Energy levels of a non local functional

G. Bellettini* A. De Masi† E. Presutti‡§

Abstract

We study the critical points of the non local free energy functional considered in [8]. The functional has two minimizers (ground states) $m^{(\pm)}$ with zero energy. We prove that there is a first excited state identified as the instanton $\hat{m}_L$ of [5], [6], [10], and that above the energy of the instanton there is a gap. We also characterize parts of the basin of attraction of $m^{(\pm)}$ and $\hat{m}_L$ under a dynamics associated to the free energy functional, [7]. The result completes the analysis in [3] of tunnelling from $m^{(-)}$ to $m^{(+)}$.

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1 Introduction

In this paper we consider a non local free energy functional $\mathcal{F}_L$ on $L^\infty([-L,L];[-1,1])$ defined by

$$\mathcal{F}_L(m) = \int_{-L}^{L} \phi_\beta(m)\,dx + \frac{1}{4}\int_{-L}^{L}\int_{-L}^{L} J^{\rm neum}(x,x')[m(x)-m(x')]^2 dx\,dx' \tag{1.1}$$

where $\beta > 1$,

$$\phi_\beta(m) = \tilde{\phi}_\beta(m) - \min_{|s|\le 1} \tilde{\phi}_\beta(s), \qquad \tilde{\phi}_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta}\mathcal{S}(m)$$

$$\mathcal{S}(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}$$

*Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica 00133 Roma, Italy. E-mail: belletti@mat.uniroma2.it

†Dipartimento di Matematica, Università di L’Aquila, via Vetoio, loc. Coppito, 67100 l’Aquila, Italy. E-mail: demasi@univaq.it

‡Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica 00133 Roma, Italy. E-mail: presutti@mat.uniroma2.it

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and

$$J^{\text{neum}}(x,y) = J(x,y) + J(x, R_L(y)) + J(x, R_{-L}(y))$$

with $R_\xi(y) = \xi - (y-\xi)$ the reflection of y around ξ and $J(x,y)$, $(x,y) \in \mathbb{R} \times \mathbb{R}$, a smooth, symmetric, translational invariant probability kernel supported in $|y-x| \le 1$. We also assume that $J(0,x)$ is a non increasing function whenever restricted to $x \ge 0$. The functional (1.1) arises in statistical mechanics in the analysis of Ising systems with Kac potentials, [7], and it has been extensively studied in the recent years.

The critical points of $\mathcal{F}_L$ are defined as the functions $m \in L^\infty([-L,L];[-1,1])$ which solve in the whole of $[-L,L]$ the "non local mean field equation"

$$m = \tanh\{\beta J^{\text{neum}} * m\}. \tag{1.2}$$

(1.2) has also a dynamical interpretation as the equation for the stationary points of the evolution equation

$$u_t = g_L(u) := -u + \tanh\{\beta J^{\text{neum}} * u\} \tag{1.3}$$

which is derived from Glauber dynamics in the Ising systems with Kac potentials mentioned above, [7].

We call "energy levels" of $\mathcal{F}_L$ its values at the critical points and borrowing from the terminology on linear operators, we call "spectrum" of $\mathcal{F}_L$ the set of all its energy levels. Many physical properties are determined by the structure of the spectrum at its bottom and its investigation is the main aim of this paper. We start by observing that the spectrum of $\mathcal{F}_L$ is contained in the positive axis, as both terms on the r.h.s. of (1.1) are non negative. The spectrum actually starts at 0. Indeed, since J^{neum} is a probability kernel, the solutions of (1.2) which are spatially homogeneous, namely $m(x) = u \in [-1,1]$ for all $x \in [-L,L]$, satisfy the "mean field equation" $u = \tanh\{\beta u\}$. As $\beta > 1$, there are three solutions $u = \pm m_\beta$, $m_\beta \in (0,1)$, and $u = 0$. The two critical points $m^{(\pm)}(x) = \pm m_\beta$, $x \in [-L,L]$, have zero energy, $\mathcal{F}_L(m^{(\pm)}) = 0$, and are called "ground states", while $\mathcal{F}_L(0) = 2\phi_\beta(0)L > 0$.

Thus the spectrum starts at 0 and 0 has degeneracy 2 as there is no other m besides $m^{(\pm)}$ with 0 energy. In fact the second term on the r.h.s. of (1.1) forces a minimizer to be spatially homogeneous and we have already seen that, among the spatially homogeneous functions, $m^{(\pm)}$ are the only ground states. We also know from the literature that the state $m \equiv 0$ is not the first excited state (i.e. with minimal positive energy), at least for L large enough. In fact its energy $2\phi_\beta(0)L$ becomes, as L increases, larger than the energy of the instanton $\hat{m}_L$

(a spatially non homogeneous critical point), which instead remains bounded as $L \to \infty$, [5], [6], [10]. In this paper we will prove:

Theorem 1.1. *For all L large enough, $\mathcal{F}_L(\hat{m}_L)$ is the first energy level above 0, it is not degenerate and above $\mathcal{F}_L(\hat{m}_L)$ there is a spectral gap, namely there is $\epsilon > 0$ so that if m is a critical point and $\mathcal{F}_L(m) \le \mathcal{F}_L(\hat{m}_L) + \epsilon$, then $m \in \{m^{(-)}, m^{(+)}, \hat{m}_L\}$.*

Such a characterization of the spectrum was the missing element in [3] in the analysis of tunnelling from $m^{(-)}$ to $m^{(+)}$, which therefore with the help of Theorem 1.1 is now complete. We can also use Theorem 1.1 to determine partially the basin of attraction of $m^{(\pm)}$ and $\hat{m}_L$:

Theorem 1.2. *For all L large enough, there is $\epsilon > 0$ so that for any m such that $\mathcal{F}_L(m) \le \mathcal{F}_L(\hat{m}_L) + \epsilon$*

$$\lim_{t \to \infty} S_t(m) \in \{m^{(-)}, m^{(+)}, \hat{m}_L\} \tag{1.4}$$

where $S_t(m)$ is the solution of (1.3) starting from m at time 0.

Theorem 1.2 is a corollary of Theorem 1.1 (but the converse is also true and indeed we will prove Theorem 1.1 as a consequence of Theorem 1.2). In fact, the semi-group $S_t(m)$ decreases the energy, in the sense that $\mathcal{F}_L(S_t(m))$ is a strictly decreasing function of t, unless m is a critical point. Moreover, by compactness and continuity the limit points of an orbit $S_t(m)$ are critical points and by lower semi-continuity of $\mathcal{F}_L$ their energy is smaller than $\mathcal{F}_L(m)$, hence (1.4).

The information on the basin of attraction of $\{m^{(-)}, m^{(+)}, \hat{m}_L\}$ contained in Theorem 1.2 is sufficient for the analysis of tunnelling, because the control parameter is the energy, but in many applications in statistical physics natural neighborhoods are in the L^∞ topology. In such neighborhoods the energy may grow proportionally to L (the domain size) and new criteria for being attracted to $\{m^{(-)}, m^{(+)}, \hat{m}_L\}$ are needed. To formulate the results we need some more notation and definition and we postpone the issue to the next section at the end of which we give an outline on the content of the paper. A preliminary version of the present paper is [4].

2 Definitions and results

For ease of notation it is sometimes convenient to work in the whole line rather than in $[-L, L]$, this is possible provided we suitably restrict the space of functions. $m \in L^\infty(\mathbb{R}; [-1, 1])$ is symmetric around ξ if $m(\xi + x) = m(\xi - x)$, $x \in \mathbb{R}$; it is called L-symmetric if it is symmetric around all points $(2n+1)L$, $n \in \mathbb{Z}$. The symmetric extension of $m \in L^\infty([-L, L]; [-1, 1])$ is then the L-symmetric function on $\mathbb{R}$ which agrees with m in $[-L, L]$ and

$$\int_{-L}^{L} \phi_\beta(m) dx + \frac{1}{4} \int_{-L}^{L} \int_{\mathbb{R}} J(x, x')(m(x) - m(x'))^2 dx\, dx' \tag{2.1}$$

is equal to $\mathcal{F}_L(m\mathbf{1}_{|x| \le L})$. Moreover if m is L-symmetric and $S_t(m)$ solves

$$u_t = g(u) := -u + \tanh\{\beta J * u\}, \quad u(\cdot, 0) = m, \tag{2.2}$$

then $S_t(m)$ is L-symmetric and its restriction to $[-L, L]$ solves (1.3). It is therefore equivalent to consider (2.1) and (2.2) on the space of L-symmetric functions or to work in the context of Section 1, and we take advantage of this equivalence by setting each problem in the one of the two contexts which is more convenient for the specific purpose. To have more compact notation we sometimes use the same symbol for a function and its L-symmetric extension. Natural neighborhoods in statistical mechanics are defined in terms of "closeness in the average" and of "coarse graining" transformations. We briefly recall the main notion adapted to the present context.

Definition 2.1. *(Coarse graining).*
Let $\mathcal{D}^{(\ell)}$, $\ell > 0$, be the partition of $\mathbb{R}$ into the intervals $[n\ell, (n+1)\ell)$, $n \in \mathbb{Z}$, and $I_x^{(\ell)}$ the interval in $\mathcal{D}^{(\ell)}$ which contains the point x. Then the ℓ-coarse grained image of a L^∞ function m is

$$m^{(\ell)}(x) := \fint_{I_x^{(\ell)}} m(y)\, dy, \quad \fint_{\Lambda} m(y)\, dy := \frac{1}{|\Lambda|} \int_{\Lambda} m(y)\, dy, \tag{2.3}$$

Definition 2.2. *(Phase indicators).* Given an "accuracy parameter" $\zeta > 0$, let

$$\eta^{(\zeta,\ell)}(m; x) = \begin{cases} \pm 1 & \text{if } |m^{(\ell)}(x) \mp m_\beta| \le \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Denoting ℓ_- and ℓ_+ two values of the parameter ℓ, with ℓ_+ an integer multiple of ℓ_-, we then define the "phase indicator"

$$\Theta^{(\zeta,\ell_-,\ell_+)}(m; x) = \begin{cases} \pm 1 & \textit{if } \eta^{(\zeta,\ell_-)}(m; \cdot) = \pm 1 \textit{ in } [-L, L] \cap \left(I_{x-\ell_+}^{(\ell_+)} \cup I_x^{(\ell_+)} \cup I_{x+\ell_+}^{(\ell_+)} \right), \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 2.3. *(Contours).*

The contours of a function $m \in L^\infty$ are defined as the connected components of the set $\{x : \Theta^{(\zeta,\ell_-,\ell_+)}(m;x) = 0\}$. $\Gamma = [x_-, x_+)$ is a plus contour if $\eta^{(\zeta,\ell_-)}(m;x_\pm) = 1$, a minus contour if $\eta^{(\zeta,\ell_-)}(m;x_\pm) = -1$, otherwise it is called mixed. The parameters $(\zeta, \ell_-, \ell_+, L)$ are called compatible with $(\zeta_0, c_1, \kappa) \in \mathbb{R}^3_+$ if $\zeta \in (0, \zeta_0)$, $\ell_- \le \kappa\zeta$, $1/\ell_- \le \ell_+$; L is a multiple integer of ℓ_+ and ℓ_+ a multiple integer of ℓ_-.

Definition 2.4. *(Good choice of parameters).*

In the sequel we take $\zeta \le \zeta_0$, $\zeta_0 > 0$ suitably small, $L > \zeta^{-8}$ and $\ell_\pm$ determined by L and ζ as follows. ℓ_+ is the smallest number $\ge \zeta^{-4}$ such that $L = n\ell_+$, $n \in \mathbb{N}$; ℓ_- is the largest number $\le \zeta^2$ such that $\ell_+ = p\ell_-$, $p \in \mathbb{N}$, $p > 1$. To have more compact notation we will omit the dependence on (ζ, ℓ_-, ℓ_+), unless ambiguities may arise.

The regions where $\Theta^{(\zeta,\ell_-,\ell_+)}(m;x) = 1$ are said to be in plus equilibrium (or in the plus phase), those where $\Theta^{(\zeta,\ell_-,\ell_+)} = -1$ are in the minus equilibrium and the mixed contours which separate plus and minus regions represent an interface. The equilibrium interface (defined by an optimization problem, see [1], [2]) is represented, when the minus phase is to the left, by the instanton $\bar{m}(x)$ which is a stationary solution of (2.2)

$$\bar{m}(x) = \tanh\{\beta(J * \bar{m})(x)\}, \qquad x \in \mathbb{R}, \tag{2.4}$$

with asymptotic behavior

$$\lim_{x \to \pm\infty} \bar{m}(x) = \pm m_\beta. \tag{2.5}$$

As recalled in Section 3 below, $\bar{m}(x)$ is unique (modulo translations) and it is a $\mathcal{C}^\infty$, strictly increasing, antisymmetric function. We denote by $\bar{m}_\xi$ a translation of the instanton, namely

$$\bar{m}_\xi(x) = \bar{m}(x - \xi), \qquad \xi \in \mathbb{R} \tag{2.6}$$

which is interpreted as the equilibrium interface located at ξ.

Definition 2.5. *(Neighborhoods of pure phases and interfaces).* Given $k \in \mathbb{N}$ and $(\zeta, \ell_-, \ell_+, L)$ as in Definition 2.4, setting $\Theta = \Theta^{(\zeta,\ell_-,\ell_+)}$, we define

$$U_- = \big\{m \in L^\infty([-L,L];[-1,1]) : \Theta(m,\cdot) < 1,\ |\{\Theta(m,\cdot) = 0\}| \le k\ell_+\big\};$$

$$U_+ = \big\{m : -m \in U_-\big\};$$

$U_{-,+} = \big\{m \in L^\infty([-L,L];[-1,1]) : m$ has a unique mixed contour Γ, $|\Gamma| \le k\ell_+$, and there exists $\xi \in \Gamma$ such that $\mathrm{dist}(\xi, [-L,L] \setminus \Gamma) \ge \ell_+/2$, and $|m^{(\ell_-)} - \bar{m}_\xi^{(\ell_-)}| \le 2\zeta$ on $[-L,L]\big\}$;

$$U_{+,-} = \{m : -m \in U_{-,+}\}.$$

Observing that U_-, U_+, $U_{-,+}$, $U_{+,-}$ are pairwise disjoint, we finally define

$$U = U_- \cup U_{-,+} \cup U_{+,-} \cup U_+.$$

A first step in the proof of Theorem 1.2 is the following result, proved in Section 6.

Theorem 2.6. *If $\zeta > 0$ is small enough, there is $k \in \mathbb{N}$ such that, for L large enough*

$$\big\{m \in L^\infty([-L,L];[-1,1]) : \mathcal{F}_L(m) < \mathcal{F}_L(\hat{m}_L) + \zeta^{100}\big\} \subset U$$

(recall that U depends on k and ζ).

Of course, the number 100 in the above statement can be replaced by any number sufficiently large.

The profiles in $U_- \cup U_+$ are attracted by $m^{(\pm)}$, more precisely in Section 7 we prove the following Theorem:

Theorem 2.7. *If $\zeta > 0$ is small enough, there is $k \in \mathbb{N}$ such that, for L large enough*

$$\lim_{t\to\infty} \|S_t(m) - m^{(\pm)}\|_\infty = 0 \qquad \textit{for all } m \in U_\pm$$

so that if $m \in U_\pm$ is stationary then $m = m^{(\pm)}$.

If $m \in U \setminus (U_- \cup U_+)$, then m may either be attracted by $m^{(\pm)}$ as in Theorem 2.7 but it could also be attracted by $\hat{m}_L$:

Theorem 2.8. *Given any $r \in (0,1)$, for all L large enough the set $\{m \in U_{-,+} : \Gamma \cap (-L,-rL] \ne \emptyset\}$ (Γ the contour of $m \in U_{-,+}$) is attracted by $m^{(+)}$; the set $\{m \in U_{-,+} : \Gamma \cap [rL, L) \ne \emptyset\}$ by $m^{(-)}$; any other $m \in U_{-,+}$ is attracted either by $m^{(\pm)}$ or by $\hat{m}_L$.*

By symmetry under sign change, Theorem 2.8 extends to its analogue in $U_{+,-}$, we do not state explicitly the result. Theorem 2.8 is proved in two steps. We show that any element in $U_{-,+}$ evolves after a finite time to a function which is in sup norm close to an instanton (restricted to $[-L, L]$); we then prove that functions close to an instanton in sup norm are attracted to $m^{(\pm)}$ if the center of the instanton is not too close to the origin, otherwise they may as well converge to $\hat{m}_L$.

Outline of the paper. In Section 3 we recall results known in the literature and which are used later in the proofs. Therefore in a first reading Section 3 may be skipped: specific references to the results stated in Section 3 will be given when needed. In Section 4 we prove lower bounds on $\mathcal{F}_L$ analogous to the Peierls bounds in statistical mechanics. In Section 5 we prove lower bounds on the infinite volume version $\mathcal{F}$ of the free energy functional. With such a background, in Section 6 we then prove the localization statement in Theorem 2.6. The second part of the paper contains the analysis of the critical points in U: in Section 7 we prove Theorem 2.7. In Section 8 we prove that functions close to an instanton in sup norm are attracted to $m^{(\pm)}$ if the center of the instanton is not too close to the origin. This proof exploit the existence of a travelling subsolution that converge to $m^{(\pm)}$ as proved in Section 9 and Appendix A. Finally in Section 10 we complete the proof of Theorem 2.8.

3 General background

In this section we collect some results in the literature which will be useful in the sequel. Some statements are not really explicit in the literature, in particular those about the finite volume instantons (see Subsection 3.3) which often have been proved in the presence of an external magnetic field, but they are all close enough to omit their proofs. The assumption that $J(0, x)$ is a non increasing function of $x \geq 0$ is used only in Subsection 3.3, all the other statements hold without such an assumption and in more generality.

In Subsection 3.1 below we state results on the spirit of the Peierls estimates for the free energy functional. To this end we need some definitions.

If Λ is a finite union of intervals contained in $[-L, L]$, we write $\Lambda^c := [-L, L] \setminus \Lambda$ for its complement in $[-L, L]$ and m_Λ for the restriction of m to Λ. We define

$$\mathcal{F}_{L;\Lambda}(m_\Lambda) = \int_\Lambda \phi_\beta(m_\Lambda)dx + \frac{1}{4}\int_\Lambda\int_\Lambda J^{\text{neum}}(x, x')(m_\Lambda(x) - m_\Lambda(x'))^2 dx\, dx'$$

$$\mathcal{F}_{L;\Lambda}(m_\Lambda|m_{\Lambda^c}) = \mathcal{F}_{L;\Lambda}(m_\Lambda) + \frac{1}{2}\int_\Lambda\int_{\Lambda^c} J^{\rm neum}(x,x')(m_\Lambda(x) - m_{\Lambda^c}(x'))^2 dx\, dx' \qquad (3.1)$$

We let $\mathcal{F}(m)$ be the functional on $L^\infty(\mathbb{R};[-1,1])$ with values in $[0,\infty]$ defined by (1.1) with $L=\infty$ and J in place of $J^{\rm neum}$.

3.1 Properties of the free energy functional (see [12])

We suppose that ζ_0, c_1 and $0<\kappa<1$ are small enough and that the parameters (ζ,ℓ_-,ℓ_+,L) are compatible with ζ_0, c_1 and κ.

- There is $\omega>0$ so that for any interval $\Lambda=[x',x'')\subset[-L,L]$, union of intervals belonging to $\mathcal{D}^{(\ell_+)}$, and for any m such that $\eta^{(\zeta,\ell_-)}(m;\cdot)=1$ on Λ, there is ψ with the following properties. $\mathcal{F}_L(m)\ge\mathcal{F}_L(\psi)$; $\psi=m$ on $[x'+1,x''-1]^c$; $\eta^{(\zeta,\ell_-)}(\psi;\cdot)=1$ on Λ;

$$\psi = \tanh\{\beta J^{\rm neum} * \psi\}, \qquad \text{on } [x'+1,x''-1] \qquad (3.2)$$

$$|\psi(x) - m_\beta| \le c_2 e^{-\omega\, {\rm dist}(x,\Lambda^c)}, \qquad x\in[x'+1,x''-1]. \qquad (3.3)$$

  The equation (3.2) supplemented by the condition $\psi=m$ outside $[x'+1,x''-1]$, has unique solution. The analogous result holds for $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot)=-1$ on Λ and $-m_\beta$ replaced by m_β and when $L=\infty$ (in which case $[x',x'')$ may also be unbounded).

- Let $\Gamma=[x',x'')$ be a contour for $m\in L^\infty([-L,L];[-1,1])$. Then

$$\mathcal{F}_{L;\Gamma}(m_\Gamma|m_{\Gamma^c}) \ge 2c_1\zeta^2\frac{\ell_-}{\ell_+} \qquad (3.4)$$

- The functional $\mathcal{F}_L(m)$ is continuous both in L^2 and L^∞. Then (lower semicontinuity) if m_n converges to m in $L^2_{\rm loc}(\mathbb{R})$,

$$\liminf_{n\to\infty}\mathcal{F}(m_n)\ge\mathcal{F}(m)$$

We now state results on the dynamics (2.2) which will be used in the sequel. We start with the following Theorem that summarizes general properties of the evolution whose proof can be found for instance in [12].

Theorem 3.1. *The following holds.*

1. *Super and subsolutions.* $z(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, *is a sub-solution (respectively a super-solution) of* (1.3) *if* $z_t - g_L(z) \leq 0$ *[respectively* ≥ 0*]. Then* $z(\cdot,t) \leq S_t(m)$, *if* $z(\cdot,0) \leq m$ *[respectively* $z(\cdot,t) \geq S_t(m)$, *if* $z(\cdot,0) \geq m$*].*

2. *Barrier Lemma. There is a constant* $C > 0$ *so that, setting* $m_i(\cdot,t) = S_t(m_i)$, $i = 1,2$, *for any* $V \geq e^2\beta$ *and for any* $t > 0$,

$$\sup_{s\leq t} \big|m_1(0,s) - m_2(0,s)\big| \leq e^{(\beta-1)t} \sup_{|r|\leq Vt} \big|m_1(r,0) - m_2(r,0)\big| + \; Ce^{-tV\log\frac{V}{e\beta}} \tag{3.5}$$

3. *Basin of attraction of* $m^\pm$*. The sets* $\{m \in L^\infty(\mathbb{R};[-1,1]) : \lim_{t\to\infty} \|S_t(m) - m^\pm\|_\infty = 0\}$ *are open both in* L^2 *and* L^∞ *and they contain the functions which are strictly positive, respectively negative.*

We call instantons stationary solutions of (2.2) and (1.3) which are increasing, antisymmetric and bounded in modulus by m_β. In particular we denote by $\bar m : \mathbb{R} \to (-m_\beta, m_\beta)$ an instanton on the whole of $\mathbb{R}$ and by $\hat m_L : [-L,L] \to (-m_\beta, m_\beta)$ a finite volume instanton, thus

$$\bar m(x) = \tanh\left\{\beta J * \bar m(x)\right\}, \quad x \in \mathbb{R} \qquad \hat m_L(x) = \tanh\left\{\beta J^{\text{neum}} * \hat m_L(x)\right\}, \quad x \in [-L, L]$$

Properties like existence, uniqueness, stability of instantons have been widely studied and, in the next two Subsections, we summarize those needed in the sequel.

3.2 Properties of the instanton (see [12], [9])

There is a unique antisymmetric $\bar m(x)$ solution of (2.4) and (2.5). Furthermore $\bar m(x)$ is a $\mathcal{C}^\infty$, strictly increasing and antisymmetric function.

1. *Asymptotic behavior of the instanton.* For $\beta > 1$, let $\alpha > 0$ be such that

$$\beta(1 - m_\beta^2) \int_{\mathbb{R}} J(0,y) e^{\alpha y}\, dy = 1 \tag{3.6}$$

Then there are $a > 0$, $\alpha_0 > \alpha$ and $c > 0$ so that for all $x \geq 0$

$$\left|\bar m(x) - (m_\beta - ae^{-\alpha x})\right| + \left|\bar m'(x) - a\alpha e^{-\alpha x}\right| + \left|\bar m''(x) + a\alpha^2 e^{-\alpha x}\right| \leq ce^{-\alpha_0 x} \tag{3.7}$$

where $\bar m'$ and $\bar m''$ are respectively the first and second derivatives of $\bar m$.

2. *Basin of attraction of the instantons manifold.* All the translations of $\bar{m}$ are solutions of (2.4) and, recalling (2.6), we have that the set $\{\bar{m}_\xi : \xi \in \mathbb{R}\}$ attracts the set

$$\mathcal{N} := \Big\{ m \in L^\infty(\mathbb{R}; [-1,1]) : \limsup_{x\to-\infty} m(x) < 0, \ \ \liminf_{x\to\infty} m(x) > 0 \Big\} \tag{3.8}$$

Namely, if $m \in \mathcal{N}$, then there is $\xi \in \mathbb{R}$ so that

$$\lim_{t\to\infty} \|S_t(m) - \bar{m}_\xi\|_\infty = 0 \tag{3.9}$$

3. ξ is a center of m if

$$\left(m - \bar{m}_\xi, \bar{m}'_\xi\right)_\xi = 0 \tag{3.10}$$

where $(\cdot,\cdot)_\xi$ denotes the scalar product in $L^2(\mathbb{R}, d\nu_\xi)$, and

$$\frac{d\nu_\xi(x)}{dx} = p_\xi(x)^{-1}, \quad p_\xi(x) = \beta\,[1 - \bar{m}_\xi(x)^2] \tag{3.11}$$

4. Any $m \in \mathcal{N}$ has a center. Moreover, there are positive constants c and δ so that if $v = m - \bar{m}_{\xi_0}$, $\|v\|_\infty < \delta$ then m has a unique center ξ and letting

$$N_{v,\xi} = \frac{(v, \bar{m}')_\xi}{(\bar{m}', \bar{m}')_\xi}, \qquad N_{v,0} = N_v$$

we have

$$\left|\xi - (\xi_0 - N_{v,\xi_0})\right| \le c\|v\|_\infty^2, \qquad |N_{v,\xi_0}| \le c\|v\|_\infty$$

5. Let Ω_ξ be the linear operator on $L^\infty(\mathbb{R})$ or $L^2(\mathbb{R}, d\nu_\xi)$

$$\Omega_\xi \psi = -\psi + p_\xi J * \psi \tag{3.12}$$

obtained by linearizing (2.2) around $\bar{m}_\xi$. Ω_ξ has eigenvalue 0 with eigenvector $\bar{m}'_\xi$ and a strictly positive spectral gap both as an operator in $L^\infty(\mathbb{R})$ and in $L^2(\mathbb{R}, d\nu_\xi)$. Thus, there is $B > 0$ so that

$$(v, \Omega_\xi v)_\xi \le -B(v,v)_\xi, \qquad (v, \bar{m}'_\xi)_\xi = 0 \tag{3.13}$$

3.3 Properties of finite volume instantons (see [5], [6], [9], [10])

1. The instanton $\hat{m}_L$ is an antisymmetric function, and there are ς and C so that

$$|\hat{m}_L(x) - m_\beta| \le Ce^{-\varsigma(L-x)}, \qquad 0 < x < L \tag{3.14}$$

Moreover, given any $\epsilon > 0$, if L is large enough

$$\sup_{|x|\le L} \left|\hat{m}_L(x) - \bar{m}(x)\right| \le \epsilon$$

2. Let $\hat{\mathcal{L}}$ be the operator on $L^2([-L, L], d\hat{\nu}_L)$, $\frac{d\hat{\nu}_L}{dx} = [\beta(1-\hat{m}_L^2)]^{-1}$,

$$\hat{\mathcal{L}}\psi = -\psi + \beta(1-\hat{m}_L^2)J * \psi$$

obtained by linearizing (1.3) around $\hat{m}_L$. If L is large enough, $\hat{\mathcal{L}}$ has a positive eigenvalue λ, $c_- e^{-2L} \le \lambda \le c_+ e^{-2L}$, $c_\pm$ positive constants, with eigenvector $e_{\hat{m}_L}(x)$, $|x| \le L$, which is a strictly positive, regular symmetric function. Moreover, there is $B' > 0$ so that for all L large enough

$$(v, \hat{\mathcal{L}}v)_{\hat{m}_L} \le -B'(v,v)_{\hat{m}_L}, \qquad (v, e_{\hat{m}_L})_{\hat{m}_L} = 0$$

$(\cdot,\cdot)_{\hat{m}_L}$ the scalar product on $L^2([-L, L], d\hat{\nu}_L)$.

3. Given any $\epsilon > 0$ small enough and $r \in (0,1)$, define

$$B_{\epsilon,r} := \left\{ m \in L^\infty([-L,L];[-1,1]) : \exists \xi \in [-rL, rL] : \|m - \bar{m}_\xi \mathbf{1}_{|x|\le L}\|_\infty \le \epsilon \right\} \tag{3.15}$$

$$\Sigma^\pm_{\epsilon,r} := \left\{ m \in L^\infty([-L,L];[-1,1]) : \|m - \bar{m}_\xi \mathbf{1}_{|x|\le L}\|_\infty \le \epsilon \ \text{ for } \xi = \pm rL \right\}$$

Then, for L large enough, if m is in $B_{\epsilon,r}$ either $\lim_{t\to\infty} \|S_t(m) - \hat{m}_L\|_\infty = 0$ or else there is a time t when $S_t(m) \in \Sigma^-_{2\epsilon,r} \cup \Sigma^+_{2\epsilon,r}$ while $S_s(m) \in B_{2\epsilon,r}$ for all $s \le t$.

4. There exist two manifolds, $w^{(\pm)}(x,s)$, $x \in [-L, L]$, $s \in \mathbb{R}$, such that $\mathcal{F}_L(w^\pm(\cdot,s)) < \mathcal{F}_L(\hat{m}_L)$,

$$S_t(w^{(\pm)}(\cdot,s)) = w^{(\pm)}(\cdot, s+t)$$

$w^{(\pm)}(\cdot,\cdot) \in (-m_\beta, m_\beta)$ and

$$\lim_{s\to-\infty} w^{(\pm)}(\cdot,s) = \hat{m}_L \qquad \lim_{s\to\infty} w^{(\pm)}(\cdot,s) = m^{(\pm)}$$

(the limits can be taken both in L^∞ and in L^2). Moreover, for any $s \in \mathbb{R}$, $w^{(+)}(x,s)$ is a non decreasing function of $x \in [-L, L]$, $w^{(-)}(x,s) = -w^{(+)}(-x,s)$ and $w^{(+)}(x,s) > w^{(-)}(x,s)$.

4 Lower bounds for $\mathcal{F}_L$

The main result of this section is Theorem 4.2 below where we estimate the cost in energy of the contours. This result extends (3.4) and its proof is mainly taken from [13].

We first prove the following preliminary proposition.

Proposition 4.1. *There are $c > 0$ and $\omega > 0$ so that for all L large enough*

$$\left|\mathcal{F}(\bar{m}) - \mathcal{F}_L(\hat{m}_L)\right| \le ce^{-\omega L} \tag{4.1}$$

Proof. The orbit $S_t(\bar{m}(x)\mathbf{1}_{|x|\le L})$ is made of antisymmetric functions since antisymmetry is preserved by the dynamics. Thus any limit point of the orbit is antisymmetric and it is a stationary solution because $\mathcal{F}_L$ is strictly decreasing at points which are not stationary solutions. Then by item 3 of Subsection 3.3, $S_t(\bar{m}(x)\mathbf{1}_{|x|\le L})$ converges to $\hat{m}_L$ so that

$$\mathcal{F}_L\big(\bar{m}\mathbf{1}_{|x|\le L}\big) \ge \mathcal{F}_L(\hat{m}_L) \tag{4.2}$$

On the other hand,

$$\begin{aligned}\mathcal{F}_L\big(\bar{m}\mathbf{1}_{|x|\le L}\big) \le \mathcal{F}(\bar{m}) + \frac{1}{2}\int_{[-L,L]}\int_{[-L,L]^c} J(x,y)\{\ (\bar{m}(x) - \bar{m}(R(y))^2 \\ -(\bar{m}(x) - \bar{m}(y))^2\ \}dy\,dx\end{aligned} \tag{4.3}$$

where $R(y)$ denotes the point obtained by reflecting y around $-L$ or L respectively when $y > L$ and $y < -L$. By (4.2), (4.3) and (3.7) we get $\mathcal{F}_L(\hat{m}_L) \le \mathcal{F}(\bar{m}) + ce^{-\omega L}$ for L large enough.

To prove the lower bound, we denote by $\tilde{m}(x)$ the function equal to $\hat{m}_L$ for $|x| \le L$ and to $\pm m_\beta$, for $x > L$ and respectively $x < -L$. Then

$$\begin{aligned}\mathcal{F}_L(\hat{m}_L) = \mathcal{F}(\tilde{m}) + \frac{1}{2}\int_{[-L,L]}\int_{[-L,L]^c} J(x,y)\ \{\ (\hat{m}_L(x) - \hat{m}_L(R(y))^2 \\ -(\hat{m}_L(x) \mp m_\beta)^2\ \}\ dy\,dx\end{aligned}$$

where $\mp$ is minus if $x > 0$ and plus if $x < 0$. By (3.9), $\lim_{t\to\infty} S_t(\tilde{m}) = \bar{m}$ and since $\mathcal{F}$ is lower semicontinuous, $\mathcal{F}(\tilde{m}) \ge \mathcal{F}(\bar{m})$. Then, by (3.14), we conclude the proof. □

Theorem 4.2. *There are positive constants ζ_0, c_1, κ, c_2, and ω so that if $(\zeta, \ell_-, \ell_+, L)$ is compatible with (ζ_0, c_1, κ), then*

$$\mathcal{F}_L(m) \ge \sum_{\Gamma \text{ contour of } m} w_{\zeta,\ell_-,\ell_+}(\Gamma) \qquad \forall m \in L^\infty([-L,L];[-1,1]), \tag{4.4}$$

where

$$w_{\zeta,\ell_-,\ell_+}(\Gamma) = 2c_1\zeta^2\frac{\ell_-}{\ell_+}|\Gamma| \text{ if } \Gamma \text{ is a plus or a minus contour;}$$

$$w_{\zeta,\ell_-,\ell_+}(\Gamma) = \max\left\{2c_1\zeta^2\tfrac{\ell_-}{\ell_+}|\Gamma|\ ;\ \mathcal{F}(\bar{m}) - c_2e^{-\omega\ell_+}\right\} \text{ if } \Gamma \text{ is a mixed contour.}$$

Proof. Let G_+, respectively G_-, be the collection of all plus, minus, contours for m. By (3.1) and (3.4), for any $\Gamma \in G_+ \cup G_-$,

$$\mathcal{F}_L(m) = \mathcal{F}_{L;\Gamma^c}(m_{\Gamma^c}) + \mathcal{F}_{L;\Gamma}(m_\Gamma | m_{\Gamma^c}) \geq \mathcal{F}_{L;\Gamma^c}(m_{\Gamma^c}) + 2c_1\zeta^2 \frac{\ell_-}{\ell_+}|\Gamma|$$

Denoting by Δ the complement in $[-L, L]$ of $G_+ \cup G_-$, iterating we deduce

$$\mathcal{F}_L(m) \geq \mathcal{F}_{L;\Delta}(m_\Delta) + 2c_1\zeta^2 \frac{\ell_-}{\ell_+} \sum_{\Gamma \in G_+ \cup G_-} |\Gamma|$$

Let $\Gamma = [x', x'')$ be a mixed contour possibly still present in Δ. We could bound it using the above procedure, but we could also use the following alternative, which will be adopted when it gives a better bound. We have $\eta^{(\zeta,\ell_-)}(m;\cdot) = -1$ on $[x' - \ell_+, x' + \ell_+)$ and $\eta^{(\zeta,\ell_-)}(m;\cdot) = 1$ on $[x'' - \ell_+, x'' + \ell_+)$ or viceversa. Suppose for instance that the former holds. By (3.2),(3.3), there is ψ equal to m_Δ outside $[x' - \ell_+ + 1, x' + \ell_+ - 1]$ such that $\eta^{(\zeta,\ell_-)}(\psi;\cdot) = -1$ on $[x' - \ell_+, x' + \ell_+)$, $\psi = -m_\beta$ on $[x' - 1, x']$ and

$$\mathcal{F}_{L;\Delta}(m_\Delta) \geq \mathcal{F}_{L;\Delta}(\psi) - c' e^{-\omega \ell_+}$$

By repeating the argument relative to the interval $[x'' - \ell_+ + 1, x'' + \ell_+ - 1]$, we conclude that there is ϕ equal to ψ outside $[x'' - \ell_+ + 1, x'' + \ell_+ - 1]$ such that $\eta^{(\zeta,\ell_-)}(\phi;\cdot) = 1$ on $[x'' - \ell_+, x'' + \ell_+)$, $\phi = m_\beta$ on $[x'' - 1, x'' + 1]$ and

$$\mathcal{F}_{L;\Delta}(m_\Delta) \geq \mathcal{F}_{L;\Delta}(\phi) - 2c' e^{-\omega \ell_+}$$

We then have

$$\mathcal{F}_{L;\Delta}(m_\Delta) \geq \mathcal{F}_{L;\Delta\setminus\Gamma}(m_{\Delta\setminus\Gamma}) + \mathcal{F}_\Gamma(\phi_\Gamma | m_\beta \mathbf{1}_{x > x''} - m_\beta \mathbf{1}_{x < x'}) - 2c' e^{-\omega \ell_+}$$

Let $u(x)$, $x \in \mathbb{R}$, be equal to ϕ on Γ and $= \pm m_\beta$ to the right and left of Γ, then

$$\mathcal{F}_\Gamma(\phi_\Gamma | m_\beta \mathbf{1}_{x > x''} - m_\beta \mathbf{1}_{x < x'}) = \mathcal{F}(u) \geq \mathcal{F}(\bar{m})$$

by (3.9) and because $\mathcal{F}(S_t(u))$ is decreasing and lower semicontinuous. Thus

$$\mathcal{F}_{L;\Delta}(m_\Delta) \geq \mathcal{F}_{L;\Delta\setminus\Gamma}(m_{\Delta\setminus\Gamma}) + \max\{\mathcal{F}(\bar{m}) - 2c' e^{-\omega \ell_+}; 2c_1\zeta^2 \frac{\ell_-}{\ell_+}|\Gamma|\}$$

By iterating the argument over all the other, possibly still present, mixed contours, we conclude the proof. □

5 Lower bounds for $\mathcal{F}$

In this section we prove that the free energy of a profile $m \in L^\infty(\mathbb{R};[-1,1])$ which is asymptotically strictly positive as $x \to \infty$ and negative as $x \to -\infty$ increases quadratically with the distance of m from $\bar{m}$, the precise statement is given in Proposition 5.2 below. We need preliminarily a regularization result:

Proposition 5.1. *There is $c > 0$ and for all ℓ small enough and such that $L = n\ell$, $n \in \mathbb{N}$, there is a (regularizing) map $\mathcal{R}$ from $L^\infty([-L,L];[-1,1])$ into itself, continuous in the L^∞ norm, such that $\mathcal{F}_L(m) \geq \mathcal{F}_L\big(\mathcal{R}(m)\big)$,*

$$\Big(\mathcal{R}(m)\Big)^{(\ell)} = m^{(\ell)},$$

$\mathcal{R}(m)$ is differentiable at all points of $[-L,L] \setminus \ell\mathbb{Z}$, and

$$\Big|\frac{d\mathcal{R}(m)(x)}{dx}\Big| \leq \beta \|J'\|_\infty, \quad x \in [-L,L] \setminus \ell\mathbb{Z} \tag{5.1}$$

Proof. Given m we denote by

$$M_i := \fint_{I_i} m(y)\,dy$$

where $I_i \in \mathcal{D}^{(\ell)}$ is the i-th element of the partition of $[-L,L]$.
We want to prove that there is a unique function ψ_i which verifies the following properties:

1. $\psi_i(x) = m(x)$ for all $x \in I_i^c$
2. $\fint_{I_i} \psi_i(y)\,dy = M_i$
3. If φ verifies (1) and (2), then $\mathcal{F}_L(\varphi|m_{I_i^c}) \geq \mathcal{F}_L(\psi_i|m_{I_i^c})$

To prove the above we introduce a Lagrange multiplier h and define

$$\mathcal{F}_{L,h}(\varphi|m_{I_i^c}) := \mathcal{F}_L(\varphi|m_{I_i^c}) - h\int_{I_i} \varphi(y)\,dy$$

The inf of $\mathcal{F}_{L,h}(\cdot|m_{I_i^c})$ over $L^\infty([-L,L];[-1,1])$ is reached on functions u such that $A_h(u) = u$, where

$$A_h(u) = \begin{cases} \tanh\{\beta[J * u + h]\} & \text{on } I_i \\ \\ m_{I_i^c} & \text{on } I_i^c \end{cases}$$

(see for instance [12]). For ℓ such that $\beta\|J\|_\infty \ell < 1/2$, A_h is a contraction:

$$\|A_h(\phi) - A_h(\psi)\|_\infty \le \beta\|J\|_\infty \ell \|\phi - \psi\|_\infty$$

Denote by $\phi^{(h)}$ its unique fixed point: $\phi^{(h)} = A_h(\phi^{(h)})$, $\phi^{(h)} = \lim_{n\to\infty} A_h^n(u)$, where, for instance $u \equiv 0$. From such a representation we deduce that, for each x, $\phi^{(h)}(x)$ is differentiable in h. By differentiating the fixed point equation, we find for $x \in I_i$,

$$\frac{d\phi^{(h)}}{dh} = p_h\{J * \frac{d\phi^{(h)}}{dh} + 1\}, \quad p_h = \frac{\beta}{\cosh^2\{\beta[J * \frac{d\phi^{(h)}}{dh} + h]\}}$$

with $\dfrac{d\phi^{(h)}(x)}{dh} = 0$ when $x \notin I_i$. For ℓ as above, this identifies uniquely $\dfrac{d\phi^{(h)}}{dh}$ and shows that when $x \in I_i$, the derivative is strictly positive. Then

$$a(h) := \fint_{I_i} \phi^{(h)}(x)\, dx$$

is a strictly increasing, continuous function of h; since $a(h) \to \pm 1$ as $h \to \pm\infty$, there is h^* such that, $a(h^*) = M_i$ and h^* depends continuously on M_i and since M_i is a continuous function of m, $\phi^{(h^*)}$ is a continuous function of m.

The function $\psi_i := \phi^{(h^*)}$ verifies (1) and (2) above. Let φ be any other function which verifies (1) and (2). Then, unless $\varphi = \psi_i$ almost everywhere,

$$\mathcal{F}_L(\varphi|m_{I_i^c}) = \mathcal{F}_{L,h^*}(\varphi|m_{I_i^c}) + h^*\ell M_i > \mathcal{F}_{L,h^*}(\psi_i|m_{I_i^c}) + h^*\ell M_i = \mathcal{F}_L(\psi_i|m_{I_i^c})$$

Finally $\phi^{(h^*)}(x)$ is differentiable w.r.t. x in the interior of I_i and its derivative is bounded as in (5.1), as it follows by differentiating the fixed point equation and integrating by parts using regularity of J.

By repeating the argument for the other intervals I_j's we conclude the proof of the proposition.

□

Since the proof is local, Proposition 5.1 holds also when $[-L, L]$ is replaced by $\mathbb{R}$.

Proposition 5.2. *There is $c > 0$ so that for all ℓ small enough,*

$$\mathcal{F}(m) \ge \mathcal{F}(\bar{m}) + c\,\ell\Big(\inf_{\xi\in\mathbb{R}} \|m^{(\ell)} - \bar{m}_\xi^{(\ell)}\|_\infty\Big)^2, \qquad m \in \mathcal{N} \tag{5.2}$$

with $\mathcal{N}$ defined in (3.8).

Proof. Set

$$w_\ell(m) = \inf_{\xi\in\mathbb{R}} \|m^{(\ell)} - \bar m_\xi^{(\ell)}\|_\infty. \tag{5.3}$$

We start with an outline of the proof. By expanding $\mathcal{F}(m)$ around $\bar m_\xi$, and calling $v = m - \bar m_\xi$,

$$\mathcal{F}(m) = \mathcal{F}(\bar m_\xi) - (v, \Omega_\xi v)_\xi + \text{ remainder terms} \tag{5.4}$$

with Ω_ξ defined in (3.12) and the scalar product right after (3.10). By (3.13), there is $B > 0$ so that if $(v, \bar m'_\xi)_\xi = 0$ then $(v, \Omega_\xi v)_\xi \le -B\,(v,v)_\xi$. To prove (5.2) we thus have to take care of the remainder terms in (5.4), fit in the condition $(v, \bar m'_\xi)_\xi = 0$ and finally relate the bound $(v,v)_\xi$ to the quantity $w_\ell(m)$.

We will prove that there are $\delta_0 \in (0, 1/2)$ and $c' > 0$ so that

$$\inf_{\delta\le\delta_0} \delta^{-2} \inf_{m\in\mathcal{N}: w_\ell(m)\ge\delta} [\mathcal{F}(m) - \mathcal{F}(\bar m)] \ge c'\ell \tag{5.5}$$

We claim that (5.2) will then hold with $c = c'\delta_0^2/4$. We first prove the claim for a function $m^* \in \mathcal{N}$ such that $\delta^* := w_\ell(m^*) \le \delta_0$. Then, the inf on the l.h.s. of (5.5) is not larger than the value at $\delta = \delta^*$, so that it is $\le \delta^{*-2}[\mathcal{F}(m^*) - \mathcal{F}(\bar m)]$, hence $c'\ell w_\ell(m^*)^2 \le [\mathcal{F}(m^*) - \mathcal{F}(\bar m)]$, which yields (5.2) with c replaced by c'. Finally, $c' > c'\delta_0^2/4$ because $\delta_0 < 1/2$.

If instead $w_\ell(m^*) > \delta_0$, then (5.5) yields $\delta_0^{-2}[\mathcal{F}(m^*) - \mathcal{F}(\bar m)] \ge c'\ell$ hence

$$\mathcal{F}(m^*) - \mathcal{F}(\bar m) \ge c'\ell\, w_\ell(m^*)^2 \frac{\delta_0^2}{w_\ell(m^*)^2}$$

which concludes the proof of the claim because $w_\ell(m^*) \le 2$.

Thus (5.2) will be proved, once we show that if $\tilde m \in \mathcal{N}$ and $w_\ell(\tilde m) \ge \delta$, then

$$\mathcal{F}(\tilde m) \ge \mathcal{F}(\bar m) + c'\ell\delta^2 \tag{5.6}$$

We first suppose that $w_\ell(\tilde m) \le \delta_0$. By Proposition 5.1, there is m in $\mathcal{N}$ such that

$$\mathcal{F}(\tilde m) \ge \mathcal{F}(m), \qquad \tilde m^{(\ell)} = m^{(\ell)},\ w_\ell(m) = w_\ell(\tilde m) \ge \delta \tag{5.7}$$

with the derivative m' of m, almost everywhere well defined and such that

$$\|m'\|_\infty \le \beta\|J'\|_\infty$$

Then, given any ξ,

$$\|m^{(\ell)} - \bar m_\xi^{(\ell)}\|_\infty \ \le\ \|m - \bar m_\xi\|_\infty \ \le\ \|m^{(\ell)} - \bar m_\xi^{(\ell)}\|_\infty + \beta\|J'\|_\infty \ell$$

Since $\|m^{(\ell)} - \bar{m}_\xi^{(\ell)}\|_\infty = \|\tilde{m}^{(\ell)} - \bar{m}_\xi^{(\ell)}\|_\infty$, if δ_0 and ℓ are small enough, by (3.12), m has a unique center ξ and

$$\delta_0 + \beta \|J'\|_\infty \ell =: \epsilon \geq \|m - \bar{m}_\xi\|_\infty \tag{5.8}$$

There is $a > 0$ so that $\phi_\beta(m) \geq a\big(|m| - m_\beta\big)^2$ and, since $\phi_\beta(m) \in L^1(\mathbb{R})$ (because $\mathcal{F}(m) < \infty$), $m \mp m_\beta \in L^2(\mathbb{R}_\pm)$. Then, with ξ the center of m,

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}_\xi) = -\frac{1}{\beta} \int \mathcal{S}(m(x)) - \mathcal{S}(\bar{m}_\xi(x))\, dx$$
$$-\frac{1}{2} \int\int J(x,y)\{m(x)m(y) - \bar{m}_\xi(x)\bar{m}_\xi(y)\}\, dx\, dy$$

Calling $v = m - \bar{m}_\xi$ and $\rho =$max$(|m|, |\bar{m}_\xi|)$,

$$-\Big(\mathcal{S}(m) - \mathcal{S}(\bar{m}_\xi)\Big) \geq -\mathcal{S}'(\bar{m}_\xi)v + \frac{1}{1-\bar{m}_\xi^2} v^2 - \frac{2}{1-\rho^2}|v|^3$$

By (5.8), $\rho \leq m_\beta + \epsilon$ and by letting δ and ℓ small enough, $1 - \rho^2 \geq \dfrac{1 - m_\beta^2}{2}$ so that

$$-\Big(\mathcal{S}(m) - \mathcal{S}(\bar{m}_\xi)\Big) \geq -\mathcal{S}'(\bar{m}_\xi)v + \frac{1}{1-\bar{m}_\xi^2} v^2 - \frac{8}{1-m_\beta^2}|v|^3 \tag{5.9}$$

By (5.9) and since $-\beta^{-1}\mathcal{S}'(\bar{m}_\xi) - J * \bar{m}_\xi = 0$,

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}_\xi) \geq -(v, \Omega_\xi v)_\xi - \frac{8\|v\|_\infty}{1 - m_\beta^2}(v,v)_\xi \tag{5.10}$$

Since $(v, \bar{m}'_\xi)_\xi = 0$, because ξ is the center of m, by (3.13),

$$\mathcal{F}(m) - \mathcal{F}(\bar{m}_\xi) \geq (v,v)_\xi \Big(B - \frac{8\|v\|_\infty}{1 - m_\beta^2}\Big) \tag{5.11}$$

By letting δ small enough we conclude that there is $c' > 0$ so that, recalling (5.7),

$$\mathcal{F}(\tilde{m}) - \mathcal{F}(\bar{m}_\xi) \geq c'(v,v)_\xi \geq c'\ell\delta^2 \tag{5.12}$$

The last inequality is proved as follows. Since $w_\ell(m) = \delta$, there is an interval of $\mathcal{D}^{(\ell)}$ where $|m^{(\ell)} - \bar{m}_\xi^{(\ell)}| \geq \delta$. By using Cauchy-Schwartz in such an interval we then get (5.12).
This proves (5.6) when $w_\ell(\tilde{m}) \leq \delta_0$. This is what needed in the text and we omit, for brevity, the proof of the full statement in the theorem, which uses that if $m \in \mathcal{N}$, then $S_t(m) \to \bar{m}_\xi$ as $t \to \infty$, while $\mathcal{F}(S_t(m)) \leq \mathcal{F}(m)$. The result then follows by showing that $w_\ell(S_t(m))$ is a continuous function of t, so that there is $s > 0$ so that $w_\ell(S_s(m)) = \delta$ with $\delta \leq \delta_0$. The proposition is proved. □

6 Localization of the energy sub-levels

In this section we prove the localization property of the free energy sub-levels stated in Theorem 2.6, indeed the proof of Theorem 2.6 follows directly from Lemma 6.1 and Lemma 6.2 below.

Recalling the Definitions 2.3 and 2.4, we choose L and $\ell_\pm$ as functions of ζ so that (ζ, ℓ_-, ℓ_+) is a good choice of the parameters. We also fix the positive constants ζ_0, c_1, κ and c_2 of Theorem 4.2 and we write $\Theta(m;\cdot)$ for $\Theta^{(\zeta,\ell_-,\ell_+)}(m;\cdot)$.

The following preliminary Lemmas are in fact corollaries of Theorem 4.2.

Lemma 6.1. *There is $\zeta < \zeta_0$ so that if $m \in L^\infty([-L,L];[-1,1])$ is such that $\mathcal{F}_L(m) \le \mathcal{F}_L(\hat{m}_L) + \zeta^{100}$, then the following holds:*

$$\big|\{\Theta(m;\cdot) = 0\}\big| \le k\ell_+, \quad k = \textit{smallest integer} \ge \frac{1}{c_1\zeta^2\ell_-}[\mathcal{F}_L(\hat{m}_L) + \zeta^{100}] \tag{6.1}$$

Furthermore if there are two points $x_\pm \in [-L,L]$, $x_+ > x_-$, where $\Theta(m;x_+)\Theta(m;x_-) = -1$, then there is a unique contour Γ and $|\Gamma| \le k\ell_+$, k as in (6.1)*.*

Proof. By (4.4) we have

$$c_1\zeta^2\frac{\ell_-}{\ell_+}\sum_\Gamma |\Gamma| \le \mathcal{F}_L(\hat{m}_L) + \zeta^{100}$$

so that

$$\big|\{\Theta(m;\cdot) = 0\}\big| \le \frac{\ell_+}{c_1\zeta^2\ell_-}[\mathcal{F}_L(\hat{m}_L) + \zeta^{100}]$$

hence (6.1).

Assume now that there are $x_\pm$ as in the statement. Then there is at least one mixed contour, that we denote by Γ. Suppose by contradiction that there is another contour Γ'. By using Theorem 4.2, the inequality (4.1) and the fact that $|\Gamma'| > \ell_+$ we get

$$\mathcal{F}_L(m) \ge \mathcal{F}(\bar{m}) - c_2e^{-\omega\ell_+} + c_1\zeta^2\ell_- \ge \mathcal{F}_L(\hat{m}_L) - ce^{-\omega L} - c_2e^{-\omega\ell_+} + c_1\zeta^2\ell_- \tag{6.2}$$

Since $L > \zeta^{-8}$, if ζ is small enough, then the r.h.s. of (6.2) is larger than $\mathcal{F}_L(\hat{m}_L) + \zeta^{100}$, hence the desired contradiction, which shows that there is a unique contour Γ with (6.1) holding. $\square$

Lemma 6.2. *Let m, x_-, x_+ be as in Lemma 6.1 and let $\Gamma = [x_-, x_+)$ be the unique contour of m. Suppose $\Theta(m;x) = \pm 1$ for $x > x_+$, respectively $x < x_-$. Then there is $\xi \in [x_- + \ell_+/2, x_+ - \ell_+/2]$, so that, for all ζ small enough,*

$$\|m^{(\ell_-)} - \bar{m}_\xi^{(\ell_-)}\|_\infty \le 2\zeta$$

Proof. If $x > x_+$ then $\Theta(m; x) = 1$ and for any $\xi \in \Gamma$, $\text{dist}(\xi, \Gamma^c) \geq \ell_+/2$ from (3.7) we get

$$|m^{(\ell_-)}(x) - \bar{m}_\xi^{(\ell_-)}(x)| \leq |m^{(\ell_-)}(x) - m_\beta| + ce^{-\alpha\ell_+/2}$$

Since $\eta(m; x) = 1$ by taking ζ small enough we then get

$$|m^{(\ell_-)}(x) - \bar{m}_\xi^{(\ell_-)}(x)| \leq \zeta + ce^{-\alpha\ell_+/2} \leq 2\zeta, \qquad \text{for all } x > x_+$$

An analogous argument applies in the case $x < x_-$ for which $\Theta(m; x) = -1$. Thus we only need to prove that there is $\xi \in [\xi_-, \xi_+]$, where $\xi_\pm = x_\pm \mp \ell_+/2$, so that,

$$\sup_{x \in \Gamma} |m^{(\ell_-)}(x) - \bar{m}_\xi^{(\ell_-)}(x)| \leq 2\zeta \tag{6.3}$$

We suppose by contradiction that (6.3) is not verified by any $\xi \in [\xi_-, \xi_+]$ and want to deduce that $\mathcal{F}_L(m) > \mathcal{F}_L(\hat{m}_L) + \zeta^{100}$. The strategy for the proof is to reduce to a similar problem on the whole line where we can use Proposition 5.2.

By item 1 in Subsection 3.1, there is ψ which is equal to m on Γ, $\mathcal{F}_L(\psi) \leq \mathcal{F}_L(m)$ and

$$\sup_{x \in [-L, -L+1]} |\psi(x) + m_\beta| \leq c_2 e^{-\omega\ell_+}, \qquad \sup_{x \in [L-1, L]} |\psi(x) - m_\beta| \leq c_2 e^{-\omega\ell_+} \tag{6.4}$$

We denote by ϕ the function on $\mathbb{R}$, equal to ψ on $[-L, L]$ and to m_β on $x > L$ and to $-m_\beta$, in $x < -L$. Then from (6.4) it follows that $\mathcal{F}_L(\phi) \geq \mathcal{F}(\phi) - ce^{-\omega\ell_+}$ that implies

$$\mathcal{F}_L(m) \geq \mathcal{F}(\phi) - c' e^{-\omega\ell_+}$$

Since $\phi^{(\ell_-)}(x) = m^{(\ell_-)}(x)$, for all $x \in \Gamma$ and all $\xi \in [\xi_-, \xi_+]$, by the contradiction assumption

$$\sup_{x \in \Gamma} |\phi^{(\ell_-)}(x) - \bar{m}_\xi^{(\ell_-)}(x)| > 2\zeta \qquad \text{for all } \xi \in [\xi_-, \xi_+]$$

Note now that if $\xi \leq \xi_-$ and $x = \xi_- + \ell_+/4$, then $|\phi^{(\ell_-)}(x) - \bar{m}_\xi^{(\ell_-)}(x)| \geq 2m_\beta - 2c_2 e^{-\omega\ell_+/4} - ce^{-\alpha\ell_+/8} > 2\zeta$. An analogous argument applies for $\xi \geq \xi_+$, so that

$$\inf_{\xi \in \mathbb{R}} \|\phi^{(\ell_-)} - \bar{m}_\xi^{(\ell_-)}\|_\infty > 2\zeta$$

Then, by Proposition 5.2, $\mathcal{F}(\phi) \geq \mathcal{F}(\bar{m}) + c\ell_-(2\zeta)^2$. Moreover, by (4.1), $\mathcal{F}(\bar{m}) \geq \mathcal{F}_L(\hat{m}_L) - ce^{-\omega L}$, so that $\mathcal{F}_L(m) > \mathcal{F}_L(\hat{m}_L) + 2\zeta$ for ζ small enough, hence the desired contradiction. □

7 On the basin of attraction of the pure phases

In this section we will prove Theorem 2.7. By symmetry it suffices to prove the statements relative to U_+.

We start with the following preliminary proposition whose proof is taken from [13]. Given $b > 0$, $R > 0$ let

$$\mathcal{M}_{b,R} = \{m \in L^\infty(\mathbb{R}; [-1,1]) : m(x) \geq b, |x| \geq R\}.$$

Proposition 7.1. *For any ϵ, b, R positive there is $\tau_{\epsilon,b,R}$ so that*

$$S_t(m) \geq m_\beta - \epsilon \quad \textit{for all } m \in \mathcal{M}_{b,R} \textit{ and } t \geq \tau_{\epsilon,b,R} \tag{7.1}$$

Furthermore

$$\lim_{t\to\infty} \|S_t(m) - m^{(+)}\|_\infty = 0 \quad \textit{for all } m \in \mathcal{M}_{b,R} \tag{7.2}$$

Proof. By item 3 of Theorem 3.1, $\lim_{t\to\infty} S_t(-1) = -m_\beta$, where we shorthand by $S_t(a)$, $a \in \mathbb{R}$, the function $S_t(m)$ with $m \equiv a$. By item 1 of Theorem 3.1, for any $\epsilon_- > 0$ there is $t_- > 0$ so that for all m, $S_{t_-}(m) \geq S_{t_-}(-1) \geq -m_\beta - \epsilon_-$.

Since $\lim_{t\to\infty} S_t(b) = m_\beta$ by the Barrier Lemma (see item 2 of Theorem 3.1), for any $\epsilon' > 0$, there are $t_0 = t_0(b, \epsilon')$ and $a = a(b, R, \epsilon')$, $a > R$, so that, for any $m \in \mathcal{M}_{b,R}$,

$$S_{t_0}(m) \geq S_{t_0}(b) - \frac{\epsilon'}{2} \geq m_\beta - \epsilon', \quad |x| \geq a$$

Thus

$$S_{t_0}(m) \geq \psi, \quad \psi(x) = \begin{cases} m_\beta - \epsilon' & \text{for } |x| \geq a \\ -m_\beta - \epsilon' & \text{otherwise} \end{cases} \tag{7.3}$$

Let $d > 0$ be such that $\bar{m}(x) \leq -m_\beta + \epsilon'$ for all $x \leq -d$. Then

$$\bar{m}_{d+a}(x) \leq -m_\beta + \epsilon' \quad \text{for all } x \leq a \tag{7.4}$$

so that, by (7.3), $\psi(x) \geq \bar{m}_{d+a}(x) - 2\epsilon'$ for all $x \leq a$ and $\psi \geq \bar{m}_{d+a} - \epsilon'$ for all $x \geq a$. Hence

$$\psi \geq \bar{m}_{d+a} - 2\epsilon' \quad \text{for all } x \tag{7.5}$$

By item 2 of Subsection 3.2, if $\epsilon' > 0$ is small enough, there is ξ so that

$$\lim_{s\to\infty} \|S_s(\bar{m}_{d+a} - 2\epsilon') - \bar{m}_\xi\|_\infty = 0$$

Then any limit point ψ^* of $S_s(\psi)$ (under uniform convergence on the compacts) verifies $\psi^* \ge \bar{m}_\xi$. The set $A := \{\xi' \in \mathbb{R} : \psi^* \ge \bar{m}_{\xi'}\}$ is thus non empty and closed. In Section 5 of [12] it is proved that if m^* is stationary, $m^* \ge \bar{m}_\xi$ and $m^* \ne \bar{m}_\xi$ then there is $\xi' > \xi$ such that $m^* \ge \bar{m}_{\xi'}$. It then follows that either $A = \mathbb{R}$ or there is ξ'' so that $\psi^* = \bar{m}_{\xi''}$. The second alternative cannot be verified: denoting by $\mathcal{R}$ the reflection operator around 0, $\mathcal{R}\psi = \psi$, $S_t(\psi) = S_t(\mathcal{R}\psi) = \mathcal{R}S_t(\psi)$ and by $\{t_n\}$ the time sequence defining ψ^*,

$$\psi^* = \lim_{t_n \to \infty} S_{t_n}(\psi) = \mathcal{R} \lim_{t_n \to \infty} S_{t_n}(\psi) = \mathcal{R}\psi^* \ge -\bar{m}_{-\xi}$$

hence $\psi^* \ge \bar{m}_{\xi'}$ for all ξ'. Then ψ^* is strictly positive, and since ψ^* is stationary, $\psi^* = m_\beta$. Thus there is t_1 so that $S_{t_1}(\psi) \ge m_\beta - \epsilon$ and $S_{t_0+t_1}(m) \ge m_\beta - \epsilon$ for any $m \in \mathcal{M}_{b,R}$.
From item 1 of Theorem 3.1 and from (7.1) it follows that

$$S_t(m_\beta - \epsilon) \le S_{t+\tau_{\epsilon,b,R}}(m) \le S_t(1). \tag{7.6}$$

Since $S_t(1)$ and $S_t(m_\beta - \epsilon)$ converge exponentially fast to m_β as $t \to \infty$, there are c and ω positive so that, for any $m \in \mathcal{M}_{b,R}$,

$$\|S_{t+\tau_{\epsilon,b,R}}(m) - m_\beta\|_\infty \le ce^{-\omega t} \tag{7.7}$$

□

By the Barrier Lemma (item 2 of Theorem 3.1) it is possible to weaken the condition $m(x) \ge b$ for all $|x| \ge R$ in Proposition 7.1. The extension is needed for applications to L-symmetric functions, which cannot be in $\mathcal{M}_{b,R}$ unless $m(x) \ge b$ for all x.
Let C and $V = e^2\beta$ be as in the Barrier Lemma, ζ, $\zeta' > \zeta$ and R all positive with ζ' small enough. Let

$$\tau^+_{\zeta,m_\beta-\zeta',R} = \max\{\tau_{\zeta,m_\beta-\zeta',R}, t^*\}, \quad Ce^{-t^*V} = \zeta \tag{7.8}$$

Notice that

$$Ce^{-\tau^+ V \log\{V/(e\beta)\}} \le \zeta, \quad \tau^+ = \tau^+_{\zeta,m_\beta-\zeta',R} \tag{7.9}$$

By Proposition 7.1 we have $S_{\tau^+}(m) \ge m_\beta - \zeta$ for all $m \in \mathcal{M}_{m_\beta-\zeta',R}$. Moreover by items 1 and 2 of Theorem 3.1,

$$S_t(m) \ge m_\beta - \zeta' - \zeta, \quad \text{for } |x| \ge R + V\tau^+ \text{ and } t \le \tau^+ \tag{7.10}$$

Lemma 7.2. *For any $\zeta' > \zeta > 0$ small enough, any $R > 0$ and ψ such that $\psi(x) > m_\beta - \zeta'$ for all $R \le |x| \le R + 2V\tau^+ + 1$, $\tau^+ = \tau^+_{\zeta, m_\beta - \zeta', R}$, we have*

$$S_{\tau^+}(\psi) > m_\beta - 2\zeta \quad \text{for all } |x| \le R + V\tau^+ + 1 \tag{7.11}$$

$$S_t(\psi) > m_\beta - \zeta' - \zeta \quad \text{for all } t \le \tau^+ \text{ and all } |x| \in [R + V\tau^+, R + V\tau^+ + 1] \tag{7.12}$$

Proof. We define

$$\psi^*(x) = \begin{cases} \psi(x) & \text{if } |x| \le R + 2V\tau^+ + 1 \\ m_\beta - \zeta' & \text{otherwise} \end{cases}$$

By (3.5) and (7.9),

$$S_t(\psi) \ge S_t(\psi^*) - \zeta \quad \text{for all } |x| \le R + V\tau^+ + 1 \text{ and } t \le \tau^+ \tag{7.13}$$

and by (7.1)

$$S_{\tau^+}(\psi^*)(x) \ge m_\beta - \zeta, \qquad \text{for all } x \in \mathbb{R}$$

and (7.11) follows. Set

$$\tilde{\psi}(x) = \begin{cases} \psi(x) & \text{for } |x| \in [R, R + 2V\tau^+ + 1] \\ m_\beta - \zeta' & \text{elsewhere} \end{cases}$$

so that $\tilde{\psi} \ge m_\beta - \zeta'$ everywhere. By the Barrier Lemma, $S_t(\psi) \ge S_t(\tilde{\psi}) - \zeta$ for all $|x| \in [R+V\tau^+, R+V\tau^+ +1]$ and all $t \le \tau^+$. Then (7.12) follows from the inequality $S_t(\tilde{\psi}) \ge m_\beta - \zeta'$. □

In the next lemma we will require ζ so small that

$$\beta \|J'\|_\infty \ell_- < \frac{\zeta}{2}, \quad Ce^{-\ell_+} < \zeta, \quad C \text{ as in the Barrier Lemma} \tag{7.14}$$

(recall that ℓ_- and ℓ_+ are determined by ζ, see Definition 2.4).

Lemma 7.3. *Let $\zeta > 0$ be small enough and such that* (7.14) *holds. Then for all $t \le \ell_+/V$, $V = e^2\beta$, and all $m \in L^\infty(\mathbb{R}, [-1,1])$,*

$$|S_t(m) - m_\beta| \le 3\zeta, \qquad \text{for all } x \text{ such that } \Theta(m; x) = 1 \tag{7.15}$$

where $\Theta \equiv \Theta^{(\zeta, \ell_-, \ell_+)}$.

Proof. Calling $\psi(x,t) = S_t(m) - e^{-t}m$, by differentiating w.r.t. x the integral version of (1.3),

$$|\frac{d}{dx}\psi(x,t)| \le \beta \|J'\|_\infty$$

Let $I_x^{(\ell_-)}$ be the interval $[n\ell_-, (n+1)\ell_-)$ which contains x; we have

$$\left|\psi(x,t) - \fint_{I_x^{(\ell_-)}} \psi\right| \le \beta \|J'\|_\infty \ell_- \le \frac{\zeta}{2}$$

In [13] it is proved that for ζ small enough, $\Theta(S_t(m);\cdot) \equiv 1$ for all $t > 0$ if it holds at $t = 0$. Suppose $\Theta(m;\cdot) \equiv 1$, then, taking $t = \log(4\zeta)$,

$$|S_t(m)(x) - m_\beta| \le 2e^{-t} + \left|\psi(x,t) - \fint_{I_x^{(\ell_-)}} \psi\right| + \left|\fint_{I_x^{(\ell_-)}} S_t(m) - m_\beta\right| \le 2\zeta$$

We will next use the Barrier Lemma to drop the assumption $\Theta(m;\cdot) \equiv 1$. Let $[a,b)$ be a maximal connected component of $\{x : \Theta(m;x) = 1\}$ and $G = [a - \ell_+, b + \ell_+)$. By definition of Θ, $\eta(m;x) = 1$ for all $x \in G$. Set $m^* = m$ in G and $m^* = m_\beta$ in G^c and $t^* = \ell_+/V$. Then, if $x \in [a,b)$ and $t \le t^*$,

$$|S_t(m)(x) - m_\beta| \le |S_t(m^*)(x) - m_\beta| + Ce^{-t^*V} \le 3\zeta$$

□

Lemma 7.4. *For all $\zeta' > 0$ small enough the following holds. Suppose there are m, an interval $[a,b)$ and $T > 0$ such that $m \ge m_\beta - \zeta'$ in $[a,b)$ and $S_t(m) \ge m_\beta - \zeta'$ in $\{[a-1,a) \cup [b,b+1)\} \times [0,T]$. Then*

$$S_t(m) \ge m_\beta - \zeta' \quad \textit{for all } x \in [a,b) \textit{ and all } t \le T \tag{7.16}$$

Proof. Denote by ψ the restriction of $S_t(m)$ to $[a,b)$, so that ψ is a function on $[a,b) \times [0,T]$. It then solves the equation

$$\frac{d\psi(x,t)}{dt} = -\psi(x,t) + \tanh\{h(x,t) + \int_{[a,b)} J(x,y)\psi(y,t)\}, \quad x \in [a,b), t \in [0,T]$$

with

$$h(x,t) = \int_{[a,b)^c} J(x,y)m(y,t)$$

Notice that in the above integral $m(y,t) \ge m_\beta - \zeta'$ because J has range 1.

By the comparison theorem (whose validity extends to the present situation) $\psi(x,t) \ge \psi^*(x,t)$ for any ψ^* which solves the above equation with $h^* \le h$ and $\psi(x,0) \ge \psi^*(x,0)$, hence (7.16).

□

As a corollary of the above lemmas we have the following result.

Proposition 7.5. *For ζ small enough and L large enough, there is $\tau > 0$ such that if $m \in U^+$ then $S_\tau(m)(x) \geq m_\beta - 4\zeta$ for all $x \in [-L, L]$.*

Proof. We need some preliminary definitions. Recalling (7.8) we shorthand

$$\tau_R^+ = \tau_{\zeta, m_\beta - 3\zeta, R}^+ \tag{7.17}$$

For any interval $I = [a, b)$, denote

$$I^* = [a', b'), \quad b' = b + 2\tau_{(b-a)/2}^+ + 1 \; ; \; a' = a - 2\tau_{(b-a)/2}^+ - 1 \tag{7.18}$$

Call connected two intervals I_1 and I_2 if $I_1^* \cap I_2^* \neq \emptyset$. Given a sequence $\{I_i\}$ of disjoint intervals, let $\{I_i'\}$ be the sequence of intervals obtained as follows. For any maximal connected component $\{I_{i_k}\}$ of $\{I_i\}$, let $[a, b)$ be the minimal interval which contains all I_{i_k}; the set of all such intervals defines $\{I_i'\}$.

We apply the above to the set of contours of m. By an abuse of notation, we still denote by m its L-symmetric extension, see the notation at the beginning of Section 2. Since $m \in U^+$, the contours $\{\Gamma_i\}$ of the L-symmetric extension of m are such that

$$\sum_{\Gamma_i} |\Gamma_i \cap [-L, L]| \leq \kappa \ell_+ \tag{7.19}$$

We then define iteratively $\{I_{i;n}\}$ by setting $\{I_{i;1}\} = \{\Gamma_i\}$ and, for $n > 1$, $\{I_{i;n}\} = \{I_{i;n-1}'\}$. Since m is L-symmetric, there is N so that $\{I_{i;n}\} = \{I_{i;N}\}$ for all $n \geq N$. Moreover by (7.19), for L large enough all $I_{i;N}$ are bounded and mutually disconnected, in particular they do not cover $\mathbb{R}$. We still need a few notation. Recalling (7.17), given an interval I we call $I^+ = \{x : \text{dist}(x, I) \leq \tau_{|I|/2}^+\}$, and $\partial I^+ = \{x \notin I^+ : \text{dist}(x, I^+) \leq 1\}$. Denote by $\{t_1, t_2, \ldots, t_k\}$ the set $\{\tau_{|I_{i;N}|/2}^+\}$, where t_i are in increasing order, $t_1 < t_2 \cdots < t_k$. Let $\mathcal{I}_j = \{I_{i,N} : \tau_{|I_{i;N}|/2}^+ = t_j\}$ and

$$\Lambda_j = \bigcup_{I \in \mathcal{I}_j} I_j^+, \quad \Lambda = \bigcup \Lambda_j \tag{7.20}$$

$$\partial \Lambda_j = \bigcup_{I \in \mathcal{I}_j} \partial I_j^+ \tag{7.21}$$

By Lemma 7.3, for ζ small enough there is s_0 so that

$$|S_{s_0}(m)(x) - m_\beta| \le 3\zeta \qquad \text{for all } x \in G := \{x : \Theta(m;x) = 1\} \tag{7.22}$$

Define $\zeta' := 3\zeta$ and $\psi := S_{s_0}(m)$. By (7.22) we have $\psi(x) \ge m_\beta - \zeta'$ for all $x \in \Lambda^c$. By Lemma 7.2 $S_{t_j}(\psi) \ge m_\beta - 2\zeta$ on Λ_j and $S_t(\psi) \ge m_\beta - \zeta - \zeta'$ on $\partial\Lambda_j$ for $t \le t_j$.
By Lemma 7.4, $S_{t_1}(\psi) \ge m_\beta - \zeta - \zeta'$ on $\Lambda^c \cup \Lambda_1$; $S_{t_2}(\psi) \ge m_\beta - \zeta - \zeta'$ on $\Lambda^c \cup [\Lambda_1 \Lambda_2]$ and, by iteration, $S_{t_k}(\psi) \ge m_\beta - \zeta - \zeta'$ everywhere.

□

Proof of Theorem 2.7. From Proposition 7.5 it follows that for any $m \in U_+$ and any $t \ge 0$,

$$S_{t+\tau}(m_\beta - 3\zeta) \le S_{t+\tau}(m) \le S_{t+\tau}(1)$$

Both $S_t(m_\beta - 3\zeta)$ and $S_t(1)$ converge exponentially to $m^{(+)}$ as $t \to \infty$, hence Theorem 2.7 is proved. □

8 Interface motion

In this section we prove the following result. For any $\epsilon > 0$ and $r \in (0,1)$, define $\Lambda_r^+ := (-L, -rL]$, $\Lambda_r^- := [rL, L)$ and

$$\mathcal{M}^{\pm}_{\epsilon,r} = \Big\{ m \in L^\infty([-L,L];[-1,1]) : \exists \xi \in \Lambda_r^{\pm} : \|m - \bar{m}_\xi\|_\infty \le \epsilon \Big\}$$

Theorem 8.1. *For any $\epsilon > 0$ and $r \in (0,1)$ there is $L(\epsilon, r)$ so that for $L > L(\epsilon, r)$,*

$$\lim_{t\to\infty} \|S_t(m) - m^{(\pm)}\|_\infty = 0 \qquad \textit{for all } m \in \mathcal{M}^{\pm}_{\epsilon,r} \tag{8.1}$$

It is enough to prove (8.1) when $m \in \mathcal{M}^+_{\epsilon,r}$, because, by symmetry, the result extends to $m \in \mathcal{M}^-_{\epsilon,r}$. The scheme of the proof is the following.
We will prove that at the end of an initial time layer, $S_t(m)$ is above a function "much closer" than ϵ to an instanton which is "only slightly" shifted to the right of the original one (to which m is close as an element of $\mathcal{M}^+_{\epsilon,r}$). We will then complete the proof by showing that this is above a travelling sub-solution of (1.3) which converges to $m^{(+)}$.

For any m close to an instanton we will construct a sub-solution which starts below m and which, after a suitably long time, is much closer to a new instanton than the original m. We introduce below the parameters which will be used in the construction of the sub-solution, their justification lies in the proof of Proposition 8.2 below.

Let

$$b_\xi^+ = ce^{-\alpha(L-\xi)},\ \xi \le 0; \qquad c = \sup_{L\ge 1,\xi\le 0}\ \max_{x\in[L-1,L],y\in[L,L+1]} |\bar m_\xi(x) - \bar m_\xi(y)| \tag{8.2}$$

c being finite by (3.7). Let p_ξ be as in (3.11). Since $\lim_{|x|\to\infty} p_\xi(x) = \beta(1-m_\beta^2) < 1$, if L is large enough there is $N > 0$ (large enough) so that, for all $|\xi - x| > N$,

$$p_\xi(x) + Cb_0^+ \le p < 1, \qquad C = \beta^2 \sup_{x\in\mathbb{R}} |\tanh''(x)| \tag{8.3}$$

We finally define the positive parameters ω and T_ξ as:

$$\omega = \frac{1-p}{2}, \qquad \frac{1-p}{4}\, 2\epsilon e^{-\omega T_\xi} = b_\xi^+ + C{b_\xi^+}^2 \tag{8.4}$$

T_ξ being well defined for L large enough, as the r.h.s. in the second equality vanishes as $L \to \infty$.

Proposition 8.2. *With ω, N as above and $\epsilon > 0$ sufficiently small, let ξ_0 be such that*

$$\xi_0 \le -\eta, \qquad \eta = \frac{2(\omega+1)\epsilon}{\omega[\min_{|x|\le N} \bar m'(x)]}. \tag{8.5}$$

Then, for all L large enough and any $t \in [0, T_{\xi_0}]$ the function

$$u(x,t) := \bar m_{\xi(t)}(x)\mathbf{1}_{|x|\le L} - \delta(t), \quad \delta(t) = 2\epsilon e^{-\omega t} \tag{8.6}$$

$$\xi(t) = \xi_0 + \eta[1 - e^{-\omega t}] \tag{8.7}$$

is a sub-solution of (1.3), namely

$$u_t - g_L(u) \le 0 \tag{8.8}$$

Proof. The time derivative of $u(x,t)$ is

$$u_t = -\xi_t \mathbf{1}_{|x|\le L}\, \bar m'_{\xi(t)} + \omega\delta(t) \tag{8.9}$$

We will see that for $x \approx \xi(t)$, $|g_L(u)| \le c\delta(t)$, $c > 0$ a suitable constant. Since $\bar m'$ has the order of unity and, by (8.7), ξ_t is larger than $(\omega + c)\delta(t)$, (8.8) will hold for $x \approx \xi(t)$. For $|x - \xi(t)|$ large, instead, $\omega\delta(t)$, which is dangerously positive, becomes dominant in (8.9), but, as we will see, $g_L(u)$ will contrast it.

For $|x| \le L-1$, $J^{\rm neum}(x,\cdot) = J(x,\cdot)$, so that

$$g_L(u) = \delta + \tanh\{\beta[J * \bar m_\xi - \delta(t)]\} - \tanh\{\beta J * \bar m_\xi\} \tag{8.10}$$

and, by a Taylor expansion,

$$g_L(u) \ge -[p_{\xi(t)} - 1]\delta(t) - \frac{C}{2}\,\delta(t)^2 \tag{8.11}$$

with C as in (8.3) and p_ξ as in (3.11).

Define $b_\xi(x) := J^{\rm neum} * \bar m_\xi - J * \bar m_\xi$ and notice that $b_\xi(x) = 0$ for $|x| \le L-1$. Since $\bar m_\xi(x)$ is an increasing function of x, for $x \in [-L, -L+1]$, $b_\xi(x) \ge 0$, and we get the same bound as in (8.11):

$$g_L(u) = \delta + \tanh\{\beta[J * \bar m_\xi + b_\xi - \delta]\} - \tanh\{\beta J * \bar m_\xi\} \ge -[p_\xi - 1]\delta - \frac{C}{2}\,\delta(t)^2$$

For $x \in [L-1, L]$, instead, $b_\xi(x) \le 0$ and the previous inequality fails. Recalling (8.2), for $\xi \le 0$, $b_\xi^+ \ge \max_{x \in [L-1,L]} |b_\xi(x)|$, and, since $\xi(t) \le \xi_0 + \eta \le 0$,

$$g_L(u) \ge -[p_\xi - 1 + C b_0^+]\delta + p_\xi b_\xi - \frac{C}{2}\,\left(\delta(t)^2 + {b_\xi^+}^2\right)$$

For $|x - \xi(t)| \le N$, we bound $g_L(u) \ge \delta(t)$ (see (8.10)) while, for $|x - \xi(t)| > N$, we use (8.3) and (8.4) getting

$$u_t - g_L(u) \le \begin{cases} -\xi_t\,[\min\limits_{|x|\le N} \bar m'(x)] + \omega\delta(t) + \delta(t) & |x - \xi(t)| \le N \\ \dfrac{p-1}{2}\,\delta(t) + b_\xi^+ + \dfrac{C}{2}\,\left(\delta(t)^2 + {b_\xi^+}^2\right) & |x - \xi(t)| \ge N \end{cases} \tag{8.12}$$

By (8.7) the first line in (8.12) is non positive. By the second inequality in (8.4), for $|x - \xi(t)| \ge N$ and $t \le T_{\xi_0}$,

$$u_t - g_L(u) \le -\frac{1-p}{4}\delta(t) + \frac{C}{2}\,\delta(t)^2 \le 0$$

having supposed ϵ so small that $\dfrac{1-p}{4} \ge \dfrac{C}{2}\,2\epsilon$. □

Let $m \in \mathcal{M}^+_{\epsilon,r}$; then there is $\xi_0 \in (-L, -rL]$ such that $\|m - \bar m_{\xi_0}\|_{L^\infty([-L,L])} \le \epsilon$. Given any $r \in (0,1)$ for L large enough we have that $-rL < -\eta$ so that by (8.6), $m \ge u(\cdot, 0)$. Then, by Proposition 8.2,

$$S_{T_{\xi_0}}(m) \ge \bar m_{\xi(T_{\xi_0})} - 2\epsilon e^{-\omega T_{\xi_0}}.$$

Since $\bar m_\xi$ is a decreasing function of ξ, there is a positive constant c' so that

$$S_{T_{\xi_0}}(m) \ge \bar m_{\xi_0+\eta} - c' e^{-\alpha(L-\xi_0)}, \qquad \eta \text{ as in } (8.5) \tag{8.13}$$

We will prove (see (8.17) below) that the r.h.s. of (8.13) is bounded from below by a function $z_{\xi'}$, $\xi' > \xi_0 + \eta$, where z_ξ has the following structure

$$z_\xi(x) = \bar m_\xi(x) - a e^{-\alpha(2L-\xi-x)} + a e^{-\alpha(2L+\xi+x)} + \Psi_\xi(x) \tag{8.14}$$

with α and a positive constants defined in (3.7), while the function Ψ_ξ is small (see (8.16) below) at least when ξ is away from the boundaries. The important point is that there is a decreasing function $\xi(t)$ so that $z_{\xi(t)}$ is a travelling sub-solutions of (1.3).

Theorem 8.3. *For any $r \in (0,1)$ and $\delta > 0$, there is $L_0(r,\delta)$ such that, for any $L > L_0(r,\delta)$, $z_{\xi(t)}$ is a sub-solution of* (1.3) *for all t such that $\xi(t) \in [-(1-r)L, -rL]$ and provided*

$$\frac{d\xi(t)}{dt} = -\sqrt{\mu} K(1-\delta) e^{-2\alpha\xi(t)} \tag{8.15}$$

with $\sqrt{\mu}$ defined in (9.13), *K in* (9.23). *Moreover $|z_\xi(x)| < m_\beta$ for all $|x| \le L$ and*

$$|\Psi_\xi(x)| \le \delta e^{-\alpha|\xi - x|} \tag{8.16}$$

Theorem 8.3 will be proved in Section 9, were we will see that δ can be chosen exponentially small in L.

Proposition 8.4. *Given any $r \in (0,\bar r)$, for all L large enough and any $\xi_0 \in (-L, -\bar r L]$ there is $\xi' \in [-(1-r)L, -rL]$ so that*

$$\bar m_{\xi_0+\eta}(x) - c' e^{-\alpha(L-\xi_0)} \ge z_{\xi'}(x), \quad |x| \le L \tag{8.17}$$

Proof. Since $\bar m_{\xi_0+\eta}(x) - c' e^{-\alpha(L-\xi_0)}$ is a decreasing function of ξ_0, without loss of generality we may suppose $\xi_0 \in [-(1-\bar r)L, -\bar r L]$. To simplify notation, we let $\xi = \xi_0 + \eta$ and will prove (8.17) with $\xi' = \xi + R$, R being a "large" constant independent of L. We will do that by deriving a lower bound for $\bar m_\xi(x) - \bar m_{\xi+R}(x)$, an upper bound for $z_{\xi+R}(x) - \bar m_{\xi+R}(x)$, with the former larger than the latter.
Recalling (8.14)-(8.16),

$$z_{\xi+R}(x) - \bar m_{\xi+R}(x) \le a e^{-\alpha(2L+\xi+R+x)} + \delta e^{-\alpha|\xi+R-x|} \tag{8.18}$$

Let $N>0$ be such that $ce^{-\alpha_0 N}=\frac{a}{2}\,e^{-\alpha N}$, where a, α, α_0 and c are as in (3.7) (without loss of generality we may and will suppose that $c>a$). Call $m'=\bar m(N)$, choose arbitrarily $m''\in(m',m_\beta)$, define R_0 so that $\bar m(R_0)=m''$ and require $R\ge 2R_0$. Then,

$$\bar m_\xi(x)-\bar m_{\xi+R}(x)\ge m'',\quad x\in[\xi,\xi+R]$$

as it follows by bounding $\bar m_\xi(x)\ge 0$, $\bar m_{\xi+R}(x)\le -m''$ for $x\in[\xi,\xi+R/2]$ and using an analogous bound for $x\in[\xi+R/2,\xi+R]$. Moreover

$$\bar m_\xi(x)-\bar m_{\xi+R}(x)\ge m''-m',\quad x\in[\xi-N,\xi]$$

because $\bar m_\xi(x)\ge -m'$ and $\bar m_{\xi+R}(x)\le -m''$. For $x\le\xi-N$, using that $\bar m$ is antisymmetric and (3.7), we write

$$\begin{aligned}\bar m_\xi(x)-\bar m_{\xi+R}(x)&=[\bar m(x-\xi)+m_\beta]+[\bar m(\xi+R-x)-m_\beta]\\&\ge ae^{-\alpha(\xi-x)}-ce^{-\alpha_0(\xi-x)}-ae^{-\alpha(\xi+R-x)}-ce^{-\alpha_0(\xi+R-x)}\end{aligned}$$

Using the definition of N we then get

$$\bar m_\xi(x)-\bar m_{\xi+R}(x)\ge\frac{a}{2}\,e^{-\alpha(\xi-x)}-\frac{3a}{2}\,e^{-\alpha(\xi+R-x)},\quad x\le\xi-N$$

Similarly

$$\bar m_\xi(x)-\bar m_{\xi+R}(x)\ge m''-m',\quad x\in[\xi+R,\xi+R+N]$$

$$\bar m_\xi(x)-\bar m_{\xi+R}(x)\ge-\frac{3a}{2}\,e^{-\alpha(x-\xi)}+\frac{a}{2}\,e^{-\alpha(x-[\xi+R])},\quad x\ge\xi+R+N$$

Using the above upper bounds and by taking $\delta\le a/4$ in Theorem 8.3 it is not difficult to check that for L and R sufficiently large,

$$\bar m_\xi(x)-\bar m_{\xi+R}(x)-c'e^{-\alpha(L-\xi)}\ge ae^{-\alpha(2L+\xi+R+x)}+\delta e^{-\alpha|\xi+R-x|}$$

This, together with (8.18) concludes the proof of the Proposition. □

By (8.13) and (8.17), pointwise,

$$S_{T_\xi}(m)\ge z_{\xi'}\tag{8.19}$$

We will prove convergence of $S_t(z_{\xi+\eta+R})$ to $m^{(+)}$ in L^∞ and this will prove that any limit point ϕ of $S_t(m)$ is not smaller than $m^{(+)}$. Since any limit point is stationary, $\phi=\tanh\{\beta J^{\rm neum}*\phi\}$; then $b:=\max_{|x|\le L}\phi(x)$ must be such that $b\ge\tanh\{\beta b\}$, hence $\phi(x)\le m_\beta$ for all x, which will then conclude the proof of (8.1) of Proposition 8.1.

Proposition 8.5. *For any positive $r < \bar{r}$ there is a constant $L_2(r,\epsilon)$ so that for any $L \geq L_2(r,\epsilon)$, if $m \in \mathcal{M}^+_{\epsilon,\bar{r}}$ then there is a time t^* such that*

$$S_{t^*}(m) \geq z_{-(1-r)L} \tag{8.20}$$

Proof. For L large enough, (8.19) (recalling from Proposition 8.4 that $\xi' \in [-(1-r)L, -rL]$) and Theorem 8.3 yield (8.20).

□

Let $N > 0$ be such that $\bar{m}(x) > m_\beta - \epsilon$ for all $x \geq N$ (ϵ as in $\mathcal{M}^+_{\epsilon,\bar{r}}$). Let

$$w(x) = \begin{cases} -m_\beta & x < -(1-r)L - N \\ z_{-(1-r)L} & |x + (1-r)L| \leq N \\ m_\beta - 2\epsilon & x > -(1-r)L + N \end{cases}$$

Then, for L large enough, $z_{-(1-r)L}(x) \geq w(x)$, $|x| \leq L$ and $|w(x) - \bar{m}_{-(1-r)L}(x)| \leq 2\epsilon$ for all $|x| \leq L$.

We will prove that there is a time T such that

$$S_T(w)(x) \geq \delta > 0 \quad \text{for all } |x| \leq L \tag{8.21}$$

so that also $S_T(z_{-(1-r)L}) > \delta$. Since the strictly positive functions are in the basin of attraction of $m^{(+)}$ (because a strictly positive function is bounded in between two positive constants, and all constant, positive functions are attracted by $m^{(+)}$), it then follows that the limit points of $S_t(z_{-(1-r)L})$ are above $m^{(+)}$ which is what we wanted to prove.

Thus we are left with the proof of (8.21). It is now convenient to regard (1.3) as an equation on the whole line, with $J^{\text{neum}}(x,y)$ replaced by $J(x,y)$, namely we will consider the equation (2.2). We thus need to show that there are δ and T positive such that

$$u(x,T) > \delta \quad \text{for all } |x| \leq L,\ u(\cdot,t) \text{ solving (2.2) with } u(\cdot,0) = w(\cdot)$$

where, by an abuse of notation, we have denoted by w the symmetric extension of w.

Let

$$k = -(1-r)L - N - (-L) = rL - N$$

so that $w(x) = -m_\beta$ when x is in the interval $[-L-k, -L+k]$, as well as in all its translates by $4nL$, $n \in \mathbb{Z}$. We then introduce a function w^* as follows. Let $k' = 11k$ and define $w^*(x) =$

$-m_\beta$ for all $x \in [-L-k', -L+k]$. Let then $w^*(x) = w(x+10k)$ for $x \in [-L-k'-2N, -L-k']$. We then set $w^*(x+4nL) = w^*(x)$, for all $x \in [-L-k'-2N, -L+k]$ and all $n \in \mathbb{Z}$. We complete the definition of w^* by setting $w^*(x) = w(x)$ at all other points $x \in \mathbb{R}$. It follows from this construction that $w^*(x) \le w(x)$ for all $x \in \mathbb{R}$ and that $w^*(x)$ is invariant under reflections around the points $(-L-5k)+4nL$, $n \in \mathbb{Z}$, so that, modulo a translation, we are still in the context of (1.3).

By applying the analysis in Proposition 8.2, Theorem 8.3 and Proposition 8.4 to the evolution starting from w^*, we conclude that there is a time T so that, denoting by z_ξ the symmetric extension of the function defined in (8.14), then

$$S_T(w^*)(x) \ge z_{-(1-r)L}(x+5k)$$

which shows that for L large enough, $S_T(w^*)(x) > 0$ for $x \in [-L, L]$. This concludes the proof. □

9 A travelling sub-solution

In this section we will prove Theorem 8.3. As the analysis will focus on what happens around the left end of the spatial domain, it is convenient to change coordinates, considering the interval $[0, 2L]$ instead of $[-L, L]$. By an abuse of notation, we will not change symbols, thus writing

$$u_t(x,t) = g_L(u(x,t)), \quad x \in [0, 2L], \qquad g_L(u) = -u + \tanh\left\{\beta J^{\text{neum}} * u\right\} \tag{9.1}$$

with J^{neum} now defined with reflections at 0 and $2L$; $S_{t;L}(m)$ is here the flow solution of (9.1) with initial datum m.

As discussed in the introduction we may regard (9.1) as an equation on the whole of $\mathbb{R}$, writing

$$u_t(x,t) = g(u(x,t)), \quad x \in \mathbb{R}, \qquad g(u) = -u + \tanh\left\{\beta J * u\right\} \tag{9.2}$$

Call "symmetric" the functions invariant under all reflections R_{n2L}, $n \in \mathbb{Z}$, then the solution of (9.2) starting from a symmetric function, is symmetric and its restriction to $x \in [0, 2L]$ solves (9.1). Thus (9.2) restricted to the space of symmetric functions is equivalent to our (9.1) and we will shift back and forth in the sequel between the two representations.

The construction of a sub-solution is done by adding "corrections" to the instanton like function $\bar m_\xi(x)$, $x \in [0, 2L]$. $\bar m_{\xi(t)}(x)$ alone, in fact, does not do the job: indeed, $g(\bar m_{\xi(t)}) = 0$

for $x \in [1, 2L-1]$ while

$$\frac{d}{dt}\bar{m}_{\xi(t)}(x) = -\bar{m}'_{\xi(t)}(x)\xi_t(t) > 0, \quad \xi_t(t) < 0$$

The instanton however must have an important role, as it solves the stationary equation in the limit $L \to \infty$ and it is natural to expect that, when L is finite, the boundary effects are responsible for the motion of the "instanton". To catch the effect, we "correct" the instanton into a function $m^0_\xi(x)$, which is the symmetric extension of

$$m^0_\xi(x) = \bar{m}_\xi(x) + A_\xi(x), \quad x \in [0, 2L]; \qquad A_\xi(x) = -ae^{-\alpha(4L-\xi-x)} + ae^{-\alpha(\xi+x)} \tag{9.3}$$

Recalling (3.7), we have the following nice interpretation of A_ξ, by remarking that the terms $ae^{-\alpha(x+\xi)}$ and $-ae^{-\alpha(4L-\xi-x)}$ are the corrections to the asymptotic behavior of $\bar{m}(\xi - x)$, which is the reflection of the original instanton through the origin, and, respectively, to $-\bar{m}(2L - \xi - x)$, which is the reflection through the point $2L$.

In the end, we will show that the true sub-solution, denoted by $z_{\xi(t)}(x)$, differs from $m^0_{\xi(t)}$ by higher order terms, for L large. These are however necessary to have $g(z_\xi)(x) > 0$ for all x, and all $\xi \in [rL, (1-r)L]$; then since $dz_\xi/d\xi$ is uniformly bounded, $z_t - g(z_{\xi(t)}) < 0$ if $d\xi(t)/dt$ is negative, but sufficiently small, hence $z_{\xi(t)}$ is a sub-solution. By a more careful analysis we will also find the right speed.

We will write

$$z_\xi = m^0_\xi + \psi_\xi + \psi^*_\xi$$

where ψ_ξ and ψ^*_ξ are two symmetric functions to be determined. To this end, we write, for $x \in [0, 2L]$,

$$g(z_\xi) = -[\bar{m}_\xi + A_\xi + \psi_\xi + \psi^*_\xi] + \tanh\{\beta J * [\bar{m}_\xi + A_\xi + B_\xi + \psi_\xi + \psi^*_\xi]\} \tag{9.4}$$

where

$$B_\xi(x) = m^0_\xi(x) - \{\bar{m}_\xi(x) + A_\xi(x)\} \tag{9.5}$$

namely B_ξ denotes the difference between the symmetric function $m^0_\xi(x)$ and the non symmetric one, $\bar{m}_\xi + A_\xi$. Thus $B_\xi(x) = 0$ for $x \in [0, 2L]$ and $\neq 0$ elsewhere and for this reason its contribution appears only in the convolution term and it is different from zero except for some $x \in [0, 1]$ and $x \in [2L-1, 2L]$.

Recalling that $\bar{m}_\xi = \tanh\{\beta J * \bar{m}_\xi\}$, we will next rewrite (9.4) via a Taylor expansion to second order around $\bar{m}_\xi$. Let

$$\Omega_\xi m = -m + p_\xi J * m$$

where $p_\xi(x)$ is here the symmetric function equal to $\beta(1-\bar{m}_\xi(x)^2)$ when $x \in [0, 2L]$. We regard Ω_ξ as an operator on $L^\infty(\mathbb{R})$, noticing that it maps symmetric functions into symmetric ones. We then get from (9.4) the following equality, valid for all x in $[0, 2L]$,

$$\begin{aligned} g(m^0_\xi + \psi_\xi + \psi^*_\xi) = \Omega_\xi[A_\xi + \psi_\xi + \psi^*_\xi] + p_\xi J * B_\xi \\ +\rho_\xi\big[J * (A_\xi + B_\xi + \psi_\xi + \psi^*_\xi)\big]^2 + 0_3 \end{aligned} \tag{9.6}$$

where

$$\rho_\xi = \frac{\beta^2}{2}\tanh''\left\{\beta J * \bar{m}_\xi\right\} \tag{9.7}$$

and 0_3 is the remainder term in the expansion: there is a constant c (proportional to β^3) so that

$$|0_3(\psi_\xi, \psi^\star_\xi, A_\xi, B_\xi)| \le c \sum_{i_1+i_2+i_3=3} |J * \psi_\xi|^{i_1} |J * \psi^\star_\xi|^{i_2} |J * (A_\xi + B_\xi)|^{i_3}$$

In the sequel we will denote by the same symbol the symmetric extension of each term on the r.h.s of (9.6), so that equality holds on the whole of $\mathbb{R}$. Thus, for instance, by $\rho_\xi(x)$ we mean the expression (9.7) if $x \in [0, 2L]$, and its symmetric extension elsewhere. The same applies to all the other terms of (9.6).

By an explicit computation which uses (3.6), we get for $x \in [0, 2L]$,

$$a_\xi = \Omega_\xi A_\xi = -e^{-2\alpha(2L-\xi)} k^+_\xi + e^{-2\alpha\xi} k^-_\xi, \qquad k^\pm_\xi(x) := a e^{\pm\alpha(x-\xi)} \frac{m^2_\beta - \bar{m}_\xi(x)^2}{1 - m^2_\beta} > 0 \tag{9.8}$$

which, as said above, is then regarded as a function on the whole of $\mathbb{R}$ by symmetric extension. To specify ψ_ξ and ψ^*_ξ, we introduce the operator

$$\mathcal{T}_\xi = \sum_{n=0}^{N} (1 + \Omega_\xi)^n, \qquad N = \text{integer part of } L^2$$

and define

$$\psi_\xi = \mathcal{T}_\xi\Big(a_\xi + p_\xi J * B_\xi + \rho_\xi\big[J * (A_\xi + B_\xi)\big]^2\Big) \tag{9.9}$$

$$\psi^\star_\xi = \mathcal{T}_\xi\Big(2\rho_\xi[J * \psi_\xi][J * \big(A_\xi + B_\xi\big)] + \rho_\xi\big[J * \psi_\xi\big]^2\Big) \tag{9.10}$$

where, according to the previous convention, all terms on which $\mathcal{T}_\xi$ acts are symmetric functions.

Lemma 9.1. *Ω_ξ as an operator on the L^∞ symmetric functions is invertible and*

$$\Omega_\xi \mathcal{T}_\xi m = -m + (1+\Omega_\xi)^{N+1} m \tag{9.11}$$

Proof. The invertibility statement is proved in [9], the remaining statements are well known algebraic identities. □

With the choice (9.9)-(9.10) and using (9.11), we get from (9.6)

$$\begin{aligned} g(m_\xi^0 + \psi_\xi + \psi_\xi^*) &= (1+\Omega_\xi)^{N+1}\Big(a_\xi + p_\xi J * B_\xi + \rho_\xi \big[J * (A_\xi + B_\xi)\big]^2\Big) \\ &+ (1+\Omega_\xi)^{N+1}\Big(2\rho_\xi [J * \psi_\xi][J * \big(A_\xi + B_\xi\big)] + \rho_\xi \big[J * \psi_\xi\big]^2\Big) \\ &+ \rho_\xi \Big([J * \psi_\xi^*]^2 + 2[J * \psi_\xi^*] \big\{ [J * \psi_\xi] + [J * \big(A_\xi + B_\xi\big)] \big\} \Big) + 0_3 \end{aligned}$$

so that all terms, except those in the last line, which we will prove to be suitably small, are in the range of the operator $(1+\Omega_\xi)^{N+1}$ which, as we are going to see, has a nice behavior. In [9] it has been proved that there is $s^\star > 0$ so that for all $\xi \in [s^\star, 2L - s^\star]$, a Perron-Frobenius theorem holds for $1 + \Omega_\xi$ (with domain the space of symmetric L^∞ functions). It is shown that Ω_ξ has a strictly positive eigenvalue λ_ξ with strictly positive eigenvector v_ξ; $\|\Omega_\xi\|_\infty = \lambda_\xi$ and the remaining part of the spectrum is made by complex numbers whose real part is not larger than $-\omega$, $\omega > 0$. Let then π_ξ be the linear functional on $L^\infty(\mathbb{R})$, defined by

$$\pi_\xi(m) := \int_0^{2L} m(x) v_\xi(x) \, \frac{dx}{p_\xi(x)} \tag{9.12}$$

and normalize v_ξ so that

$$\pi_\xi(v_\xi) = 1$$

while we call $\mu > 0$ the normalizing constant when $L = \infty$, i.e.

$$\mu \int_{\mathbb{R}} \bar m_\xi'(x)^2 \frac{dx}{1 - \bar m_\xi(x)^2} \, dx = 1 \tag{9.13}$$

We recall some results we need in the sequel, see [9, Thms. 2.1, 2.3, 2.4], and also [10, Thm. 3.2]. There are constants $c_\pm > 0$, $c' > 0$ and $\delta > 0$ so that for $\xi \in [s^\star, 2L - s^\star]$

$$c_-[e^{-2\alpha\xi} + e^{-2\alpha(2L-\xi)}] \le \lambda_\xi \le c_+[e^{-2\alpha\xi} + e^{-2\alpha(2L-\xi)}] \tag{9.14}$$

$$|v_\xi(x) - \sqrt{\mu}\, \bar m_\xi'(x)| \le c_+[e^{-2\alpha\xi} + e^{-2\alpha(2L-\xi)}] e^{\alpha|\xi - x|} \xi^4 \qquad \text{for } |\xi - x| \le \frac{\xi}{2} \tag{9.15}$$

$$|v_\xi(x) - \sqrt{\mu}\, a\alpha e^{-\alpha\xi}\big(e^{-\alpha x} + e^{\alpha x}\big)| \le c_+ e^{-(\alpha+\delta)(\xi - x)} \qquad \text{for } x \in [0, \frac{\xi}{2}] \tag{9.16}$$

$$|v_\xi(x) - \sqrt{\mu}\, a\alpha e^{-\alpha(2L-\xi)}\big(e^{-\alpha x} + e^{\alpha x}\big)| \le c_+ e^{-(\alpha+\delta)(x-\xi)} \quad \text{for } x \in [\frac{3\xi}{2}, 2L] \tag{9.17}$$

Lemma 9.2. *There are c and ω positive so that for any L^∞ symmetric function m,*

$$\|(1+\Omega_\xi)^{N+1} m - (1+\lambda_\xi)^{N+1}\pi_\xi(m) v_\xi\|_\infty \le c e^{-\omega N}\|m\|_\infty$$

Proof. It follows from Theorem 2.4 in [9]. □

Notice that $e^{-\omega N} \le e^{-\omega(L^2-1)}$, which will be a negligible error. Moreover, by (9.14), for ξ large enough, $(1+\lambda_\xi)^{N+1} \sim 1 + \lambda_\xi(N+1)$ and to leading order ~ 1, so that we may think that $(1+\Omega_\xi)^{N+1} m$ is essentially given by $\pi_\xi(m) v_\xi$. We then rewrite (9.6) as

$$g(m^0_\xi + \psi_\xi + \psi^*_\xi) = d_\xi v_\xi + E_\xi + 0_3 + 0_{\exp} \tag{9.18}$$

where

$$\begin{aligned} d_\xi = (1+\lambda_\xi)^{N+1}\pi_\xi\Big(a_\xi + p_\xi J * B_\xi + \rho_\xi\Big[[J*(A_\xi + B_\xi)]^2 \\ +2\{J*\psi_\xi\}\{J*(A_\xi+B_\xi)\} + \big[J*\psi_\xi\big]^2\Big]\Big) \end{aligned} \tag{9.19}$$

$$E_\xi = \rho_\xi\Big\{[J*\psi^\star_\xi]^2 + 2[J*\psi^\star_\xi][J*\psi_\xi + J*(A_\xi+B_\xi)]\Big\} \tag{9.20}$$

and $0_{\exp}$ is the error term, for replacing $(1+\Omega_\xi)^{N+1}(\,\cdot\,)$ by $(1+\lambda_\xi)^{N+1}\pi_\xi(\,\cdot\,)v_\xi$.

Lemma 9.3. *There is a constant c' so that*

$$|0_{\exp}| \le c' N^2 e^{-\omega N} \tag{9.21}$$

Proof. Since $\|a_\xi\|_\infty \le c''$, $\|A_\xi\|_\infty \le c''$ and $\|B_\xi\|_\infty \le c''$, $\|\mathcal{T}_\xi\|_\infty \le N+1$, then $\|\psi_\xi\|_\infty \le C(N+1)$ and $\|\psi^*_\xi\|_\infty \le C'(N+1)^2$ and (9.21) follows. □

Lemma 9.4. *There is $c_1 > 0$ such that $\|k^\pm_\xi\|_\infty \le c_1$ and moreover*

$$\big|\pi_\xi(a_\xi) - \sqrt{\mu} K\{e^{-2\alpha\xi} - e^{-2\alpha(2L-\xi)}\}\big| \le c_1[e^{-5\alpha/2(2L-\xi)} + e^{-5\alpha/2\xi}] \tag{9.22}$$

where

$$K = \frac{a}{\beta(1-m^2_\beta)}\int_{\mathbb{R}} dx\, \frac{e^{\alpha x}\bar m'(x)(m^2_\beta - \bar m^2(x))}{1-\bar m^2(x)} > 0 \tag{9.23}$$

Proof. By (3.7),

$$0 \le k_\xi^\pm(y) \le \frac{2m_\beta}{1-m_\beta^2}[1+e^{-(\alpha_0-\alpha)|y-\xi|}], \qquad y \in [0,2L]$$

Then

$$\pi_\xi(k_\xi^\pm) = \int_{|x-\xi|\le \xi/2} \frac{v_\xi(x)}{p_\xi(x)} k_\xi^\pm(x) + O(e^{-\alpha\xi/2})$$

From (9.15) we then get that for a suitable constant $c > 0$,

$$\Big|\pi_\xi(k_\xi^\pm) - \int_{|y|\le\xi/2} \frac{\sqrt{\mu}\bar m'(y)}{p(y)} k_0^\pm(y)\Big| \le c e^{-2\alpha\xi} e^{\alpha\xi/2}\xi^4$$

On the other hand, using (3.7) and recalling the definition (9.23) of K we have

$$\sqrt{\mu}\,K = \sqrt{\mu}\int dx\,\frac{\bar m'(x)}{p_{\bar m}(x)} k^\pm(x) = \int_{|y|\le\xi/2} \frac{\sqrt{\mu}\bar m'(y)}{p(y)} k_0^\pm(y) + O\big(e^{-\alpha\xi/2}\big)$$

so that

$$\big|\pi_\xi(k_\xi^\pm) - \sqrt{\mu}\,K\big| \le C e^{-\alpha\xi/2}$$

Hence (9.22) follows. □

By using Lemma 9.4 on the right hand side of (9.19), we then get from (9.18)

$$g(m_\xi^0 + \psi_\xi + \psi_\xi^\star) = (1+\lambda_\xi)^{N+1}\sqrt{\mu}K\{e^{-2\alpha\xi} - e^{-2\alpha(2L-\xi)}\}v_\xi + D_\xi v_\xi + E_\xi + 0_3 + 0_{\text{exp}} \quad (9.24)$$

where

$$D_\xi = (1+\lambda_\xi)^{N+1}[\pi_\xi(a_\xi) - \sqrt{\mu}K\{e^{-2\alpha\xi} - e^{-2\alpha(2L-\xi)}\}] + (1+\lambda_\xi)^{N+1}\pi_\xi\Big(p_\xi J * B_\xi$$
$$+\rho_\xi[J*(A_\xi+B_\xi)]^2 + 2\{J*\psi_\xi\}\{J*(A_\xi+B_\xi)\} + \rho_\xi\big[J*\psi_\xi\big]^2\Big) \quad (9.25)$$

The analysis proceeds by showing that $|D_\xi|$ is negligible w.r.t. the coefficient multiplying v_ξ in the first term on the r.h.s. of (9.24). We will then show that $|E_\xi| \le \epsilon e^{-2\alpha\xi-\alpha|\xi-x|}$ with $\epsilon \to 0$ as $L \to \infty$, while $\|0_3\|_\infty < ce^{-2\alpha L}$ and 0_{exp} can be also bounded in the same way. The proof of such statements requires a careful bound of each one of the terms which appear in the above expressions and uses essentially the following estimate proved by equations (6.28) and (6.29) of [9]: for any $\zeta \in (0,\alpha)$ there is a constant c so that, denoting by $\mathcal{T}_\xi(x,y)+\delta(x-y)$ the kernel of $\mathcal{T}_\xi$ (the δ function coming from $n=0$),

$$|\mathcal{T}_\xi(x,y)| \le cN^2 e^{-\zeta|x-y|}, \quad x \ne y$$

We will thus bound

$$|\mathcal{T}_\xi m(x)| \le cN^2\{|m(x)| + \int_0^{2L} e^{-\zeta|x-y|}|m(y)|\,dy\} \tag{9.26}$$

To use (9.26) we need bounds on $|m(y)|$, which, in the case of B_ξ, are proved below.

Lemma 9.5. *There are positive constants c_1 and c_2 so that*

$$|p_\xi J * B_\xi(x)| \le c_1\big[(e^{-\alpha_0\xi} + e^{-\alpha(4L-\xi)})\ \mathbf{1}_{x\in[0,1]} + (e^{-\alpha_0(2L-\xi)} + e^{-\alpha(2L+\xi)})\ \mathbf{1}_{x\in[2L-1,2L]}\big] \tag{9.27}$$

$$\Big|\pi_\xi\big(p_\xi J * B_\xi\big)\Big| \le c_2\big[e^{-(\alpha_0+\alpha)\xi} + e^{-(\alpha_0+\alpha)(2L-\xi)} + e^{-\alpha 4L}\big] \tag{9.28}$$

with α_0 as in (3.7).

Proof. By (9.5), $B_\xi(x) = 0$ for $x \in [0, 2L]$ and we just need to specify its values in the two intervals $[-1, 0]$ and $[2L, 2L+1]$. We write $B^0_\xi(x) = B_\xi(x)\ \mathbf{1}_{x\in[-1,0]}$ and $B^+_\xi(x) = B_\xi(x)\ \mathbf{1}_{x\in[2L,2L+1]}$. Explicitly

$$\begin{aligned} B^0_\xi(x) &= \Big[\{\bar m(-x-\xi) - ae^{-\alpha(4L+x-\xi)} + ae^{-\alpha(-x+\xi)}\} \\ &\qquad -\{\bar m(x-\xi) - ae^{-\alpha(4L-x-\xi)} + ae^{-\alpha(x+\xi)}\}\Big]\ \mathbf{1}_{x\in[-1,0]} \\ &= \Big(ae^{-\alpha(4L-\xi)}[e^{\alpha x} - e^{-\alpha x}] + \{\bar m(-x-\xi) - [-m_\beta + ae^{-\alpha(x+\xi)}]\} \\ &\qquad -\{\bar m(x-\xi) - [-m_\beta + ae^{-\alpha(\xi-x)}]\}\Big)\ \mathbf{1}_{x\in[-1,0]} \end{aligned}$$

Since J has support equal to one, by using (3.7), we get

$$|p_\xi J * B^0_\xi(x)| \le c\big(e^{-\alpha(4L-\xi)} + e^{-\alpha_0\xi}\big)\ \mathbf{1}_{x\in[0,1]}$$

Analogously we write

$$\begin{aligned} B^+_\xi(x) &= \Big[\{\bar m(4L-x-\xi) - ae^{-\alpha(x-\xi)} + ae^{-\alpha(4L-x+\xi)}\} \\ &\qquad -\{\bar m(x-\xi) - ae^{-\alpha(4L-x-\xi)} + ae^{-\alpha(x+\xi)}\}\Big]\ \mathbf{1}_{x\in[2L,2L+1]} \\ &= \Big(ae^{-\alpha(2L+\xi)}[e^{\alpha(2L-x)} - e^{-\alpha(2L-x)}] + \{\bar m(4L-x-\xi) - m_\beta + ae^{-\alpha(4L-x-\xi)}]\} \\ &\qquad -\{\bar m(x-\xi) - m_\beta + ae^{-\alpha(x-\xi)}]\}\Big)\ \mathbf{1}_{x\in[2L,2L+1]} \end{aligned}$$

By using again (3.7), we then get

$$|p_\xi J * B^+_\xi(x)| \le c\big(e^{-\alpha_0(2L-\xi)} + e^{-\alpha(2L+\xi)}\big)\ \mathbf{1}_{x\in[2L-1,2L]}$$

hence (9.27) follows. Applying the definition (9.12) of π_ξ, we bound $|\pi_\xi\big(p_\xi J * B_\xi\big)|$ using the l.h.s. of (9.27). We then get (9.28). □

The remaining part of the proof of Theorem 8.3 is just computational, we thus outline the way computations are performed and state the results, a few details are reported in Appendix A. The result holds for L large enough: given r we need to choose ζ (ζ as in (9.26)) sufficiently close to α, in a way which depends on r, and which then determines how large L should be. Recall that $\xi \in [(1-r)L, rL]$.

We need first to estimate $\psi_\xi(x)$, which is given by (9.9). We use (9.26), expressing a_ξ as in (9.8); bound $p_\xi J * B_\xi$ using (9.27); we are then left with integrals of functions which are piecewise pure exponentials. The result is then bounded proportionally to the value of the integrand computed at the endpoints of the intervals where it is a pure exponential. The bound is reported in (A.1). The same strategy is used for $\psi^\star_\xi$, the result is (A.4).

Next step is to prove that for any $\epsilon > 0$, if L is large enough, then $|D_\xi| e^{2\alpha\xi} \le \epsilon$, so that the first term on the r.h.s. of (9.24) dominates the second one, with D_ξ. The expression (9.25) for D_ξ involves the action of the linear functional π_ξ, namely an integral with the function v_ξ. We then use (9.15), (9.16), (9.17) and reduce to integrals which are like those for ψ_ξ and ψ^*_ξ.

By direct inspection, given any $\epsilon > 0$ and L large enough, $e^{2\alpha\xi} e^{\alpha|\xi - x|} |E_\xi(x)| \le \epsilon$. Also $e^{-2\alpha(2L-\xi)} \le e^{-2\alpha\xi}\epsilon$, because $2L - \xi \ge Lr$, it then follows from (9.24) that

$$g(m^0_\xi + \psi_\xi + \psi^\star_\xi) \ge \sqrt{\mu} K e^{-2\alpha\xi}(1-\epsilon) v_\xi$$

We then have

$$\begin{aligned} z_t(x,t) - g(z(x,t)) \le -[\bar m'_\xi(x) + a\alpha e^{-\alpha(x+\xi)} - \frac{\partial \psi_\xi(x)}{\partial \xi} - \frac{\partial \psi^\star_\xi(x)}{\partial \xi}]\, \xi_t \\ -\sqrt{\mu} K e^{-2\alpha\xi}(1-\epsilon) v_\xi(x) \end{aligned}$$

Since the square bracket which multiplies ξ_t is bounded, the whole expression is negative with ξ_t negative but sufficiently small, hence z_ξ is a sub-solution. A further analysis shows that for any $\epsilon > 0$ if L is large enough, then

$$e^{\alpha|\xi - x|} \Big|\frac{\partial \psi_\xi(x)}{\partial \xi}\Big| \le \epsilon; \quad e^{\alpha|\xi - x|} \Big|\frac{\partial \psi^\star_\xi(x)}{\partial \xi}\Big| \le \epsilon$$

which proves (8.15) since we can take ξ_t arbitrarily close to $-\sqrt{\mu} K e^{-2\alpha\xi}$. The proof of (8.16) is given in Appendix A, see (A.6) and (A.7). The proof of Theorem 8.3 is completed.

10 Proof of Theorem 2.8

By Definition 2.5, for any $m \in U_{-,+}$ there exists ξ such that $L-|\xi| \geq \ell_+/2$ and $\sup_{|x|\leq L} |m^{(\ell_-)} - \bar{m}_\xi^{(\ell_-)}| \leq 2\zeta$.

Lemma 10.1. *There is $c > 0$ so that for all L large enough, the following holds. For any $m \in U_{-,+}$ and any $t \geq 0$,*

$$\sup_{|x|\leq L} |S_t(m) - \bar{m}_\xi| \leq e^{\beta t}(2\zeta + c\ell_-) + 2e^{-t} \tag{10.1}$$

Proof. Let $u(\cdot,t) = S_t(m) - \bar{m}_\xi$, then, with $u^{(\ell_-)}$ as in (2.3),

$$\frac{d}{dt}u^{(\ell_-)} = -u^{(\ell_-)} + \tanh\{\beta J^{(\ell_-)} * m^{(\ell_-)}\} - \tanh\{\beta J^{(\ell_-)} * \bar{m}_\xi^{(\ell_-)}\} + R \tag{10.2}$$

where, calling I_x the interval in $\mathcal{D}^{(\ell_-)}$ which contains x,

$$J^{(\ell_-)}(x,y) = \fint_{I_x} \fint_{I_y} J^{\text{neum}}(x',y')\, dx'dy'$$

while the remainder R such that $\sup_{|x|\leq L} |R| \leq c'(\ell_- + e^{-\alpha\ell_+/2})$, for a suitable constant c'. The term $e^{-\alpha\ell_+/2}$ takes into account the fact that $\bar{m}_\xi \neq \tanh\{J^{\text{neum}} * \bar{m}_\xi\}$ when $|x| \geq L-1$, but the error is exponentially small by (3.9).

From (10.2) we then get

$$\sup_{|x|\leq L} |u^{(\ell_-)}(x,t)| \leq e^{(\beta-1)t}2\zeta + \frac{\|R\|_\infty}{\beta-1}e^{(\beta-1)t}$$

(10.1) is then a consequence of the inequality $\|\bar{m}'\|_\infty \leq \beta\|J'\|_\infty$ and

$$|\frac{d}{dx}\{m(x,t) - e^{-t}m(x,0)\}| \leq \beta\|J'\|_\infty$$

which follows straightly from the integral version of (2.2). Then $|u(x,t)| \leq |u^{(\ell_-)}(x,t)| + 2e^{-t} + c''\ell_-$ and Lemma 10.1 is proved.

□

Proof of Theorem 2.8. Given $\epsilon > 0$ let t_0 be such that $2e^{-t_0} = \epsilon/2$ and ζ so small that $e^{\beta t_0}(2\zeta + c\ell_-) = \epsilon/2$. Then by (10.1)

$$\sup_{|x|\leq L} |S_{t_0}(m) - \bar{m}_\xi| \leq \epsilon \qquad \text{for some } \xi \equiv \xi(m) \in (-L, L)$$

We fix $r \in (0,1)$ as in Proposition 8.1 and we consider the set $B_{\epsilon,r}$ defined in (3.16). It then follows that either $S_{t_0}(m) \in \mathcal{M}^+_{\epsilon,r} \cup \mathcal{M}^-_{\epsilon,r}$ or $S_{t_0}(m) \in B_{\epsilon,r}$. In the first case, from Proposition 8.1 it follows that $S_{t_0+s}(m)$ converges to $m^{(+)}$ or to $m^{(-)}$ as $s \to \infty$. In the second case, using [5], we have that either $S_{t_0+s}(m)$ converges to $\hat{m}_L$ as $s \to \infty$, or else, at some time s_1, $S_{t_0+s_1}(m) \in \{\Sigma^-_{2\epsilon,r} \cup \Sigma^+_{2\epsilon,r}\}$, see (3.16) for the definition of the latter. Since $\{\Sigma^-_{2\epsilon,r} \cup \Sigma^+_{2\epsilon,r}\}$ is a subset of $\mathcal{M}^+_{2\epsilon,r} \cup \mathcal{M}^-_{2\epsilon,r}$, from Proposition 8.1 it follows that $S_{t_0+s_1+s}(m)$ converges to $m^{(+)}$ or to $m^{(-)}$ as $s \to \infty$, concluding the proof of Theorem 2.8.

A Some details on the travelling sub-solution

Analysis of $\psi_\xi(x)$

We have

$$|\psi_\xi(x)| \le n_-(x)\mathbf{1}_{x\le\xi} + n_+(x)\mathbf{1}_{x\ge\xi} \tag{A.1}$$

where

$$n_-(x) = cL^4\{e^{-2\alpha\xi} + e^{-\alpha_0\xi-\zeta x}\}$$
$$n_+(x) = cL^4\{e^{-2\alpha(2L-\xi)} + e^{-2\alpha\xi-\zeta|x-\xi|} + (e^{-\alpha_0(2L-\xi)} + e^{-\alpha(2L+\xi)})\, e^{-\zeta(2L-x)}\}$$

Recalling its definition (9.9) we start by examining $\mathcal{T}_\xi a_\xi$, with a_ξ given by (9.8). We have

$$|k_\xi^+(y)| \le c\{\mathbf{1}_{y\ge\xi} + \mathbf{1}_{y<\xi}e^{-2\alpha(\xi-y)}\}, \quad |k_\xi^-(y)| \le c\{e^{-2\alpha(y-\xi)}\mathbf{1}_{y\ge\xi} + \mathbf{1}_{y<\xi}\}$$

We will use (9.26), so that, calling $a_\xi^\pm$ the contributions to a_ξ with $k_\xi^\pm$, and recalling that $\zeta < \alpha$,

$$|\mathcal{T}_\xi a_\xi^+| \le cN^2 e^{-2\alpha(2L-\xi)} \begin{cases} e^{-\zeta|\xi-x|} & x \le \xi \\ 1 & x > \xi \end{cases}$$
$$|\mathcal{T}_\xi a_\xi^-| \le cN^2 e^{-2\alpha\xi} \begin{cases} 1 & x \le \xi \\ e^{-\zeta|x-\xi|} & x > \xi \end{cases}$$

Since $\xi \le L$,

$$|\mathcal{T}_\xi a_\xi| \le cL^4 \begin{cases} e^{-2\alpha\xi} & x \le \xi \\ e^{-2\alpha(2L-\xi)} + e^{-2\alpha\xi}e^{-\zeta|x-\xi|} & x > \xi \end{cases}$$

Recalling (9.27), and since $\alpha_0 < 2\alpha$,

$$|\mathcal{T}_\xi(p_\xi J * B_\xi)| \le cL^4\Big(e^{-\alpha_0\xi}e^{-\zeta x} + (e^{-\alpha_0(2L-\xi)} + e^{-\alpha(2L+\xi)})e^{-\zeta(2L-x)}\Big)$$

and same bounds (even better) holds for $|\mathcal{T}_\xi(\rho_\xi[J * B_\xi]^2)|$.
Recalling the definition (9.3) of A_ξ, we have

$$|\mathcal{T}_\xi\rho_\xi[J * A_\xi]^2| \le cL^4\Big(e^{-2\alpha(2L-\xi)}e^{-\zeta(2L-x)} + e^{-\zeta x}e^{-2\alpha\xi}\Big) \tag{A.2}$$

(A.1) is thus proved. $\square$

Analysis of $\psi^*_\xi(x)$

We have

$$\begin{aligned}
|\mathcal{T}_\xi(\rho_\xi\big[J*\psi_\xi\big]^2)| &\le V_1^-(x)\mathbf{1}_{x\le\xi} + V_1^+(x)\mathbf{1}_{x\ge\xi} = V_1(x)\\
|\mathcal{T}_\xi(\rho_\xi[J*\psi_\xi]\,[J*A_\xi])| &\le V_2^-(x)\mathbf{1}_{x\le\xi} + V_2^+(x)\mathbf{1}_{x\ge\xi} = V_2(x)\\
|\mathcal{T}_\xi(\rho_\xi[J*\psi_\xi]\,[J*B_\xi])| &\le V_3(x)
\end{aligned} \tag{A.3}$$

so that

$$|\psi^*_\xi(x)| \le V_1(x) + V_2(x) + V_3(x) \tag{A.4}$$

where (bounding $e^{-(2L-\xi)(\alpha-\zeta)} \le 1$)

$$\begin{aligned}
V_1^- &= cL^4\Big(e^{-4\alpha\xi} + e^{-2\alpha_0\xi-\zeta x}\Big)\\
V_1^+ &= cL^4\Big(e^{-\zeta(2L-x)}n_+(2L)^2 + n_+(x)^2 + e^{-\zeta(x-\xi)}[n_+(\xi)^2 + n_-(\xi)^2] + e^{-\zeta x}n_-(0)^2\Big)\\
V_2^- &= cL^4\Big(e^{-\zeta x-(\alpha_0+\alpha)\xi} + e^{-\zeta(2L-x)-\alpha(2L-\xi)}\Big)\\
V_2^+ &= cL^4\Big(e^{-\zeta(2L-x)}n_+(2L)A_\xi(2L) + n_+(x)A_\xi(x)\\
&\qquad\qquad + e^{-\zeta|x-\xi|}[n_+(\xi) + n_-(\xi)]A_\xi(\xi) + e^{-\zeta x}n_-(0)A_\xi(0)\Big)\\
V_3 &= cL^4\Big(e^{-\zeta x}n_-(0)(e^{-\alpha_0\xi} + e^{-\alpha(4L-\xi)}) + e^{-\zeta(2L-x)}n_+(2L)(e^{-\alpha_0(2L-\xi)} + e^{-\alpha(2L+\xi)})\Big)
\end{aligned}$$

Bounds on D_ξ

We will prove that for any $\epsilon > 0$ there is L_ϵ so that for any $L \ge L_\epsilon$

$$e^{2\alpha\xi}|D_\xi| \le \epsilon$$

Since $(1+\lambda_\xi)^{N+1} \le 2$, using (9.22), the contribution of the first term on the r.h.s. of (9.25) is bounded by

$$2c_1 e^{2\alpha\xi}[e^{-(5/2)\alpha)(2L-\xi)} + e^{-(5/2)\alpha\xi}]$$

which, for $\xi \ge rL$ is exponentially small in L.

By (9.28), the contribution of the second term on the r.h.s. of (9.25) is bounded by

$$2e^{2\alpha\xi}c_2\big[e^{-(\alpha_0+\alpha)\xi} + e^{-(\alpha_0+\alpha)(2L-\xi)} + e^{-\alpha 4L}\big]$$

which is again exponentially small. Moreover

$$e^{2\alpha\xi}\pi_\xi\Big(\rho_\xi(J*B_\xi)^2\Big) \le ce^{2\alpha\xi}\Big[e^{-2\alpha_0\xi}e^{-\alpha\xi} + \{e^{-2\alpha_0(2L-\xi)} + e^{-2\alpha(2L+\xi)}\}\, e^{-\alpha(2L-\xi)}\Big]$$

Observe that the next bounds can be recovered from the corresponding ones with $\mathcal{T}_\xi(\cdot)$, provided the latter are computed at $x = \xi$ and with ζ replaced by α, the term L^4 being dropped. Thus, from (A.2),

$$\begin{aligned} e^{2\alpha\xi}\pi_\xi\Big(\rho_\xi(J*A_\xi)^2\Big) &\le ce^{2\alpha\xi}\Big[e^{-2\alpha\xi}e^{-\alpha\xi} + e^{-2\alpha(2L-\xi)}e^{-\alpha(2L-\xi)}\Big] \\ &\le e^{-\alpha\xi} + e^{-\alpha(2L-\xi)} \end{aligned}$$

The same strategy applies to the terms

$$e^{2\alpha\xi}\pi_\xi\Big(\rho_\xi(J*\psi_\xi)(J*A_\xi)\Big), \quad e^{2\alpha\xi}|\pi_\xi(\rho_\xi[J*\psi_\xi][J*B_\xi)])|, \quad e^{2\alpha\xi}|\pi_\xi(\rho_\xi\big[J*\psi_\xi\big]^2)|$$

which are estimated in terms of (A.3). They also vanish as $L\to\infty$. In particular, the first one is bounded by $e^{-(\alpha+\alpha')(2L-\xi)+2\alpha\xi}$ which vanishes by the assumption that $\xi > rL$, $r>0$.

Bounds on E_ξ

We will prove that for any $\epsilon > 0$ there is L_ϵ so that for any $L \ge L_\epsilon$

$$\sup_{x\in[0,2L]} e^{2\alpha\xi}e^{\alpha|\xi-x|}|E_\xi(x)| \le \epsilon \tag{A.5}$$

We bound the terms in (9.20), using the bounds on ψ_ξ, ψ^*_ξ, A_ξ and B_ξ already obtained. We distinguish the intervals $x\le\xi$ and $x>\xi$. In each of them the dependence on x is of the form e^{bx}, with b depending on the term under consideration. Thus the max of the function to bound will be achieved at one of the endpoints. The bound at $x=0$ is

$$\begin{aligned} e^{3\alpha\xi}\rho_\xi[J*\psi^\star_\xi]^2\Big|_0 \le cL^8e^{3\alpha\xi}\Big\{&e^{-2\alpha_0\xi} + [e^{-(\alpha_0+\alpha)\xi} + e^{-\zeta 2L-\alpha(2L-\xi)}] \\ &+[n_-(0)(e^{-\alpha_0\xi} + e^{-\alpha(4L-\xi)}) + e^{-\zeta 2L}n_+(2L)(e^{-\alpha_0(2L-\xi)} + e^{-\alpha(2L+\xi)})]\Big\}^2 \end{aligned}$$

$$\begin{aligned} e^{3\alpha\xi}2\rho_\xi[J*\psi^\star_\xi][J*\psi_\xi]\Big|_0 \le cL^8e^{3\alpha\xi}\Big\{&e^{-2\alpha_0\xi} + [e^{-(\alpha_0+\alpha)\xi} + e^{-\zeta 2L-\alpha(2L-\xi)}] \\ &+[n_-(0)(e^{-\alpha_0\xi} + e^{-\alpha(4L-\xi)}) + e^{-\zeta 2L}n_+(2L)(e^{-\alpha_0(2L-\xi)} + e^{-\alpha(2L+\xi)})]\Big\}n_-(0) \end{aligned}$$

$$\begin{aligned} e^{3\alpha\xi}2\rho_\xi[J*\psi^\star_\xi][J*A_\xi]\Big|_0 \le cL^8e^{3\alpha\xi}\Big\{&e^{-2\alpha_0\xi} + [e^{-(\alpha_0+\alpha)\xi} + e^{-\zeta 2L-\alpha(2L-\xi)}] \\ &+[n_-(0)(e^{-\alpha_0\xi} + e^{-\alpha(4L-\xi)}) + e^{-\zeta 2L}n_+(2L)(e^{-\alpha_0(2L-\xi)} \\ &+e^{-\alpha(2L+\xi)})]\Big\}[e^{-\alpha(4L-\xi)} + e^{-\alpha\xi}] \end{aligned}$$

$$e^{3\alpha\xi}2\rho_\xi[J*\psi_\xi^\star][J*B_\xi]\Big|_0 \le cL^8 e^{3\alpha\xi}\Big\{e^{-2\alpha_0\xi}+[e^{-(\alpha_0+\alpha)\xi}+e^{-\zeta 2L-\alpha(2L-\xi)}]$$
$$+[n_-(0)(e^{-\alpha_0\xi}+e^{-\alpha(4L-\xi)})+e^{-\zeta 2L}n_+(2L)(e^{-\alpha_0(2L-\xi)}$$
$$+e^{-\alpha(2L+\xi)})]\Big\}e^{-\alpha_0\xi}$$

All these terms, with $\xi < L$, are bounded by ϵ for L large enough.
When $x=\xi$, we have from (A.5),

$$e^{2\alpha\xi}\rho_\xi[J*\psi_\xi^\star]\Big|_\xi \le cL^8 e^{2\alpha\xi}\Big(e^{-\alpha 4\xi}+e^{-(2\alpha_0+\zeta)\xi}+e^{-(\zeta+\alpha+\alpha_0)\xi}+e^{-(\alpha+\zeta)(2L-\xi)}\Big)$$

which vanishes as $L\to\infty$ and $(1-r)L > \xi > rL$. It then follows that the contribution in (A.5) of $x=\xi$ can be made as small as desired, by taking L large.
When $x=2L$,

$$e^{2\alpha\xi+\alpha(2L-\xi)}\rho_\xi[J*\psi_\xi^\star]\Big|_{2L} \le cL^8 e^{2\alpha\xi+\alpha(2L-\xi)}\{e^{-(\alpha+\alpha_0)(2L-\xi)}\}$$

This term alone does not vanish as $L\to\infty$, but once multiplied by $J*\psi_\xi$, $J*A_\xi$, $J*B_\xi$ the contribution to (A.5) becomes as small as desired, for L large.

Asymptotic behavior of the sub-solution

We will first prove that for any $\epsilon>0$ there is L_ϵ so that for $L>L_\epsilon$ and $\xi\in[rL,(1-r)L]$,

$$\sup_{x\in[0,2L]} e^{\alpha|\xi-x|}|\psi_\xi(x)| \le \epsilon \tag{A.6}$$

For $x\le\xi$, $e^{\alpha|\xi-x|}|\psi_\xi(x)|$ is bounded by cL^4 times

$$e^{\alpha|\xi-x|}n_-(x) \le e^{\alpha(\xi-x)-2\alpha\xi}+e^{\alpha(\xi-x)-\alpha_0\xi-\zeta x} \le e^{-\alpha(\xi+x)}+e^{-(\alpha+\zeta)x-(\alpha_0-\alpha)\xi}$$

which vanishes as $L\to\infty$. For $x\ge\xi$, $|\psi_\xi(x)|\le n_+(x)$ which is the sum of three terms, thus $e^{\alpha|\xi-x|}|\psi_\xi(x)|$ is bounded by cL^4 times the sum of the following three terms.

$$e^{\alpha|x-\xi|}e^{-2\alpha(2L-\xi)} \le e^{-\alpha(2L-\xi)}$$
$$e^{\alpha(x-\xi)}e^{-2\alpha\xi-\zeta(x-\xi)} \le e^{(\alpha-\zeta)(2L-\xi)-2\alpha\xi}$$
$$e^{\alpha(x-\xi)}e^{-\zeta(2L-x)}[e^{-\alpha_0(2L-\xi)}+e^{-\alpha(2L+\xi)}] \le e^{-(\alpha_0-\alpha)(2L-\xi)}+e^{-2\alpha\xi}$$

We will next prove that for any $\epsilon>0$ there is L_ϵ so that for $L>L_\epsilon$ and $\xi\in[rL,(1-r)L]$,

$$\sup_{x\in[0,2L]} e^{\alpha|\xi-x|}|\psi_\xi^*(x)| \le \epsilon \tag{A.7}$$

Recalling (A.4), we have, with $x \leq \xi$,

$$e^{\alpha(\xi-x)}(V_1^- + V_2^-) \leq cL^4\Big(\{e^{-2\alpha\xi} + e^{-(2\alpha_0-\alpha)\xi}\} + \{e^{-\alpha_0\xi} + e^{2\alpha\xi-(\alpha+\zeta)2L}\}\Big)$$
$$e^{\alpha(\xi-x)}V_3(x) \leq cL^4 e^{\alpha\xi}V_3(0)$$

which vanish as $L \to \infty$.
For $x \geq \xi$,

$$e^{\alpha(x-\xi)}[V_1^+(x) + V_2^+(x) + V_3^+(x)] \leq e^{\alpha(2L-\xi)}[V_1^+(2L) + V_2^+(2L) + V_3^+(2L)]$$

which vanishes as $L \to \infty$.

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