Stochastic Processes and their Applications 42 (1992) 1-30
North-Holland

Nonequilibrium fluctuations in particle systems modelling reaction-diffusion equations

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Received 14 December 1989
Revised 18 March 1991

Dedicated to Roland L'vovich Dobrushin on his 60th birthday

We consider a class of stochastic evolution models for particles diffusing on a lattice and interacting by creation-annihilation processes. The particle number at each site is unbounded. We prove that in the macroscopic (continuum) limit the particle density satisfies a reaction-diffusion PDE, and that microscopic fluctuations around the average are described by a generalized Ornstein-Uhlenbeck process, for which the covariance kernel is explicitely exhibited.

interacting particle systems * fluctuation fields * reaction-diffusion equations

Introduction

In recent years considerable attention has been devoted to the derivation of reaction-diffusion (r.d.) equations from discrete particle models with stochastic dynamics [1, 2, 3, 7, 8, 11, 12, 17]. Reaction-diffusion equations arise in different fields of science, and may describe very different phenomena, such as chemical reactions, population dynamics, economic processes, etc. The stochastic evolution of point particles with local interaction can be, in general no more than a rough caricature of the 'physical' or 'real' evolution. Nevertheless it appears that the collective behavior that is relevant in the macroscopic continuum limit ('hydrodynamic',

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Research partially supported by CNR-MMAIT and M.P.I. research funds.

0304-4149/92/$05.00

'kinetic', or other) depends only on very general features of the evolution, and the stochastic evolution models give actually a very good account of the physical phenomena.

Perhaps the most natural models leading to r.d. equations are those in which particles perform independent identically distributed random walks on a lattice, and are subject to some local interaction producing particle creation or annihilation. The continuum limit in which the r.d. equation holds is obtained as the lattice constant ε goes to 0, and the random walk is speeded up by a factor ε^{-2}, as in the usual diffusive scaling, while the creation and annihilation rates are kept constant. Models of this type have been studied in recent years by several authors (see [1, 2, 3, 11] and references therein). In the present paper we consider essentially the same model as in the paper [3]. It can be described as follows. On the lattice $\mathbb{Z}$ particles perform independent random walks. The jump times are independently distributed with Poisson distribution of intensity ε^{-2}, where $\varepsilon \to 0$ is the parameter that controls the macroscopic continuum limit. In addition, at each site $x \in \mathbb{Z}$ particles are created and destroyed with rates $q_+(\eta_t(x))$, $q_-(\eta_t(x))$, depending on the occupation numbers $\eta_t(x)$. For simplicity it is assumed that the functions q_+, q_- are nonnegative polynomials in the occupation number, such that the degree of the annihilation rate q_- is larger than that of the creation rate q_+. A condition of this type is needed to ensure that the average occupation number is bounded. (For large particle densities annihilation will dominate over creation.)

In the paper [3] it was proved that as $\varepsilon \to 0$ the process approximates, in a suitable sense, the solution of the r.d. equation

$$\frac{\partial}{\partial t}\rho_t(r) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\rho_t(r) - V'(\rho_t(r))$$

where $V'(u)$ is a function depending on q_+, q_-. In the present paper we give another proof of this result, and in addition we study the space-time behavior of the fluctuation field. The main difference with [3] is that we consider here processes defined on a periodic lattice $\mathbb{Z}_\varepsilon$ with ε^{-2} sites. We prove that the limiting fluctuation field is a generalized Ornstein-Uhlenbeck process, and that the covariance kernel is given by the solution of a linear equation with coefficients depending on the particle density $\rho_t(r)$.

Though the process that we consider here is intuitively simple, one has to face the technical difficulty that there is no a priori bound on the occupation numbers $\eta_t(x)$. This gives serious trouble in dealing with fluctuations, in spite of the fact that the average values are bounded for all times. So the program sketched in [3], to push the methods developed there to obtain the fluctuation theory, is apparently very hard to carry out, if one looks for results valid for any finite time interval.

In the present paper we use a new technical tool, a class of suitable correlation functions called the *v-functions*, which, roughly speaking measure the distance of the distribution from an independent Poisson distribution. The v-functions, together with an iteration scheme, have been used in [10] for a model in which the occupation numbers are bounded. We modify the iteration procedure to deal with unbounded occupation numbers.

The idea consists in proving propagation of chaos in the following way. The evolution is decomposed into steps, each step corresponding to a time interval long enough for the random motion to smooth out the initial data, and at the same time so short that the occupation numbers $\eta_t(x)$ cannot grow very much, with overwhelming probability. On such intermediate time scale we can prove that if the v-functions are initially small, they stay small, i.e. the distribution remains close to an independent Poisson distribution, provided that one throws away a 'bad set' at each step. One then gets control on the whole process by proving that the total probability of the 'bad sets' goes to 0 fast enough as $\varepsilon \to 0$.

A similar use of the v-functions for a related class of models with no a priori bound is made in the paper [5].

For particle evolution models in which the particle number is preserved the nonequilibrium fluctuation theory has been carried out only in a few examples, as the symmetric zero range [13] and the weakly asymmetric exclusion process [10]. The main ingredient is the proof of the following *principle*, first understood and proven in equilibrium situations by Brox and Rost [4]: the fluctuation fields of nonconserved quantities change on a time scale much faster than the conserved ones. The principle can hold in the 'kinetic'-limit also for models with no conservation laws, as it was proved, for instance in the papers [7, 8] for a bounded spin model. Our proof is modelled on that of [7, 8], with due account with the fact that we have to deal with unbounded variables. The essential feature in the proof is that the interaction changes the occupation numbers at a very slow rate with respect to diffusion, which conserves the particle number, and drives the system towards local equilibrium. One can say that on the time scale that is relevant for the principle the particle number is preserved.

In a recent paper, using different techniques, Dittrich [12] could prove convergence of the fluctuation field to the limiting O.U. process for small times and bounded volumes, in a particular model, the main feature of which is that only annihilation of particles takes place. This allows simple upper bounds on the correlation functions.

The paper is organized as follows: in Section 2 we state our results, the proofs are given in Section 3 and 4.

2. Definitions and results

The configuration space is $X_\varepsilon = \mathbb{N}^{\mathbb{Z}^\varepsilon}$, with $\mathbb{Z}_\varepsilon \equiv \mathbb{Z} \pmod{2K+1}$, where $\mathbb{Z}$ is the set of the integers and $K = [\varepsilon^{-2}]$ ($[\cdot]$ denotes integer part). A point of X_ε is a periodic sequence $\eta \equiv \{\eta(x), x \in \mathbb{Z}_\varepsilon\}$, $\eta(x) \in \mathbb{N}$ is the occupation number at the site $x \in \mathbb{Z}_\varepsilon$.

The symbol ξ will denote fixed configurations, independent of ε with finite support $E = E(\xi) \equiv \{x\colon \xi(x) > 0\}$. The cardinality of ξ, or total particle number is denoted by $|\xi| \equiv \sum_{x \in \mathbb{Z}_\varepsilon} \xi(x)$. Sometimes it is convenient to consider ξ not as a function over

$\mathbb{Z}_\varepsilon$ but as a collection of particles, that is, as a collection of sites $x \in \mathbb{Z}_\varepsilon$ with multiplicity $\xi(x)$ (i.e. $\boldsymbol{X}_\varepsilon$ is identified with $\bigcup_k ((\mathbb{Z}_\varepsilon)^k)_S$, where $(\cdot)_S$ denotes symmetrization). One can then speak of subsets of ξ, which are again elements of $\boldsymbol{X}_\varepsilon$, and the meaning of notation like $x \in \xi$, $\xi' \subset \xi$ is clear.

As in paper [3] an important tool in dealing with the dynamics are the polynomials $Q(\xi, \eta)$ defined by the relation

$$Q(\xi, \eta) \equiv \prod_{x \in E} Q_{\xi(x)}(\eta(x)), \quad Q(\emptyset, \eta) = 1, \qquad \eta \in \boldsymbol{X}_\varepsilon, \tag{2.1a}$$

where

$$Q_k(n) \equiv n(n-1) \cdots (n-k+1), \quad n, k \in \mathbb{N}, \; n \geq k. \tag{2.1b}$$

The dynamics is defined via its generator. Given $\varepsilon > 0$ we define the operator

$$L^\varepsilon = \varepsilon^{-2} L_0 + L_G \tag{2.2a}$$

which acts on the cylinder functions g on $\boldsymbol{X}_\varepsilon$ as follows:

$$L_0 g(\eta) = \tfrac{1}{2} \sum_x \eta(x)[g(\eta^{x,x+1}) + g(\eta^{x,x-1}) - 2g(\eta)], \tag{2.2b}$$

$$L_G g(\eta) = \sum_x \{Q_{k_+}(\eta(x))[g(\eta^{x,+}) - g(\eta)]$$
$$+ Q_{k_-}(\eta(x))[g(\eta^{x,-}) - g(\eta)]\}, \tag{2.2c}$$

where k_+, k_- are nonegative integers, $k_+ < k_-$, and

$$\eta^{x,x\pm 1}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, x \pm 1, \\ \eta(x) - 1 & \text{if } y = x, \\ \eta(x) + 1 & \text{if } y = x \pm 1, \end{cases} \tag{2.3a}$$

$$\eta^{x,\pm}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ \eta(x) \pm 1 & \text{if } y = x. \end{cases} \tag{2.3b}$$

That is, our choice is $q_+ = Q_{k_+}$, $q_- = Q_{k_-}$. We could as well take a linear combination of Q_k, $k = 0, \ldots, k_+$ ($k = 0, \ldots, k_-$) for q_+ (q_-), the proofs will be the same, except for heavier notation. The evolution is defined for any initial measure. We shall prove that if the initial measure is suitably close to a Poisson measure, the measure that describes the system at a later time is also close to a Poisson measure. Observe that the class of Poisson measures is invariant under the dynamics with generator L_0 (by Doob's theorem, since L_0 corresponds to independent random walks).

The path space is denoted by

$$\Omega = \mathscr{D}([0, \infty), \boldsymbol{X}_\varepsilon) \tag{2.4a}$$

and a path (or trajectory) in Ω by

$$\underline{\eta} \equiv \{\eta_t\}_{t \geq 0}, \quad \eta_t \in \boldsymbol{X}_\varepsilon. \tag{2.4b}$$

For any ξ, ξ', $|\xi| = |\xi'| = n$, $\pi_t^\varepsilon(\xi \to \xi') \equiv \mathrm{e}^{\varepsilon^{-2} L_0 t}(\xi, \xi')$ denotes the 'free' transition probability, corresponding to independent random walks. If $\pi_t^\varepsilon(x \to y)$, $x, y \in \mathbb{Z}^n$, is the usual single particle transition probability for the random walk, $\pi_t^\varepsilon(\xi \to \xi')$ can be written as

$$\pi_t^\varepsilon(\xi \to \xi') = \left(\prod_{x \in \xi} \pi_t^\varepsilon(x \to y(x)) \right)_{\mathrm{S}} \tag{2.5}$$

where $(\cdot)_{\mathrm{S}}$ denotes symmetrization, more precisely average with equal weights over all possible choices of the one-to-one maps $x \to y(x)$ between ξ and ξ'.

Note that for ε small enough,

$$\pi_t^\varepsilon(x \to y) < 2\varepsilon/\sqrt{t}. \tag{2.6}$$

For any initial measure ν we consider the function

$$f_t(\xi \mid \nu) \equiv \boldsymbol{E}_\nu(Q(\xi, \eta_t)), \tag{2.7}$$

where $\boldsymbol{E}_\nu$ denotes expectation with respect to the measure on path space corresponding to the initial measure ν. In particular for an atomic initial measure $\nu = \delta_\eta$ we write $f_t(\xi \mid \eta)$.

Given a function $\rho : \mathbb{Z}_\varepsilon \mapsto \mathbb{R}$ we define the 'centered' polynomials (see (2.1))

$$V(\xi, \eta; \rho) = \sum_{\xi' \subseteq \xi} Q(\xi', \eta)(-1)^{|\xi \setminus \xi'|} \prod_{x \in \xi \setminus \xi'} \rho(x). \tag{2.8}$$

The centering functions which we consider are solutions of the discrete integral equation

$$\rho_t(x; g) = \sum_y \pi_t^\varepsilon(x \to y) g(y) + \int_0^t \mathrm{d}s \sum_y \pi_{t-s}^\varepsilon(x \to y)(\rho_s(y; g)^{k_+} - \rho_s(y; g)^{k_-}) \tag{2.9}$$

where $g : \mathbb{Z}_\varepsilon \mapsto \mathbb{R}_+$ is the initial data.

Given an initial measure ν and a solution ρ_t of (2.9), the functions

$$v_t(\xi, \rho_t \mid \nu) = \boldsymbol{E}_\nu(V(\xi, \eta_t; \rho_t)) \tag{2.10}$$

(hereafter to be called v-functions) give a measure of how far the distribution of η_t is from the independent Poisson distribution with parameter ρ_t. We are interested in the v-functions for $\nu = \delta_\eta$, and centered around the solution $\rho_t(x; \eta)$ of (2.9) with initial data $g(x) = \eta(x)$, for a 'typical' $\eta \in \boldsymbol{X}_\varepsilon$. They will be denoted by $v_t(\xi \mid \eta)$.

The initial data of the limiting partial differential equation and the initial family of measures are related as follows.

Definition 2.1. The initial data of the limiting (continuous) problem is a bounded function $\rho_0(r) \in C^2(\mathbb{R})$, $\rho_0(r) \geq 0$, $r \in \mathbb{R}$.

Let $H(r)$ be a C^∞ function with values in $[0, 1]$ with support in $(-1, 1)$ and such that $H(r) = 1$ for $r \in [-1 + a, 1 - a]$, $0 < a < 1$. The function $\rho_0^{(\varepsilon)}(r) = \rho_0(r) H(\varepsilon^2 r)$ can be extended to a periodic C^2-function with period ε^{-2}. The initial family of measures $\{\mu^\varepsilon, \varepsilon > 0\}$ is a family of probability measures on X_ε such that $E_{\mu^\varepsilon}(\eta_0(x)) = \rho_0^\varepsilon(x) \equiv \rho_0^{(\varepsilon)}(\varepsilon x)$, and, for some positive function c and any fixed $k \geq 1$,

$$\sup_{\xi: |\xi| = k} |v_0(\xi, \rho_0^\varepsilon | \mu^\varepsilon)| < c(k) \varepsilon^{\lceil (k+1)/2 \rceil}. \tag{2.11}$$

In what follows ρ_t^ε will denote the solution of (2.9) with initial data ρ_0^ε and μ_t^ε the evolution of the initial measure μ^ε associated to the Markov process generated by L^ε. We also write $v_t(\xi | \mu^\varepsilon)$ for $v_t(\xi, \rho_t^\varepsilon | \mu^\varepsilon)$.

Next we define the density field and the density fluctuation field.

Definition 2.2. For $\phi \in \mathscr{S}(\mathbb{R})$ (the Schwartz space of rapidly decreasing functions), we define for any $t \geq 0$ the density field

$$X_t^\varepsilon(\phi) = \varepsilon \sum_x \phi(\varepsilon x) \eta_t(x). \tag{2.12}$$

We consider $X_t^\varepsilon(\phi)$, $t \geq 0$, $\phi \in \mathscr{S}(\mathbb{R})$, as a stochastic process on $\mathscr{D}([0, \infty), \mathscr{S}'(\mathbb{R}))$ with the distribution induced by the process $\{\eta_t, t \geq 0\}$ for initial measure μ^ε of Definition 2.1.

The fluctuation field $Y_t^\varepsilon(\phi)$, $t \geq 0$, $\phi \in \mathscr{S}(\mathbb{R})$, is the stochastic process on $\mathscr{D}([0, \infty), \mathscr{S}'(\mathbb{R}))$ defined by

$$Y_t^\varepsilon(\phi) = \sqrt{\varepsilon} \sum_x \phi(\varepsilon x)[\eta_t(x) - E_{\mu^\varepsilon}^\varepsilon(\eta_t(x))]. \tag{2.13}$$

We denote by P^ε the law of the fluctuation field in $\mathscr{D}([0, \infty), \mathscr{S}'(\mathbb{R}))$.

Observe that if $t = 0$ by the law of large numbers we have

$$\lim_{\varepsilon \to 0} X_o^\varepsilon(\phi) = \int \mathrm{d}r \, \phi(r) \rho_0(r) \tag{2.14}$$

and the fluctuation for small ε is known to be Gaussian with δ-like correlations. The results of the present paper can be summarized in the following two theorems, which hold under the hypotheses on the initial measure μ_0^ε and the initial datum ρ_0 listed in Definition 2.1 and 2.2.

Theorem 1. *Let $\{\mu^\varepsilon, \varepsilon > 0\}$ be as in Definition 2.1. Then for any $\phi \in \mathscr{S}$,*

$$\lim_{\varepsilon \to 0} X_t^\varepsilon(\phi) = \int \mathrm{d}r \, \phi(r) \rho(r, t) \tag{2.15}$$

where $\rho(\cdot, \cdot)$ is the solution of the following reaction-diffusion equation

$$\begin{aligned} &\partial_t \rho = \tfrac{1}{2} \partial_r^2 \rho - V'(\rho), \\ &\rho(r, 0) = \rho_0(r), \end{aligned} \tag{2.16}$$

where $V'(\rho) = \rho^{k_-} - \rho^{k_+}$. Moreover if $\nu_{\rho(\varepsilon \cdot, t)}$ denotes the product of Poisson measures with parameters $\rho(\varepsilon x, t)$, $x \in \mathbb{Z}$, then the difference $\mu_t^\varepsilon - \nu_{\rho(\varepsilon \cdot, t)}$ tends to 0 as $\varepsilon \to 0$, in any metric corresponding to weak convergence on X_ε.

By (2.15) we mean that the distribution of $X^\varepsilon(\cdot)$ converges weakly in $\mathscr{D}([0,\infty), \mathscr{S}(\mathbb{R}))$ to the degenerate law with support on the right-hand side of (2.15).

A similar statement was made in Corollary 1.3 of [3]. A simplified version of the proof (which corrects some infinite volume estimates of [3]), is given in [9].

Theorem 1 follows from Lemma 1 below. A short sketch of the proof is given at the end of this section.

Our main result is the following:

Theorem 2. *The law P^ε of the fluctuation field converges weakly to the law of the mean zero generalized Ornstein–Uhlenbeck process with covariance kernel $C^*_{t,s}(r',r)$ given by the solution of the equation*

$$\partial_s C^*_{t,t+s}(r',r) = \tfrac{1}{2}\partial_r^2 C^*_{t,t+s}(r',r) - V''(\rho(r,t+s))C^*_{t,t+s}(r',r), \tag{2.17a}$$

$$C^*_{t,t}(r',r) = C_t(r',r) + \delta(r-r')\rho(r,t), \tag{2.17b}$$

where $C_t(r',r)$ satisfies the equation

$$\begin{aligned}\partial_t C_t(r',r) = {} & \tfrac{1}{2}(\partial_r^2 C_t + \partial_{r'}^2 C_t) - [V''(\rho(r,t)) + V''(\rho(r',t))]C_t(r',r)\\ & + \delta(r-r')2[k_+\rho(r,t)^{k_+} - (k_- - 1)\rho(r,t)^{k_-}]\end{aligned} \tag{2.18}$$

with initial condition $C_0(r',r) = 0$. Here V'' is the derivative of V', given in the statement of Theorem 1 *and ρ is the solution of* (2.16).

The proof of Theorem 2 is given in Section 3. For the proof we need to control the v-functions, which is done by the following fundamental lemma.

Lemma 1. *Let $\{\mu^\varepsilon, \varepsilon > 0\}$ be as in Definition* 2.1. *Then for any $T \geq 0$, and $k > 2$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \sup_{0<t\leq T} \sup_{|\xi|=k} |v_t(\xi \mid \mu^\varepsilon)| = 0 \tag{2.19}$$

and moreover there is a constant $c(T)$ such that

$$\sup_{|\xi|=1,2} |v_t(\xi \mid \mu^\varepsilon)| < c(T)\varepsilon. \tag{2.20}$$

The proof of Lemma 1 is technically complicated, and will be given in Section 4.

Proof of Theorem 1. It is not hard to prove that in our hypotheses the solution of the discrete equation (2.9) with initial data $g = \rho_0^\varepsilon$ computed at the lattice point $[\varepsilon^{-1}r]$, $\rho_t([\varepsilon^{-1}r]), \rho_0^\varepsilon)$ converges, as $\varepsilon \to 0$, uniformly in any compact set of $r \in \mathbb{R}$, to the unique solution of (2.16). Theorem 1 then follows from inequality (2.20) of Lemma 1. □

3. Proof of Theorem 2

In this section we assume Lemma 1 and we prove the Gaussian structure of the fluctuation field.

We follow closely the proofs given in [13] where the Holley and Stroock theory [14, 15, 16], for generalized Ornstein-Uhlenbeck processes is used.

Theorem 2 is a consequence of the following three statements.

S1. *The family* $\underline{Y}^\varepsilon \equiv \{Y_t^\varepsilon(\phi), t \geq 0, \phi \in \mathscr{S}(\mathbb{R})\}$ *is tight in* $D([0,\infty), \mathscr{S}'(\mathbb{R}))$ *and any limiting point has support in* $C^0([0,\infty), \mathscr{S}'(\mathbb{R}))$.

S2. *Any limiting point* $\underline{Y}$ *of* $\underline{Y}^\varepsilon$ *satisfies the following 'martingale equation'. For any* $\phi \in \mathscr{S}(\mathbb{R})$ *and any* $F \in C_0^\infty(\mathbb{R})$,

$$F(Y_t(\phi)) - \int_0^t \mathrm{d}s\, F'(Y_s(\phi))\, Y_s(A_s\phi) - \int_0^t \mathrm{d}s\, F''(Y_s(\phi))\tfrac{1}{2}\|B_s\phi\|^2 \tag{3.1}$$

is a martingale with respect to the canonical filtration in $C^0([0,\infty), \mathscr{S}'(\mathbb{R}))$.

In (3.1), F' *and* F'' *denote the first and the second derivative of the function* F, *while the operators* A_s *and* B_s *are given by*

$$(A_s\phi)(r) = \tfrac{1}{2}\partial_r^2\phi - V''(\rho(r,s))\phi(r), \tag{3.2a}$$

$$V''(\rho) = k_+\rho^{k_+-1} - k_-\rho^{k_--1}, \tag{3.2b}$$

$$\|B_s\phi\|^2 = \int_{-\infty}^{\infty} \mathrm{d}r\, \{\phi'(r)^2\rho(r,s) + \phi(r)^2[\rho(r,s)^{k_+} + \rho(r,s)^{k_-}]\}, \tag{3.3}$$

and ρ *is the solution of* (2.16).

S3. *The law of* $\underline{Y}_0$ *(i.e., the law of the limiting process at* $t=0$) *is Gaussian with covariance kernel* $C_0^*(r,r')$ *given by* (2.17b) *for* $t=0$. □

Statement S3 is an imediate consequence of the properties of the initial measure μ^ε. Statements S1 and S2 can be proven as follows.

First notice that the expression

$$F(Y_t^\varepsilon(\phi)) - \int_0^t \mathrm{d}s\, [L^\varepsilon F(Y_s^\varepsilon(\phi)) + \partial_s F(Y_s^\varepsilon(\phi))] \tag{3.4}$$

is a martingale. In (3.4) L^ε is the generator defined by (2.2), and ∂_s is the derivative computed by taking into account only the dependence on s through the average values.

We have $L^\varepsilon F = \varepsilon^{-2}L_0F + L_GF$ hence

$$\varepsilon^{-2}L_0F(Y_s^\varepsilon(\phi)) = \varepsilon^{-2}\sum_x \tfrac{1}{2}\eta_s(x)[F(Y_s^\varepsilon(\phi)^{x,x+1}) + F(Y_s^\varepsilon(\phi)^{x,x-1}) - 2F(Y_s^\varepsilon(\phi))], \tag{3.5a}$$

$$L_GF(Y_s^\varepsilon(\phi)) = \sum_x \{Q_{k_+}(\eta_s(x))[F(Y_s^\varepsilon(\phi)^{x,+}) - F(Y_s^\varepsilon(\phi))] + Q_{k_-}(\eta_s(x))[F(Y_s^\varepsilon(\phi)^{x,-}) - F(Y_s^\varepsilon(\phi))]\}, \tag{3.5b}$$

where $F(Y_s^\varepsilon(\phi)^{x,x\pm1})$ and $F(Y_s^\varepsilon(\phi)^{x,\pm})$ are computed as follows. First observe that

$$
\begin{aligned}
&F(Y_s^\varepsilon(\phi)^{x,x\pm1})\\
&\quad = F\Bigg(\sqrt{\varepsilon}\sum_{y\neq x,x\pm1}\phi(\varepsilon y)(\eta_s(y)-\langle\eta_s(y)\rangle)\\
&\qquad\qquad +\sqrt{\varepsilon}[\phi(\varepsilon x)(\eta_s(x)-1-\langle\eta_s(x)\rangle)\\
&\qquad\qquad\qquad +\phi(\varepsilon(x+1))(\eta_s(x\pm1)+1-\langle\eta_s(x\pm1)\rangle)]\Bigg)\\
&\quad = F(Y_s^\varepsilon(\phi))+\sqrt{\varepsilon}\,F'(Y_s^\varepsilon(\phi))[\phi(\varepsilon(x\pm1))-\phi(\varepsilon x)]\\
&\qquad +\tfrac{1}{2}\varepsilon F''(Y_s^\varepsilon(\phi))[\phi(\varepsilon(x\pm1))-\phi(\varepsilon x)]^2\\
&\qquad + R_1^\pm(\varepsilon, x, s)
\end{aligned}
\tag{3.6}
$$

where $\langle\cdot\rangle$ denotes averaging. Notice that

$$
\begin{aligned}
\left|\varepsilon^{-2}\sum_x \eta_s(x)R_1^\pm(\varepsilon,x,s)\right| &\leqslant \left|\varepsilon^{-2}\varepsilon^{3/2}\sum_x[\phi(\varepsilon(x\pm1))-\phi(\varepsilon x)]^3\eta_s(x)\right|\\
&\leqslant \varepsilon^{3/2}\varepsilon\sum_x|\phi'(\varepsilon x)|^3\eta_s(x)
\end{aligned}
\tag{3.7}
$$

and by Theorem 1 the right-hand side of (3.7) goes to zero as $\varepsilon\to0$.

From (3.5a) and (3.6) it follows that

$$
\begin{aligned}
&\varepsilon^{-2}L_0F(Y_s^\varepsilon(\phi))\\
&\quad = F'(Y_s^\varepsilon(\phi))\sqrt{\varepsilon}\sum_x\varepsilon^{-2}[\tfrac{1}{2}\phi(\varepsilon(x+1))+\tfrac{1}{2}\phi(\varepsilon(x-1))-\phi(\varepsilon x)]\eta_s(x)\\
&\qquad + F''(Y_s^\varepsilon(\phi))\varepsilon\sum_x\tfrac{1}{4}\varepsilon^{-2}\{[\phi(\varepsilon(x+1))-\phi(\varepsilon x)]^2\\
&\qquad\qquad +[\phi(\varepsilon(x-1))-\phi(\varepsilon x)]^2\}\eta_s(x)\\
&\qquad +\varepsilon^{-2}-\tfrac{1}{2}\sum_x[R_1^+(\varepsilon,x,s)+R_1^-(\varepsilon,x,s)]\eta_s(x).
\end{aligned}
\tag{3.8}
$$

Therefore using (3.7) and the fact that $\phi\in\mathscr{S}(\mathbb{R})$, we have

$$
\begin{aligned}
\varepsilon^{-2}L_0F(Y_s^\varepsilon(\phi)) &= F'(Y_s^\varepsilon(\phi))\sqrt{\varepsilon}\sum_x\tfrac{1}{2}\phi''(\varepsilon x)\eta_s(x)\\
&\qquad +\tfrac{1}{2}F''(Y_s^\varepsilon(\phi))\varepsilon\sum_x\phi'(\varepsilon x)^2\eta_s(x)+\bar{R}_1(\varepsilon)
\end{aligned}
\tag{3.9a}
$$

where

$$
\lim_{\varepsilon\to0}\bar{R}_1(\varepsilon)=0. \tag{3.9b}
$$

Next we compute

$$F(Y_s^\varepsilon(\phi)^{(x,\pm)}) = F\Bigg(\sqrt{\varepsilon}\sum_{y\neq x}\phi(\varepsilon y)[\eta_s(y)-\langle\eta_s(y)\rangle]$$

$$+\sqrt{\varepsilon}\,\phi(\varepsilon x)[\eta_s(x)\pm 1-\langle\eta_s(x)\rangle]\Bigg)$$

$$= F(Y_s^\varepsilon(\phi)) \pm F'(Y_s^\varepsilon(\phi))\sqrt{\varepsilon}\,\phi(\varepsilon x)$$

$$+\tfrac{1}{2}F''(Y_s^\varepsilon(\phi))\varepsilon\phi(\varepsilon x)^2 + R_2^\pm(\varepsilon, x, s) \tag{3.10a}$$

where by Lemma 1,

$$\left|\sum_x Q_{k_\pm}(\eta_s(x))R_2^\pm(\varepsilon, x, s)\right| \leqslant \sqrt{\varepsilon}\left|\varepsilon\sum_x \phi(\varepsilon x)^3 Q_{k_\pm}(\eta_s(x))\right| \to 0. \tag{3.10b}$$

Moreover we have that

$$\partial_s F((Y_s^\varepsilon(\phi)) = -F'((Y_s^\varepsilon(\phi))\sqrt{\varepsilon}\sum_x \phi(\varepsilon x)E_{\mu^\varepsilon}^\varepsilon((\varepsilon^{-2}L_0+L_G)\eta_s(x)). \tag{3.11}$$

Therefore from (3.4), (3.5), (3.9), (3.10) and (3.11) we get that

$$F(Y_t^\varepsilon(\phi)) - \int_0^t \mathrm{d}s\, F'(Y_s^\varepsilon(\phi))\gamma_1^\varepsilon(s,\phi) - \int_0^t \mathrm{d}s\,\tfrac{1}{2}F''(Y_s^\varepsilon(\phi))\gamma_2^\varepsilon(s,\phi) + R_\varepsilon \tag{3.12}$$

is a martingale and

$$\lim_{\varepsilon\to 0} R_\varepsilon = 0, \tag{3.13}$$

$$\gamma_1^\varepsilon(s,\phi) \equiv Y_s^\varepsilon(\tfrac{1}{2}\phi'') + Z_s^\varepsilon(\phi), \tag{3.14}$$

$$Z_s^\varepsilon(\phi) = \sqrt{\varepsilon}\sum_x \phi(\varepsilon x)[Q_{k_+}(\eta_s(x)) - E_{\mu^\varepsilon}^\varepsilon(Q_{k_+}(\eta_s(x)))$$

$$-(Q_{k_-}(\eta_s(x)) - E_{\mu^\varepsilon}^\varepsilon(Q_{k_-}(\eta_s(x))))], \tag{3.15}$$

$$\gamma_2^\varepsilon(s,\phi) \equiv \varepsilon\sum_x \phi'(\varepsilon x)^2\eta_s(x) + \varepsilon\sum_x \phi(\varepsilon x)^2[Q_{k_+}(\eta_s(x)) + Q_{k_-}(\eta_s(x))]. \tag{3.16}$$

From Lemma 1 it follows that (cf. (3.3))

$$\lim_{\varepsilon\to 0}\gamma_2^\varepsilon(s,\phi) = \int \mathrm{d}r\,[\phi'(r)^2\rho(r,s)^2 + \phi(r)^2(\rho(r,s)^{k_+} + \rho(r,s)^{k_-})]$$

$$= \|B_s\phi\|^2. \tag{3.17}$$

From the above calculations we expect that the term $\gamma_1^\varepsilon(s,\phi)$ defined in (3.14) is close to $Y_s(A_s\phi)$ (cf. (3.1) and (3.2)). The problem here is $Z_s^\varepsilon(\phi)$: this is the fluctuation field of a quantity which is not a density field. Using arguments introduced in [13] (see also in [7] and [10]), we show that it is close to a density fluctuation field in the following sense.

Proposition 3.1. *For any choice of $\phi \in \mathcal{S}(\mathbb{R})$, t_0 such that $t_0 > 0$, the following holds:*

$$\lim_{T\to\infty} \lim_{\varepsilon\to 0} \sup_{0<t<t_0} \boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\left(\frac{1}{\varepsilon^2 T}\int_t^{\varepsilon^2 T+t} [Z^{\varepsilon}_s(\phi) - Y^{\varepsilon}_s(-V''(\rho^{\varepsilon}_s(\cdot))\phi)]\,\mathrm{d}s\right)^2\right) = 0. \quad (3.18)$$

We postpone the proof of Proposition 3.1 to the end of this section. We first prove that from (3.12) and (3.18) statements S1 and S2 follow easily.

Proof of S1. We have to prove tightness in the space $\mathcal{D}([0,\infty), \mathcal{S}'(\mathbb{R}))$. To this purpose, following Theorem 2.3 of [13], we observe the following. Although the functions $F(r) = r$ and $F(r) = r^2$ are not bounded, it is easy to see that relation (3.12) holds, i.e. the quantities

$$M^{\varepsilon}_t(\phi) \equiv Y^{\varepsilon}_t(\phi) - \int_0^t \gamma^{\varepsilon}_1(s, \phi)\,\mathrm{d}s + R^{\varepsilon}_1(t) \quad (3.19)$$

and

$$N^{\varepsilon}_t(\phi) \equiv M^{\varepsilon}_t(\phi)^2 - \int_0^t \gamma^{\varepsilon}_2(s, \phi)\,\mathrm{d}s + R^{\varepsilon}_2(t) \quad (3.20)$$

are martingales. The functions $\gamma^{\varepsilon}_i(s, \phi)$, $i = 1, 2$, are defined in (3.14) and (3.16) while $R^{\varepsilon}_i(t)$, $i = 1, 2$, are easily recovered from the calculations and go to 0 as $\varepsilon \to 0$. Then, as stated in Theorem 2.3 of [13], and proven in [18], in order to prove S1 we only need to prove the following relations. For any $\phi \in \mathcal{S}(\mathbb{R})$, and $t_0 > 0$,

(i) $$\sup_{\varepsilon} \sup_{0<t<t_0} \boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}(Y^{\varepsilon}_t(\phi)^2) < \infty, \quad (3.21)$$

$$\sup_{\varepsilon} \sup_{0<t<t_0} \boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}(\gamma^{\varepsilon}_i(t, \phi)^2) < \infty, \quad i = 1, 2, \quad (3.22)$$

(ii) there is some $\delta(t_0, \phi, \varepsilon)$ such that $\lim_{\varepsilon\to 0}\delta(t_0, \phi, \varepsilon) = 0$ and

$$\lim_{\varepsilon\to 0} P^{\varepsilon}\left(\sup_{0\le t'\le t_0} |Y^{\varepsilon}_{t'}(\phi) - Y^{\varepsilon}_{t'-}(\phi)| \ge \delta(t_0, \phi, \varepsilon)\right) = 0. \quad (3.23)$$

Equations (3.21) and (3.22) are easy consequences of Lemma 1 and Proposition 3.1, and (3.23) follows from the fact that the probability that there is more than one jump in a small time interval goes to zero faster than the length of the interval. (This is the main estimate in the proof of the existence of the process, cf. Notes to Chapter 4 of [9].) □

Proof of S2. Once (3.18) is established the fact that any limiting point Y or Y^{ε} satisfies the martingale equation (3.1) is easy, we refer to [6] for details. □

Proof of Proposition 3.1. This part is also standard and is similar to the corresponding proof in Theorem 0.4 of [13] and Theorem 4 in [7]. Therefore we only sketch the various steps.

The first step is to rewrite the fluctuation field Z_t^ε as a sum of fluctuation fields for the single site V-functions $V_k(\eta(x), \rho_t^\varepsilon(x))$, defined as follows (cf. (2.8)):

$$V_k(n, u) = \sum_{l=0}^{k} Q_l(n) u^{k-l} (-1)^{k-l} \binom{k}{l}.$$

The field for $k=1$ cancels $Y_t^\varepsilon(-V''(\rho_t^\varepsilon(\cdot))\phi)$, and using Proposition A.1 of Appendix A, we find

$$\begin{aligned} W_t^\varepsilon(\phi) &\equiv Z_t^\varepsilon(\phi) - Y_t^\varepsilon(-V''(\rho_t^\varepsilon(\cdot))\phi) \\ &= \sqrt{\varepsilon} \sum \phi(\varepsilon x) \sum_{k=2}^{k_-} c_k(\rho_t^\varepsilon(x)) \\ &\qquad \times (V_k(\eta_t(x), \rho_t^\varepsilon(x)) - \boldsymbol{E}_{\mu^\varepsilon}^\varepsilon(V_k(\eta_t(x), \rho_t^\varepsilon(x)))) \end{aligned} \tag{3.24}$$

where the c_k's are polynomially bounded functions. In (3.24) the expected value $\boldsymbol{E}_{\mu^\varepsilon}^\varepsilon(V_k(\eta_t(x), \rho_t^\varepsilon(x))$ is a v-function with $|\xi| = k \geqslant 2$. Hence, by Lemma 1, it is $\mathcal{O}(\varepsilon)$ and gives no contribution to the limit (3.18). Therefore in order to achieve the proof of (3.18) we have to estimate the following function $A_\varepsilon(k, k')$, $k, k' \in \{2, \ldots, k_-\}$,

$$\begin{aligned} A_\varepsilon(k, k') \equiv \frac{2}{\varepsilon^2 T} \int_t^{\varepsilon^2 T+t} \mathrm{d}s \frac{1}{\varepsilon^2 T} \int_0^{\varepsilon^2 T+t-s} \mathrm{d}s'\, \varepsilon \sum_x \phi(\varepsilon x) \sum_z \phi(\varepsilon z) \\ \times \boldsymbol{E}_{\mu^\varepsilon}^\varepsilon(V_k(\eta_s(x), \rho_s^\varepsilon(x)) V_{k'}(\eta_{s+s'}(z), \rho_{s+s'}^\varepsilon(z))). \end{aligned} \tag{3.25}$$

We can write the expectation in (3.25) as

$$\boldsymbol{E}_{\mu^\varepsilon}^\varepsilon[V_k(\eta_s(x), \rho_s^\varepsilon(x)) \boldsymbol{E}_{\mu^\varepsilon}^\varepsilon(V_{k'}(\eta_{s+s'}(z), \rho_{s+s'}^\varepsilon(z)) \,|\, \eta_s)]. \tag{3.26}$$

Using the *integration by parts formula* ((A.18), in Appendix A) for the expectation inside (3.26), we obtain

$$\begin{aligned} &\boldsymbol{E}_{\mu^\varepsilon}^\varepsilon(V_{k'}(\eta_{s+s'}(z), \rho_{s+s'}^\varepsilon(z)) \,|\, \eta_s) \\ &\quad = \sum_{\xi'} \pi_{s'}^\varepsilon(\xi(z) \to \xi') V(\xi', \eta_s; \rho_s^\varepsilon) + \mathrm{O}(\varepsilon^{2-\gamma}) \end{aligned} \tag{3.27}$$

where $\xi(z)$ denotes by abuse of notation the configuration at the site z and γ is a positive number less than 1. The term $\mathrm{O}(\varepsilon^{2-\gamma})$ in the right-hand side of (3.27) gives a vanishing contribution, and, using (3.26) and (3.27) we get

$$\begin{aligned} \Bigg| A_\varepsilon - \frac{2}{\varepsilon^2 T} \int_t^{\varepsilon^2 T+t} \mathrm{d}s \frac{1}{\varepsilon^2 T} \int_0^{\varepsilon^2 T+t-s} \mathrm{d}s'\, \varepsilon \sum_{x,z} \phi(\varepsilon x) \phi(\varepsilon z) \sum_{\xi'} \pi_{s'}^\varepsilon(\xi(z) \to \xi') \\ \times \boldsymbol{E}_{\mu^\varepsilon}^\varepsilon\, (V_k(\eta_s(x), \rho_s^\varepsilon(x)) V(\xi', \eta_s; \rho_s^\varepsilon)) \Bigg| \to 0. \end{aligned} \tag{3.28}$$

If $k' \equiv |\xi'| = k$, the sum for $\xi' \equiv \xi'(x)$, i.e. for the configurations ξ' which are made of k copies of x, is bounded by (cf. (2.6)),

$$\begin{aligned} &\text{const.} \frac{1}{\varepsilon^2 T} \int_t^{\varepsilon^2 T+t} \mathrm{d}s \frac{1}{\varepsilon^2 T} \int_0^{\varepsilon^2 T+t-s} \mathrm{d}s' \frac{\varepsilon^2}{\sqrt{s'}} \sum_{x,z} \pi_{s'}^\varepsilon(z \to x) \phi(\varepsilon x) \phi(\varepsilon z) \\ &\leqslant \text{const.}\, \varepsilon \frac{1}{\varepsilon^2 T} \int_t^{\varepsilon^2 T+t} \mathrm{d}s \frac{1}{\varepsilon^2 T} \int_0^{\varepsilon^2 T+t-s} \mathrm{d}s' \frac{1}{\sqrt{s'}} = \frac{\text{const.}}{\sqrt{T}}. \end{aligned} \tag{3.29}$$

For $x \notin \xi'$, observe that $V_k(\eta_s(x), \rho_s^\varepsilon) V(\xi', \eta_s; \rho_s^\varepsilon) = V(\xi(x) \cup \xi', \eta_s; \rho_s^\varepsilon)$ and $|\xi(x) \cup \xi'| > 2$. Therefore since by Lemma 1, $|\boldsymbol{E}^\varepsilon_{\mu^\varepsilon}(V_k(\eta_s(x), \rho_s^\varepsilon(x)) V(\xi', \eta_s; \rho_s^\varepsilon))| = \mathrm{o}(\varepsilon)$, it follows

$$\left| \varepsilon \sum_{x,z;\xi'} \pi^\varepsilon_{s'}(\xi(z) \to \xi') \phi(\varepsilon x) \phi(\varepsilon z) \boldsymbol{E}^\varepsilon_{\mu^\varepsilon}(V_k(\eta_s(x), \rho_s^\varepsilon(x)) V(\xi', \eta_s; \rho_s^\varepsilon)) \right| \to 0.$$

Finally if $\xi' = \xi'(x) \cup \xi''$ with $0 < |\xi''| < |\xi'|$, then

$$\begin{aligned} &V_k(\eta_s(x), \rho_s^\varepsilon) V(\xi', \eta_s; \rho_s^\varepsilon) \\ &\quad = V(\xi'', \eta_s; \rho_s^\varepsilon) \sum_{h=0}^{k+\xi'(x)} c_h(\rho_s^\varepsilon(x)) V_h(\eta_s(x), \rho_s^\varepsilon(x)), \end{aligned}$$

and

$$\begin{aligned} &\left| \varepsilon \sum_{x,z} \sum_{\xi'} \pi^\varepsilon_{s'}(\xi(z) \to \xi') \phi(\varepsilon x) \phi(\varepsilon z) \boldsymbol{E}^\varepsilon_{\mu^\varepsilon}(V_h(\eta_s(x), \rho_s^\varepsilon(x)) V(\xi'', \eta_s; \rho_s^\varepsilon)) \right| \\ &\quad \leq \text{const.} \frac{\varepsilon^3}{\sqrt{s'}} \sum_{x,z} |\phi(\varepsilon z)| |\phi(\varepsilon x)|. \end{aligned}$$

Performing the integration as above (cf. (3.29)) relation (3.18) follows. $\square$

4. Proof of Lemma 1

The proof of the fundamental Lemma 1 will be made in several steps. It is essentially based on a control of the behavior of the system for a short time $\tau = \varepsilon^\beta$, $\beta > 0$. In this section λ^ε will denote the measure on the path space Ω, T is a fixed time and $t_k = k\tau$, $0 \leq k \leq [T/\tau]$.

The first step is a simple lemma.

Lemma 4.1. *For any choice of $\gamma \in (0, \beta)$, and of the positive integers n_0, p, one can find a set $\Omega_1 \equiv \Omega_1(\gamma) \subset \Omega$, $\lambda^\varepsilon(\Omega_1) > 1 - \varepsilon^p$ such that for $\underline{\eta} \equiv \{\eta_t\}_{t \geq 0} \in \Omega_1$ the following inequalities hold for ε small enough*:

$$\sup_{t \geq 0} \rho_t(x; \eta_{t_k}) < \varepsilon^{-\gamma}, \tag{4.1a}$$

$$\max_{|\xi| = n} \sup_{t \in [0,\tau]} f_t(\xi \mid \eta_{t_k}) < \varepsilon^{-\gamma n}, \tag{4.1b}$$

where $\rho_t(x; \eta_{t_k})$ is the solution of (2.9) for with initial data η_{t_k}.

Proof. By a result of [3] we know that

$$\max_{|\xi| = n} f_t(\xi \mid \mu^\varepsilon) \leq \bar{c}(n, T), \quad 0 \leq t \leq T, \tag{4.2}$$

so that by the Chebyshev inequality (of power m),

$$\lambda^\varepsilon(\eta_{t_k} > \varepsilon^{-\zeta}) < c'(m, T) \varepsilon^{\zeta m}$$

for $\zeta < \gamma$ and any $m \geq 0$. Taking $m > (4+p)/\zeta$ and setting

$$\bar{\Omega} \equiv \left\{ \underline{\eta} : \max_{k \leq [T/\tau]} \max_{x \in \mathbb{Z}_\varepsilon} \eta_{t_k}(x) \leq \varepsilon^{-\zeta} \right\}$$

one has

$$\lambda^\varepsilon(\bar{\Omega}) > 1 - \varepsilon^{p+1}, \tag{4.3}$$

whence (4.1a) follows by the maximum principle for the solution of (2.9) with initial data η_{t_k}.

By formula (2.2) of [3],

$$f_t(\xi \mid \eta_{t_k}) = \sum_{\xi_1} \pi_t^\varepsilon(\xi \to \xi_1) Q(\xi_1, \eta_{t_k})$$
$$+ \int_0^t \mathrm{d}s \sum_{\xi_1} \pi_{t-s}^\varepsilon(\xi \to \xi_1) \boldsymbol{E}(L_G Q(\xi_1, \eta_s) \mid \eta_{t_k}).$$

By the previous result the first term on the right-hand side is bounded by $\varepsilon^{-\zeta n}$ for $\underline{\eta} \in \bar{\Omega}$. For the other term we have

$$\left| \int_0^t \mathrm{d}s \sum_{\xi_1} \pi_{t-s}^\varepsilon(\xi \to \xi_1) \boldsymbol{E}(L_G Q(\xi_1, \eta_s) \mid \eta_{t_k}) \right|$$
$$\leq \int_0^T \mathrm{d}s \sum_{\xi_1} \pi_{t-s}^\varepsilon(\xi \to \xi_1) \left| \boldsymbol{E}(L_G Q(\xi_1, \eta_s) \mid \eta_{t_k}) \right| \equiv R(\xi).$$

Using again (4.2) and the Chebyshev inequality for some power $N > 0$, we have for ε small enough $\lambda^\varepsilon(R(\xi) > 1) \leq \varepsilon^{\beta N} c'(n, N)$, and the result follows by taking N large enough and $\Omega_1 = \bar{\Omega} \cap \{\max_{\xi : |\xi| \leq n_0} R(\xi) < 1\}$. $\square$

Proposition 4.1. *On the same set* Ω_1, *for any* $n \leq n_0$ *and any* $\delta' < \delta \equiv (1/(2k-1))(\frac{1}{2} - \frac{1}{4}\beta)$, *the following inequalities hold for* ε *small enough*:

$$\max_{|\xi| = n} |v_t(\xi \mid \eta_{t_k})| < \varepsilon^{\delta' n}, \quad \tau \leq t \leq 2\tau, \tag{4.4}$$

for all $k = 0, \ldots, [T/\tau]$.

Proof. The basic tool in the proof is the iteration formula for the v-functions ((A.18) of Appendix A):

$$v_s(\xi \mid \eta_{t_k}) = \sum_{\xi_1} \pi_s^\varepsilon(\xi \to \xi_1) V(\xi_1, \eta_{t_k}; \eta_{t_k})$$
$$+ \int_0^s \mathrm{d}s' \sum_{\xi_1} \pi_{s-s'}^\varepsilon(\xi \to \xi_1)$$
$$\times \sum_{x_1 \in \xi_1} \sum_{h=-k_-+1}^{k_-} c_h(\xi_1(x_1), \rho_{s'}(x_1 \mid \eta_{t_k})) v_{s'}(\xi_1(x_1, h) \mid \eta_{t_k}) \tag{4.5}$$

where $\xi(x, h)$ is obtained by adding to ξ $h-1$ 'copies' of x (actually subtracting if $h - 1 < 0$).

We substitute for $v_{s'}(\xi_1(x_1, h)\,|\,\eta_{t_k})$ in the right-hand side of (4.5) its expression given by the same (4.5) and iterate the procedure N times, obtaining

$$v_s(\xi\,|\,\eta_{t_k}) = \sum_{l=0}^{N} \boldsymbol{C}_l(s, \xi) + \boldsymbol{R}_{N+1}(s, \xi) \tag{4.6}$$

where $C_l(s, \xi)$, $\boldsymbol{R}_{N+1}(s, \xi)$ denote the lth integral obtained in the iteration and the Nth residual term. We shall prove that for any positive integer N,

$$\max_{|\xi|=n} |\boldsymbol{R}_N(t, \xi)| < c_2(n, N)\varepsilon^{(\beta-\gamma(2k_-+n/N))N}, \quad \tau \leqslant t \leqslant 2\tau, \tag{4.7}$$

$$\max_{|\xi|=n} \sum_{l=0}^{N} |\boldsymbol{C}_l(t, \xi)| < c_1(n, N)\varepsilon^{(\delta-\gamma)n}, \quad \tau \leqslant t \leqslant 2\tau. \tag{4.8}$$

Proof of (4.7). Inequality (4.7) follows immediately from (4.1b) and formula (4.5), which imply (on the good set Ω_1)

$$|\boldsymbol{R}_N(t, \xi)| < \text{const. } \varepsilon^{\beta N}\varepsilon^{-\gamma(2k_-+n/N)N}.$$

Proof of (4.8). We have from (4.5),

$$\begin{aligned}
\boldsymbol{C}_l(t, \xi) = \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{l-1}} \mathrm{d}s_l \sum_{\xi_1} \pi^{\varepsilon}_{t-s_1}(\xi \to \xi_1) \\
\times \sum_{x_1 \in \xi_1} \sum_{h_1}{}^{*} c_{h_1} \sum_{\xi_2} \pi^{\varepsilon}_{s_1-s_2}(\xi_1(x_1, h_1) \to \xi_2) \cdots \\
\times \sum_{x_{l-1} \in \xi_{l-1}} \sum_{h_{l-1}}{}^{*} c_{h_{l-1}} \sum_{\xi_1} \pi^{\varepsilon}_{s_{l-1}-s_l}(\xi_{l-1}(x_{l-1}, h_{l-1}) \to \xi_l) \\
\times \sum_{x_l \in \xi_l} \sum_{h_l} c_{h_l} \sum_{\xi_{l+1}} \pi^{\varepsilon}_{s_l}(\xi_1(x_l, h_l) \to \xi_{l+1}) V(\xi_{l+1}, \eta_{t_k}; \eta_{t_k}),
\end{aligned} \tag{4.9}$$

where the notation $\sum^*_{h_i}$, $1 \leqslant i \leqslant l-1$, denotes that we exclude the term corresponding to the case in which all particles are at x_i and disappear, i.e. $\xi_i(x_i, h_i) = \emptyset$, or $h_i - 1 + |\xi_{i-1}(x_{i-1}, h_{i-1})| = 0$ (they are included in the term $\boldsymbol{C}_i$), and we dropped the arguments of the functions c_{h_j} for shortness.

The coefficients $c_h(m, u)$ have the following properties as shown in Proposition A.2 of Appendix A:

$$c_h(m, u) = 0 \quad \text{unless } m \geqslant m(h)$$

where

$$m(h) = \begin{cases} 1, & h \geqslant 1, \\ \max(2, -h+1), & h \leqslant 0. \end{cases}$$

That is, particles can disappear at x only if there are at least two particles, and no more particles can disappear than there are at x before interaction.

It is convenient to rewrite the integration function in expression (4.9) by inverting the sums

$$\sum_{\xi'} \pi^{\varepsilon}(\xi \to \xi') \sum_{x \subset \xi'} \sum_{h=-k_-+1}^{k_-}{}^{*} c_h$$

$$= \sum_{x \in \mathbb{Z}_\varepsilon} \sum_{h=-k_-+1}^{k_-}{}^{*} \sum_{\xi'} \pi^{\varepsilon}(\xi \to \xi') \chi(\xi'(x) \geq m(h)) |\xi'(x)| c_h$$

where χ denotes the indicator function and $\sum^*$ denotes as before that we exclude the case $-h+1=|\xi|$.

Using (A.9) for the coefficients c_{h_i} and Lemma 4.1, we find that the contribution of the sum in expression (4.9) for fixed choices of the interaction times $s_1, \ldots, s_l$, of the interaction sites $x_1, \ldots, x_l$, and of the 'generation numbers' $h_1, \ldots, h_l$, is bounded by

$$\begin{aligned} &\text{const.}\ \varepsilon^{-\gamma l k_-} \sum_{\xi_1} \chi(\xi_1(x) \geq m_1) \pi^{\varepsilon}_{t-s_1}(\xi \to \xi_1) \\ &\times \sum_{\xi_2} \chi(\xi_2(x) \geq m_2) \pi^{\varepsilon}_{s_1-s_2}(\xi_1(x_1, h_1) \to \xi_2) \cdots \\ &\times \sum_{\xi_l} \chi(\xi_l(x) \geq m_l) \pi^{\varepsilon}_{s_{l-1}-s_l}(\xi_{l-1}(x_{l-1}, h_{l-1}) \to \xi_l) \\ &\times \sum_{\xi_{l+1}} \pi^{\varepsilon}_{s_l}(\xi_l(x_l, h_l) \to \xi_{l+1}) |V(\xi_{l+1}, \eta_{t_k}; \eta_{t_k})|. \end{aligned} \tag{4.10}$$

For each interaction time s_i the configuration before interaction can be decomposed as $\xi_i = \{x_i\}^{m_i} \cup \xi'_i$, and the configuration arising after interaction as $\xi_i^{(x_i, h_i)} = \{x_i\}^{p_i} \cup \xi'_i$, where $\{x\}^m$ is the configuration made of m copies of x, and $p_i \equiv m_i + h_i - 1$ is the number of the particles at x_i after interaction. If we fix the subset $\xi^{(1)} \subset \xi$ that goes into $\{x_1\}^{m_1}$ the first sum in expression (4.10) reduces to

$$\pi^{\varepsilon}_{t-s_1}(\xi^{(1)} \to \{x_1\}^{m_1}) \sum_{\xi'_1} \pi^{\varepsilon}_{t-s_1}(\xi \backslash \xi^{(1)} \to \xi'_1).$$

The next transition probability $\pi^{\varepsilon}_{s_1-s_2}(\{x_1\}^{p_1} \cup \xi'_1 \to \xi_2)$, where $\xi_2 = \{x_2\}^{m_2} \cup \xi'_2$, is a sum of products of transition probabilities for all possible choices of the subset of $\{x_1\}^{p_1}$ and of the subset $\xi'_{12} \subset \xi'_1$ that go into $\{x_2\}^{m_2}$. Let $\xi^{(2)}$ denote the subset of $\xi \backslash \xi^{(1)}$ that go into ξ'_{12}, and set $\xi^{+}_{12} = \xi'_1 \backslash \xi'_{12}$, $n = |\xi'_{12}| \leq m_2$. By summing over ξ'_{12}, ξ''_{12} we obtain the composition of the corresponding transition probabilities (π's), so that the contribution of the first two sums in expression (4.10), for fixed choices of $\xi^{(1)}$, $\xi^{(2)}$ and n, is

$$\begin{aligned} &\pi^{\varepsilon}_{t-s_1}(\xi^{(1)} \to \{x_1\}^{m_1}) \pi^{\varepsilon}_{t-s_2}(\xi^{(2)} \to \{x_2\}^{m_2}) (\pi^{\varepsilon}_{s_1-s_2}(\{x_1\}^{m_2-n} \to \{x_2\}^{m_2-n})) \\ &\times \sum_{\xi''_2, \xi'''_2} \pi^{\varepsilon}_{t-s_2}(\xi \backslash (\xi^{(1)} \cup \xi^{(2)}) \to \xi'''_2) \pi^{\varepsilon}_{s_1-s_2}(\{x_1\}^{p_1-m_2+n} \to \xi''_2) \end{aligned}$$

where $\xi'''_2 = \xi'_2 \backslash \bigcup \xi''_2$.

One then goes on, specifying for each interaction site x_i the 'birthplace' of the particles that go into $\{x_i\}^{m_i}$, and ends up by specifying the birthplace of the final particles ξ_{l+1}. As a result expression (4.10), is split into a finite sum of terms, each term corresponding to a different choice of the numbers of the bonds connecting the interaction and the final sites. The total number of terms is bounded by some combinatorial factor depending on l, $|\xi|$, and on the generation numbers h_i. Since (Proposition A.1) $V(\xi, \eta; \eta)=0$ if $\xi(x)=1$ for some x, and moreover

$$|V(\xi, \eta; \eta)| \leq \text{const.} \left(\max_{x \in \xi} \eta(x) \right)^{[(|\xi|+1)/2]},$$

it follows that each term of the sum is bounded by

$$\text{const. } \varepsilon^{-\gamma(lk_- + [(M+1)/2])} \Gamma_{t; s_1, \ldots, s_l}(\xi; x_1, \ldots, x_l; \xi_{l+1}) \tag{4.11}$$

where Γ denotes the sum of transition probabilities (π's), and $M \equiv |\xi_{l+1}|$ is the number of the final particles.

The term Γ can be represented by a graph with vertices at the points (t, x), $x \in \xi$ (initial vertices), (O, z), $z \in \xi_{l+1}$ (final vertices), and (s_i, x_i), $i=1, \ldots, l$ (interaction vertices). Each bond $\{(s, x), (s', x')\}$, $s > s'$ corresponds to a factor $\pi^{\varepsilon}_{s-s'}(x \to x')$, the usual random walk transition probability.

Since the number of bonds connecting any pair of vertices of the graph is specified the function Γ in expression (4.11) is a function of the positions $x_1, \ldots, x_l, z_1, \ldots, z_r$ of the interaction and of the final vertices. The main point in the proof of (4.7) is the estimate of the sum

$$A_t(s_1, \ldots, s_l) \equiv \sum_{x_1, \ldots, x_l} \sum_{z_1, \ldots, z_r} \Gamma_{t; s_1, \ldots, s_l}(\xi; x_1, \ldots, x_l; z_1, \ldots, z_r). \tag{4.12}$$

In Appendix B we prove, using graph techniques that

$$\int_0^t \mathrm{d}s_1 \cdots \int_0^{s_{l-1}} \mathrm{d}s_l \, A_t(s_1, \ldots, s_l) \leq \text{const. } \varepsilon^{\delta|\xi|} \varepsilon^{\beta l} \tag{4.13}$$

where $\delta = (1/(2k_-))(\frac{1}{2} - \frac{1}{4}\beta)$.

The final result is obtained by summing up the contributions of all graphs with given $h_1, \ldots, h_l$ (which is a finite number). Since no more than k_- particles can be produced at each interaction, $M \equiv |\xi_{l+1}| \leq |\xi| + lk_-$, therefore inequality (4.13) and (4.11) implies that expression (4.9) is bounded by

$$\text{const. } \varepsilon^{|\xi|(\delta-\gamma)} \varepsilon^{l(\beta - 2k - \gamma)}.$$

(We take of course β and γ such that $\delta > \beta > 2k_- \gamma$.) Summing from $l=0$ up to $l=N$ we get inequality (4.8).

The proposition now follows by taking N so large that $\beta > \delta n/N + \gamma(2k_- + n/N)$. $\square$

Before going to the proof of Lemma 1 we need a result on the smoothing properties of the free dynamics (with generator L_0) which produces the fact that the various

$\rho_{t-t_k}(x;\eta_{t_k})$ are close to $\rho_t^\varepsilon(x)$, where $\rho_t^\varepsilon(x)$ is the solution of (2.9) with initial data $\rho_0^\varepsilon(x)=\boldsymbol{E}_{\mu_\varepsilon}(\eta_0(x))$ (cf. Definition 2.2). This of course will be true only on a large set and when t is a bit larger than t_k for the smoothing property to apply.

From now on we always intend that $\delta\equiv(1/(2k_-))(\frac{1}{2}-\frac{1}{4}\beta)$ and β and γ such that $2k_-\gamma<\beta<\delta$.

Proposition 4.2. *If δ' is chosen as in Proposition 4.1, then for any choice of $\delta''<\delta'-\beta$ and of $p>0$ one can find a set Ω_2, $\lambda^\varepsilon(\Omega_2)>1-\varepsilon^p$ such that for $\underline{\eta}\in\Omega$,*

$$\sup_{t_{k+1}\leq t\leq T}\ \max_x|\rho_{t-t_k}(x;\eta_{t_k})-\rho_t^\varepsilon(x)|<\varepsilon^{\delta''}. \tag{4.14}$$

for all $k=0,1,\ldots,[T/\tau]$.

Proof. We first estimate

$$\Delta_0(t,x)=\rho_t(x;\eta_0)-\rho_t^\varepsilon(x).$$

Since both $\rho_{t-t_k}(x;\eta_{t_k})$ and $\rho_t^\varepsilon(x)$ are solutions of (2.9) we have

$$\begin{aligned}\Delta_0(t,x)={}&\sum_y \pi_t^\varepsilon(x\to y)\hat{\eta}_0(y)\\&+\int_0^t \mathrm{d}s\sum_y \pi_{t-s}^\varepsilon(x\to y)\sum_{h=1}^{k_-}a_h(\rho_s^\varepsilon(y))(\Delta_0(s,y))^h,\end{aligned} \tag{4.15}$$

where

$$a_h(\rho)=\binom{k_+}{h}\rho^{k_+-h}-\binom{k_-}{h}\rho^{k_--h},$$

and $\hat{\eta}=\eta-\boldsymbol{E}\eta$.

Let $m_0=\max_x \boldsymbol{E}(\eta_0(x))$. By the maximum principle $\rho_t^\varepsilon(x)\leq\max(m_0,1)$, and in Ω_1 (cf. Lemma 4.1) the integration function in (4.15) does not exceed in absolute value $M_0\varepsilon^{-\gamma k_-}$, $M_0=K_1(\max(m_0,1))^{k_-}$, where K_1 is an absolute constant. Hence for $t=s_0\equiv\varepsilon^{\delta'}<\varepsilon^\beta$ the integral in (4.15) is bounded by $M_0\varepsilon^{\delta'-\gamma k_-}$.

To estimate the first term

$$\mathcal{T}_0\equiv\sum_y \pi_{s_0}^\varepsilon(x\to y)\hat{\eta}_0(y)$$

we take the $2N$th moment

$$\boldsymbol{E}_{\mu^\varepsilon}(\mathcal{T}_0)^{2N}=\sum_{y_1,\ldots,y_{2N}}\prod_i \pi_{s_0}^\varepsilon(x\to y_i)\boldsymbol{E}_{\mu^\varepsilon}\left(\prod_i\hat{\eta}_0(y_i)\right). \tag{4.16}$$

The product $\prod_{x\in\xi}\hat{\eta}_0(x)$ coincides with $V(\xi,\eta_0;\boldsymbol{E}\eta_0)$ if all particles of ξ are at different sites. If some sites coincide, the expected value of the product of $\hat{\eta}_0$ does not have to be small, and we have to rely, as usual, on the fact that the probability of having coincidences is small. The relation between products of $\hat{\eta}$ and V-functions is given by the following formula, obtained after simple algebraic manipulations. If $g:\mathbb{Z}_\varepsilon \to \mathbb{R}_+$ is a nonnegative function, and $\tilde{\eta}(x)=\eta(x)-g(x)$,

$$\prod_{x\in\xi}\tilde{\eta}(x)=\sum_{\xi'\subset\hat{\xi}} d(\xi,\xi';g)\,V(\xi\backslash\xi',\eta_0;g) \tag{4.17}$$

where $\hat{\xi}$ is the subset of the elements of ξ that are repeated more than once, and d has the properties

$$d(\xi,\emptyset;g)=1,\qquad 0\leqslant d(\xi,\xi';g)\leqslant \boldsymbol{d}(|\xi|)\left(\max_x g(x)\right)^{|\xi'|}.$$

We use this and insert (4.17) into (4.16), for $g(x)=\boldsymbol{E}(\eta_0(x))\equiv\rho_0^\varepsilon(x)$. Observe that in performing the sum over ξ for a fixed value of $|\xi'|=r$, we can extract at least $[\frac{1}{2}(r+1)]$ factors $\varepsilon/\sqrt{s_0}$ (cf. (2.6)), and we find

$$\boldsymbol{E}\mathcal{T}_0^{2N}\leqslant c_N m_0^{2N}\varepsilon^{2N\delta^*}, \tag{4.18}$$

where $\delta^*=\min(\delta',\frac{1}{2}-\frac{1}{4}\delta')$ and c_N is an absolute constant. By the Chebyshev inequality $\mu_\varepsilon(|\mathcal{T}_0|>\varepsilon^{\delta^*-\zeta})\leqslant\varepsilon^{p+2}$ for any $\zeta>0$, N large enough and ε small enough. Hence for any $\delta_1<\delta^*$ $\max_x|\Delta_0(s_0,x)|<\varepsilon^{\delta_1}$, on the proper set.

Clearly by (4.14) $\Delta_0(t,x)$ is continuous in t. Let $\bar{t}=\sup\{t\geqslant s_0:\max_x|\Delta_0(t,x)|<1\}$. Until time $\bar{t}$, $|\Delta_0(t,x)|$ is bounded by the solution of the integral equation

$$y(t)=\varepsilon^{\delta_1}+M_0\int_{s_0}^t \mathrm{d}s\,y(s)$$

i.e. $y(t)=\varepsilon^{\delta_1}\exp(M_0(t-s_0))$. Hence for ε small enough $\bar{t}\geqslant s_0+(\delta_1\log\varepsilon^{-1})/M_0>T$, and we find

$$\max_x|\Delta_0(t,x)|\leqslant\varepsilon^{\delta_1}\mathrm{e}^{M_0T},\quad s_0\leqslant t\leqslant T, \tag{4.19}$$

which implies

$$\max_x\sup_{t\geqslant s_0}\rho_t(x\,|\,\eta_0)\leqslant m_0+\varepsilon^{\delta_1}\exp(M_0T)\equiv m_1. \tag{4.20}$$

We then repeat the procedure for

$$\Delta_k(t,x)=\rho_{t-t_k}(x;\eta_{t_k})-\rho_{t-t_{k-1}}(x;\eta_{k-1}).$$

For $k=1$, we have, in analogy with (4.15),

$$\Delta_1(t,x)=\sum_y\pi^\varepsilon_{t-t_1}(x\to y)(\eta_{t_1}(y)-\rho_{t_1}(y;\eta_0))$$
$$+\int_{t_1}^t\mathrm{d}s\sum_y\pi^\varepsilon_{t-s}(x\to y)\sum_{h=1}^{k_-}a_h(\rho_s(y;\eta_0))(\Delta_1(s,y))^h. \tag{4.21}$$

Using (4.20), for $t=t_1+s_0$, the integral on the right-hand side is bounded by $M_1\varepsilon^{\delta'-\gamma k_-}$ with $M_1=K_1((\max(m_1,1))^{k_-}$. To estimate the other term, denoted by $\mathcal{T}_1$, observe that, by (4.17), we have

$$\boldsymbol{E}_{\eta_0}\prod_{x\in\xi}(\eta_{t_1}(x)-\rho_{t_1}(x;\eta_0))=\sum_{\xi'\subset\hat{\xi}}d(\xi,\xi';\rho_{t_1}(\cdot\,;\eta_0))v_{t_1}(\xi\backslash\xi'\,|\,\eta_0).$$

Using Proposition 4.1, and repeating the arguments that lead to (4.18), we find, for η_0 in the good set Ω_1,

$$E_{\eta_0}(\mathcal{T}_1)^{2N} \leq c_N \varepsilon^{2N\delta^*} m_1^{2N}.$$

By the Chebyshev inequality, we get, as above, that on the proper set

$$\max_x |\Delta_1(t,x)| < \varepsilon^{\delta_1} \exp(M_1 T), \quad t_1 + s_0 \leq t \leq T,$$

and we can take $M_1 \leq K_1(\max(m_1, 1))^{k_-}$.

In repeating the procedure for $\Delta_2, \dots$, we have only to check that the constants m_i (and hence $M_i \equiv K_1(\max(m_i, 1))^{k_-}$) are bounded. Setting $\hat{m}_i \equiv m_i - m_0$, $b_i \equiv \max(1, m_i)$ we find the recursive inequality

$$\hat{m}_{j+1} \leq \hat{m}_j + \varepsilon^{\delta_1} A^{b_j^{k_-}},$$

where $A = e^{K_1 T}$.

Let $j_0 = \max\{j\colon \hat{m}_j < m_0\}$. Then, for $j \leq j_0$, $\hat{m}_j \leq a_j$, where a_j is the solution of the recursive equation

$$a_{j+1} = a_j + \varepsilon^{\delta_1} B, \quad B = A^{(\max(2m_0, 1))^{k_-}},$$

with $a_0 = 0$. Hence $j_0 \geq \text{const.}\ \varepsilon^{-\delta_1} > T\varepsilon^{-\beta}$ if ε is small enough.

We get finally

$$|\rho_{t-t_k}(x;\eta_{t_k}) - \rho_t^\varepsilon(x)| \leq \sum_{j=0}^{k} |\Delta_k(t,x)| \leq K_2 T\, e^{K_3 T} \varepsilon^{\varepsilon_1 - \beta}$$

where K_2, K_3 are absolute constants. Proposition 4.2 then follows, maybe by redefining p. □

Proof of Lemma 1. Let $\eta_t \in \Omega_1 \cap \Omega_2 \equiv \bar{\Omega}$. We have, by Propositions 4.1 and 4.2, for $t_{k+1} \leq t \leq t_{k+2}$,

$$\begin{aligned}
&|E_{\eta_{t_t}} V(\xi, \eta_t; \rho_t^\varepsilon)| \\
&\quad \leq \sum_{\xi' \subset \xi} |E_{\eta_{t_k}} V(\xi, \eta_t; \rho_{t-t_k}(\cdot\,; \eta_{t_k}))| \left| \prod_{x \in \xi \setminus \xi'} (\rho_t^\varepsilon(x) - \rho_{t-t_k}(x; \eta_{t_k})) \right| \\
&\quad = \sum_{\xi' \subset \xi} |v_{t-t_k}(\xi' | \eta_{t_k})| \prod_{x \in \xi \setminus \xi'} |(\rho_t^\varepsilon(x) - \rho_{t-t_k}(x; \eta_{t_k})| \\
&\quad \leq (\varepsilon^{\delta'} + \varepsilon^{\delta''})^{|\xi|}.
\end{aligned} \tag{4.22}$$

Since $V(\xi, \eta_t; \rho_t^\varepsilon)$ is in L^1 and $\lambda^\varepsilon(\bar{\Omega}^c) < \varepsilon^p$, if p is chosen large enough we get from (4.22) and the Chebyshev inequality, maybe by redefining δ'',

$$\sup_{\varepsilon^\beta \leq t \leq T} \max_{\xi: |\xi| = n} |v_t(\xi | \mu_\varepsilon)| < \varepsilon^{\delta'' n}. \tag{4.23}$$

For $t \leq \varepsilon^\beta$, (4.23) is obtained in the same way as in Proposition 4.1.

Let $k_0=[2/\delta'']+1$. By (4.23) $|v_t(\xi\,|\,\mu_\varepsilon)|=o(\varepsilon)$ for $|\xi|\geq k_0$. Observe that, since $\delta''<\frac{1}{2}$, $k_0>5$. Take ξ such that $|\xi|=k_0-1$. By iterating the recursive integral equation for the v-functions, we find, by the properties of the initial measure μ^ε,

$$|v_t(\xi\,|\,\mu_\varepsilon)|\leq \varepsilon^{[k_0/2]}+\left|\int_0^t \mathrm{d}s \sum_{\xi_1} \pi^\varepsilon_{t-s}(\xi\to\xi_1)\right.$$
$$\left.\times \sum_{x_1\in\xi_1}\sum_{h=-k_-+1}^{k_-} c_h(\xi_1(x_1),\rho^\varepsilon_{s'}(x_1))v_{s'}(\xi_1,h)\,|\,\mu_\varepsilon)\right|.$$

The sum for $h>1$ gives $o(\varepsilon)$ by the definition of k_0. For $h=0$ we get an integral of $v_s(\xi'\,|\,\mu_\varepsilon)$ with $|\xi'|=k_0-2$, which is bounded by const $\varepsilon\sqrt{t}\,\varepsilon^{(k_0-2)\delta''}=o(\varepsilon)$. Similarly the integral of the term with $h=-1$ is $\mathcal{O}(\varepsilon^{1+(k_0-3)\delta''})=o(\varepsilon)$. The terms for $h<-1$ are nonzero only if $\xi_1(x_1)\geq 3$, and the corresponding integral is bounded by const.$(\varepsilon^{1+\alpha}+\varepsilon^2\int_{\varepsilon^{1+\alpha}}^t(1/t)\,\mathrm{d}t)=o(\varepsilon)$.

We get therefore, for $y_t=\max_\xi|v_t(\xi\,|\,\mu^\varepsilon)|$ the inequality

$$y_t\leq o(\varepsilon)+\text{const.}\int_0^t y_s\,\mathrm{d}s, \tag{4.24}$$

which implies $v_t(\xi\,|\,\mu_\varepsilon)=o(\varepsilon)$. We can then repeat exactly the above arguments for k_0-2, with k_0-1 playing the role of k_0, and the result follows for $|\xi|\geq 3$.

For $|\xi|=2$ the only change is that the integral of the term with $h=-1$ is now $\mathcal{O}(\varepsilon)$, which gives, in general $v_t(\xi\,|\,\mu_\varepsilon)=\mathcal{O}(\varepsilon)$. For $|\xi|=1$, the integration by parts formula allows us to reduce the estimate to the one for $|\xi|=2$. We omit the details.

Lemma 1 is proved. □

Appendix A: Some algebra

For $\xi\in\mathbb{Z}_\varepsilon$ we denote by $E(\xi)=\{x\colon\ \xi(x)>0\}$ the support of ξ.

Proposition A.1 (Properties of the V-functions). *The following properties hold.*

(i) (*Factorization*). *For any function $\rho:\mathbb{Z}_\varepsilon\to\mathbb{R}_+$ we have*

$$V(\xi,\eta;\rho)=\prod_{x\in E(\xi)} V_{\xi(x)}(\eta(x),\rho(x)) \tag{A.1}$$

with

$$V_k(n,u)=\sum_{l=0}^{k} Q_l(n)u^{k-l}(-1)^{k-l}\binom{k}{l},\quad k,n\in\mathbb{N},\ u\in\mathbb{R}. \tag{A.2}$$

(ii) $V(\xi,\eta;\eta)=0$ *if $\xi(x)=1$ for some x, and in general*

$$|V(\xi,\eta;\eta)|<\text{const.}\left(\max_{x\in E(\xi)}\eta(x)\right)^{[(|\xi|+1)/2]}. \tag{A.3}$$

$$\text{(iii)}\qquad V_k(n,u)V_m(n,u)=\sum_{r=0}^{k+m}c_r(k,m;u)V_r(n,u) \tag{A.4}$$

with

$$|c_r(k,m;u)|\le c(k,m)(\max(|u|,1))^{k+m}. \tag{A.5}$$

Proof. Property (i) holds for a single site and in general it can be proven by iteration.

The first assertion of (ii) follows immediately from property (i). It follows also that the second assertion is a consequence of the following inequality

$$|V_k(n,n)|\le \text{const. } n^{[(k+1)/2]}. \tag{A.6}$$

The proof of (A.6) goes as follows. Clearly there are coefficients $d_h(l)$ for which

$$Q_l(n)=\sum_{h=0}^{l-1}d_h(l)n^{l-h},\quad d_0=1, \tag{A.7}$$

and substituting into (A.2) for $u=n$, inverting the summation order (with the convention $d_h(l)=0$ for $h\ge l$),

$$V_k(n,n)=\sum_{h=0}^{k-1}\sum_{l=0}^{k}n^{k-h}(-1)^{k-l}\binom{k}{l}d_h(l). \tag{A.8}$$

It is easy to see that $d_h(l)$ is a polynomial in l of degree at most $2h$, hence it can be written as a linear combination of $Q_r(l)$, $r=0,\dots,2h$, with coefficients $c_r(h)$. Substituting into (A.8) we see that the coefficient of n^{k-h} is

$$\sum_{r=0}^{2h}c_r(h)\sum_{l=0}^{k}Q_r(l)(-1)^{k-l}\binom{k}{l}=\sum_{r=0}^{2h}c_r(h)\left(\frac{d}{dx}\right)^r(x-1)^k\bigg|_{x=1}$$

and is zero if $2h<k$, which proves assertion (ii).

To prove assertion (iii) observe that the relation

$$Q_k(n)Q_m(n)=\sum_{j=0}^{\min(k,m)}\binom{k}{j}\binom{m}{j}j!\,Q_{k+m-j}(n) \tag{A.9}$$

holds for $m=0$, and is proved in general by iteration, using the recursion relation $nQ_k(n)=Q_{k+1}(n)+kQ_k(n)$. It follows that

$$\begin{aligned}V_k(n,u)V_m(n,u)&=\sum_{j_1=0}^{k}\sum_{j_2=0}^{m}\binom{k}{j_1}\binom{m}{j_2}(-u)^{k+m-j_1-j_2}Q_{j_1}(n)Q_{j_2}(u)\\&=\sum_{j_1=0}^{k}\sum_{j_2=0}^{m}\binom{k}{j_1}\binom{m}{j_2}(-1)^{k+m-j_1-j_2}\\&\qquad\times\sum_{l}\binom{j_1}{l}\binom{j_2}{l}l!\,Q_{j_1+j_2-l}(n).\end{aligned} \tag{A.10}$$

To transform the right-hand side of (A.10) into a linear combination of V-functions, we introduce a linear mapping $\mathscr{F}$ of the vector space spanned by the polynomials (in n) Q_k into the vector space of the usual polynomials in $z\in\mathbb{R}$, by setting

$\mathscr{F}(Q_k)=z^k$. Clearly $\mathscr{F}$ is a vector space isomorphism. A simple check shows that $\mathscr{F}(V_k(n,u))=(z-u)^k$. Applying $\mathscr{F}$ to the right-hand side of (A.10), and writing z^k as $(z-u+u)^k$, one finds

$$\sum_{j_1=0}^{k}\sum_{j_2=0}^{m}\binom{k}{j_1}\binom{m}{j_2}(-1)^{k+m-j_1-j_2} \times\sum_l\sum_r\binom{j_1}{l}\binom{j_2}{l}l!\binom{j_1+j_2-l}{r}u^{k+m-l-r}(z-u)^r. \tag{A.11}$$

Transforming back by $\mathscr{F}^{-1}$ one finds that (A.4) holds with

$$c_r(k,m;u)=\sum_{j_1=0}^{k}\sum_{j_2=0}^{m}\binom{k}{j_1}\binom{m}{j_2}(-1)^{k+m-j_1-j_2}\sum_l\binom{j_1}{l}\binom{j_2}{l}l!\binom{j_1+j_2-2l}{r}u^{k+m-l-r}. \quad\square$$

Proposition A.2. *For any solution of* (2.9) ρ_t, *and any measure* ν *such that* $f_t(\xi\,|\,\nu)<\infty$ *for any* ξ, *the following relation holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}v_t(\xi,\rho_t\,|\,\nu)=\sum_{x\in\xi}\Delta_x v_t(\xi,\rho_t\,|\,\nu)+\sum_{x\in\xi}\sum_{h=-k_-+1}^{k_-}c_h(\xi(x),\rho_t)v_t(\xi^{(x,h)},\rho_t\,|\,\nu) \tag{A.12}$$

where Δ_x *is the discrete Laplacian*

$$\Delta_x f(\xi)=\tfrac{1}{2}(f(\xi^{+,x})+f(\xi^{-,x})-2f(\xi)),$$

$$\xi^{\pm,x}(y)=\begin{cases}\xi(y) & \textit{if } y\neq x,\ x\pm1,\\ \xi(x)-1 & \textit{if } y=x,\\ \xi(x\pm1)+1 & \textit{if } y=x\pm1,\end{cases}$$

and

$$mc_h(m,u)=\sum_{r,j:j-r=h}m(m-1)\cdots(m-r) \times\left[\binom{k_+}{r}\binom{k_+}{j}u^{k_+-j}-\binom{k_--1}{r}\binom{k_-}{j}u^{k_--j}\right] -\delta_{h,0}(u^{k_+}-u^{k_-}),\quad m-1+h\geqslant0, \tag{A.12'}$$

$$c_h(m,u)=0,\quad m-1+h<0 \tag{A.12''}$$

($\binom{k}{n}=0$ *for* $n>k$ *or* $n<0$).

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}v_t(\xi,\rho_t\,|\,\nu)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\xi'\cup\xi''=\xi}f_t(\xi'\,|\,\nu)(-1)^{|\xi''|}\prod_{x\in\xi''}\rho_t(x)\right)$$
$$=\sum(-1)^{|\xi''|}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f_t(\xi'\,|\,\nu)\prod_{x\in\xi''}\rho_t(x) +f_t(\xi'\,|\,\nu)\sum_{x\in\xi''}\frac{\mathrm{d}\rho_t(x)}{\mathrm{d}t}\prod_{y\in\eta''\setminus\{x\}}\rho_t(y)\right\}$$
$$\equiv E_\nu(\text{‘}L_0\text{’}V(\xi,\eta_t;\rho_t)+\text{‘}L_G\text{’}V(\xi,\eta_t;\rho_t)) \tag{A.13}$$

with

$$
\begin{aligned}
&`L_0\text{'} V(\xi, \eta; \rho_t) \\
&\quad = \sum (-1)^{|\xi''|} \Big\{ (L_0 Q(\xi', \eta)) \prod_{x \in \xi''} \rho_t(x) \\
&\qquad\qquad + Q(\xi', \eta) \sum_{x \in \xi''} \Delta_x \rho_t(x) \prod_{y \in \xi'' \setminus \{x\}} \rho_t(y) \Big\},
\end{aligned} \tag{A.14}
$$

$$
\begin{aligned}
&`L_G\text{'} V(\xi, \eta; \rho_t) \\
&\quad = \sum (-1)^{|\xi''|} \Big\{ L_G Q(\xi', \eta) \prod_{x \in \xi''} \rho_t(x) \\
&\qquad\qquad + Q(\xi', \eta) \sum_{x \in \xi''} (\rho_t^{k_+}(x) - \rho_t^{k_-}(x)) \prod_{y \in \xi'' \setminus \{x\}} \rho_t(y) \Big\}.
\end{aligned} \tag{A.14'}
$$

One can easily show that

$$
L_0 Q(\xi, \eta) = \sum_{x \in \xi} \Delta_x Q(\xi, \eta)
$$

and that

$$
E_\nu(`L_0\text{'} V(\xi, \eta_t; \rho_t)) = \sum_{x \in \xi} \Delta_x v_t(\xi, \rho_t \mid \nu).
$$

For the second term observe that

$$
\begin{aligned}
L_G Q(\xi, \eta) &= \sum_{x \in E(\xi)} Q(\xi_{(x)}, \eta) L_G Q_{\xi(x)}(\eta(x)) \\
&= \sum_{x \in E(\xi)} Q(\xi_{(x)}, \eta) \xi(x) \{ Q_{k_+}(\eta(x)) Q_{\xi(x)-1}(\eta(x)) \\
&\qquad - Q_{k_-}(\eta(x)) Q_{\xi(x)-1}(\eta(x)-1) \}
\end{aligned} \tag{A.15}
$$

where $\xi_{(x)}$ is the restriction of ξ to $E(\xi) \setminus \{x\}$. Therefore

$$
`L_G\text{'} V(\xi, \eta; \rho) = \sum_{x \in E} V(\xi_{(x)}, \eta; \rho) `L_G\text{'} V_{\xi(x)}(\eta(x), \rho(x)) \tag{A.16}
$$

with

$$
\begin{aligned}
&`L_G\text{'} V_m(n, u) \\
&\quad = \sum_{p=0}^{m} \binom{m}{p} (-1)^{m-p} \{ L_G Q_p(n) u^{m-p} \\
&\qquad\qquad + Q_p(n)(m-p) u^{m-p-1}(u^{k_+} - u^{k_-}) \} \\
&\quad = \sum_{p=0}^{m} \binom{m}{p} (-1)^{m-p} L_G Q_p(n) u^{m-p} - m V_{m-1}(n, u)(u^{k_+} - u^{k_-}).
\end{aligned} \tag{A.17}
$$

Using relations (A.9) and (A.15) one finds

$$\begin{aligned} L_G Q_k(n) &= k\{Q_{k_+}(n)Q_{k-1}(n) - Q_{k_-}(n)Q_{k-1}(n-1)\} \\ &= \sum_{j=0}^{k_+} \binom{k_+}{j} k(k-1)\cdots(k-j)Q_{k+k_+-1-j}(n) \\ &\quad - \sum_{j=0}^{k_-} \binom{k_- - 1}{j} k(k-1)\cdots(k-j)Q_{k+k_--1-j}(n). \end{aligned}$$

Substituting into the first term of the second line of (A.17) and applying the mapping $\mathscr{F}$ introduced in the proof of Proposition A.1, we get a first term

$$\begin{aligned} &\sum_{r=0}^{k_+} \binom{k_+}{r} z^{k_+} \sum_{p=0}^{m} \binom{m}{p} (-1)^{m-p} u^{m-p} p(p-1)\cdots(p-r) z^{p-1-r} \\ &= \sum_{r=0}^{k_+} \binom{k_+}{r} z^{k_+} \frac{\mathrm{d}^{r+1}}{\mathrm{d}z^{r+1}} (z-u)^m \\ &= \sum_{r,j=0}^{k_+} \binom{k_+}{r}\binom{k_+}{j} u^{k_+-j} m(m-1)\cdots(m-r)(z-u)^{m-r-1+j} \end{aligned}$$

and a corresponding term

$$- \sum_{r,j=0}^{k_-} \binom{k_- - 1}{r}\binom{k_-}{j} u^{k_--j} m(m-1)\cdots(m-r)(z-u)^{m-r-1+j}.$$

Transforming back by $\mathscr{F}^{-1}$ and substituting into (A.17) we find

$$\text{`}L_G\text{'}\, V_m(n, u) = \sum_{h=-k_-+1}^{k_-} m c_h(m, u) V_{m-1+h}(n, u)$$

which implies the result. □

Remark A.1. As a consequence of Proposition A.2 the following 'integration by parts' formula holds.

$$\begin{aligned} v_s(\xi, \rho_s \mid \nu) &= \sum_{\xi_1} \pi_s^\varepsilon(\xi \to \xi_1) v_0(\xi_1, \rho_0 \mid \nu) \\ &\quad + \int_0^s \mathrm{d}s' \sum_{\xi_1} \pi_{s-s'}^\varepsilon(\xi \to \xi_1) \sum_{x_1 \in \xi_1} \sum_{h=-k_-+1}^{k_-} c_h(\xi_1(x_1), \rho_{s'}(x_1 \mid \eta_{t_k})) \\ &\quad \times v_{s'}(\xi_1(x_1), \rho_{s'} \mid \nu) \end{aligned} \tag{A.18}$$

where ν is any initial measure and ρ_t is a solution of (2.9).

Appendix B: Proof of (4.13)

The term Γ in (4.12) can be represented by a graph with vertices at the points (t, x), $x \in \xi$ (initial vertices), (O, y), $y \in \xi_{l+1}$ (final vertices), and (s_i, x_i), $i = 1, \ldots, l$

(interaction vertices). Each bond $\{(s,x),(s',x')\}$, $s>s'$ corresponds to a factor $\pi^{\varepsilon}_{s-s'}(x\to x')$. $s-s'$ is the 'length' of the bond. A bond is incoming (outgoing) at (s,x) if the other endpoint (s',x') has a higher (lower) time coordinate s'.

The number of the incoming bonds, m, and of the outgoing bonds, p, at an interaction vertex depend on the interaction parameter h, and we distinguish vertices of the following types:

$$
\begin{array}{llll}
h=1, & m=1, & p=1 & \text{(type (o))},\\
h>1, & m=1, & p=h & \text{(type (i) creation)},\\
h=0, & m=2, & p=1 & \text{(type (ii) destruction of 1 particle)},\\
h\leqslant -1, & m=-h+1, & p=0 & \text{(type (iii) destruction of } -h+1 \text{ particles)}.
\end{array}
$$

The interaction vertices are graphically represented in Figure 1. A typical graph is shown in Figure 2.

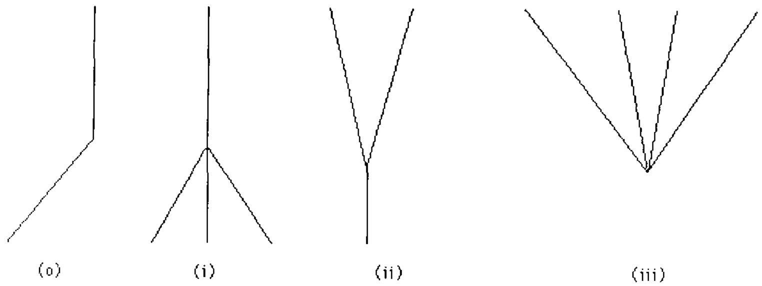

Fig. 1.

Since the initial vertices are kept fixed, we shall always consider that we have as many distinct initial vertices as there are particles in ξ, even though the positions of the particles may coincide (in which case one may think of drawing them on the graph as 'close' points).

Clearly the sum (4.12) decomposes into a product of factors corresponding to the connected subgraphs. Hence it is enough to consider any connected graphs (i.e. graphs for which any two vertices can be connected by a sequence of bonds belonging to the graph).

The estimate is done in two steps. The first step consists in eliminating the vertices of type (o), and then the vertices of type (ii) and (iii), which prescribe multiplicities $m_i>1$. The sum (4.12) is then estimated by a corresponding sum which involves a simpler ('reduced') graph $\bar{\Gamma}$ which has an interaction vertices only a subset of the vertices of type (i) of Γ. The final step is the estimate of the contribution of the reduced graph.

Step 1. Reduced graphs. We first sum expression (4.12) over the positions of the vertices of type (o). If (s,x) is such a vertex, $h=1$ and $\xi(x,h)=\xi(x)$. Summing over x we obtain by the composition rule

$$\sum_x \pi^{\varepsilon}_{s_j-s}(x_j\to x)\,\pi^{\varepsilon}_{s-s_k}(x\to x_k)=\pi^{\varepsilon}_{s_j-s_k}(x_j\to x_k).$$

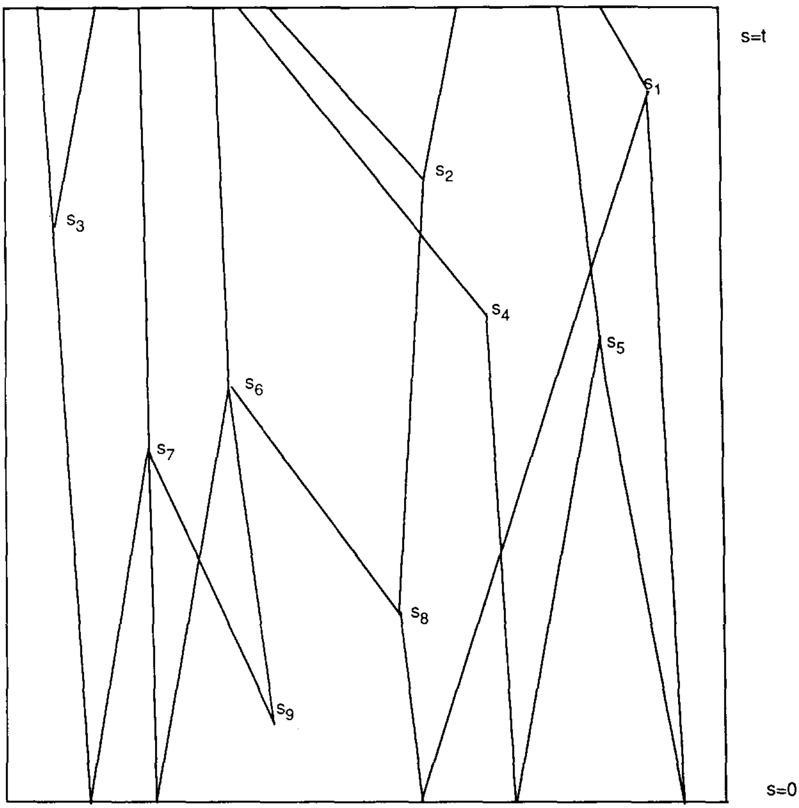

Fig. 2. A graph.

Here $(s, x) = (s_i, x_i)$ for some i, and $0 \leq j < i < k < l+1$ (for $j = 0$, $s_0 = t$, $x_0 \in \xi$, and for $j = l+1 - s_{l+1} = 0$, $x_{l+1} \in \xi_{l+1}$). The result is then expressed as a sum over vertices of a new graph which is obtained from Γ by canceling the vertex (s, x) and substituting the two bonds $\{(s_j, x_j), (s, x)\}$ and $\{(s, x), (s_k, x_k)\}$ with the single bond $\{(s_j, x_j), (s_k, x_k)\}$.

Next we estimate the sum over the positions of the vertices of type (ii) ($h_i = 0$: two incoming and one outgoing bond). Let (s^*, x^*) be the one with lowest value of s. We estimate the factor $\pi^{\varepsilon}_{s_i - s^*}(x_i \to x^*)$ corresponding to the incoming bond of minimal length (to one of the two if they have equal length) by $2\varepsilon/\sqrt{s_i - s^*}$. The product of the two π's that is left is summed over x^* using the composition rule. The graph is modified by removing the bond $\{(s_i, x_i), (s^*, x^*)\}$ and joining the two bonds at (s^*, x^*) that survive into a single one, as before. The vertex at the other end (s_i, x_i) of a bond that is canceled can be either an initial vertex, or a vertex of type (i) or (ii). In the first case by canceling the bond we cancel a particle from the initial configuration ξ. If it is a vertex of type (i), and by canceling the bond it becomes of type (o), we get rid of it by summing over the position x_i as described before. If (s_i, x_i) has more than two outgoing bonds it remains a vertex of type (i) with one less outgoing bond. Finally if (s_i, x_i) is a vertex of type (ii) then it becomes a vertex of type (iii). We then repeat the procedure for the vertices of type (ii) that are left (in the modified graph), in order of increasing s, each time modifying the graph as described.

Finally we consider the vertices of type (iii) in the graph thus obtained. Let $(\hat{s}, \hat{x})$ be the one with lowest s value. We estimate the factor corresponding to one of the bonds with minimal length by $2\varepsilon/\sqrt{s_i - \hat{s}}$, and estimate the sum over the vertex position $\hat{x}$ of the remaining product of π's by 1. The graph is modified by canceling all the bonds going into $(\hat{s}, \hat{x})$. Each time that a bond is canceled the vertex at the other end is treated as described in the discussion for vertices of type (ii) above. We then proceed with the vertices of type (iii) that are left in order of increasing s.

The net result is given by the following inequality:

$$A_t(s_1, \ldots, s_l) \leq (2\varepsilon)^{l_1} \prod_{j=1}^{l_1} \frac{1}{\sqrt{s_{i_j - r_j} - s_{i_j}}} \bar{A}_t(s_1'', \ldots, s_{l_2}'') \tag{B.1}$$

with

$$\bar{A}_t(s_1'', \ldots, s_{l_2}'') \equiv \sum_{x_1'', \ldots, x_{l_2}''} \sum_{z_1, \ldots, z_r} \bar{\Gamma}_{t, s_1'', \ldots, s_{l_2}''}(\xi'; x_1'', \ldots, x_{l_2}''; z_1, \ldots, z_r) \tag{B.2}$$

where l_1 is the total number of vertices of type (ii) and (iii), $\{s_{ij}\}_{j=1}^{l_1}$ are their time coordinates, $s_{i_j - r_j} - s_{i_j}$ is the length of the bonds corresponding to the π factors that have been estimated, l_2 is the number of vertices of type (i) that survived, $\{(x_i'', s_i'')\}_{i=1}^{l_2}$ are their coordinates, and $z_1, \ldots, z_r$ are the positions of the final vertices.

Figure 3 shows the reduced graph $\bar{\Gamma}$ corresponding to the graph Γ of Figure 2.

Step 2. Estimate of reduced graphs. The following inequality holds:

$$\bar{A}_t(s_1'', \ldots, s_{l_2}'') \leq (2\varepsilon/\sqrt{t})^{[(|\xi'|+1)/2]} \tag{B.3}$$

Proof of (B.3). We divide the bonds of $\bar{\Gamma}$ into three types: the type I bonds are the bonds of length t (noninteracting particles), the type II bonds are the bonds connecting interaction vertices to final vertices, and the type III bonds are those connecting initial and interaction vertices.

In estimating the contribution of $\bar{\Gamma}$ a crucial role is played by the fact that all final vertices have at least two incoming bonds, otherwise the V functions are zero. The final vertices are also divided into two types. The type (a) vertices with I-bonds only, the type (b) vertices with at least one II-bond.

$\bar{\Gamma}$ is not necessarily connected if Γ is connected. Clearly the sum in (B.3) decomposes into a product of factors corresponding to the connected components. Therefore we carry out the estimate only for connected graphs $\bar{\Gamma}$.

Since each III-bond has its own distinct initial vertex, either $\bar{\Gamma}$ has a single (a)-vertex or it has no (a)-vertex. Let k denote the number of I-bonds in $\bar{\Gamma}$.

Case A: a single (a)-vertex. Let z denote its position. We write $(2\varepsilon/\sqrt{t})^{k-1}$ for the product of $k-1$ π factors, and sum over z the π that is left, i.e. the contribution of the graph is bounded by $(2\varepsilon/\sqrt{t})^{k-1}$.

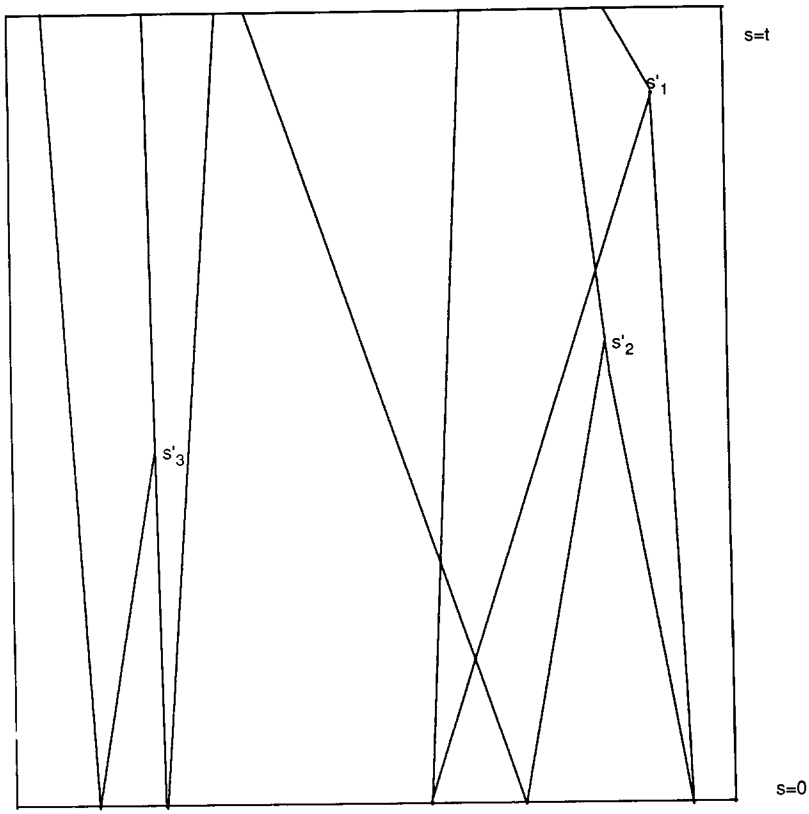

Fig. 3. The reduced graph corresponding to the graph of Figure 2.

Case B: *no (a)-vertex*. We estimate each I-bond by $2\varepsilon/\sqrt{t}$, obtaining a factor $(2\varepsilon/\sqrt{t})^k$. The sum over positions of all interaction and final vertices of the π factors corresponding to II- and III-bonds is clearly bounded by 1.

Inequality (B.3) is proved.

Proof of (4.13). We set $s'_j = s_{i_j}, j = 1, \ldots, l_1$, and $\hat{s}_j = s_{i_j - r_j}$. By inequalities (B.1) and (B.3),

$$\int_0^t \mathrm{d}s_1 \cdots \int_0^{s_{l-2}} \mathrm{d}s_{l-1} \int_0^{s_{l-1}} \mathrm{d}s_l \, A_t(s_1, \ldots, s_l)$$
$$\leqslant \text{const.} \left(\frac{\varepsilon}{\sqrt{t}}\right)^{[(|\xi'|+1)/2]} \varepsilon^{l_1} \int_0^t \mathrm{d}s_1 \cdots \int_0^{s_{l-2}} \mathrm{d}s_{l-1} \int_0^{s_{l-1}} \mathrm{d}s_l \prod_{j=1}^{l_1} \frac{1}{\sqrt{|\hat{s}_j - s'_j|}}. \tag{B.4}$$

We extend the integration up to time t in all variables. By integrating first over the variables s'_j, we get factors $2(\hat{s}_j^{1/2} + (t - \hat{s}_j)^{1/2}) \leqslant 4\sqrt{t}$. Performing the integration over the remaining $l - l_1$ variables, we obtain that the right-hand side of (B.4), for $t \leqslant \varepsilon^{\beta}$, is bounded by

$$\text{const. } t^{l}\left(\frac{\varepsilon}{\sqrt{t}}\right)^{[(|\xi'|+1)/2]+l_1} \leqslant \text{const. } \varepsilon^{\beta l}\varepsilon^{(1-\beta/2)([(|\xi'|+1)/2]+l_1)}. \tag{B.5}$$

Since $|\xi'|+l_1k_- \geqslant |\xi|$, because no more than k_- particles can disappear at each interaction, we see that expression (B.5) is bounded by the right-hand side of inequality (4.13). $\square$

Acknowledgments

We thank Errico Presutti for suggestions and discussions. One of us (A.P.) thanks Pierre Picco for the hospitality at Centre de Physique Theorique, Marseille Luminy.

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