta. \tag{5.9}$$

$a_1 > 0$ is a parameter which controls $a(t)$. By the continuity of the motion for any m verifying (5.9), $t_{a_1}(m)$ can be made as large as desired by letting $\delta \ll a_1$. One condition on a_1 is that $a_1 < \epsilon_0$ so that we can apply lemma 5.3.

$\tau^* > 0$ denotes the initial time layer, after which regularity properties of time evolution apply (see lemma C.2).

$\mu_0 \in (0, 1)$ is a control parameter for $\|\phi(t)\|_2$.

We choose $\tau^* > \tau_\beta$ (τ_β as in (C.23)) so large that together with μ_0 they satisfy

$$2\mathrm{e}^{-\tau^*/\beta} + c_M(2\mu_0^2)^{1/3} < \min\left\{\frac{\hat{\omega}}{4c_M}, \frac{\omega}{10^2 C_2}\right\} \tag{5.10}$$

(the ratio $\omega/(10C_2)$ appears in proposition 3.10).

We also assume that $\tau^* < t_{\epsilon_0}(m)$ for all m satisfying (5.9). This because $t_{\epsilon_0}(m) > t_{a_1}(m)$ can be made as large as desired by letting ϵ_0 and therefore δ sufficiently small.

To control strength and location of the external force fluctuations we introduce for any $t > \tau^*$ and $\delta > 0$,

$$A_{t,\delta} = \left\{x \in \left[-\frac{L}{2}, \frac{L}{2}\right] : \int_{t-\tau^*}^{t} K(x,s)^2 < \delta\right\}.$$

By the Chebyshev inequality,

$$|A_{t,\delta}^c| \leqslant \frac{1}{\delta}\int_{A_{t,\delta}^c}\int_{t-\tau^*}^{t} K(x,s)^2\,\mathrm{d}s\,\mathrm{d}x \leqslant \frac{\delta^2}{\delta}. \tag{5.11}$$

Let finally

$$\alpha^* = c^*\mathrm{e}^{\|J\|_\infty \tau^*}(\sqrt{\delta} + \delta)^{1/2}, \tag{5.12}$$

with c^* the constant in (C.19).

Before stating smallness conditions on δ and a_1, we prove the following lemma.

Lemma 5.4. *There exists $c'_M > 0$ so that if K and m satisfy (5.9) then*

$$\sup_{x\in A_{t,\delta}} |\phi(x,t)| \leqslant \alpha^* + 2\mathrm{e}^{-\frac{\tau^*}{\beta}} + c'_M(2\|\phi(t)\|_2^2 + 2L(\alpha^*)^2 + 8\delta)^{1/3} \qquad \forall t \in (\tau^*, t_{\epsilon_0}(m)), \tag{5.13}$$

with

$$\|\phi(t)\|_2^2 \leqslant \mathrm{e}^{8c_M\tau^*}\left(\|\phi(t-\tau^*)\|_2^2 + \frac{2\sqrt{L}}{\sqrt{c_M}}\delta + a_t^{+\,2}\right) \qquad \forall t \in (\tau^*, t_{\epsilon_0}(m)), \tag{5.14}$$

where $a_t^+ := \max\{|a(S_s^K(m))| : s \in [t-\tau^*, t]\}$.

Proof. (5.14) follows from lemma 5.3, so that we only need to prove (5.13). By (C.19),

$$\sup_{x\in A_{t,\delta}} |S_t^K(m)(x) - S_{\tau^*}(m(t-\tau^*))(x)| \leqslant c^*\mathrm{e}^{\|J\|_\infty\tau^*}(\sqrt{\delta}+\delta)^{1/2} = \alpha^* \tag{5.15}$$

Since $\|S_t^K(m) - S_{\tau^*}(m(t-\tau^*))\|_\infty \leqslant 2$, (5.15) implies that

$$\|S_t^K(m) - S_{\tau^*}(m(t-\tau^*))\|_2^2 \leqslant L(\alpha^*)^2 + 4|A_{t,\delta}^c|. \tag{5.16}$$

Calling $\psi(t) := S_{\tau^*}(m(t-\tau^*)) - [\hat{m}_L + a(t)\hat{e}]$,

$$\phi(t) = [S_t^K(m) - S_{\tau^*}(m(t-\tau^*))] + \psi(t), \tag{5.17}$$

so that

$$\sup_{x\in A_{t,\delta}} |\phi(x,t)| \leqslant \alpha^* + \sup_{x\in A_{t,\delta}} |\psi(x,t)|. \tag{5.18}$$

By lemmas C.2 and C.1,

$$\|\psi(t)\|_\infty \leqslant 2\mathrm{e}^{-\tau^*/\beta} + c\|\psi(t)\|_2^{2/3}$$

By (5.17) and (5.16),

$$\begin{aligned}\tfrac{1}{2}\|\psi\|_2^2 &\leqslant \|\phi\|_2^2 + \|S_t^K(m) - S_{\tau^*}(m(t-\tau^*))\|_2^2 \\ &\leqslant \|\phi\|_2^2 + (\alpha^*)^2 L + 4|A_{t,\delta}^c|,\end{aligned}$$

so that

$$\|\psi(t)\|_\infty \leqslant 2\mathrm{e}^{-\tau^*/\beta} + c(2\|\phi(t)\|_2^2 + 2L(\alpha^*)^2 + 8|A_{t,\delta}^c|)^{1/3},$$

which together with (5.18) and (5.11) proves (5.13). □

5.1. *A first set of conditions on δ and a_1*

Given $a_1 \in (0, \epsilon_0)$ we will require δ so small that for any m satisfying (5.9), $t_{a_1}(m) > \tau^*$.

Another set of requirements (to which others will be added later) is the following one:

$$\mathrm{e}^{8c_M\tau^*}\left(\delta^2 + \frac{2\sqrt{L}}{\sqrt{c_M}}\delta + a_1^2\right) + \frac{2c_M}{\hat{\omega}}(4^3\delta + \tilde{c}_+\mu_0 a_1^2) + \frac{4\delta}{\sqrt{\hat{\omega}}} < \mu_0^2, \tag{5.19}$$

$$\alpha^* + 2\mathrm{e}^{-\tau^*/\beta} + c_M'(2\mu_0^2 + 2L(\alpha^*)^2 + 8\delta)^{1/3} < \min\left\{\frac{\hat{\omega}}{4c_M}, \frac{\omega}{10^2 C_2}\right\}, \tag{5.20}$$

with α^* as in (5.12) and τ^* and μ_0 that satisfies (5.10).

Theorem 5.5. *There exists $c > 0$ so that for any $a_1 \in (0, \mu_0)$ small enough and any $\delta > 0$ small enough the following holds. If K and m satisfy (5.9), then*

$$\sup_{t\leqslant t_{a_1}(m)} \|\phi(t)\|_2 \leqslant ca_1. \tag{5.21}$$

Moreover $t_{a_1}(m) > \tau^*$ and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi,\phi\rangle \leqslant -\frac{\hat{\omega}}{2}\langle\phi,\phi\rangle + 4^3 c_M\delta + c_M\tilde{c}_+\|\phi\|_2 a(t)^2 + |\langle\phi, K\rangle| \qquad \forall t \in [\tau^*, t_{a_1}(m)), \tag{5.22}$$

$$\sup_{x\in A_{t,\delta}} |\phi(x,t)| \leqslant \frac{\omega}{10^2 C_2}, \qquad |A_{t,\delta}^c| \leqslant \delta \quad \forall t \in [\tau^*, t_{a_1}(m)). \tag{5.23}$$

Finally

$$t_{a_1}(m) < \infty \Rightarrow \|\phi(t_{a_1})\|_2 \leqslant \frac{\epsilon_4^2}{10^2}\, a_1 \tag{5.24}$$

(ϵ_4 as in proposition 3.10).

Proof. For $t \geqslant \tau^*$ (5.7) yields (with $\tilde{c}_+ \geqslant \|\hat{e}^2\|_\infty$),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_2^2 \leqslant -\hat{\omega}\|\phi\|_2^2 + 4^3 c_M\delta + c_M[\langle|\phi|, \mathbf{1}_{A_{t,\delta}}|\phi|^2\rangle + \tilde{c}_+\|\phi\|_2 a(t)^2] + |\langle\phi, K\rangle|. \tag{5.25}$$

Call S the first time when $\|\phi(s)\|_2 = \mu_0$ (we will show later that $S > t_{a_1}(m)$). Then by (5.13) and (5.20), for all $t \leqslant S$,

$$c_M\langle|\phi|, \mathbf{1}_{A_{t,\delta}}|\phi|^2\rangle \leqslant \frac{\hat{\omega}}{4}\|\phi\|_2^2$$

so that from (5.25) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi, \phi\rangle \leqslant -\frac{\hat{\omega}}{2}\langle\phi, \phi\rangle + 4^3 c_M\delta + c_M\tilde{c}_+\mu_0 a(t)^2 + |\langle\phi, K\rangle|. \tag{5.26}$$

Thus for $t \in [\tau^*, \min\{t_{a_1}, S\}]$ we get an improvement of (5.14), namely recalling that $a_t^+ = \max\{|a(S_s^K(m))|, s \in [t-\tau^*, t]\}$,

$$\|\phi(t)\|_2^2 \leqslant \mathrm{e}^{-\hat{\omega}(t-\tau^*)}\|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}}(4^3\delta + \tilde{c}_+\mu_0 a_t^{+\,2}) + \frac{4\delta}{\sqrt{\hat{\omega}}}, \tag{5.27}$$

having used (5.8) to bound

$$\int_{\tau^*}^t \mathrm{e}^{-\hat{\omega}(t-s)}|\langle\phi, K\rangle| \leqslant 4\int_{\tau^*}^t \mathrm{e}^{-\hat{\omega}(t-s)}k(s) \leqslant 4(2\hat{\omega})^{-1/2}\delta.$$

Since $a_t^+ \leqslant a_1$

$$\|\phi(t)\|_2^2 \leqslant \mathrm{e}^{-\hat{\omega}(t-\tau^*)}\|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}}(4^3\delta + \tilde{c}_+\mu_0 a_1^2) + \frac{4\delta}{\sqrt{\hat{\omega}}} \tag{5.28}$$

By (5.5) and (5.9)

$$\|\phi(\tau^*)\|_2^2 \leqslant \mathrm{e}^{8c_M\tau^*}\left(\delta^2 + \frac{2\sqrt{L}}{\sqrt{c_M}}\delta + a_1^2\right) \tag{5.29}$$

and by (5.19) and (5.28), $\|\phi(t)\|_2^2 < \mu_0^2$. Hence for all $t \leqslant t_{a_1}(m)$, $\|\phi(t)\|_2^2 < \mu_0^2$ and (5.26) and (5.27) hold for all $t \in [\tau^*, t_{a_1}(m)]$. Thus (5.22) is proved.

From (5.28) it follows that for $\delta < a_1^2$ and sufficiently small, there is c so that

$$\|\phi(t)\|_2^2 \leqslant ca_1^2, \qquad \forall t \in [0, t_{a_1}(m)]$$

and (5.21) is proved. (5.23) follows from (5.13), $\|\phi(t)\|_2 \leqslant \mu_0$ and (5.20).

Thus it only remains to prove (5.24). Going back to (5.22) we use (5.21) to bound $\|\phi\|_2$ in the term containing $a(t)^2$. We then have the following improvement of (5.27):

$$\|\phi(t)\|_2^2 \leqslant \mathrm{e}^{-\hat{\omega}(t-\tau^*)}\|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}}(4^3\delta + \tilde{c}_+(ca_1)a_1^2) + \frac{4\delta}{\sqrt{\hat{\omega}}}.$$

By (5.29) if $t-\tau^*$ is large enough, the first term becomes smaller than $\epsilon_4^2 a_1/10^3$. This is possible if $t_{a_1}(m)$ is larger than such a value of t, a condition which can be achieved by supposing a_1 small enough and then δ consequently small. By taking δ and a_1 small enough also the second and third term can be made smaller than $\epsilon_4^2 a_1/10^3$, and we then get (5.24). □

As already remarked, the external force even if small may win if the initial condition is too close to $\hat{m}_L$ thus determining the future outcome of the orbit, which may either stay always close to $\hat{m}_L$ or leave it. The next theorem shows that if we know that $a(S_t^K(m))$ reaches a (still very small) positive or negative value then the orbit goes to $m^{(+)}$ or $m^{(-)}$ according to the sign of $a(S_t^K(m))$, following closely along the way, $\mathcal{W}_+$ or, respectively, $\mathcal{W}_-$.

We will use the following properties of $v^{(\pm)}$ that are consequences of (1.9), (1.10) and of the results of section 4. We state them only for $v^{(+)}$, for $v^{(-)}$ the statements are completely analogous.

We have chosen the parametrization of $\mathcal{W}_+$ in such a way that

$$a(v^{(+)}(0)) = a_0 < \epsilon_4 \tag{5.30}$$

with ϵ_4 as in proposition 3.10.

For $\zeta > 0$ let $s^-(\zeta)$ be the (unique) number such that

$$a(v^{(+)}(s^-(\zeta))) = \zeta. \tag{5.31}$$

From (5.30) it follows that if $\zeta < a_0$ then $s^-(\zeta) < 0$. By (1.9) there exists a number $s^+(\zeta) > 0$ such that

$$\sup_{t \geqslant s^+(\zeta)} \|v^{(+)}(t) - m^{(+)}\|_2 \leqslant \zeta. \tag{5.32}$$

In order to obtain uniqueness, one can choose the smallest number that satisfies (5.32), but this will not be important later on.

Definition 5.6. *Given $a_1 > 0$ and $\delta \in (0, a_1)$ we define*

$$\mathcal{C}^+_{a_1,\delta} := \{m \in L^2((-L/2, L/2)) : \|m - \hat{m}_L\|_2 < \delta, t_{a_1}(m) < \infty,\ a(S^K_{t_{a_1}}(m)) > 0\}. \tag{5.33}$$

The set $\mathcal{C}^-_{a_1,\delta}$ is defined in a similar way with $a(S^K_{t_{a_1}}(m)) < 0$.

Theorem 5.7. *Given $\zeta \in (0, a_0)$ there exist $a^* > 0$, a positive function $\delta^*(\cdot)$ such that for any $a_1 \in (0, a^*)$ and $\delta \in (0, \delta^*(a_1))$, if $\|K\| < \delta$ then*

$$t_\zeta(m) < \infty \qquad \forall m \in \mathcal{C}^+_{a_1,\delta},$$

and

$$\sup_{0 \leqslant t \leqslant s^+(\zeta)+|s^-(\zeta)|} \|S^K_{t_\zeta(m)+t}(m) - v^{(+)}(s^-(\zeta) + t)\|_2 \leqslant \zeta,$$

$$\sup_{t > s^+(\zeta)+|s^-(\zeta)|} \|S^K_{t_\zeta(m)+t}(m) - m^{(+)}\|_2 \leqslant \zeta. \tag{5.34}$$

If $m \in \mathcal{C}^-_{a_1,\delta}$, then (5.34) holds with $v^{(-)}$ in place of $v^{(+)}$ and $m^{(-)}$ in place of $m^{(+)}$.

Remark 5.8. From theorem 5.5 it follows that $\|S^K_t(m) - \hat{m}_L\|_2 \leqslant c\zeta$ for all $t \leqslant t_\zeta(m)$ if $\|K\|$ is small enough.

If instead the external force is absent, i.e. $K = 0$ (and of course theorems 5.5 and 5.7 still apply) we can prove that $\mathcal{B}^+$, the basin of attraction of $m^{(+)}$, in a neighbourhood of $\hat{m}_L$ contains a triangular shape with vertex $\hat{m}_L$.

Theorem 5.9. *Assume $K \equiv 0$. Given $c > 0$, for any $\epsilon > 0$ small enough and any m such that $a(m) \in (0, \epsilon)$ and $\|m - v_{a(m)}\|_2 \leqslant ca(m)$, we have*

$$\lim_{t\to\infty} \|S_t(m) - m^{(+)}\|_2 = 0.$$

5.2. Proof of theorem 5.7

Without loss of generality we will suppose ζ small enough and such that

$$\zeta^{2/3} < \frac{\hat{\lambda}}{32c_0'} \frac{\epsilon_4}{10^6} \tag{5.35}$$

(the ratio $\hat{\lambda}/c_0'$ appears for the first time in lemma 3.2).

By using the stability of $m^{(+)}$, there are ζ, $t_0 > s^+(\zeta)$, $\zeta_0 \in (0, \zeta)$ and δ so that if $\|K\| < \delta$

$$\sup_{t \geqslant 0} \|S_t^K(u) - m^{(+)}\|_2 \leqslant \frac{\zeta}{10} \qquad \text{for all } u \text{ such that } \|u - v^{(+)}(t_0)\|_2 \leqslant \tfrac{\zeta}{10^2}, \tag{5.36}$$

$$\sup_{0 \leqslant t \leqslant t_0 + |s^-(\zeta)|} \|S_t^K(u) - v^{(+)}(s^-(\zeta) + t)\|_2 \leqslant \frac{\zeta}{10^2}, \qquad \text{for all } u : \|u - v^{(+)}(s^-(\zeta))\|_2 \leqslant \zeta_0. \tag{5.37}$$

We finally choose $a_1 \leqslant \zeta_0^2$ (other conditions will be given later) and then any $b > 0$ so small that

$$\sup_{s^-(\frac{1}{2}a_1) \leqslant t \leqslant s^-(2\zeta)} \|v_b(\theta(a_0; b) + t) - v^{(+)}(t)\|_2 \leqslant \frac{\zeta_0}{10^2} \tag{5.38}$$

(this is possible because of theorem 4.1, see (4.2)).

For $a_1 < \zeta_0^2 < \epsilon_4$, we now consider an $m \in \mathcal{C}^+_{a_1,\delta}$ and we prove the following. We call

$$m_1 = S^K_{t_{a_1}}(m), \qquad m_1 = \hat{m}_L + a_1 \hat{e} + \phi(0) = v_{a_1} + \phi(0), \qquad \langle \phi(0), \hat{e} \rangle = 0$$

and we observe that by theorem 5.5,

$$\|\phi(0)\|_2 \leqslant \frac{\epsilon_4^2}{10^2} a_1. \tag{5.39}$$

Recalling definition 3.1, we denote by $a(t) = a(S_t^K(m_1))$, the a-coordinate of $S_t^K(m_1)$, letting

$$S_t^K(m_1) = v_{a(t)} + \phi(t), \qquad \langle \phi(t), \hat{e} \rangle = 0, \tag{5.40}$$

and we denote by $\tau(t)$ the t-coordinate of $S_t^K(m_1)$ relative to $\{v_b(\cdot)\}$.

We observe that since $\epsilon_4 \leqslant \epsilon_3$, by proposition 3.8 $\tau(0)$ is well defined and we call s^* the largest time such

$$\|S_t^K(m_1) - v_{a(t)}\|_2 = \|\phi(t)\|_2 < \epsilon_4 a(t), \qquad \text{for all } t < s^* \tag{5.41}$$

so that $\tau(t)$ is well defined for all $t \leqslant s^*$.

We also denote by $a(\tau(t); b) := \langle v_b(\tau(t)), \hat{m}_L \rangle$ the a-coordinate of $v_b(\tau(t))$. We finally call

$$t^* := \inf\{t : a(\tau(t); b) = 2\zeta\}, \qquad t^* := \infty \text{ if } a(\tau(t); b) < 2\zeta \qquad \text{for all } t$$

Observe that from (3.46) it follows that

$$\tfrac{1}{2} a(t) < a(\tau(t); b) < 2a(t), \qquad \forall t \in [0, s^*] \tag{5.42}$$

We are going to prove that

$$t^* < \infty, \qquad s^* \geqslant t^* \tag{5.43}$$

and for δ small enough

$$\|S_t^K(m_1) - v_b(\tau(t))\|_2 \leqslant \frac{\epsilon_4}{8} \zeta_0^2 + C^* \delta \leqslant \frac{\epsilon_4}{4} \zeta_0^2, \qquad \forall t \leqslant t^*. \tag{5.44}$$

We conclude the proof of the theorem by using (5.43) and (5.44) that we will prove afterwards.

Proof that $t_\zeta(m) < t^* < \infty$. From (5.42) and (5.43) it follows that $a(t^*) \in [\zeta, 4\zeta]$ and since $a(0) = a_1 < \zeta$ by continuity of $a(t)$ we get that $t_\zeta(m) \leqslant t^* < \infty$.

Proof of (5.34). From (5.36) and (5.37) it follows that we only need to prove that

$$\|S^K_{t_\zeta(m)}(m) - v^{(+)}(s^-(\zeta))\|_2 \leqslant \zeta_0 \tag{5.45}$$

We first observe that the a-coordinate of an L^2-function u denoted by $a(u)$ is a continuous function, more precisely there is c^* so that

$$|a(u) - a(v)| \leqslant c^* \|u - v\|_2. \tag{5.46}$$

Thus, from (5.38) we get that for $t \in \left(s^-(\frac{1}{2}a_1), s^-(2\zeta)\right)$

$$a(v_b(\theta(a_0; b) + t)) \in \left(\frac{a_1}{2} - c^*\frac{\zeta_0}{10^2}, 2\zeta + c^*\frac{\zeta_0}{10^2}\right) \tag{5.47}$$

and from (5.44) we get that

$$a(\tau(t_\zeta(m) : b)) \in \left(\zeta - c^*\frac{\epsilon_4}{4}\zeta_0^2, \zeta + c^*\frac{\epsilon_4}{4}\zeta_0^2\right) \tag{5.48}$$

In proposition 3.6 it has been proved that $a(t; b)$ is a strictly increasing function of t so from (5.47) and (5.48) we get that (for ϵ_4 and ζ_0 small enough)

$$t^*_\zeta := \tau(t_\zeta(m)) - \theta(a_0; b) \in (s^-(a_1/2), s^-(2\zeta)). \tag{5.49}$$

From (5.38) and (5.49) we get that

$$\|v_b(\tau(t_\zeta(m))) - v^{(+)}(t^*_\zeta)\|_2 \leqslant \frac{\zeta_0}{10^2} \tag{5.50}$$

$$a(v^{(+)}(t^*_\zeta)) \in \left(\zeta - c^*\frac{\zeta_0}{10^2}, \zeta + c^*\frac{\zeta_0}{10^2}\right). \tag{5.51}$$

Since $a(v^+(t))$ is a strictly increasing function of t (see theorem 4.2) from (5.51) we get that for a suitable constant c,

$$\|v^{(+)}(t^*_\zeta) - v^{(+)}(s^-(\zeta))\|_2 \leqslant c\frac{\zeta_0}{10^2} \tag{5.52}$$

Thus

$$\begin{aligned}\|S^K_{t_\zeta(m)}(m) - v^{(+)}(s^-(\zeta))\|_2 &\leqslant \|S^K_{t_\zeta(m)}(m) - v_b(\tau(t_\zeta(m)))\|_2 \\ &\quad + \|v_b(\tau(t_\zeta(m))) - v^{(+)}(t^*_\zeta)\| + \|v^{(+)}(t^*_\zeta) - v^{(+)}(s^-(\zeta))\|_2 \\ &\leqslant \frac{\epsilon_4}{4}\zeta_0^2 + \frac{\zeta_0}{10^2} + c\frac{\zeta_0}{10^2},\end{aligned}$$

which gives (5.45) for ζ_0 and ϵ_4 small enough.

Proof of (5.44). We set $u(t) := S^K_t(m_1) - v_b(\tau(t))$. With computations similar to (3.69) and (3.71) we get

$$\dot\tau - 1 = A^K(S^K_t(m_1); b), \tag{5.53}$$

$$A^K(m; b) = A(m; b) + \frac{\langle K, e_{v_b(\tau)}\rangle}{\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle - \left\langle u, \dfrac{\delta e_{v_b(\tau)}}{\delta v_b(\tau)} f(v_b(\tau))\right\rangle}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = B^K(S^K_t(m_1); b), \tag{5.54}$$

$$B^K(m; b) = B(m; b) - [A^K(m; b) - A(m; b)]\langle u, f_L(v_b(\tau))\rangle + \langle u, K\rangle, \tag{5.55}$$

where $A(m; b)$ is defined in (3.70) and $B(m; b)$ in (3.72).

To bound $A(S_t^K(m_1); b)$ and $B(S_t^K(m_1); b)$ we use proposition 3.10. We thus have to split $u(t) = S_t^K(m_1) - v_b(\tau(t))$ into $u(t) = u_1(t) + u_2(t)$. We choose $u_2(t) = \phi(x,t)\mathbf{1}_{x\in A^c_{t,\delta}}$, recalling that $\phi(t) = S_t^K(m_1) - v_{a(t)}$, $a(t) = a(S_t^K(m_1))$, and the parameter ζ of proposition 3.10 equals δ (this ζ is obviously not the ζ of theorem 5.7). Thus

$$u_1(t) = \phi(t)\mathbf{1}_{x\in A_{t,\delta}} + v_{a(t)} - v_b(\tau(t)).$$

We then use (5.13) bounding on its right-hand side the term $\|\phi(t)\|_2^2 \leqslant [\epsilon_4 a(t)]^2$ because $t \leqslant s^*$, getting

$$\|u_1(t)\|_\infty \leqslant \alpha^* + 2\mathrm{e}^{-\tau^*/\beta} + c'_M(2[\epsilon_4 a(t)]^2 + 2L(\alpha^*)^2 + 8\delta)^{1/3} \\ + |a(t) - a(\tau(t); b)|\|\hat{e}\|_\infty + \frac{4c'_0}{\hat{\lambda}} a(\tau(t); b)^{5/3}, \tag{5.56}$$

where the last term comes from (3.27).

By (3.48) and (5.10) we then have $\|u_1(t)\|_\infty \leqslant \omega/10C_2$ as required in (3.74).

Denoting by $k(t)^2 = \int K(x,t)^2\,\mathrm{d}x$ and using (3.73) and (3.77), we then have

$$|A^K(S_t^K(m); b)| \leqslant 2\left\{4c_1\|u(t)\|_2 + \frac{C_2\|e_{v_b(\tau)}\|_\infty \|u(t)\|_2^2}{|\langle f_L(v_b(\tau)), e_{v_b(\tau)}\rangle|}\right\} + \frac{4k(t)}{\|f_L(v_b(\tau))\|_2} \tag{5.57}$$

and from (3.75) we have

$$B^K(S_t^K(m); b) \leqslant -\frac{\omega}{2}\|u(t)\|_2^2 + 8C_2\delta + \frac{4k(t)}{\|f_L(v_b(\tau))\|_2}\|u(t)\|_2\|f_L(v_b(\tau))\|_2 + \|u(t)\|_2 k(t)$$

Thus, since $\|u(t)\|_2 \leqslant 2$,

$$B^K(S_t^K(m); b) \leqslant -\frac{\omega}{2}\|u(t)\|_2^2 + 8C_2\delta + 5\|u(t)\|_2 k(t) \\ \leqslant -\frac{\omega}{2}\|u(t)\|_2^2 + 8C_2\delta + 10k(t) \tag{5.58}$$

that, together with (5.54) implies that

$$\|u(t)\|_2^2 \leqslant \mathrm{e}^{-\omega t}\|u(0)\|_2^2 + C^*\delta, \qquad C^* := \frac{1}{\omega}(16C_2 + 10).$$

By (3.64) and (5.39) we have that

$$\|u(0)\|_2 \leqslant 360\frac{c'_0}{\hat{\lambda}}a_1^{5/3} + 81\|m - v_{a_1}\|_2 \leqslant 360\frac{c'_0}{\hat{\lambda}}a_1^{5/3} + 81\frac{\epsilon_4^2}{10^2}a_1 \leqslant \frac{\epsilon_4}{8}a_1.$$

Therefore from our choice of a_1 we get that

$$\|u(t)\|_2^2 \leqslant \frac{\epsilon_4}{8}\zeta_0^2 + C^*\delta, \tag{5.59}$$

which proves (5.44).

Proof that $t^* < \infty$. We need to exclude the fact that $a(\tau(t); b)$ decreases reaching a value less than a_1. To this purpose we call $t^{**} \leqslant \infty$ the first time when $a(\tau(t^{**}); b) = a_1/2$ and $\sigma^* = \min\{t^*, t^{**}\}$ so that

$$\frac{a_1}{2} \leqslant a(\tau(t)) \leqslant 2\zeta, \qquad \text{for all } t \leqslant \sigma^* \tag{5.60}$$

For $t \leqslant \sigma^*$, using proposition 3.6, see (3.31), we get from (5.57)

$$|A^K(S_t^K(m); b)| \leqslant 2\left\{4c_1\|u(t)\|_2 + \frac{C_2\|e_{v_b(\tau)}\|_\infty\|u(t)\|_2^2}{\hat{\lambda}a_1/2}\right\} + \frac{4k(t)}{\hat{\lambda}a_1/2} \tag{5.61}$$

By (5.59) supposing ζ_0 and δ small enough,

$$|A^K(S_t^K(m); b)| \leqslant \frac{1}{4} + \frac{8k(t)}{\hat{\lambda} a_1} \tag{5.62}$$

From (3.28), (3.46), (5.53), (5.60) and (5.62), assuming that $\hat{\lambda} - \tilde{c}_2(2\zeta)^{2/3} \geqslant \frac{1}{2}$ and using the Cauchy–Schwartz inequality, we get for all $t \leqslant \sigma^*$

$$a(\tau(t); b) - a_1 \geqslant a(\tau(0); b) - a_1 + \int_0^t a(\tau(s); b)[\hat{\lambda} - \tilde{c}_2(2\zeta)^{2/3}]\dot{\tau}(s)\,\mathrm{d}s$$
$$\geqslant -\frac{a_1}{2} + \frac{a_1}{4}\frac{3t}{4} - \frac{16\delta\sqrt{t}}{\hat{\lambda}}.$$

Let h be a small positive number. Given h, suppose $\delta > 0$ so small that $16\delta < \hat{\lambda}\sqrt{h}\frac{a_1}{16}$. Then

$$a(\tau(h); b) - a_1 \geqslant -\frac{a_1}{2} + \frac{a_1}{8}h,$$

which shows that $h < t^{**}$. Moreover $a(\tau(h); b) \geqslant \frac{a_1}{2} + \frac{a_1}{8}h$. By using the same estimates we get

$$a(\tau(2h); b) - a(\tau(h); b) \geqslant \frac{a_1}{8}h$$

Thus $a(\tau(2h); b) \geqslant \frac{a_1}{2} + \frac{a_1}{4}h$. By iteration we then get that $t^* < t^{**}$ is finite.

Proof of the inequality $s^* \geqslant t^*$. Since

$$\|\phi(t)\|_2^2 = \inf_a \|S_t^K(m_1) - v_a\|^2,$$

from (3.27) and (5.35) we get for $t \leqslant t^*$

$$\|\phi(t)\|_2 \leqslant \|u(t)\|_2 + \|v_b(\tau(t)) - v_{a(\tau(t);b)}\|_2 \leqslant \|u(t)\|_2 + \frac{4c_0'}{\hat{\lambda}}a(\tau(t); b)^{5/3}$$
$$\leqslant \|u(t)\|_2 + \frac{4c_0'}{\hat{\lambda}}(2\zeta)^{2/3}a(\tau(t); b) \leqslant \|u(t)\|_2 + \frac{\epsilon_4}{10^6}a(\tau(t); b)$$

From (3.48), (5.42) and (5.35) we have that for $t \leqslant t^*$

$$|a(\tau(t); b) - a(t)| \leqslant 80\|S_t^K(m_1) - v_{a(t)}\|_2 + 80\frac{4c_0'}{\hat{\lambda}}a(t)^{5/3} \leqslant 80\|\phi(t)\| + 80\frac{4c_0'}{\hat{\lambda}}(4\zeta)^{2/3}a(t)$$
$$\leqslant 80\|\phi(t)\| + 80\frac{\epsilon_4}{10^6}a(t)$$

Thus since $1 - 80\frac{\epsilon_4}{10^6} > \frac{1}{2}$, from (5.44) we get

$$\|\phi(t)\|_2 \leqslant 2\|u(t)\|_2 + 2\frac{\epsilon_4}{10^6}\left[1 + 80\frac{\epsilon_4}{10^6}\right]a(t) \leqslant 2\frac{\epsilon_4}{4}\zeta_0^2 + \frac{\epsilon_4}{2}a(t)$$

Thus for ζ_0 small enough we get that $\|\phi(t)\|_2 < \epsilon_4 a(t)$ for all $t < t^*$ namely $s^* > t^*$.

Theorem 5.7 is thus proved. □

5.3. *Proof in the case of no external force*

In this section we prove theorem 5.9. Let m be as in the statement and denote by $a(0) := a(m) > 0$. We call $a(t) := a(S_t(m))$, the a-coordinate of $S_t(m)$, thus

$$S_t(m) = \hat{m}_L + a(t)\hat{e} + \phi(t), \qquad \langle\phi(t), \hat{e}\rangle = 0.$$

We then compute the time derivative of $a(t)$ getting

$$\dot{a} = \hat{\lambda} a + \langle R, \hat{e} \rangle, \qquad R = f_L(S_t(m)) - f_L(\hat{m}_L) - \hat{\mathcal{L}}(S_t(m) - \hat{m}_L).$$

Let c_M be as in the proof of lemma 5.3 and let $a_1 > a(0)$ be as in theorem 5.7 and small enough (this is possible by taking $\epsilon > a(0)$ small enough). Then the time derivative $\dot{a}(t)$ satisfies for all $t \leqslant t_{a_1}(m)$

$$\dot{a} \geqslant \hat{\lambda} a - c_M(a^2 + \|\phi\|_2^2) \tag{5.63}$$

and we want to prove that $\dot{a} > 0$. Let t^* be the largest time such that $\dot{a}(t) > 0$ for $t \leqslant t^*$ and let $t^{**} := \min\left(t^*, t_{a_1}(m)\right)$. Recalling the definition of τ^* given in (5.10), we first consider $t \leqslant \min\{\tau^*, t^{**}\}$ and use (5.5) to bound

$$\|\phi(t)\|_2^2 \leqslant e^{8c_M\tau^*}(c+1)a(t)^2 \tag{5.64}$$

having bounded $\|\phi(0)\|_2^2 \leqslant ca(0)^2 \leqslant ca(t)^2$. By (5.63) and (5.64),

$$\dot{a} \geqslant \hat{\lambda} a - c_M[1 + e^{8c_M\tau^*}(c+1)]a^2.$$

Since $a(t) < a_1$ for all $t < \min\{\tau^*, t^{**}\}$, by choosing a_1 small enough we can bound

$$c_M[1 + e^{8c_M\tau^*}(c+1)]a \leqslant c_M[1 + e^{8c_M\tau^*}(c+1)]a_1 < \tfrac{1}{2}.$$

Thus

$$\dot{a} \geqslant \frac{\hat{\lambda}}{2} a. \tag{5.65}$$

Thus $t^* > \tau^*$. For $t \geqslant \tau^*$ and $t \leqslant t^{**}$, we use (5.27) to bound

$$\begin{aligned}\|\phi(t)\|_2^2 &\leqslant \|\phi(\tau^*)\|_2^2 + \frac{2c_M}{\hat{\omega}} \tilde{c}_+ \mu_0 a(t)^2 \\ &\leqslant \left\{ e^{8c_M\tau^*}(c+1) + \frac{2c_M}{\hat{\omega}} \tilde{c}_+ \mu_0 \right\} a(t)^2.\end{aligned}$$

For a_1 small enough we then get as in (5.65) the bound $\dot{a} \geqslant \hat{\lambda} a/2$. We thus conclude that till when $a(t) < a_1$, $\dot{a} \geqslant \hat{\lambda} a/2$ so that the time when $a(t) = a_1$ is finite and theorem 5.9 follows from theorem 5.7. □

6. Energy bounds

Let Σ be the basin of attraction of $\hat{m}_L$, namely

$$\Sigma := \left\{ u \in L^\infty((-L/2, L/2); [-1, 1]) : \lim_{t\to\infty} \|S_t(u) - \hat{m}_L\|_2 = 0 \right\}.$$

Theorem 6.1. *There exists* $c_\Sigma > 0$ *such that*

$$F_L(m) - F_L(\hat{m}_L) \geqslant c_\Sigma \|m - \hat{m}_L\|_2^2 \qquad \text{for any } m \in \Sigma.$$

We will first reduce the proof to the case $\|m\|_\infty < 1$.

We denote by v^0 the solution of (C.22). Given $a \in (m_\beta, 1)$ we let t_a be such that $v^0(t_a) = a$ (since $v^0(t)$ is strictly decreasing, t_a is well defined).

Lemma 6.2. *For any a sufficiently close to 1 and any* $m \in L^\infty((-L/2, L/2); [-1, 1])$ *we have*

$$\sup_{t \geqslant t_a} \|S_t(m)\|_\infty \leqslant a, \tag{6.1}$$

and

$$\inf_{t \leqslant t_a} \|S_t(m) - \hat{m}_L\|_2 \geqslant \frac{1 - m_\beta}{30} \|m - \hat{m}_L\|_2. \tag{6.2}$$

Proof. Given any $m \in L^\infty((-L/2, L/2); [-1,1])$, we set $D_+ := \{x \in (-L/2, L/2) : m(x) \geqslant a\}$, $D_- := \{x \in (-L/2, L/2) : m(x) \leqslant -a\}$, $D := D_+ \cup D_-$ and $D^c := (-L/2, L/2) \setminus D$. Define

$$\psi(t) := S_t(m) - \hat{m}_L, \qquad \psi_1(t) := \mathbf{1}_{D^c}\, \psi(t), \qquad \psi_2(t) := \mathbf{1}_D\, \psi(t).$$

Since v^0 is a super-solution and $-v^0$ a sub-solution of (1.1), $\|S_{t_a}(m)\|_\infty \leqslant a$. Since $a > m_\beta$, a is a super-solution and $-a$ a sub-solution of (1.1), so that $\|S_t(S_{t_a}(m))\|_\infty \leqslant a$ for all $t \geqslant 0$, and (6.1) is proved.

- Proof of (6.2) under the assumption that

$$\|\psi_2(0)\|_2 > \tfrac{1}{2}\|\psi_1(0)\|_2. \tag{6.3}$$

We postpone the proof that if a is sufficiently close to 1 then

$$|S_{t_a}(m)| \geqslant 1 - 3(1-a) \qquad \text{in } D. \tag{6.4}$$

By (6.4) it follows $\|\psi_2(t_a)\|_2^2 \geqslant [1 - 3(1-a) - m_\beta]^2 |D|$. Since $|D| \geqslant \frac{1}{4}\int_D \psi_2(0)^2\, \mathrm{d}x$ we then get

$$\|\psi_2(t_a)\|_2^2 \geqslant \tfrac{1}{4}[1 - 3(1-a) - m_\beta]^2\ \|\psi_2(0)\|_2^2.$$

By (6.3), $\|\psi_2(0)\|_2 \geqslant \|\psi(0)\|_2/3$. We choose a so close to 1 that $[1 - 3(1-a) - m_\beta] \geqslant (1 - m_\beta)/2$, thus deriving (6.2) when (6.3) holds and pending the validity of (6.4), which we will prove next.

The solution $w(t)$ of

$$w_t = -1 - \frac{1}{\beta}\mathrm{arctanh}(w), \qquad w(0) = a \tag{6.5}$$

is a sub-solution of the equation satisfied by $S_t(m)$ restricted to D_+, considering $S_t(m)\mathbf{1}_{D_+^c}$ as a known term. Obviously,

$$w(t_a) \geqslant a - \left\{1 + \frac{1}{\beta}\ \mathrm{arctanh}(a)\right\} t_a$$

By (C.22) and the definition of t_a we have that

$$a = v^0(t_a) = 1 + \int_0^{t_a} [1 - \beta^{-1}\ \mathrm{arctanh}(v^0(s))]\, \mathrm{d}s.$$

Since $v^0(s) > a$ for $s < t_a$, we get

$$t_a \leqslant \frac{1-a}{\beta^{-1}\ \mathrm{arctanh}(a) - 1} \tag{6.6}$$

Let a be so close to 1 that $\frac{\mathrm{arctanh}(a)+\beta}{\mathrm{arctanh}(a)-\beta} \leqslant 2$. Then $w(t_a) \geqslant a - 2(1-a) = 1 - 3(1-a)$. The same argument applies in D_- and (6.4) is proved.

- Proof of (6.2) when (6.3) does not hold.

Suppose that a so close to 1 that $\mathrm{arctanh}\,(a) > \beta$. Then a is a super-solution of the equation $u_t = 1 - \beta^{-1}\mathrm{arctanh}\,(u)$ and since $m\mathbf{1}_{D^c} \leqslant a$, it follows $S_t(m)\mathbf{1}_{D^c} \leqslant a$. Similarly, $S_t(m)\mathbf{1}_{D^c} \geqslant -a$. Therefore

$$\|S_t(m)\mathbf{1}_{D^c}\|_\infty \leqslant a. \tag{6.7}$$

By (6.7),

$$\frac{1}{2}\frac{\mathrm{d}\|\psi_1(t)\|_2^2}{\mathrm{d}t} \geqslant -|\langle \psi_1(t), J^{\mathrm{neum}} * [\psi_1(t) + \psi_2(t)]\rangle| - \frac{\|\psi_1(t)\|_2^2}{\beta(1-a^2)},$$

hence

$$\frac{1}{2}\frac{\mathrm{d}\|\psi_1(t)\|_2^2}{\mathrm{d}t} \geqslant -\|\psi_1(t)\|_2^2 - \|\psi_1(t)\|_2\|\psi_2(t)\|_2 - \frac{\|\psi_1(t)\|_2^2}{\beta(1-a^2)}. \tag{6.8}$$

Let t_a^* be the maximal $\tau \leqslant t_a$ such that

$$\|\psi_2(t)\|_2 \leqslant \|\psi_1(t)\|_2 \qquad \text{for all } t \leqslant \tau. \tag{6.9}$$

We have $t_a^* > 0$, because (6.3) does not hold. Inequality (6.8) then gives

$$\frac{\mathrm{d}\|\psi_1(t)\|_2^2}{\mathrm{d}t} \geqslant -2\left\{2 + \frac{1}{\beta(1-a^2)}\right\}\|\psi_1(t)\|_2^2, \qquad t \leqslant t_a^*,$$

hence

$$\|\psi_1(t)\|_2^2 \geqslant \mathrm{e}^{-2\left\{2+\frac{1}{\beta(1-a^2)}\right\}t}\|\psi_1(0)\|_2^2, \qquad t \leqslant t_a^*. \tag{6.10}$$

For a sufficiently close to 1, $\mathrm{arctanh}(a) - \beta \geqslant |\log(1-a)|/4$ and by (6.6),

$$t_a \leqslant \frac{4\beta(1-a)}{|\log(1-a)|},$$

so that the exponent in (6.10) is bounded by

$$-2\left\{2 + \frac{1}{\beta(1-a^2)}\right\}t_a \geqslant -8\frac{2\beta(1-a^2)+1}{(1+a)|\log(1-a)|}.$$

Hence, for a sufficiently close to 1, (6.10) yields

$$\|\psi_1(t)\|_2^2 \geqslant \left\{1 - \frac{5}{|\log(1-a)|}\right\}\|\psi_1(0)\|_2^2, \qquad t \leqslant t_a^*. \tag{6.11}$$

We will next show that $t_a^* = t_a$. Indeed, in D_+, $m_t(x,t) < 0$ for all $t \leqslant t_a$: in fact

$$m(x,t) \geqslant w(t), \qquad w \text{ as in (6.5)}$$

and for a sufficiently close to 1,

$$\mathrm{arctanh}(w(t_a)) > \beta.$$

Hence $m_t(x,t) < 0$. An analogous argument shows that $m_t(x,t) > 0$ in D_-. Since $w(t_a) \geqslant m_\beta \geqslant |\hat{m}|$, $|\psi_2(x,t)| \leqslant |\psi_2(x,0)|$. The condition (6.9) is then satisfied with $t = t_a$. Suppose in fact by contradiction that $t_a^* < t_a$, then, by (6.11) and with a sufficiently close to 1,

$$\|\psi_1(t_a^*)\|_2 > \frac{\|\psi_1(0)\|_2}{2} \geqslant \|\psi_2(0)\|_2 \geqslant \|\psi_2(t_a^*)\|_2,$$

so that, by continuity, (6.9) is satisfied also for $t > t_a^*$ against the assumption of maximality of the latter.

Thus (6.11) holds for all $t \leqslant t_a$. Writing $\epsilon = 5/|\log(1-a)|$ in (6.11), (6.1) follows from

$$\|\psi(t)\|_2 \geqslant \|\psi_1(t)\|_2 \geqslant (1-\epsilon)^{1/2}\|\psi_1(0)\|_2 \geqslant \frac{2(1-\epsilon)^{1/2}}{3}\|\psi(0)\|_2.$$

Having used that from the fact that (6.3) does not hold we have

$$\|\psi(0)\|_2 \leqslant \|\psi_1(0)\|_2 + \|\psi_2(0)\|_2 \leqslant \tfrac{3}{2}\|\psi_1(0)\|_2.$$

The proof of lemma 6.2 is concluded. □

6.1. *Proof of theorem 6.1*

We will prove equivalently that there exists a constant $c > 0$ such that for any $\delta > 0$ small enough,

$$\inf_{m\in\Sigma:\|m-\hat{m}_L\|_2\geqslant\delta} F_L(m) \geqslant F_L(\hat{m}_L) + c\delta^2. \tag{6.12}$$

Given m as above, let t^* be the first time t when $\|S_t(m) - \hat{m}_L\|_2 = \delta/30$. By lemma 6.2 we have $t^* \geqslant t_a$, so that $m^* := S_{t^*}(m)$ verifies (6.1). Calling $v(\cdot, t) := S_t(m^*) - \hat{m}_L$, we have by definition

$$\|v(\cdot, 0)\|_2 = \frac{\delta}{30}. \tag{6.13}$$

Recalling that $f_L(\hat{m}_L) = 0$, we compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_2^2 = \langle v(t), f_L(S_t(m^*)) - f_L(\hat{m}_L)\rangle.$$

Note that

$$f_L(S_t(m^*))(x, t) - f_L(\hat{m}_L)(x) = J^{\mathrm{neum}} * v(x, t) - \frac{1}{\beta}\operatorname{arctanh}'(\xi)v(x, t),$$

where ξ is in the interval with endpoints $S_t(m^*)(x, t)$ and $\hat{m}_L(x)$. Since $\|\hat{m}_L\|_\infty \leqslant m_\beta$ and $\|S_t(m^*)\|_\infty \leqslant a$, there is a constant $c_1 = c_1(a)$ so that

$$\frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_2^2\right| \leqslant c_1\|v(t)\|_2^2,$$

which yields, for any $t > 0$,

$$\mathrm{e}^{-c_1 t}\|v(0)\|_2 \leqslant \|v(t)\|_2 \leqslant \mathrm{e}^{c_1 t}\|v(0)\|_2. \tag{6.14}$$

Since $F_L(m) \geqslant F_L(S_t(m^*))$, recalling the definition of $\hat{\mathcal{L}} = \mathcal{L}_{\hat{m}_L}$ given in (3.1) and (2.11), we have

$$F_L(m) - F_L(\hat{m}_L) \geqslant F_L(S_t(m^*)) - F_L(\hat{m}_L) = -\frac{1}{2}\langle v(t), \hat{\mathcal{L}}v(t)\rangle$$
$$+\frac{1}{3!\beta}\int_{-\frac{L}{2}}^{\frac{L}{2}} \operatorname{arctanh}''(\xi)v(x, t)^3\,\mathrm{d}x,$$

where $\xi = \xi(x, t)$ is in the interval having endpoints $F_L(S_t(m^*(x, t)))$ and $\hat{m}_L(x)$. From lemmas C.2 and C.1 we have that there are ρ and C, with

$$\rho \leqslant 2\mathrm{e}^{-t/\beta}, \qquad C \leqslant \frac{3}{\sqrt{8}}(\beta\|J'\|_\infty)^{1/3},$$

such that $\|v(t)\|_\infty \leqslant \rho + C\|v(t)\|_2^{2/3}$. Then, for a suitable constant $c_2 > 0$ we have

$$F_L(m) - F_L(\hat{m}_L) \geqslant -\tfrac{1}{2}\langle v(t), \hat{\mathcal{L}}v(t)\rangle - c_2(\rho + C\|v(t)\|_2^{2/3})\|v(t)\|_2^2. \tag{6.15}$$

We denote by $v_\perp(t)$ and $v_\parallel(t)$ the components of $v(\cdot, t)$ perpendicular and parallel to $\hat{e}$ respectively. According to definition 3.1, we have

$$v_\parallel(t) = a(S_t(m^*))\hat{e}, \qquad \langle v_\perp(t), \hat{e}\rangle = 0. \tag{6.16}$$

From (2.13) and since $\hat{L}\hat{e} = \hat{\lambda}\hat{e}$, we get

$$\langle v(t), \hat{\mathcal{L}}v(t)\rangle = \langle v_\perp(t), \hat{\mathcal{L}}v_\perp(t)\rangle + \langle v_\parallel(t), \hat{\mathcal{L}}v_\parallel(t)\rangle \leqslant \hat{\omega}\|v_\perp(t)\|_2^2 + \hat{\lambda}\|v_\parallel(t)\|_2^2.$$

This inequality together with (6.14) and (6.15) gives

$$F_L(m) - F_L(\hat{m}_L) \geqslant \left(\frac{\hat{\omega}}{2} - c_2(\rho + C\mathrm{e}^{(2/3)c_1 t}\|v(0)\|_2^{2/3}\right)\|v_\perp(t)\|_2^2$$

$$-\left(\frac{\hat{\lambda}}{2} + c_2(\rho + C\mathrm{e}^{(2/3)c_1 t}\|v(0)\|_2^{2/3}\right)\|v_\|(t)\|_2^2.$$

We then choose $t = \tau$ and τ such that

$$c_2\,(2\mathrm{e}^{-\tau/\beta}) \leqslant \frac{\hat{\omega}}{16},$$

and recalling (6.13) we choose δ in (6.12) so small that

$$c_2 C\mathrm{e}^{(2/3)c_1\tau}\delta^{2/3} \leqslant \frac{\hat{\omega}}{16}. \tag{6.17}$$

Thus

$$F_L(m) - F_L(\hat{m}_L) \geqslant \frac{\hat{\omega}}{4}\|v_\perp(\tau)\|_2^2 - \left(\frac{\hat{\lambda}}{2} + \frac{\hat{\omega}}{8}\right)\|v_\|(\tau)\|_2^2.$$

We claim that for δ small enough

$$\frac{\hat{\omega}}{4}\|v_\perp(\tau)\|_2^2 \geqslant 2\left(\frac{\hat{\lambda}}{2} + \frac{\hat{\omega}}{8}\right)\|v_\|(\tau)\|_2^2.$$

Assume by contradiction that $\|v_\perp(\tau)\|_2^2 < B\|v_\|(\tau)\|_2^2$ with $B = 2\left[\frac{\hat{\lambda}}{2} + \frac{\hat{\omega}}{8}\right]\frac{4}{\hat{\omega}}$. Then, recalling (6.16), we have for a suitable constant c,

$$\|S_\tau(m^*) - [\hat{m}_L + a(S_\tau(m^*))\hat{e}]\|_2 \leqslant c a(S_\tau(m^*)).$$

From (6.14) and (6.13) we have that for all $\epsilon > 0$ and for $\delta < \epsilon$ small enough

$$a(S_\tau(m^*)) = \langle v(\tau), \hat{e}\rangle \leqslant \mathrm{e}^{c_1\tau}\frac{\delta}{30} < \epsilon.$$

Thus theorem 5.9 implies that $S_t(S_\tau(m^*))$ converges in L^2, to $m^{(+)}$ as $t \to \infty$, thus contradicting the assumption that $m \in \Sigma$. Hence

$$F_L(m) - F_L(\hat{m}_L) \geqslant \frac{\hat{\omega}}{8}\|v_\perp(\tau)\|_2^2 \geqslant \frac{\kappa^2\hat{\omega}}{8(1+\kappa^2)}\|v(\tau)\|_2^2,$$

where

$$\kappa^2 := 2\left(\frac{\lambda}{2} + \frac{\hat{\omega}}{8}\right)\frac{4}{\hat{\omega}},$$

which, using (6.14), proves (6.12). □

7. Extensions

In this section we discuss several extensions of the above results mainly in view of applications to tunnelling. They involve a repetition of the proofs we have done so far, just an adaptation to the new contexts without any really new ideas. For this reason we merely state the results without proofs.

7.1. Glauber dynamics

By this we refer to the evolution (1.11) (indeed the scaling limit of Glauber dynamics in Ising systems with Kac potentials gives rise to (1.11)) and denote by T_t the corresponding flow. Stationary solutions are the same for T_t and S_t but invariant sets may differ. Indeed the invariant manifolds $v^{(\pm)}$ are not invariant for T_t. However, there are invariant manifolds that we denote by $v_{\text{gl}}^{(\pm)}$ for T_t as well satisfying (1.9)–(1.10). Actually their existence has been proved prior to the present paper, see [4]. The proof in [4] is similar but not exactly as in the present paper, as it exploits the existence of a spectral gap in L^∞. The stability properties in section 5 require some additional properties on the external force as one should also impose that the solution has values in $[-1, 1]$; such a condition is automatically satisfied for the equation (5.1). To avoid complications we thus state that theorems 5.5, 5.7 and 5.9 retain their validity for S_t^K replaced by T_t, i.e. without the additional force K, and with $v_{\text{gl}}^{(\pm)}$. We just state here from [2] the following result.

Theorem 7.1. *There is $\epsilon > 0$ and for any $r \in (0, 1)$ there is $L^* > 0$ so that if $L \geqslant L^*$ and $\|m - \bar{m}_\xi \mathbf{1}_{|x|\leqslant L}\|_\infty < \epsilon$, $|\xi| > rL$, then*

$$\lim_{t\to\infty} \|T_t(m) - m^{(\sigma)}\|_\infty = 0,$$

where σ is equal to the sign of ξ.

The same proofs of section 6 apply to T_t showing that:

Theorem 7.2. *There is $c > 0$ so that if $\lim_{t\to\infty} \|T_t(m) - \hat{m}_L\|_2 = 0$, then*

$$\|m - \hat{m}_L\|_2^2 \leqslant c[F_L(m) - F_L(\hat{m}_L)]$$

7.2. The two-dimensional case

The stability properties of section 5 and the lower bound on the energy of section 6 extend to the $d = 2$ case as we explain below.

Consider the spatial domain $Q_L = [-L/2, L/2]^2$ and the evolution equation with a kernel $j^{\text{neum}}(r, r')$ defined with reflections at the boundaries and supposing that the probability kernel $j(r, r')$ is smooth, depends on $|r - r'|$ and vanishes for $|r - r'| \geqslant 1$. The solution to the corresponding equation on Q_L with an initial datum $m(r)$ which depends only on $r \cdot e_1$ (e_1 the unit vector along the x-axis) preserves the planar symmetry and verifies (1.1) with

$$J(x, x') = \int_{\mathbb{R}} j(0, (x' - x)e_1 + ye_2)\,\mathrm{d}y$$

(the roles of j and J are interchanged in [3]). As a consequence the invariant manifolds $v^{(\pm)}$ are still invariant for S_t in $d = 2$. The stability properties given in section 5 use the structure of the equation which is the same in $d = 2$ and are based on spectral properties of the linearized operators.

7.2.1. Spectral analysis. The analogue of theorem 2.4 also holds in $d = 2$, as we explain next. Given any function $m \in L^2((-L/2, L/2); [-1, 1])$ we denote by $m^{(e)}$ its extension to Q_L, namely

$$m^{(e)}(r) := m(r \cdot e_1), \qquad r \in Q_L,$$

and we call *planar* a function $m \in L^2(Q_L, [-1, 1])$ that depends only on the x-coordinate. We also denote by $\mathcal{M}_L \subset L^2(Q_L, [-1, 1])$ the set that contains the instanton $\hat{m}_L^{(e)}$ and all the planar functions m such that

$$\|m - \hat{m}_L^{(e)}\|_\infty \leqslant \epsilon(L),$$

where $\epsilon(L)$ is a small number as required by theorem 7.3 below.

Given any $m \in \mathcal{M}_L$ we denote by $\mathcal{L}_m$ the linearized operator in Q_L, namely,

$$\mathcal{L}_m u(r) := -\frac{1}{p_m(r \cdot e_1)} u(r) + j^{\text{neum}} * u(r), \qquad p_m(x) := \beta[1 - m(x)^2], \qquad r \in Q_L.$$

Theorem 7.3. *There are positive constants $c_\pm$, $\hat{c}$ and a positive function $\epsilon(L)$ so that for L large enough if $m \in \mathcal{M}_L$ then the L^2 norm of the operator $\mathcal{L}_m$ is bounded by $1 + \beta^{-1}\text{arctanh}''((1 + m_\beta)/2)$ and:*

(i) $\mathcal{L}_m$ has a positive eigenvalue λ_m

$$c_- \mathrm{e}^{-2\alpha L} \leqslant \lambda_m \leqslant c_+ \mathrm{e}^{-2\alpha L},$$

with eigenfunction e_m which is planar, smooth and strictly positive;

(ii) $\mathcal{L}_m$ has a gap $\omega(L) = \hat{c}/L^2$, i.e.,

$$\langle u, \mathcal{L}_m u\rangle \leqslant -\omega(L)\langle u, u\rangle, \qquad u \in L^2(Q_L; [-1, 1]), \quad \langle u, e_m\rangle = 0; \tag{7.1}$$

(iii) let $\frac{\delta e_m}{\delta m}$ be the linear operator such that $\left.\frac{\mathrm{d}e_{m(s)}}{\mathrm{d}s}\right|_{s=0} = \frac{\delta e_m}{\delta m} \left.\frac{\mathrm{d}m(s)}{\mathrm{d}s}\right|_{s=0}$ for any smooth curve $m(s)$, $m(0) = m$. Then there is $c_1 > 0$ so that

$$\left\|\frac{\delta e_m}{\delta m}\right\|_2 \leqslant c_1.$$

Proof. The proof is the same as the one of theorem 2.4 given in appendix B since it is a consequence of the following remark. Let Ω_m, $m \in \mathcal{M}_L$ be the linearized operator of the Glauber semigroup T_t, namely

$$\Omega_m u(r) := -u(r) + p_m(r \cdot e_1) j^{\text{neum}} * u(r), \qquad r \in Q_L.$$

As explained in appendix B, by using that $\mathcal{L}_m = \frac{1}{p_m}\Omega_m$ the spectral analysis of $\mathcal{L}_m$ in $L^2(Q_L, \mathrm{d}x\, \mathrm{d}y)$ is a consequence of the spectral properties of Ω_m in $L^2(Q_L, p_m^{-1}(x)\, \mathrm{d}x\, \mathrm{d}y)$. Since the statements (i)–(iii) are proven in [3] for the operator Ω_m in $L^2(Q_L, p_m^{-1}\, \mathrm{d}x\, \mathrm{d}y)$, we get that the same holds for $\mathcal{L}_m$ as well. We omit the details. □

7.2.2. Stability under a small additive force Here we briefly comment on the proof in $d = 2$ of theorems 5.5, 5.7 and 5.9.

Consider the evolution (5.1) for $r \in Q_L$ and with the time depending force $K(r, t)$ smooth and such that

$$\|K\|^2 = \int_0^\infty k(t)^2\, \mathrm{d}t < \infty, \qquad k(t)^2 := \int_{Q_L} K(r, t)^2\, \mathrm{d}r.$$

As in (5.4), for $\|m - \hat{m}_L^{(e)}\|_\infty$ small enough, we write for any $r = (x, y) \in Q_L$,

$$S_t^K(m)(r) = v_{a(t)}(x) + \phi(r, t), \qquad \langle\phi(t), \hat{e}\rangle = 0. \tag{7.2}$$

where $a(t)$ is the a-coordinate of $S_t^K(m)$. Observing that the first function on the right-hand side of (7.2) is planar, it is possible to repeat line by line the proofs given in section 5 based on the spectral-gap property (7.1). We omit the details.

7.3. Application to tunnelling

From theorem 5.5 and theorem 5.7 in $d = 1, 2$ we get theorem 7.4 below, used in [1] for $d = 1$ and in [3] for $d = 2$.

Theorem 7.4. *For any $\tau > 0$ and for any $\zeta > 0$ there is $\delta > 0$ so that if $\|K\| < \delta$ and $\|m - \hat{m}^e_L\|_2 < \delta$ only the following two alternatives hold:*

(i) for all times $t \geqslant 0$, $\|S^K_t(m) - \hat{m}^e_L\|_2 < \zeta$;
(ii) there are $t^* > 0$ and $\sigma \in \{-, +\}$ so that $\|S^K_t(m) - \hat{m}^e_L\|_2 < 2\|v^{(\sigma)}_L(\cdot, -\tau) - \hat{m}^e_L\|_2$ for all $t \leqslant t^*$ while $\|S^K_t(m) - v^{(\sigma)}_L(\cdot, -\tau + (t - t^*))\|_2 < \zeta$ for all $t \geqslant t^*$.

Proof. Let m be such that $\|m - \hat{m}^e_L\|_2 < \delta$ with δ small enough so that $a_1 \equiv a(m)$ satisfies the hypothesis of theorems 5.5 and 5.7. Then if $t_{a_1}(m) = \infty$ from theorem 5.5 we get that (i) holds. If instead $t_{a_1}(m) < \infty$, theorem 5.7 implies that (ii) holds. □

We finally state, again without proofs, that theorem 7.2 retains its validity also in $d = 2$ with Q_L.

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Appendix A. Existence of dynamics

Let $m \in \mathcal{C}([-L/2, L/2]; (-1, 1))$, $b := \|m\|_\infty$, $s \in (0, \infty]$ and

$$\mathcal{X}(m; s) := \left\{u \in \mathcal{C}([-L/2, L/2] \times [0, s]) : u(\cdot, 0) = m, \ \sup_{t \leqslant s} \|u(\cdot, t)\|_\infty < 1\right\}.$$

Given $u \in \mathcal{X}(m; s)$ and $t \leqslant s$ define $\Psi(u)(t)$ as

$$\Psi(u)(t) := m + \int_0^t f_L(u)\, \mathrm{d}\sigma. \tag{A.1}$$

Note that there exists $a > 0$ so that for all $m \in \mathcal{C}([-L/2, L/2]; (-1, 1))$ we have $f_L(m)(x) \leqslant a$ if $m(x) \geqslant 0$ and $f_L(m)(x) \geqslant -a$ if $m(x) \leqslant 0$. Hence, given m, there is $s(m) > 0$ so that Ψ maps $\mathcal{X}(m; s(m))$ into itself. Moreover, Ψ is a contraction on $\mathcal{X}(m; s^*)$ if $s^* \leqslant s(m)$ is small enough. Hence Ψ has a fixed point that we may call $S_t(m)$ because it solves (1.1) for $t \leqslant s^*$ with initial datum m.

In addition Ψ preserves the order, so that if $b(t)$ solves the ordinary differential equation $b_t = -b + \beta^{-1} \operatorname{arctanh}(b)$, $b(0) = b_0$, then $\sup_{t \geqslant 0} b(t) < 1$ and for any $s > 0$ the set

$$\mathcal{X}^0(m; s) := \{u \in \mathcal{X}(m; \infty) : u(\cdot, 0) = m, \|u(\cdot, t)\|_\infty \leqslant b(t) \text{ for all } t \leqslant s\}$$

is well defined and left invariant by Ψ. Thus $S_t(m) \in \mathcal{X}^0(m; s^*)$ and by setting $S_{t+s^*}(m) = S_t(S_{s^*}(m))$ we can extend the evolution past s^* and by iteration to all times, thus concluding that (1.1) has a unique global solution for any initial datum $m \in \mathcal{C}([-L/2, L/2]; (-1, 1))$.

Appendix A.1. Proof of proposition 2.1

We prove the proposition only for supersolutions, the case of subsolutions being similar. Let v be a super-solution of (1.1) with $v(\cdot, 0) \geqslant m$. Since $v(\cdot, t) \geqslant \Psi(v)(t)$, the set

$\{u \in \mathcal{X}(m; s(m)) : u(\cdot, t) \leqslant v(\cdot, t)$for all $t \leqslant s(m)\}$ is left invariant by Ψ and consequently $S_t(m), t \leqslant s(m)$, is in such a set, thus proving that $S_t(m) \leqslant v(\cdot, t)$ for $t \leqslant s(m)$. The argument can be repeated starting from $s(m)$ and we then conclude that $S_t(m) \leqslant v(\cdot, t)$ for all $t \geqslant 0$.

Appendix A.2. Proof of proposition 2.2

Set $v(x, t) := S_t(m)(x) - S_t(\tilde{m})(x)$. If $v(x, t) > 0$ (respectively, < 0) then $-[\operatorname{arctanh}(S_t(m)(x)) - \operatorname{arctanh}(S_t(\tilde{m})(x))] < 0$ (respectively, > 0). Therefore,

$$|v(x, t)| \leqslant |v(x, 0)| + \int_0^t \int_{-L}^L J^{\text{neum}}(x, y)|v(y, s)|\, dy,$$

so that $\|v(\cdot, t)\|_\infty \leqslant e^t \|v(\cdot, 0)\|_\infty$.

Appendix A.3. Proof of proposition 2.3

Define $K_t^\pm(b)$ as the solutions of the Cauchy problem $b_t = \pm 1 + \beta^{-1} \operatorname{arctanh}(b)$, $b(0) = b_0$, $|b_0| < 1$. Since $|J^{\text{neum}} * m| \leqslant 1$ for any $m \in L^\infty((-L/2, L/2); (-1, 1))$, $K_t^-(-\|m\|_\infty) \leqslant S_t(m) \leqslant K_t^+(\|m\|_\infty)$. Moreover, for any $t > 0$ there exists $\lim_{b\to\pm 1} K_t^\pm(b) \in (-1, 1)$, hence proposition 2.3 follows.

Appendix B. Proof of theorem 2.4

To bound the L^2 norm of the operator $\mathcal{L}_m$ defined in (2.11), we write

$$\langle f, Jf\rangle \leqslant \frac{1}{2} \int J^{\text{neum}}(x, y)\{f^2(x) + f^2(y)\}\, dx\, dy \leqslant \langle f, f\rangle,$$

while $\operatorname{arctanh}''(m) \leqslant \operatorname{arctanh}''((1+m_\beta)/2)$ if $\epsilon(L, \delta)$ is small enough, hence the desired bound on the norm of $\mathcal{L}_m$.

Now take positive constants $a_\pm$ so that for any m as in the hypotheses of the theorem

$$\|p_m\|_\infty < a_+, \qquad \|p_m^{-1}\|_\infty < a_-.$$

Using (2.8) we get

$$\lambda_m := \sup_u \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} = \sup_u \frac{\langle u, \Omega_m u\rangle_{p_m}}{\langle u, u\rangle_{p_m}} \frac{\langle u, u\rangle_{p_m}}{\langle u, u\rangle} \leqslant a_-\lambda \leqslant a_-\hat{c}_+ e^{-2\alpha L},$$

which is the upper bound in (2.12).

Recall that e is the strictly positive eigenfunction of Ω_m (see (2.7)) corresponding to the positive eigenvalue λ. Using that $\langle e, \Omega_m e\rangle = \lambda\langle e, e\rangle$ and (2.8) we get

$$\sup_u \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} \geqslant \frac{\langle e, \mathcal{L}_m e\rangle}{\langle e, e\rangle} = \lambda \frac{\langle e, e\rangle_{p_m}}{\langle e, e\rangle} = \lambda \frac{\langle e, e\rangle_{p_m}}{\langle e, p_m e\rangle_{p_m}} \geqslant \frac{\lambda}{a_+} \geqslant \frac{\hat{c}_-}{a_+} e^{-2\alpha L},$$

which is the lower bound in (2.12).

Let e_m be such that $\langle e_m, e_m\rangle = 1$ and $\lambda_m = \langle e_m, \mathcal{L}_m e_m\rangle$. Then, if $|e_m| \neq e_m$ it follows (since $J \geqslant 0$)

$$0 \geqslant \langle |e_m|, \mathcal{L}_m |e_m|\rangle - \langle e_m, \mathcal{L}_m e_m\rangle = \int J(x, y)\{|e_m(x)e_m(y)| - e_m(x)e_m(y)\} > 0,$$

which is a contradiction. Therefore e_m is nonnegative. To see that e_m is strictly positive, let x_0 be such that $e_m(x_0) = 0$. Since $J * e_m(x_0) > 0$, we get that $\mathcal{L}_m e_m(x_0) \neq \lambda_m e_m(x_0)$. The proof of assertion (i) is concluded.

The second largest spectral point is

$$\inf_{w} \sup_{u:\langle u,w\rangle=0} \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle}.$$

Let $\omega > 0$ be as in (2.9), i.e.,

$$-\omega = \sup_{u:\langle u,v_m\rangle_{p_m}=0} \frac{\langle u, \Omega_m u\rangle_{p_m}}{\langle u, u\rangle_{p_m}}.$$

Choosing $w = p_m^{-1} e$ we have

$$\inf_{w} \sup_{u:\langle u,w\rangle=0} \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} \leqslant \sup_{u:\langle u,p^{-1}e\rangle=0} \frac{\langle u, \mathcal{L}_m u\rangle}{\langle u, u\rangle} = \sup_{u:\langle u,e\rangle_{p_m}=0} \frac{\langle u, \Omega_m u\rangle_{p_m}}{\langle u, u\rangle}$$
$$\leqslant -\omega \frac{\langle u, u\rangle_{p_m}}{\langle u, u\rangle} \leqslant -\omega a_-,$$

which proves (ii).

It remains to show (2.14). Recalling that $\frac{\delta e_m}{\delta m}$ is given by

$$\frac{\mathrm{d}e_{m(\cdot,s)}}{\mathrm{d}s}\bigg|_{s=0} = \frac{\delta e_m}{\delta m} \frac{\mathrm{d}m(\cdot,s)}{\mathrm{d}s}\bigg|_{s=0},$$

we differentiate the following identity:

$$\mathcal{L}_m e_m = \lambda_m e_m, \qquad \langle e_m, e_m\rangle = 1.$$

Thus, denoting by $\dot{u}$ derivative with respect to s at $s = 0$, we get

$$(\mathcal{L}_m - \lambda_m)\dot{e}_m = [\dot{\mathcal{L}}_m - \dot{\lambda}_m]e_m, \qquad \dot{\lambda}_m = \langle e_m, \dot{\mathcal{L}}_m e_m\rangle,$$
$$\dot{\mathcal{L}}_m e_m = -\frac{2m\dot{m}}{\beta(1-m^2)^2} e_m,$$

so that

$$\frac{\delta e_m}{\delta m}\psi = -(\mathcal{L}_m - \lambda_m)^{-1}\left[\frac{2me_m\psi}{\beta(1-m^2)^2} - \left\langle e_m, \frac{2me_m\psi}{\beta(1-m^2)^2}\right\rangle\right]. \tag{B.1}$$

From (B.1) and (2.15), (2.14) follows. □

Appendix C. Estimates on the evolution

In the following lemmas m is a measurable function on $(-L/2, L/2)$ with $\|m\|_\infty \leqslant 1$.

Lemma C.1. *Suppose there are* $\rho \geqslant 0$ *and* $c > 0$ *so that*

$$|m(x) - m(y)| \leqslant \rho + c\,|x-y| \qquad \text{for a.e.} x, y \in (-L/2, L/2). \tag{C.1}$$

If

$$c \geqslant \frac{\sqrt{2}\|m\|_2}{L^{3/2}} \tag{C.2}$$

then

$$\|m\|_\infty \leqslant \rho + C\|m\|_2^{2/3}, \qquad C := (2^{-1/2} + 2^{-5/3})c^{1/3}. \tag{C.3}$$

Proof. Let u be the function on $(-L/2, L/2)$ obtained from m by reflections around $\pm L/2$. For $\delta \in (0, L/2]$ and almost every x with $|x| < L/2$,

$$2\delta m(x) = \int_{x-\delta}^{x+\delta} u(y)\,\mathrm{d}y + \int_{x-\delta}^{x+\delta} (u(x) - u(y))\,\mathrm{d}y.$$

By Cauchy–Schwartz inequality and (C.1) it follows $2\delta|m(x)| \leqslant \sqrt{2\delta}\|m\|_2 + 2\delta\rho + \delta^2 c/2$, hence

$$|m(x)| \leqslant \frac{\|m\|_2}{\sqrt{2\delta}} + \rho + \frac{\delta c}{4}. \tag{C.4}$$

Let $\delta = \min\left\{\left(\frac{\sqrt{2}\|m\|_2}{c}\right)^{2/3}, L\right\}$. By (C.2) we have $\delta = \left(\frac{\sqrt{2}\|m\|_2}{c}\right)^{2/3}$. Then (C.3) follows from (C.4). □

We next prove that for $t > 0$ the solution $S_t(m)$ satisfies the hypothesis of lemma C.1.

Lemma C.2. *Let $\|m\|_\infty < 1$ and $m(x,t) = S_t(m)(x)$. Then for almost any $x, y \in (-L/2, L/2)$ and any $t > 0$,*

$$|m(x,t) - m(y,t)| \leqslant 2\mathrm{e}^{-t/\beta} + \beta\|J'\|_\infty\,|x-y|, \tag{C.5}$$

where $\|J'\|_\infty := \sup_x |J(0,x)|$.

Proof. Fix arbitrarily two points $x, y \in [-L/2, L/2]$ and write

$$m_t(x,t) = -\frac{\operatorname{arctanh}\{m(x,t)\}}{\beta} + g_0(t), \qquad m_t(y,t) = -\frac{\operatorname{arctanh}\{m(y,t)\}}{\beta} + g_1(t), \tag{C.6}$$

where $g_0(t) := J * m(x,t)$ and $g_1(t) := J * m(y,t)$ are regarded as 'known' terms. By the smoothness of J,

$$\|g_1 - g_0\|_\infty \leqslant \|J'\|_\infty |x-y|. \tag{C.7}$$

Let $\lambda \in [0,1]$, define $g(t;\lambda) := \lambda g_1(t) + (1-\lambda)g_0(t)$ and $u(t;\lambda)$, $t \geqslant 0$, as the solution of

$$u_t(t;\lambda) = -\frac{\operatorname{arctanh}\{u(t;\lambda)\}}{\beta} + g(t;\lambda) \tag{C.8}$$

with initial datum

$$u(0;\lambda) = \lambda m(y,0) + (1-\lambda)m(x,0).$$

Note that $u(0;\lambda)$ is a constant in $(-L/2, L/2)$, since x and y are fixed. Calling $v(t;\lambda) := \frac{\mathrm{d}}{\mathrm{d}\lambda}u(t;\lambda)$, we have

$$v_t = -\frac{v}{\beta(1-u^2)} + (g_1 - g_0) \tag{C.9}$$

whose solution is

$$v(t;\lambda) = \mathrm{e}^{-a(t,0;\lambda)}v(0;\lambda) + \int_0^t \mathrm{e}^{-a(t,s;\lambda)}[g_1(s) - g_0(s)]\,\mathrm{d}s, \tag{C.10}$$

where $a(t,s;\lambda) := \int_s^t \frac{1}{\beta(1-u^2)}$. The lemma is concluded using the inequality

$$|m(y,t) - m(x,t)| \leqslant \int_0^1 |v(t;\lambda)|\,\mathrm{d}\lambda, \tag{C.11}$$

and the facts that $|v(0,\lambda)| \leqslant 2$ and $a(t,s;\lambda) \geqslant \frac{t-s}{\beta}$. □

Lemma C.3. *Assume that* $m([-L/2, L/2])$ *is compactly contained in* $(-1, 1)$. *Then the following assertions hold:*

(i) let $\|m\|_\infty \leqslant \mathtt{a}$ *and* $\mathtt{a} \in (m_\beta, 1)$. *Then* $\|S_t(m)\|_\infty \leqslant \mathtt{a}$ *for any* $t > 0$*;*

(ii) suppose $\|m_x\|_\infty < \infty$. *Then*

$$\left\| \frac{\partial}{\partial x} S_t(m) \right\|_\infty < \infty \qquad \forall t \geqslant 0, \tag{C.12}$$

and for L *sufficiently large*

$$\|S_t(m)\|_\infty \leqslant C_t \|S_t(m)\|_2^{2/3}, \qquad C_t = \frac{3}{\sqrt{8}} (\mathrm{e}^{-t/\beta} \|m_x(\cdot, 0)\|_\infty + \beta \|J'\|_\infty)^{1/3}; \tag{C.13}$$

(iii) let $u(t) := S_t(m) - \hat{m}_L$ *with* m *as in (i), and let*

$$R(t) := \beta^{-1}\{\mathrm{arctanh}\{S_t(m)\} - \mathrm{arctanh}\{\hat{m}_L\} - \mathrm{arctanh}'\{\hat{m}_L\} u(t)\}. \tag{C.14}$$

Then

$$|R(t)| \leqslant C_2 (u(t))^2, \qquad C_2 := \frac{1}{2\beta} \mathrm{arctanh}''(\mathtt{a}). \tag{C.15}$$

Proof. Statement (i) follows from the maximum principle. Calling $v(\cdot, t) := S_t(m)(\cdot)$, by differentiating (1.1) with respect to x we get

$$v_{xt} = -\frac{1}{\beta(1 - v^2)} v_x + J_x^{\mathrm{neum}} * v,$$

the solution of which is

$$v_x(x, t) = \mathrm{e}^{\int_0^t a(x,s)\,\mathrm{d}s} v_x(x, 0) + \int_0^t \mathrm{e}^{\int_s^t a(x,s')\,\mathrm{d}s'} b(x, s)\,\mathrm{d}s, \tag{C.16}$$

where $a := -[\beta(1 - v^2)]^{-1}$, $b := J_x^{\mathrm{neum}} * v$. Recalling that $|v| < 1$, we get

$$\|v_x(\cdot, t)\|_\infty \leqslant \mathrm{e}^{-t/\beta} \|v_x(\cdot, 0)\|_\infty + \beta \|J'\|_\infty \tag{C.17}$$

which proves (C.12). To prove (C.13) we use that from (C.17) it follows that we can apply lemma C.1 to v with $\rho = 0$ and $c = \mathrm{e}^{-t/\beta} \|v_x(\cdot, 0)\|_\infty + \beta \|J'\|_\infty$. Indeed, since $\|S_t(m)\|_2 = \|v\|_2 \leqslant 2L$, then

$$\frac{\sqrt{2}\|S_t(m)\|_2}{L^{3/2}} \leqslant \frac{2\sqrt{2}}{L^{1/2}}.$$

Thus for all L such that

$$\frac{2\sqrt{2}}{L^{1/2}} \leqslant \mathrm{e}^{-t/\beta} \|v_x(\cdot, 0)\|_\infty + \beta \|J'\|_\infty,$$

we have that (C.2) is satisfied.

Finally (C.15) follows from (i). □

In the next lemma we give an estimate of the difference between $S_t(m)$ and the solution u_K of the equation

$$u_t = f_L(u) + K, \qquad u(\cdot, 0) = m, \tag{C.18}$$

where $K \in L^\infty((-L/2, L/2) \times (0, +\infty))$.

Lemma C.4. *There exists a constant* $c^* > 0$ *so that the following holds. Let* $u(\cdot, t) := S_t(m)$ *and* $u_K(\cdot, t)$ *solve (C.18). Define*

$$A_{t,\delta} := \{x \in (-L/2, L/2) : \int_0^t |K(x,s)|^2 \, \mathrm{d}s < \delta\}, \qquad t > 0, \quad \delta > 0.$$

Then

$$\sup_{x \in A_{t,\delta}} |u(x,t) - u_K(x,t)| \leqslant c^* \mathrm{e}^{\|J\|_\infty t} (\sqrt{\delta} + |A_{t,\delta}^c|)^{1/2}, \tag{C.19}$$

where $A_{t,\delta}^c := (-L/2, L/2) \setminus A_{t,\delta}$.

Proof. Set $\phi := u - u_K$, $w(s) := \sup_{x \in A_{t,\delta}} |\phi(x,s)|$. Pick $x \in A_{t,\delta}$. Then, using also that $|\phi| \leqslant 2$ and that $\phi(x,t)[\operatorname{arctanh}(u(x,t)) - \operatorname{arctanh}(u_K(x,t))] \geqslant 0$, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} (\phi(x,s)^2) \leqslant |\phi(x,s)| \|J\|_\infty (w(s) + 2|A_{t,\delta}^c|) + 2|K(x,s)|. \tag{C.20}$$

Let us integrate (C.20) over $s \in [0, t]$, and then take the supremum over $x \in A_{t,\delta}$. We get

$$w(t)^2 \leqslant 2\|J\|_\infty \int_0^t w(s)^2 \, \mathrm{d}s + 8t\|J\|_\infty |A_{t,\delta}^c| + 4t^{1/2}\sqrt{\delta},$$

and (C.19) follows. □

C.1. Estimates on the forced evolution

We now give estimates on the evolution (C.18) in order to reduce to the case when the solution has support compactly contained in the interval $(-1, 1)$.

Define

$$A_t := \left\{ x \in (-L/2, L/2) : \int_0^t |K(x,s)| \, \mathrm{d}s < \frac{1}{6}(1 - m_\beta^+) \right\},$$

$$B := \left\{ x : u(x,0) < m_\beta^+ + \frac{1}{2}(1 - m_\beta^+) \right\}.$$

Let

$$m_\beta^+ = \tanh \beta, \qquad m_\beta < m_\beta^+ < 1, \tag{C.21}$$

and $v^0(t)$, $t \geqslant 0$, be the solution of

$$v_t^0 = 1 - \frac{1}{\beta} \operatorname{arctanh}(v^0), \qquad \lim_{t \to 0} v^0(t) = 1. \tag{C.22}$$

Then $v^0(t)$ decreases to m_β^+ as $t \to \infty$, so that we can define τ_β as

$$v^0(\tau_\beta) = m_\beta^+ + \tfrac{1}{2}(1 - m_\beta^+). \tag{C.23}$$

Lemma C.5. *If* u_K *solves (C.18), then the following holds. For any* $t > 0$ *and* $x \in A_t \cap B$, *and for any* $t \geqslant \tau_\beta$ *and* $x \in A_t$, *we have* $u_K(x,s) < m_\beta^+ + \frac{2}{3}(1 - m_\beta^+)$ *for all* $s \leqslant t$.

Proof. We start from the proof of the last statement. Set $K^+ := \max\{K, 0\}$. Then the problem

$$v_t = 1 - \frac{1}{\beta} \operatorname{arctanh}(v) + K^+, \qquad \lim_{t \to 0} v(x,t) = 1 \tag{C.24}$$

has a solution v defined in $(-L/2, L/2) \times (0, +\infty)$, with $0 < v(x,t) < 1$ for any $(x,t) \in (-L/2, L/2) \times (0, +\infty)$. We claim that v is a super-solution of (C.18). Indeed, since $v \leqslant 1$ (as K is bounded),

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 1 - \frac{1}{\beta}\mathrm{arctanh}\,(v) + K^+ \geqslant J^{\mathrm{neum}} * v - \frac{1}{\beta}\mathrm{arctanh}\,(v) + K.$$

Thus $u_K \leqslant v$. On the other hand, $v(x,t) \leqslant w(x,t) := v^0(t) + \int_0^t K(x,s)^+\,\mathrm{d}s$, because w is a super-solution of (C.24). Indeed, since $w \geqslant v^0$,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 1 - \frac{1}{\beta}\,\mathrm{arctanh}\,(v^0) + K^+ \geqslant 1 - \frac{1}{\beta}\,\mathrm{arctanh}\,(w) + K^+.$$

We have thus proved that $u_K \leqslant w$, hence for $t \geqslant \tau_\beta$, and recalling that $v^0(t)$ is a decreasing function of t,

$$\begin{aligned} u_K(x,t) &\leqslant v^0(\tau_\beta) + \int_0^t K^+(x,s)\,\mathrm{d}s < v^0(\tau_\beta) + \frac{1}{6}\,(1-m_\beta^+) \\ &= m_\beta^+ + \frac{1}{2}(1-m_\beta^+) + \frac{1}{6}(1-m_\beta^+) = m_\beta^+ + \frac{2}{3}(1-m_\beta^+) \end{aligned}$$

which concludes the proof of the last statement in the lemma. The first one is proved in the same way after defining $v(t)$ as the solution of (C.24) with $v(0) = m_\beta^+ + \frac{1}{2}\,(1-m_\beta^+)$, and redefining $w(x,t) = v^0(t+\tau_\beta) + \int_0^t K^+(x,s)\,\mathrm{d}s$. □

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