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Extinction time for a random walk in a random environment

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We consider a random walk with death in $[-N, N]$ moving in a time dependent environment. The environment is a system of particles which describes a current flux from N to $-N$. Its evolution is influenced by the presence of the random walk and in turn it affects the jump rates of the random walk in a neighborhood of the endpoints, determining also the rate for the random walk to die. We prove an upper bound (uniform in N) for the survival probability up to time t which goes as $c\exp\{-bN^{-2}t\}$, with c and b positive constants.

Keywords: random walk in moving environment; survival probability

1. Introduction

We consider a random walk on the discrete interval $\Lambda_N := [-N, N]$ of $\mathbb{Z}$ which eventually dies by jumping to a final state $\varnothing$ (where it stays thereafter). Let $z \in \Lambda_N \cup \{\varnothing\}$ denote the state of the random walk, of which we say to be *alive* when $z \in \Lambda_N$ and *dead* when $z = \varnothing$. When z is alive and $|z| \le N - 2$, it moves as a simple random walk: after an exponential time of mean 1 it jumps to its right or left neighbor with probability 1/2. When $z \in I$, $I = I_+ \cup I_-$, $I_+ = \{N - 1, N\}$, $I_- = \{-N, -N + 1\}$ then, besides moving, the walk z may also die. The jump and death rates depend on the environment.

The environment is a particle configuration η on $\Lambda_N \setminus \{z\}$, z the state of the random walk (i.e., if $z = \varnothing$ then $\eta \in \{0, 1\}^{\Lambda_N}$, otherwise $\eta \in \{0, 1\}^{\Lambda_N \setminus \{z\}}$). The evolution of the environment is influenced by the motion of the random walk: it consists of jumps of the particles (as second class symmetric exclusion particles with z being first class) plus birth-death events localized in I. The precise formulation is given in the next section. We just mention here that the birth-deaths events are "rare" as their intensity is proportional to $1/N$ and we are interested in the case of large N.

When $z = \varnothing$, the environment evolves as in [3] with $K = 2$ there (K refers to the cardinality of I_+ and I_-). Namely, it is the simple symmetric exclusion process (SSEP, see [10,11]) in Λ_N plus injection of particles into I_+ and removal from I_-, from now on referred as the DPTV process:

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at rate $j/(2N)$, one tries to inject a particle at the rightmost empty site in I_+ and at the same rate there is an attempt to remove the leftmost particle in I_-, the corresponding action being aborted if I_+ is full or I_- is empty. When $z \in \Lambda_N$, the evolution of (z, η) corresponds to a coupling of two realizations of the DPTV process that differ at z and undergo the same "attempts" to create or to remove particles. Here, $j > 0$ is a fixed parameter, while we are interested in large N. Thus, when the random walk is dead, the η process describes a flux of particles from right to left and it models how currents can be induced by "current reservoirs", represented here by the injection and removal processes at I_+ and, respectively, I_-. With respect to [3], we now take $K = 2$ for simplicity. It will be clear that the arguments extend to any fixed K, the role of current reservoirs being more closely achieved as K grows.

The presence of the random walk changes the picture and the purpose of this paper is to study how long does such an influence persist: we shall prove that the survival probability of the random walk decreases exponentially in time, being bounded above by $c \exp\{-bN^{-2}t\}$, $c, b > 0$ independent of t and N. In a companion paper [6] we use the techniques and results developed here to bound the extinction time in the case of several random walks. These random walks correspond to the positions of discrepancies between two configurations that evolve according to the DPTV process mentioned before. By stochastic inequalities, the result yields a lower bound of the form bN^{-2} for the spectral gap in this process, which is the motivation for our study here.

2. Model and results

The evolution of (z, η) (random walk plus environment) is a Markov process determined by a generator L which is the sum of the generators defined below, in (2.1)–(2.8). Letting the value $\eta(x) = 1$ ($\eta(x) = 0$) indicate the presence (absence) of a particle at x, we may for convenience always take $\eta \in \{0, 1\}^{\Lambda_N}$ by requesting that $\eta(z) = 0$ whenever $z \neq \varnothing$.

We first suppose $z \neq \varnothing$ and write

$$L^0_{\mathrm{env}} f(z, \eta) = \frac{1}{2} \left\{ \sum_{x=-N}^{z-2} + \sum_{x=z+1}^{N-1} \right\} \left[f\left(z, \eta^{(x,x+1)}\right) - f(z, \eta) \right], \tag{2.1}$$

$$\begin{aligned} L^0_{\mathrm{z}} f(z, \eta) = \frac{1}{2} \big\{ & \mathbf{1}_{z<N} \left[f\left(z+1, \eta^{(z,z+1)}\right) - f(z, \eta) \right] \\ & + \mathbf{1}_{z>-N} \left[f\left(z-1, \eta^{(z-1,z)}\right) - f(z, \eta) \right] \big\}, \end{aligned} \tag{2.2}$$

where $\eta^{(x,x+1)}$ is obtained from η by interchanging the occupation values at x and $x+1$, and $\mathbf{1}_{z \in A}$ refers to the indicator function.

Denoting by $\eta^{(+,x)}$ ($\eta^{(-,x)}$), the configuration which has the value 1 (0, resp.) at x and otherwise coincides with η

$$\begin{aligned} L^+_{\mathrm{env}} f(z, \eta) = \frac{j}{2N} \big\{ & \mathbf{1}_{z<N} \left(1 - \eta(N)\right) \left[f\left(z, \eta^{(+,N)}\right) - f(z, \eta) \right] \\ & + \mathbf{1}_{z<N-1} \left(1 - \eta(N-1)\right) \eta(N) \left[f\left(z, \eta^{(+,N-1)}\right) - f(z, \eta) \right] \big\}, \end{aligned} \tag{2.3}$$

$$L^-_{\mathrm{env}} f(z,\eta) = \frac{j}{2N}\big\{\mathbf{1}_{z>-N}\eta(-N)\big[f\big(z,\eta^{(-,-N)}\big) - f(z,\eta)\big] \\ + \mathbf{1}_{z>-N+1}\eta(-N+1)\big(1-\eta(-N)\big)\big[f\big(z,\eta^{(-,-N+1)}\big) - f(z,\eta)\big]\big\}, \tag{2.4}$$

$$L^+_{\mathrm{death}} f(z,\eta) = \frac{j}{2N}\big\{\mathbf{1}_{z=N}\eta(N-1)\big[f\big(\varnothing,\eta^{(+,N)}\big) - f(z,\eta)\big] \\ + \mathbf{1}_{z=N-1}\eta(N)\big[f\big(\varnothing,\eta^{(+,N-1)}\big) - f(z,\eta)\big]\big\}, \tag{2.5}$$

$$L^-_{\mathrm{death}} f(z,\eta) = \frac{j}{2N}\big\{\mathbf{1}_{z=-N}\big(1-\eta(-N+1)\big)\big[f(\varnothing,\eta) - f(z,\eta)\big] \\ + \mathbf{1}_{z=-N+1}\big(1-\eta(-N)\big)\big[f(\varnothing,\eta) - f(z,\eta)\big]\big\}, \tag{2.6}$$

$$L^+_{\mathrm{z}} f(z,\eta) = \frac{j}{2N}\mathbf{1}_{z=N}\big(1-\eta(N-1)\big)\big[f\big(N-1,\eta^{(+,N)}\big) - f(z,\eta)\big], \tag{2.7}$$

$$L^-_{\mathrm{z}} f(z,\eta) = \frac{j}{2N}\mathbf{1}_{z=-N}\eta(-N+1)\big[f\big(-N+1,\eta^{(-,-N+1)}\big) - f(z,\eta)\big]. \tag{2.8}$$

When $z=\varnothing$, the generator L is the sum of only those in (2.1), (2.3) and (2.4) after replacing the indicator functions by 1 and putting $z=\varnothing$. It is the one considered in [3] in the special case when the sets $I_\pm$ consist of only two sites.

Denote by $(z_t,\eta_t)_{t\ge 0}$ the Markov process with the above generator and by $P_{z,\eta}$ its law starting from (z,η). We now state the main result to be proven in the next sections.

Theorem 2.1. *There exist c and b positive and independent of N so that for any initial datum (z_0,η_0), $z_0\neq\varnothing$ and any $t>0$*

$$P_{z_0,\eta_0}[z_t\neq\varnothing] \le c\mathrm{e}^{-bN^{-2}t}. \tag{2.9}$$

3. The auxiliary process

It will be useful to consider an auxiliary process $(\tilde z_t)_{t\ge 0}$. This will be a time-inhomogeneous Markov process whose jump intensities at time t are obtained by averaging those of the original process over the environment conditioned on the state of the random walk at that time. The explicit expression of the time dependent generator $\mathcal{L}_t$ is given below in (3.6) after introducing some definitions and notation. We fix hereafter arbitrarily the initial condition (z_0,η_0) at time 0, $z_0\neq\varnothing$, and denote by $\tilde P_{z_0}$ and $\tilde E_{z_0}$ the law of the auxiliary process and corresponding expectation. We shall prove that for any bounded measurable function $\phi(z,\eta)=f(z)$:

$$E_{z_0,\eta_0}\big[\phi(z_t,\eta_t)\big] = \tilde E_{z_0}\big[f(\tilde z_t)\big]. \tag{3.1}$$

By taking $f(z)=\mathbf{1}_{z\neq\varnothing}$, (3.1) shows that the distributions of the extinction time for the true and the auxiliary processes are the same. The proof of (3.1) follows from the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{z_0,\eta_0}\big[\phi(z_t,\eta_t)\big] = E_{z_0,\eta_0}\big[\mathcal{L}_t f(z_t)\big], \tag{3.2}$$

which we shall prove next.

We obviously have $L^{\pm}_{\text{env}}\phi = 0$ and, for $z \neq \varnothing$, $L^0_{\text{z}}\phi = \mathcal{L}^0 f$ with $\mathcal{L}^0$ the generator of the simple random walk on $[-N, N]$ with jumps outside $[-N, N]$ suppressed (as in the definition of L^0_{z}). Recalling (2.5)–(2.6)

$$L^+_{\text{death}}\phi = \frac{j}{2N}\big\{\mathbf{1}_{z=N}\eta(N-1)\big[f(\varnothing) - f(N)\big] + \mathbf{1}_{z=N-1}\eta(N)\big[f(\varnothing) - f(N-1)\big]\big\},$$

$$\begin{aligned} L^-_{\text{death}}\phi = \frac{j}{2N}\big\{&\mathbf{1}_{z=-N}\big(1-\eta(-N+1)\big)\big[f(\varnothing) - f(-N)\big] \\ &+ \mathbf{1}_{z=-N+1}\big(1-\eta(-N)\big)\big[f(\varnothing) - f(-N+1)\big]\big\}. \end{aligned}$$

By (2.7) and (2.8),

$$L^+_{\text{z}}\phi = \frac{j}{2N}\mathbf{1}_{z=N}\big(1-\eta(N-1)\big)\big[f(N-1) - f(N)\big],$$
$$L^-_{\text{z}}\phi = \frac{j}{2N}\mathbf{1}_{z=-N}\eta(-N+1)\big[f(-N+1) - f(-N)\big].$$

Thus, we define

$$\begin{aligned} d(N,t) &= \frac{j}{2N}E_{z_0,\eta_0}\big[\eta_t(N-1)|z_t = N\big], \\ d(N-1,t) &= \frac{j}{2N}E_{z_0,\eta_0}\big[\eta_t(N)|z_t = N-1\big], \\ d(-N,t) &= \frac{j}{2N}E_{z_0,\eta_0}\big[\big(1-\eta_t(-N+1)\big)|z_t = -N\big], \\ d(-N+1,t) &= \frac{j}{2N}E_{z_0,\eta_0}\big[\big(1-\eta_t(-N)\big)|z_t = -N+1\big] \end{aligned} \tag{3.3}$$

set $d(z,t) = 0$ if $|z| < N-1$, and let

$$\begin{aligned} a(N,t) &= \frac{j}{2N}E_{z_0,\eta_0}\big[\big(1-\eta_t(N-1)\big)|z_t = N\big], \\ a(-N,t) &= \frac{j}{2N}E_{z_0,\eta_0}\big[\eta_t(-N+1)|z_t = -N\big]. \end{aligned} \tag{3.4}$$

Given $t \geq 0$, define

$$\begin{aligned} \mathcal{L}^a_t f(z) = \mathbf{1}_{z\neq\varnothing}\mathcal{L}^0 f(z) &+ \mathbf{1}_{z=N}a(N,t)\big[f(N-1) - f(N)\big] \\ &+ \mathbf{1}_{z=-N}a(-N,t)\big[f(-N+1) - f(-N)\big] \end{aligned} \tag{3.5}$$

and

$$\mathcal{L}_t f(z) = \mathcal{L}^a_t f(z) + d(z,t)\big[f(\varnothing) - f(z)\big], \tag{3.6}$$

so that we get (3.2), and hence (3.1) at once.

The auxiliary process $\tilde{z}_t$ is thus the Markov process with time dependent generator $\mathcal{L}_t$. It is a simple random walk with extra jumps $N \to N-1$ and $-N \to -N+1$ which occur with intensities $a(\pm N, t)$ and death rates $(z \to \varnothing)$ given by $d(z,t)$. Calling $\mathcal{P}_{z_0}$ the law of the process $\tilde{z}_t$ with time dependent generator $\mathcal{L}_t^a$ (same fixed η_0 and the same initial condition z_0 at time 0) and denoting by $\mathcal{E}_{z_0}$ the corresponding expectation, one sees that (see [1], Chapter III),

$$\begin{aligned} P_{z_0,\eta_0}[z_t \neq \varnothing] = \tilde{P}_{z_0}[\tilde{z}_t \neq \varnothing] &= \mathcal{E}_{z_0}\left[\exp\left\{-\int_0^t d(\tilde{z}_s, s)\,\mathrm{d}s\right\}\right] \\ &\le \mathcal{E}_{z_0}\left[\exp\left\{-\int_0^t d(N,s)\mathbf{1}_{\tilde{z}_s=N}\,\mathrm{d}s\right\}\right], \end{aligned} \tag{3.7}$$

where the last inequality is not really necessary, brings some loss, but is just to simplify.

The proof of Theorem 2.1 follows from (3.7) and the following two statements which will be proved in the next sections.

- There are $\delta^* > 0$ and $\kappa > 0$ so that for all $t \ge T_2 = \kappa N^2$:

$$d(N,t) \ge \frac{j\delta^*}{N}. \tag{3.8}$$

- There exists a positive constant b so that calling $T^*(t)$ the total time spent at N by $\tilde{z}_s, 0 \le t$:

$$\mathcal{E}_{z_0}\big[\mathrm{e}^{-j\delta^* N^{-1} T^*(t)}\big] \le \mathrm{e}^{-bN^{-2}t}, \qquad t \ge T_2 = \kappa N^2. \tag{3.9}$$

4. Proof of (3.9)

Throughout the rest of the paper we shall write $\varepsilon \equiv N^{-1}$. With the notation introduced above and writing $\mathcal{E}_{t,\tilde{z}}$ for the conditional distribution (under $\mathcal{P}_{z_0}$) of $(\tilde{z}_s, s \ge t)$ given $\tilde{z}_t = \tilde{z}$, we prove: Given any $\delta > 0$ there is $p < 1$ so that uniformly in ε and for all non negative integers n:

$$\mathcal{E}_{t_n,\tilde{z}_{t_n}}\big[\mathrm{e}^{-X}\big] \le p, \qquad X := \varepsilon\delta \int_{t_n}^{t_{n+1}} \mathbf{1}_{\tilde{z}_s=N}\,\mathrm{d}s, \qquad t_n = 2\varepsilon^{-2}n. \tag{4.1}$$

We see that (3.9) follows at once from (4.1): taking $\delta = j\delta^*$ in the latter and using the Markov property the left-hand side of (3.9) is bounded from above by $p^{[t/(2N^2)]}$, compatible with its right-hand side. Now, the key point in proving (4.1) is the following.

Lemma 4.1. *For any $0 < c_- < c$, there is $p < 1$ (as given in (4.4) below) so that the following holds. Let (Ω, μ) be a probability space, E the expectation and $\mathcal{F}$ the set of all measurable functions $f \ge 0$ such that $E[f] \ge c_-$ and $E[f^2] \le c^2$. Then $E[\mathrm{e}^{-f}] \le p$ for any $f \in \mathcal{F}$.*

Proof. Let $f \in \mathcal{F}$, $\zeta := c_-/2$, $\gamma := \mu[f > \zeta]$. Then

$$c_- \le E[f] = E[f; f \le \zeta] + E[f; f > \zeta] \le \zeta(1-\gamma) + c\gamma^{1/2}. \tag{4.2}$$

Call $a = \gamma^{1/2}$, then (4.2) yields $\zeta(1-a^2) + ca - c_- \geq 0$, so that $a_- < a < a_+$ where $a_\pm$ are the roots of the corresponding equation with equality:

$$\zeta a^2 - ca + c_- - \zeta = 0, \qquad \text{that is, } 2\zeta a = c \pm \sqrt{c^2 - 4\zeta(c_- - \zeta)} = c \pm \sqrt{c^2 - c_-^2}.$$

Thus,

$$2\zeta a_- = c - c\sqrt{1 - \frac{c_-^2}{c^2}} \geq c - c\left(1 - \frac{1}{2}\frac{c_-^2}{c^2}\right) = \frac{c_-^2}{2c}$$

so that (since $\mu[f > \zeta] = a^2$ and $a \geq a_-$)

$$\mu[f > \zeta] \geq \left(\frac{c_-}{2c}\right)^2 \tag{4.3}$$

and

$$\begin{aligned} E\left[\mathrm{e}^{-f}\right] &\leq \mathrm{e}^{-\zeta}\mu[f > \zeta] + 1 - \mu[f > \zeta] \\ &= 1 - \mu[f > \zeta]\left(1 - \mathrm{e}^{-\zeta}\right) \leq 1 - \left(\frac{c_-}{2c}\right)^2\left(1 - \mathrm{e}^{-c_-/2}\right) =: p. \end{aligned} \tag{4.4}$$

□

To apply the lemma, we need to prove the existence of constants $0 < c_- < c$ so that for any ε, any n and $\tilde{z}_{t_n}$,

$$c_- \leq \mathcal{E}_{t_n,\tilde{z}_{t_n}}[X], \qquad \mathcal{E}_{t_n,\tilde{z}_{t_n}}\left[X^2\right] \leq c^2. \tag{4.5}$$

Proof that $\mathcal{E}_{t_n,\tilde{z}_{t_n}}[X] \geq c_-$. We claim that under $\mathcal{P}_{t_n,\tilde{z}_{t_n}}$ the time spent at N by the process $(\tilde{z}_t)$ during the time interval $[t_n, t_{n+1}]$ is stochastically larger than the time spent at N during the interval $[0, 2N^2]$ by a simple random walk (x_t) in $\mathbb{Z}$ that starts at time 0 from $\tilde{z}_{t_n}$. Since $a(N,t) < 1/2$, the intensity with which the process $(\tilde{z}_t)$ jumps from N to $N-1$ is smaller than one, which is the jump rate of (x_t). It is then easy to construct a coupling of both processes for which $|x_{t-t_n} - N| \geq |\tilde{z}_t - N|$ for all t. This is done by constructing a suitable time inhomogeneous Markov process for the pair evolution. Here are the details of the coupling, setting the jump rates at time t when $(x_{t-t_n} = x, \tilde{z}_t = \tilde{z})$ with the property that $|\tilde{z} - N| \leq |x - N|$:

- Let $x = \tilde{z} = N$, the pair $(x, \tilde{z})$ moves to $(N-1, N-1)$ with intensity $1/2$; it moves to $(N+1, N)$ with intensity $1/2 - a(N,t)$, and with intensity $a(N,t)$ it moves to $(N+1, N-1)$.
- Let $\tilde{z} = N$ and $x \neq N$. From (x, N), the pair moves to $(x, N-1)$ with intensity $1/2 + a(N,t)$; with intensity $1/2$ it moves to $(x-1, N)$ and with intensity $1/2$ it moves to $(x+1, N)$.
- Let $\tilde{z} \neq \pm N$ and $x : |\tilde{z} - N| \leq |x - N|$. With intensity $1/2$ both coordinates move by 1 away from N, and with intensity $1/2$ both move by 1 toward N.
- Let $\tilde{z} = -N$ and $x : |\tilde{z} - N| \leq |x - N|$. With intensity $1/2$ x moves by 1 toward N and $\tilde{z}$ moves to $-N+1$; with intensity $a(-N,t)$, x moves by 1 away from N and z moves to $-N+1$; with intensity $1/2 - a(-N,t)$, x moves by 1 away from N and $\tilde{z}$ stays put.

Observe that this gives a coupling of the processes and that all the jumps preserve the inequality $|\tilde{z} - N| \le |x - N|$.

Proof that $\mathcal{E}_{t_n,\tilde{z}_{t_n}}[X^2] \le c^2$. Since $\mathcal{E}_{t_n,\tilde{z}_{t_n}}[X^2] \le \mathcal{E}_{t_n,N}[X^2]$, we just need to prove the inequality when $\tilde{z}_{t_n} = N$. A simple construction, similar to the previous one, allows to couple $(\tilde{z}_t)$ and (x_t) a simple random walk that moves in $[0, N]$, that is, the jumps to -1 and $N + 1$ are suppressed, starting at N at time 0, in such a way that $\tilde{z}_t \le x_{t-t_n}$ for all $t \in [t_n, t_{n+1}]$. The details are quite simple and, therefore, omitted. As a consequence, the time spent at N by $(\tilde{z}_t)$ during $[t_n, t_{n+1}]$ is stochastically smaller than that spent at N during $[0, 2N^2]$ by this simple random walk (x_t).

The process (x_t) can be realized on the unit rate symmetric simple random walk (y_t) on $\mathbb{Z}$ (jumps ± 1 with rate $1/2$ each) by identifying sites on $\mathbb{Z}$ modulo repeated reflections around $N + 1/2$ and $-1/2$, that is, reflections that identify $N + 1$ with N, and -1 with 0 (see, e.g., [3], Proposition 4.1). Thus, calling N_i the images of N under the above reflections, we have to bound

$$2\int_0^{t_1} \mathrm{d}s \int_s^{t_1} \mathrm{d}s' \sum_{i,k} E_N[\mathbf{1}_{y_s = N_i} \mathbf{1}_{y_{s'} = N_k}]. \tag{4.6}$$

By the local central limit theorem as in [9] (see also Theorem 3 in [3]), this can be bounded in terms of Gaussian integrals, from which (4.5) is proved. Details are omitted.

5. Proof of (3.8)

We continue to write $\varepsilon := N^{-1}$, and set the following notation:

$$\pi(x, t) := P_{z_0,\eta_0}[z_t = x] = \tilde{P}_{z_0}[\tilde{z}_t = x], \qquad B(x, t) := (j\varepsilon)^{-1} d(x, t)\pi(x, t),$$

so that (3.8) is implied by

$$B(N, t) \ge \delta^* \pi(N, t), \qquad t \ge T_2 = \kappa\varepsilon^{-2}. \tag{5.1}$$

Having defined

$$T_1 = \varepsilon^{-(1-a)}, \qquad T_0 = T_1 - \varepsilon^{-(1-a)/2}, \qquad T_2 = \kappa\varepsilon^{-2}, \qquad a > 0 \text{ small enough} \tag{5.2}$$

and

$$p_t(x, y) = \text{transition probability of the simple random walk on } \Lambda_N \tag{5.3}$$

(the jumps to $\pm(N + 1)$ being suppressed), we postpone the proof of the following three bounds, for $t \ge T_2$:

- There are $b_1 > 0$ and, for any n, c_n so that

$$B(N, t) \ge b_1 \sum_z p_{T_1}(N, z)\pi(z, t - T_1) - c_n\varepsilon^n \tilde{P}_{z_0}[\tilde{z}_{t-T_2} \ne \varnothing]. \tag{5.4}$$

- There are $b_2 > 0$, and for any n, c_n so that

$$\pi(N,t) \le b_2 \sum_z p_{T_1}(N,z)\pi(z,t-T_1) + c_n\varepsilon^n \tilde{P}_{z_0}[\tilde{z}_{t-T_2} \neq \varnothing]. \tag{5.5}$$

- There is $b_3 > 0$ so that

$$\pi(N,t) \ge b_3\varepsilon^3 \tilde{P}_{z_0}[\tilde{z}_{t-T_2} \neq \varnothing]. \tag{5.6}$$

Claim. (5.1) *follows from* (5.4), (5.5), (5.6).

Proof. By (5.6), we get from (5.5)

$$\left[1 - \frac{c_n}{b_3}\varepsilon^{n-3}\right]\pi(N,t) \le b_2 \sum_z p_{T_1}(N,z)\pi(z,t-T_1), \tag{5.7}$$

and from (5.4)

$$B(N,t) \ge b_1 \sum_z p_{T_1}(N,z)\pi(z,t-T_1) - \frac{c_n}{b_3}\varepsilon^{n-3}\pi(N,t). \tag{5.8}$$

Using (5.7) and (5.8), we have

$$B(N,t) \ge \frac{b_1}{b_2}\left[1 - \frac{c_n}{b_3}\varepsilon^{n-3}\right]\pi(N,t) - \frac{c_n}{b_3}\varepsilon^{n-3}\pi(N,t), \tag{5.9}$$

which for a fixed n large enough and all ε small enough proves (5.1). □

Proof of (5.4). We need a lower bound for $B(N,t) = \frac{1}{2}E_{z_0,\eta_0}[\mathbf{1}_{z_t=N}\eta_t(N-1)]$. We condition on $\mathcal{F}_{t-T_1}$ (the canonical filtration) and denote by $E_{\bar{z},\bar{\eta},t-T_1}$ the conditional expectation given $(\bar{z},\bar{\eta})$, $\bar{z} \neq \varnothing$, the configuration at time $t-T_1$. The realizations where $z_{t-T_1} = \varnothing$ evidently do not contribute to $B(N,t)$.

Let $\mathcal{D}$ denote the event where the rate $\varepsilon j/2$ clocks at $\pm N$ (attempts to create or remove a particle) never ring in the time interval $[t-T_1,t]$, and by $P(\mathcal{D})$ its probability. Then

$$\begin{aligned} E_{\bar{z},\bar{\eta},t-T_1}\left[\mathbf{1}_{z_t=N}\eta_t(N-1)\right] &\ge E_{\bar{z},\bar{\eta},t-T_1}\left[\mathbf{1}_{\mathcal{D}}\mathbf{1}_{z_t=N}\eta_t(N-1)\right] \\ &= P[\mathcal{D}]\sum_y q_{T_1}\big(X,(\bar{z},y)\big)\bar{\eta}(y) \\ &= \mathrm{e}^{-\varepsilon^a j}\sum_y q_{T_1}\big(X,(\bar{z},y)\big)\bar{\eta}(y), \end{aligned} \tag{5.10}$$

where $X = (N, N-1)$, $Y = (y_1, y_2)$ and $q_s(X,Y)$ is the probability under the stirring process (SSEP) on Λ_N of going from X to Y in a time s; the first equality follows because the process conditioned on $\mathcal{D}$ has the law of the stirring process and the second because $P[\mathcal{D}] = \mathrm{e}^{-\varepsilon j T_1} = \mathrm{e}^{-\varepsilon^a j}$.

Writing $Y = (\bar{z}, y)$, $Z = (z_1, z_2)$, $Z^0 = (z_1^0, z_2^0)$, $z_i \in \Lambda_N$, $z_i^0 \in \Lambda_N$, $i = 1, 2$:

$$q_{T_1}(X, Y) = \sum_{Z, Z^0} Q_{T_0}\big(X, X; Z, Z^0\big) q_{T_1 - T_0}(Z, Y),$$

where T_0 is defined in (5.2) and Q refers to the law of the coupling between two stirring $(z_1(s), z_2(s))$ and two independent $(z_1^0(s), z_2^0(s))$ particles as defined in [4] (see Definitions 1 and 4 there in the particular case of two particles), with $Q_T(\cdot, \cdot)$ denoting the corresponding transition probabilities in time T. The coupling is such that $z_1(s) = z_1^0(s)$ for all $s \geq 0$, and $z_2(s)$ makes the same jumps as $z_2^0(s)$ unless $|z_1(s) - z_2(s)| = 1$ or one of the involved particles (independent and stirring) is at the boundary of Λ_N and the other is not. In particular, if starting at the same pair, independent and stirring particles move together while $|z_1(\cdot) - z_2(\cdot)| \geq 2$. Moreover, given any $\zeta > 0$, the following estimate is contained in Theorem 4.5 of [4]: for any n there is c_n so that

$$\sum_{(Z, Z^0) \in \mathcal{A}^c} Q_{T_0}\big(X, X; Z, Z^0\big) \leq c_n \varepsilon^n, \tag{5.11}$$

where

$$\mathcal{A} = \big\{\big(Z, Z^0\big) : z_1 = z_1^0; \big|z_2 - z_2^0\big| \leq \varepsilon^{-(1-a)/4 - \zeta}\big\}. \tag{5.12}$$

Remark. For the case of particles moving in $\mathbb{Z}$, this type of estimate has been proven and used since long ago (see Section 6.6 in [2]; also Section 3 in [7] or references therein): its rough content is that a pair of stirring particles can be coupled to a pair of independent random walks in a way that the first components coincide, and at time s the second components differ by at most $s^{1/4+\delta}$, except for a set of probability at most $c_k s^{-k}$, as described above, for any given $\delta > 0$. The restriction to Λ_N brings in extra nuisance, as treated in the proof of Theorem 4.5 of [4].

Let

$$\mathcal{B} = \big\{Z^0 : \big|z_1^0 - z_2^0\big| \geq \varepsilon^{-(1-a)/2 + \zeta}\big\} \tag{5.13}$$

so that

$$q_{T_1}(X, Y) \geq \sum_{(Z, Z^0) \in \mathcal{A}, Z^0 \in \mathcal{B}} Q_{T_0}\big(X, X; Z, Z^0\big) q_{T_1 - T_0}(Z, Y). \tag{5.14}$$

We write (see (5.3))

$$\sum_y q_{T_1 - T_0}\big(Z, (\bar{z}, y)\big)\bar{\eta}(y) = p_{T_1 - T_0}(z_1, \bar{z}) \sum_y p_{T_1 - T_0}(z_2, y)\bar{\eta}(y) + R(Z), \tag{5.15}$$

where

$$R(Z) = \sum_y \big[q_{T_1 - T_0}\big(Z, (\bar{z}, y)\big) - p_{T_1 - T_0}(z_1, \bar{z}) p_{T_1 - T_0}(z_2, y)\big]\bar{\eta}(y). \tag{5.16}$$

But if $(Z, Z^0) \in \mathcal{A}$ and $Z^0 \in \mathcal{B}$, then $Z \in \mathcal{B}' := \{Z : |z_1 - z_2| \ge \frac{1}{2}\varepsilon^{-(1-a)/2+\zeta}\}$ for all small ε. Also observe that if we let

$$\mathcal{C} = \Big\{ \sup_{0\le s\le T_1-T_0} \big|z_i(s) - z_i\big| \le (T_1 - T_0)^{1/2}\varepsilon^{-\zeta}, i = 1, 2\Big\}$$

then whenever $Z \in \mathcal{B}'$ and $Z(\cdot) \in \mathcal{C}$, we have for ε, a, ζ small enough

$$\big|z_1(s) - z_2(s)\big| \ge \tfrac{1}{2}\varepsilon^{-(1-a)/2+\zeta} - 2\varepsilon^{-(1-a)/4-\zeta} \ge 2, \qquad 0 \le s \le T_1 - T_0.$$

Therefore independent and stirring particles starting from Z can be coupled to evolve together while in $\mathcal{C}$, yielding

$$\mathbb{E}_Z[\mathbf{1}_{Z(T_1-T_0)=Y}\mathbf{1}_{\mathcal{C}}] = \mathbb{E}^0_Z[\mathbf{1}_{Z^0(T_1-T_0)=Y}\mathbf{1}_{\mathcal{C}}], \qquad Z \in \mathcal{B}',$$

where $\mathbb{E}_Z$ and $\mathbb{E}^0_Z$ ($\mathbb{P}_Z$ and $\mathbb{P}^0_Z$) denote the expectation (law) relative to the stirring and the independent processes both starting from Z. It follows at once from this and (5.16) that for $Z \in \mathcal{B}'$:

$$\big|R(Z)\big| \le \mathbb{P}_Z\big[\mathcal{C}^c\big] + \mathbb{P}^0_Z\big[\mathcal{C}^c\big] \le 4 \sup_{z\in\Lambda_N} \mathcal{P}^0_z\Big[\sup_{0\le s\le T_1-T_0} \big|z(s) - z\big| > (T_1 - T_0)^{1/2}\varepsilon^{-\zeta}\Big],$$

where at the last inequality we use that under $\mathbb{P}_Z$ or $\mathbb{P}^0_Z$, the components perform simple random walks in Λ_N, whose law is written as $\mathcal{P}^0$. We then easily see that for each n there exists c_n positive constant so that

$$\big|R(Z)\big| \le c_n\varepsilon^n. \tag{5.17}$$

From (5.10), (5.14) and (5.17), we then get[1]

$$\begin{aligned} E_{\bar z,\bar\eta,t-T_1}\big[\mathbf{1}_{z_t=N}\eta_t(N-1)\big] \ge{}& e^{-\varepsilon^a j} \sum_{(Z,Z^0)\in\mathcal{A},Z^0\in\mathcal{B}} Q_{T_0}\big(X, X; Z, Z^0\big) \\ &\times p_{T_1-T_0}(z_1, \bar z)\sum_{y\ne\bar z} p_{T_1-T_0}(z_2, y)\bar\eta(y) - c_n\varepsilon^n. \end{aligned} \tag{5.18}$$

Letting

$$\mathcal{G} = \Big\{(\bar z, \bar\eta) : \bar z \ne \varnothing, \inf_x \sum_{y\ne\bar z} p_{T_1-T_0}(x, y)\bar\eta(y) \ge \delta^*\Big\}, \tag{5.19}$$

we can thus write for $\tilde z \ne \varnothing$,

$$\begin{aligned} &E_{\bar z,\bar\eta,t-T_1}\big[\mathbf{1}_{z_t=N}\eta_t(N-1)\big] \\ &\quad\ge e^{-\varepsilon^a j}\delta^*\mathbf{1}_{\mathcal{G}}(\bar z, \bar\eta) \sum_{(Z,Z^0)\in\mathcal{A},Z^0\in\mathcal{B}} Q_{T_0}\big(X, X; Z, Z^0\big)p_{T_1-T_0}(z_1, \bar z) - c_n\varepsilon^n. \end{aligned} \tag{5.20}$$

[1]Changing the constants c_n.

But

$$\sum_{(Z,Z^0)\in\mathcal{A},Z^0\in\mathcal{B}} Q_{T_0}\bigl(X,X;Z,Z^0\bigr)p_{T_1-T_0}(z_1,\bar z)$$
$$\ge -Q_{T_0}\bigl(X,X;\mathcal{A}^c\bigr)+\sum_{|z_1^0-z_2^0|\ge\varepsilon^{-(1-a)/2+\zeta}} p_{T_0}\bigl(N,z_1^0\bigr)p_{T_0}\bigl(N-1,z_2^0\bigr)p_{T_1-T_0}\bigl(z_1^0,\bar z\bigr),$$

and for any z_1^0 and small ε

$$\sum_{z_2^0:|z_1^0-z_2^0|\ge\varepsilon^{-(1-a)/2+\zeta}} p_{T_0}\bigl(N-1,z_2^0\bigr)\ge\frac{1}{2},$$

so that by (5.11)

$$\sum_{Z,Z^0\in\mathcal{A},Z^0\in\mathcal{B}} Q_{T_0}\bigl(X,X;Z,Z^0\bigr)p_{T_1-T_0}(z_1,\bar z)\ge\frac{1}{2}p_{T_1}(N,\bar z)-c_n\varepsilon^n.$$

Recalling the definition of $B(N,t)$ and taking the expectation in (5.20) we have

$$B(N,t)\ge \mathrm{e}^{-\varepsilon^a j}\frac{\delta^*}{4}\sum_{z\ne\varnothing}p_{T_1}(N,z)\pi(z,t-T_1)$$
$$-\mathrm{e}^{-\varepsilon^a j}\frac{\delta^*}{2}P_{z_0,\eta_0}\bigl[\mathcal{G}^c\cap\{z_{t-T_1}\ne\varnothing\}\bigr]-c_n\varepsilon^n P_{z_0,\eta_0}[z_{t-T_1}\ne\varnothing].$$

In Section 6, we shall prove that

$$P_{z_0,\eta_0}\bigl[\mathcal{G}^c\cap\{z_{t-T_1}\ne\varnothing\}\bigr]\le c_n\varepsilon^n P_{z_0,\eta_0}[z_{t-T_2}\ne\varnothing] \tag{5.21}$$

which will then complete the proof of (5.4). □

Proof of (5.5). (The proof given below uses that the cardinality K of $I_\pm$ is 2, for $K>2$ the proof is similar but more complex.) By conditioning on $\tilde z_{t-T_1}$, we get

$$P_{z_0,\eta_0}[z_t=N]=\tilde P_{z_0}[\tilde z_t=N]=\tilde E_{z_0}\bigl[\mathbf{1}_{\tilde z_{t-T_1}\ne\varnothing}\tilde P_{t-T_1,\tilde z_{t-T_1}}[\tilde z_t=N]\bigr], \tag{5.22}$$

where $\tilde P_{t-T_1,z'}$ is the law of the auxiliary Markov process[2] $\tilde z_s$, $s\ge t-T_1$ which starts at time $t-T_1$ from $z'\ne\varnothing$. Denoting as before by $\mathcal{P}$ and $\mathcal{E}$ the law and expectation of the auxiliary process with generator $\mathcal{L}_t^a$, that is, when the death part of the generator is dropped, we have by (3.7),

$$\tilde P_{t-T_1,z'}[\tilde z_t=N]\le\mathcal{P}_{t-T_1,z'}[\tilde z_t=N]. \tag{5.23}$$

[2]Fixed z_0, η_0 at time 0 as before.

By the integration by parts formula,

$$\mathcal{P}_{t-T_1,z'}[\tilde{z}_t = N] \le \mathcal{P}^0_{T_1}\big(z', N\big) + \int_{t-T_1}^{t} \mathcal{P}^0_{t-s}(N-1, N)\frac{\varepsilon j}{2}\mathcal{P}_{t-T_1,z'}[\tilde{z}_s = N]\,\mathrm{d}s + c_k\varepsilon^k,$$

where $c_k\varepsilon^k$ bounds the contribution of trajectories that visit I_- and reach N within time T_1 and we used that the rates $a(N, s)$ of extra jumps are bounded by $\varepsilon j/2$; see (3.4). The random walk probabilities $\mathcal{P}^0_{T_1}(z', N)$ and $\mathcal{P}^0_{t-s}(N-1, N)$ can be computed with the time reverted, yielding

$$\mathcal{P}_{t-T_1,z'}[\tilde{z}_t = N] \le p_{T_1}\big(N, z'\big) + \int_{t-T_1}^{t} p_{t-s}(N, N-1)\frac{\varepsilon j}{2}\mathcal{P}_{t-T_1,z'}[\tilde{z}_s = N]\,\mathrm{d}s + c_k\varepsilon^k.$$

Iterating (and writing $s_0 = t$)

$$\begin{aligned}
&\mathcal{P}_{t-T_1,z'}[z_{T_1} = N] \\
&\quad\le \sum_{n=0}^{\infty}\left(\frac{\varepsilon j}{2}\right)^n \int_{t-T_1}^{t}\mathrm{d}s_1\int_{t-T_1}^{s_1}\mathrm{d}s_2\cdots\int_{t-T_1}^{s_{n-1}}\mathrm{d}s_n \\
&\qquad p_{t-s_1}(N, N-1)p_{s_1-s_2}(N, N-1)\cdots\big(p_{s_n-(t-T_1)}\big(N, z'\big) + c_k\varepsilon^k\big).
\end{aligned} \tag{5.24}$$

We write the nth term of the series as $R_n + R'_n$ where R_n is the term with $s_n \le t-1$ and R'_n the one with $s_n > t-1$. We start by bounding R'_n. After a change of variables ($s_i \to t - s_i$), calling $\underline{s} = (s_1, \dots, s_n)$ and $s_0 \equiv 0$,

$$\begin{aligned}
R'_n &:= \left(\frac{\varepsilon j}{2}\right)^n \int_{[0,T_1]^n, s_n<1}\left\{\prod_{i=1}^{n}\mathbf{1}_{s_i\ge s_{i-1}}p_{s_i-s_{i-1}}(N, N-1)\right\}\big(p_{T_1-s_n}\big(N, z'\big) + c_k\varepsilon^k\big)\,\mathrm{d}\underline{s} \\
&\le \left(\frac{\varepsilon j}{2}\right)^n \int_{[0,1]^n}\left\{\prod_{i=1}^{n}\mathbf{1}_{s_i\ge s_{i-1}}\right\}\big(p_{T_1-s_n}\big(N, z'\big) + c_k\varepsilon^k\big)\,\mathrm{d}\underline{s} \\
&\le \frac{1}{n!}\left(\frac{\varepsilon j}{2}\right)^n\big(e p_{T_1}\big(N, z'\big) + c_k\varepsilon^k\big).
\end{aligned} \tag{5.25}$$

To prove the last inequality, we have written

$$p_{T_1-s_n}\big(N, z'\big) = \frac{p_{s_n}(N, N)}{p_{s_n}(N, N)}p_{T_1-s_n}\big(N, z'\big) \le \frac{p_{T_1}(N, z')}{p_{s_n}(N, N)}$$

and used $p_{s_n}(N, N) > \mathrm{e}^{-1}$.

To bound R_n, we do the same change of variables as above and use the inequality

$$p_{s_i-s_{i-1}}(N, N-1) \le \frac{c}{\sqrt{s_i - s_{i-1}}}.$$

Then

$$R_n \le \left(\frac{\varepsilon j}{2}\right)^n \int_{[0,T_1]^n} \mathbf{1}_{s_n\ge 1} f(\underline{s})\big(p_{T_1-s_n}(N,z') + c_k\varepsilon^k\big)\,\mathrm{d}\underline{s},$$

where

$$f(\underline{s}) = \mathbf{1}_{0\equiv s_0\le s_1\le s_2\le\cdots\le s_n\le T_1} \prod_{i=1}^{n} \frac{c}{\sqrt{s_i - s_{i-1}}}.$$

Since $p_{s_n}(N,N) > b/\sqrt{s_n}$ (recall that $s_n \ge 1$) we get

$$\begin{aligned} R_n &\le \left(\frac{\varepsilon j}{2}\right)^n \int_{[0,T_1]^n, s_n\ge 1} f(\underline{s})\left(\frac{p_{s_n}(N,N)}{p_{s_n}(N,N)} p_{T_1-s_n}(N,z') + c_k\varepsilon^k\right)\mathrm{d}\underline{s} \\ &\le \left(\frac{\varepsilon j}{2}\right)^n \big(b^{-1}p_{T_1}(N,z') + c_k\varepsilon^k\big)\int_{[0,T_1]^n, s_n\ge 1} f(\underline{s})\sqrt{s_n}\,\mathrm{d}\underline{s}. \end{aligned}$$

We change variables: $s_i \to T_1 s_i$ and get, using Lemma 5.2 of [3],

$$\begin{aligned} \int_{[0,T_1]^n, s_n\ge 1} f(\underline{s})\sqrt{s_n}\,\mathrm{d}\underline{s} &\le T_1^{(n+1)/2}\int_{[0,1]^n} f(\underline{s})\sqrt{s_n}\,\mathrm{d}\underline{s} \\ &\le T_1^{(n+1)/2}\int_{[0,1]^n} f(\underline{s})\,\mathrm{d}\underline{s} \\ &\le C^n \mathrm{e}^{-(n/2)[\log(n/2)-1]}\varepsilon^{-(1/2)(n+1)+(a/2)(n+1)}. \end{aligned}$$

Thus,

$$R_n \le \left(\frac{Cj}{2}\right)^n \mathrm{e}^{-(n/2)[\log(n/2)-1]}\varepsilon^{(1/2)(n-1)+(a/2)(n+1)}\big(b^{-1}p_{T_1}(N,z') + c_k\varepsilon^k\big). \quad (5.26)$$

Putting together the estimates (5.25) and (5.26), we can bound the sum of $R_n + R_n'$ over n in (5.24) (by convergent series). It follows that positive constants $\tilde{c}$ and c_k can be found so that for all $z' \in \Lambda_N$ and all k

$$\mathcal{P}_{t-T_1,z'}[\tilde{z}_t = N] \le \tilde{c}\,p_{T_1}(N,z') + c_k\varepsilon^k$$

for all ε small. Now combining this into (5.22), and since $\tilde{P}_{z_0}[\tilde{z}_{t-T_1} \ne \varnothing] \le \tilde{P}_{z_0}[\tilde{z}_{t-T_2} \ne \varnothing]$ we have (5.5). □

Proof of (5.6). Let $t \ge T_2 := \kappa\varepsilon^{-2}$, then analogously to (3.7),

$$\pi(N,t) \equiv \tilde{P}_{z_0}[\tilde{z}_t = N] = \tilde{E}_{z_0}\big[\mathbf{1}_{\tilde{z}_{t-T_2}\ne\varnothing}\mathcal{E}_{t-T_2,\tilde{z}_{t-T_2}}\big[\mathrm{e}^{-\int_{t-T_2}^{t} d(z_s,s)\,\mathrm{d}s}\mathbf{1}_{z_t=N}\big]\big] \quad (5.27)$$

with $\mathcal{E}_{t,x}$ as defined in the beginning of Section 4.

We denote by $\mathcal{E}'_N$ the expectation with respect to the time-backward process, z'_s, $s \in [0, T_2]$, which starts at time 0 from N and is a simple random walk with additional jump intensity $a(\pm N, t-s)$ for the jump $\pm(N-1) \to \pm N$ at time s. We then have

$$\begin{aligned}
\pi(N,t) &= \mathcal{E}'_N\left[\pi\left(z'_{T_2}, t-T_2\right)\exp\left\{-\int_0^{T_2} d\left(z'_s, t-s\right)\mathrm{d}s\right\}\right] \\
&\geq \mathrm{e}^{-\varepsilon j}\mathcal{E}'_N\left[\pi\left(z'_{T_2}, t-T_2\right)\exp\left\{-\int_1^{T_2-1} d\left(z'_s, t-s\right)\mathrm{d}s\right\}\right] \\
&\geq \mathrm{e}^{-\varepsilon j}\mathcal{E}'_N\left[\pi\left(z'_{T_2}, t-T_2\right)\mathbf{1}_{z'_1=N-2}\exp\left\{-\int_1^{T_2-1} d\left(z'_s, t-s\right)\mathrm{d}s\right\}\right] \qquad (5.28)\\
&\geq \mathrm{e}^{-\varepsilon j}\alpha\sum_{|x|\leq N-2}\pi(x, t-T_2)\alpha'\mathcal{P}^0_{N-2}\left[x_{T_2-2}=x, \sup_{s\in[0,T_2-2]}|x_s|<N-1\right] \\
&\quad + \mathrm{e}^{-\varepsilon j}\alpha\sum_{\sigma=\pm, x\in I_\sigma}\pi(x, t-T_2)\alpha''\mathcal{P}^0_{N-2}\left[x_{T_2-2}=\sigma(N-2), \sup_{s\in[0,T_2-2]}|x_s|<N-1\right],
\end{aligned}$$

where $\mathcal{P}^0_x$ is the law of the random walk x_s with no extra jumps (just a simple random walk on Λ_N starting at x) and

$$\alpha = \mathcal{P}'_N\left[z'_1 = N-2\right] > 0, \qquad \alpha' = \min_{|x|\leq N-2}\mathcal{P}^0_N[x_{T_2-1}=x|x_{T_2-2}=x] > 0,$$

$$\alpha'' = \min_{\sigma\in\{-1,1\}}\min_{x\in\{N-1,N\}}\mathcal{P}'_N\left[z'_{T_2}=\sigma x|z'_{T_2-1}=\sigma(N-2)\right] > 0.$$

We thus need to bound from below the probability of the event $\{x_{T_2-2}=x, |x_s|\leq N-2, s\in[0, T_2-2]\}$ uniformly in $|x|\leq N-2$. The basic idea is to reduce to a single time estimate; indeed, the condition $|x_s|\leq N-2$, $s\in[0,T_2-2]$ can be dropped provided we study the process on the whole $\mathbb{Z}$ and take as initial condition the antisymmetric datum, which is obtained by assigning a weight ± 1 to the images of x under reflections around $\pm(N-1)$. The details are given in the Appendix. To have control of the plus and minus contributions, it is convenient to reduce to small time intervals; moreover, the analysis will distinguish the case where x is "close" to $\pm N$ and when it is not. Closeness here means that $N-|x|\leq N/100$ (the choice $1/100$ is just for the sake of concreteness, any "small" number would work as well).

Let us now be more specific. We split $T_2 - 2 = m\tau\varepsilon^{-2}$, m an integer and $\tau > 0$ small enough, and write

$$\begin{aligned}
&\mathcal{P}^0_{N-2}\left[x_{m\varepsilon^{-2}\tau}=x; \sup_{s\in[0,T_2-2]}|x_s|<N-1\right] \\
&\quad\geq \mathcal{P}^0_{N-2}\left[\bigcap_{i=1}^{m-1}\left\{\sup_{s\in[i-1,i]\varepsilon^{-2}\tau}|x_s|<N-1; |x_{i\varepsilon^{-2}\tau}|\leq N/100\right\}\right. \\
&\qquad\qquad\left.\cap\left\{\sup_{s\in[m-1,m]\varepsilon^{-2}\tau}|x_s|<N-1; x_{m\varepsilon^{-2}\tau}=x\right\}\right].
\end{aligned}$$

In the Appendix, we shall prove that for τ small enough there is c so that for all ε ($N = \varepsilon^{-1}$), the following bounds hold:

$$\mathcal{P}^0_{N-2}\Big[|x_{\varepsilon^{-2}\tau}| \le N/100; \sup_{s\in[0,\varepsilon^{-2}\tau]} |x_s| < N-1\Big] \ge c\varepsilon, \tag{5.29}$$

$$\inf_{|x|\le N/100} \mathcal{P}^0_x\Big[|x_{\varepsilon^{-2}\tau}| \le N/100; \sup_{s\in[0,\varepsilon^{-2}\tau]} |x_s| < N-1\Big] \ge c, \tag{5.30}$$

$$\inf_{|x|\le N/100} \inf_{|x'|\le N99/100} \mathcal{P}^0_x\Big[x_{\varepsilon^{-2}\tau} = x'; \sup_{s\in[0,\varepsilon^{-2}\tau]} |x_s| < N-1\Big] \ge c\varepsilon, \tag{5.31}$$

$$\inf_{|x|\le N/100} \inf_{N99/100\le|x'|\le N-2} \mathcal{P}^0_x\Big[x_{\varepsilon^{-2}\tau} = x'; \sup_{s\in[0,\varepsilon^{-2}\tau]} |x_s| < N-1\Big] \ge c\varepsilon^2. \tag{5.32}$$

The above bounds together with (5.28) prove (5.6). □

6. Proof of (5.21)

For any (z, η), we define the configurations $\eta^{(1)}$ and $\eta^{(2)}$ in $\{0,1\}^{\Lambda_N}$ as follows: If $z \ne \varnothing$, then $\eta^{(1)}(x) = \eta^{(2)}(x) = \eta(x)$ for any $x \in \Lambda_N \setminus z$, and $\eta^{(1)}(z) = 1$, $\eta^{(2)}(z) = 0$. If $z = \varnothing$, then $\eta^{(1)} = \eta^{(2)} = \eta$.

If $(z_t, \eta_t)_{t\ge0}$ is the process defined in Section 2, we can see that $(\eta^{(2)}_t)_{t\ge0}$ has the law of the process introduced in [3] that we are here calling DPTV for simplicity (as well as $(\eta^{(1)}_t)_{t\ge0}$, though such a property will not be used in the following). Details can be found in [6].

For any $x \in \Lambda_N$, we introduce the function $A_x(\eta)$, $\eta \in \{0,1\}^{\Lambda_N}$, by setting

$$A_x(\eta) := \sum_y p_{T_1-T_0}(x,y)\eta(y), \qquad \eta \in \{0,1\}^{\Lambda_N}. \tag{6.1}$$

Then, recalling that $\mathcal{G}$ has been defined in (5.19) and writing $\tau := t - T_1$, the left-hand side of (5.21) is equal to

$$P_{z_0,\eta_0}\Big[z_\tau \ne \varnothing, \inf_x A_x\big(\eta^{(2)}_\tau\big) \le \delta^*\Big] \le E_{z_0,\eta_0}\Big[\mathbf{1}_{z_{t-T_2}\ne\varnothing} P_{z_{t-T_2},\eta_{t-T_2}}\Big[\inf_x A_x\big(\eta^{(2)}_\tau\big) \le \delta^*\Big]\Big]$$

which is bounded from above by

$$\tilde{P}_{z_0}[\tilde{z}_{t-T_2} \ne \varnothing] \sup_{\eta\in\{0,1\}^{\Lambda_N}} \mathbf{P}_\eta\Big[\inf_x A_x(\eta_{T_2-T_1}) < \delta^*\Big],$$

where $\mathbf{P}_\eta$ is the law of the DPTV process starting from η at time 0. We thus need to prove that

$$\sup_{\eta\in\{0,1\}^{\Lambda_N}} \mathbf{P}_\eta\Big[\inf_x A_x(\eta_{T_2-T_1}) < \delta^*\Big] \le c_n\varepsilon^n.$$

Since the evolution preserves the coordinate-wise order in $\{0,1\}^{\Lambda_N}$ (see [4]) and $\inf_x A_x(\eta)$ is a non-decreasing function of η, it suffices to show that

$$\mathbf{P_0}\Big[\inf_x A_x(\eta_{T_2-T_1}) < \delta^*\Big] \le c_n \varepsilon^n, \tag{6.2}$$

with $\mathbf{0}$ the configuration with $\eta(x)=0$ for all x.

In [4], it is proved that there is $\tau^*>0$ (independent of N) so that if $t \in [N^2, \tau^* N^2 \log N]$ then for any n there is c_n so that

$$\mathbf{P_0}\Big[\inf_x \big|A_x(\eta_t) - A_x\big(\gamma(\cdot,t)\big)\big| \ge \varepsilon^{1/4}\Big] \le c_n \varepsilon^n, \tag{6.3}$$

where $\gamma(y,t)=\rho(\varepsilon y, \varepsilon^2 t)$ and $\rho(r,t)$, $r\in[-1,1]$, $t\ge 0$, is the solution of the hydrodynamic equation for the DPTV system starting from $\rho(r,0)\equiv 0$. In [5], it is proved that

$$\lim_{t\to\infty} \sup_{|r|\le 1} \big|\rho(r,t) - \rho^{\mathrm{st}}(r)\big| = 0 \tag{6.4}$$

and that $\rho^{\mathrm{st}}(r)$ is an increasing function (linear with positive slope) with $\rho^{\mathrm{st}}(-1)>0$. Thus, there is $\kappa>0$ independent of N so that for all N large enough

$$\mathbf{P}_0\left[\inf_x A_x(\eta_s) \ge \frac{\rho^{\mathrm{st}}(-1)}{2}\right] \ge 1 - c_n\varepsilon^n, \qquad \frac{\kappa}{2}N^2 \le s \le \kappa N^2 \tag{6.5}$$

which implies (6.2), provided $\delta^* < \rho^{\mathrm{st}}(-1)/2$ and $T_2 = \kappa N^2$.

Appendix

We now prove the bounds (5.29)–(5.32). The key point is the identity below for the transition probabilities for the simple random walk in an interval, absorbed at the boundaries. (The proof follows the same argument as that given for the Brownian motion case; see, e.g., Proposition 8.10 in Chapter 2 of [8].) Let $L = N-1 \ge 2$,

$$\begin{aligned}\mathcal{P}_x^0&\big[x_t = y;\, |x_s| < L, \forall s \in \big[0, \varepsilon^{-2}\tau\big]\big] \\ &= \sum_{k\in\mathbb{Z}} \big[p_t(4kL + y - x) - p_t(4kL - 2L - y - x)\big],\end{aligned} \tag{A.1}$$

where x and y in (A.1) are in $[-L+1, L-1]$, and $p_t(z)$ is the probability for a simple random walk on $\mathbb{Z}$ starting from 0 to be at z at time t, for $z\in\mathbb{Z}$.

Writing $z = L-y$ and $w = L-x$, rearranging the sum, and using the symmetry of $p_t(\cdot)$, we rewrite (A.1) as

$$\begin{aligned}\mathcal{P}_x^0&\big[x_t = y;\, |x_s| < L, \forall s \in \big[0, \varepsilon^{-2}\tau\big]\big] \\ &= p_t(z-w) - p_t(z+w) + \sum_{k=1}^{\infty}\Big(\big[p_t(4kL - z + w) - p_t(4kL - z - w)\big] \\ &\qquad - \big[p_t(4kL + z + w) - p_t(4kL + z - w)\big]\Big).\end{aligned} \tag{A.2}$$

To prove (5.29) (where $x = N - 2$), we take $w = 1$ in (A.2) and get (recall $L = N - 1$):

$$\begin{aligned} &\mathcal{P}^0_{N-2}\big[x_t = L - z; |x_s| < N - 1, \forall s \in [0,t]\big] \\ &\quad \geq p_t(z-1) - p_t(z+1) \\ &\qquad - \sum_{1\leq k\leq \varepsilon^{-b}} \sum_{\sigma=\pm 1} \big|p_t(4kL + \sigma z - 1) - p_t(4kL + \sigma z + 1)\big| \\ &\qquad - 2 \sum_{|u|\geq N\varepsilon^{-b}/2} p_t(u), \end{aligned} \tag{A.3}$$

$b > 0$ a small constant.

Given b and τ positive constants (independent of ε), since $t = \tau\varepsilon^{-2}$ there is $c > 0$ so that for all ε small enough (and, say, all $\tau \in (0, 1]$)

$$\sum_{|u|\geq N\varepsilon^{-b}/2} p_t(u) \leq \mathrm{e}^{-c\varepsilon^{-2b}}, \tag{A.4}$$

by simple tail estimate for the random walk on $\mathbb{Z}$.

We prove that $p_t(z-1) - p_t(z+1)$ is bounded from below proportionally to ε^2, so that the last sum in (A.3) will be negligible with respect to the first. The other terms on the right-hand side of (A.3) are bounded in the following proposition, and using the smallness of τ we see that their sum over $1 \leq k \leq \varepsilon^{-b}$ is a small fraction of the first term on the right-hand side of (A.3), from which (5.29) will follow.

Proposition A.1. *Recalling that* $N \equiv \varepsilon^{-1}$, $t \equiv \varepsilon^{-2}\tau$, *there are positive constants* c, C *and* b *such that for every* τ, *the following holds for all* ε *small enough*:

- *When* $N/2 < y < 2N$,

$$p_t(y) - p_t(y+2) \geq \frac{\varepsilon^2}{\sqrt{2\pi\tau}} \mathrm{e}^{-(\varepsilon y)^2/2\tau} \frac{1}{4\tau}(1 - c\varepsilon). \tag{A.5}$$

- *When* $N/2 < y < N\varepsilon^{-b}$,

$$p_t(y) - p_t(y+2) \leq \frac{\varepsilon^2}{\sqrt{2\pi\tau}} \mathrm{e}^{-(\varepsilon y)^2/2\tau} \frac{8\varepsilon y}{\tau}(1 + c\varepsilon). \tag{A.6}$$

Proof. We have

$$p_t(y) = \mathrm{e}^{-t} \sum_n{}^{*} \left(\frac{1}{2}\right)^n \frac{t^n}{n!} \binom{n}{m}, \qquad y = 2m - n,$$

where $\sum^*_n$ means that n runs over either the odd or the even integers of $\mathbb{Z}$ according to whether y is odd or, respectively, even. n is the total number of jumps, m the number of jumps to the right so that $m - (n - m) = y$.

We start by proving (A.5). For every pair y and $y' := y+2$, let m and m' be the number of the corresponding jumps to the right, so that $m' = m+1$. Then

$$\binom{n}{m} - \binom{n}{m'} = \binom{n}{m}\left(1 - \frac{n-m}{m+1}\right) = \binom{n}{m}\frac{y+1}{m+1}. \tag{A.7}$$

We bound $m = (n+y)/2 \le t$, which is valid when $n \le 2t - 2N$. Thus,

$$\begin{aligned} p_t(y) - p_t(y+2) &\ge \frac{N}{2(t+1)} e^{-t} \sum_{n \le 2t-2N} \left(\frac{1}{2}\right)^n \frac{t^n}{n!}\binom{n}{m} \\ &\ge \frac{N}{4t} e^{-t}\left[\sum_{n\ge 1}\left(\frac{1}{2}\right)^n \frac{t^n}{n!}\binom{n}{m} - \sum_{n > 2t-2N}\left(\frac{1}{2}\right)^n \frac{t^n}{n!}\binom{n}{m}\right] \end{aligned}$$

and (A.5) then follows from the local limit theorem ([9], page 58, Theorem 2.5.6), after observing that the sum over $n > 2t - 2N$ is exponentially small in t.

To prove (A.6), we proceed similarly. Since we want an upper bound, we write $m + 1 \ge n/2$, getting

$$p_t(y) - p_t(y+2) \le \frac{y+1}{t/4} e^{-t} \sum_{n \ge t/2} \left(\frac{1}{2}\right)^n \frac{t^n}{n!}\binom{n}{m}.$$

As before, (A.6) is again a consequence of the local limit theorem, and the large deviation estimate on the number of jumps for the set $n < t/2$. □

Proof of (5.29). By (A.3) and (A.4), using the above proposition,

$$\begin{aligned} &\mathcal{P}^0_{N-2}\big[x_t = y; |x_s| < L,\ \text{for all } s \in [0,t]\big] \\ &\quad \ge \frac{\varepsilon^2}{\sqrt{2\pi\tau}} e^{-(\varepsilon z)^2/2\tau} \frac{1}{4\tau}(1 - c\varepsilon) \\ &\qquad - 2 \sum_{1 \le k \le \varepsilon^{-b}} \frac{\varepsilon^2}{\sqrt{2\pi\tau}} e^{-([4k(1-\varepsilon) - \varepsilon z - \varepsilon])^2/2\tau} \frac{8\varepsilon(4k+2)}{\tau}(1 + c\varepsilon) \\ &\qquad - 2e^{-c\varepsilon^{-2b}}, \qquad \text{where } z = N - 1 - y. \end{aligned}$$

If $\tau > 0$ is sufficiently small, then for all ε small enough

$$\mathcal{P}^0_{N-2}\big[x_t = y; |x_s| < L,\ \text{for all } s \in [0,t]\big] \ge \frac{\varepsilon^2}{\sqrt{2\pi\tau}} e^{-(\varepsilon z)^2/2\tau} \frac{1}{8\tau}$$

and (5.29) is proved. □

To prove (5.30) and (5.31), we use again (A.1) and bound

$$\mathcal{P}^0_x\Big[x_t = y; \sup_{s \in [0, \varepsilon^{-2}\tau]} |x_s| < L\Big] \ge p_t(y - x) - \sum_{n \in \mathbb{Z}, n \ne 0} p_t(y_n - x) - \sum_{n \in \mathbb{Z}} p_t(y'_n - x), \tag{A.8}$$

where $y_n = y + 4nL$ and $y'_n = -y - 2L + 4nL$, which we may rewrite as

$$\mathcal{P}_x^0\Big[x_t = y; \sup_{s\in[0,\varepsilon^{-2}\tau]} |x_s| < L\Big] \geq p_t(\tilde{y}_0 - x) - \sum_{n\in\mathbb{Z},n\neq 0} p_t(\tilde{y}_n - x), \tag{A.9}$$

where $\tilde{y}_0 = y$ and the points $\tilde{y}_n$ stay at distance at least aN from each other. As before, we then may bound from below the right-hand side by

$$p_t(y-x) - \sum_{1\leq|n|\leq N\varepsilon^{-b}} p_t(\tilde{y}_n - x) - \sum_{|z|\geq N\varepsilon^{-b}} p_t(z)$$

and (5.30) and (5.31) follow using the local limit theorem and large deviations as before.

Proof of (5.32). We use the equality

$$\mathcal{P}_x^0\Big[x_{\varepsilon^{-2}\tau} = x'; \sup_{s\in[0,\varepsilon^{-2}\tau]} |x_s| < L\Big] = \mathcal{P}_{x'}^0\Big[x_{\varepsilon^{-2}\tau} = x; \sup_{s\in[0,\varepsilon^{-2}\tau]} |x_s| < N-1\Big] \tag{A.10}$$

recalling that $|x| \leq N/100$ and $N99/100 \leq |x'| \leq N-2$; we thus need to bound from below the right-hand side of (A.10) by $c\varepsilon^2$ with $c > 0$ independent of x and x' when they vary in the above sets.

We thus use (A.2) with $x \to x'$ and $y \to x$, so that on the right-hand side we must read $z = L - x$ and $w = L - x'$. Observe that $N - 1 - \frac{N}{100} \leq z \leq N - 1 + N/100$ and $w \in [1, \frac{N}{100} - 1] \cup [2N - 1 - \frac{N}{100}, 2N - 3]$. To have the same structure as in (A.3), we write

$$p_t(z-w) - p_t(z+w) = \sum_{i=1}^{w} \big[p_t(z - y_i - 1) - p_t(z - y_i + 1)\big],$$

where $y_i = w + (2i-1)$, $1 \leq i \leq w$ with the analogous decomposition for $p_t(z'-w) - p_t(z'+w)$ with $z' = 4kL \pm z$. Then

$$\begin{aligned}
&\mathcal{P}_{x'}^0\big[x_t = x; |x_s| < L, \text{ for all } s \in [0, \varepsilon^{-2}\tau]\big] \\
&\quad\geq \sum_{i=1}^{w}\Big(\big[p_t(z - y_i - 1) - p_t(z - y_i + 1)\big] \\
&\qquad - \sum_{1\leq k\leq N\varepsilon^{-b}} \sum_{\sigma=\pm 1} \big|p_t\big(4kL - \sigma(z-y) - 1\big) - p_t\big(4kL - \sigma(z-y) + 1\big)\big| \\
&\qquad - 2\sum_{|u|\geq N\varepsilon^{-b}/2} p_t(u)\Big)
\end{aligned} \tag{A.11}$$

and for each y_i we have the same bound as before, hence (5.32). □

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