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Interface Fluctuations and Couplings in the $D = 1$ Ginzburg–Landau Equation with Noise[1]

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We consider a Ginzburg–Landau equation in the interval $[-\varepsilon^{-\kappa}, \varepsilon^{-\kappa}]$, $\varepsilon > 0$, $\kappa \geqslant 1$, with Neumann boundary conditions, perturbed by an additive white noise of strength $\sqrt{\varepsilon}$. We prove that if the initial datum is close to an "instanton" then, in the limit $\varepsilon \to 0^+$, the solution stays close to some instanton for times that may grow as fast as any inverse power of ε, as long as "the center of the instanton is far from the endpoints of the interval". We prove that the center of the instanton, suitably normalized, converges to a Brownian motion. Moreover, given any two initial data, each one close to an instanton, we construct a coupling of the corresponding processes so that in the limit $\varepsilon \to 0^+$ the time of success of the coupling (suitably normalized) converges in law to the first encounter of two Brownian paths starting from the centers of the instantons that approximate the initial data.

KEY WORDS: Stochastic PDEs; Interface dynamics; invariance principle; coupling of infinite dimensional processes.

1. INTRODUCTION

In this paper we study the Ginzburg–Landau equation perturbed by an additive white noise α of strength $\sqrt{\varepsilon}$,

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 m}{\partial x^2} - V'(m) + \sqrt{\varepsilon}\,\alpha \tag{1.1}$$

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where $\varepsilon > 0$ is a small parameter that eventually goes to 0 and $V(m)$, $m \in \mathbb{R}$, is a symmetric double well potential, with minima at $m = \pm 1$. We consider this equation in the interval $\mathscr{T}_{\varepsilon, \kappa} = [-\varepsilon^{-\kappa}, \varepsilon^{-\kappa}]$, $\kappa \geqslant 1$ with Neumann boundary conditions (N.b.c.).

We call pure phases the two constant functions $m(x) = \pm 1$, $x \in \mathscr{T}_{\varepsilon, \kappa}$ and we study the interface dynamics, that is the evolution of profiles that are close to the two pure phases to the left and to the right of some point, say x_0. The deterministic Ginzburg–Landau equation (i.e., setting $\varepsilon = 0$ in (1.1) and considering the equation in the whole $\mathbb{R}$) has a stationary solution $\bar{m}(x) = \tanh x$, $x \in \mathbb{R}$ that we call "instanton". The solution $\bar{m}$ is a wavefront with speed 0, that connects the two pure phases. The set of all the translates of $\bar{m}$ is stable, that is if the initial datum is close to $\bar{m}(x - x_0)$, for some "center" x_0, then the solution of the deterministic Ginzburg–Landau converges to an instanton with center x'_0 close to x_0.

Here we prove a similar stability result for the stochastic Ginzburg–Landau equation, showing that the solution of (1.1) with initial condition close to the restriction of some instanton to $\mathscr{T}_{\varepsilon, \kappa}$, remains close to the set of translate of $\bar{m}(x - x_0)$, for times of the order of any inverse power of ε, and that its center, suitably normalized, converges to a Brownian motion.

Our motivation for studying this problem comes from the physics of the spinodal decomposition. This is the phenomenon that appears in a quenching experiment. In a quenching experiment a system is in thermodynamic equilibrium with a reservoir whose temperature is suddenly varied from above to below the critical temperature of the system. This process is in general so fast that we may suppose the state of the system unchanged at the end of the cooling, but no longer in equilibrium with the reservoir. The state is still stationary, but unstable. What happens next is called "phase separation". At a very early stage the interaction with the environment (usually modeled by a small random perturbation) is dominant because the system without perturbations would stay in its initial stationary state. As soon as the state changes, the deterministic forces internal to the system take over and drive it quickly away from the initial unstable equilibrium, see De Masi *et al.*[9] and references therein for a mathematical analysis of the phenomenon. In general equilibrium is not yet reached at this stage, at least when the system is spatially extended. In each subregion, in fact, the state will approach a stable thermodynamic phase, but since several ones are equally accessible (as the temperature is below its critical value) there is no reason why the equilibria of far away regions should be coincide. The typical picture is thus a collection of phases with interfaces in between and the next regime of phase separation describes the competition between phases. For nonconservative evolutions (that we consider here) the larger clusters grow at the expenses of the smaller ones and typically the interfaces move by mean curvature. Once the clusters become very large (i.e., the curvature very small) this process becomes extremely slow and new effects arise, in particular stochastic forces may again be dominant. As a matter of fact there are examples involving deterministic evolutions where the phase separation even stops and the system fails to reach the true equilibrium, being stuck in some locally stable but spurious equilibrium, see Friesecke and McLeod.(14) These effects are more frequent in one dimension, therefore our equation is a good model for studying these problems.

Fusco and Hale,[15] and Carr and Pego,[6] have studied the deterministic Ginzburg–Landau equation in the finite interval $\mathscr{T}_{\varepsilon,\kappa}$ with Neumann b.c. and with initial datum close to the two pure phases to the right and to the left of some point x_0 respectively. They prove that the solution relaxes in a short time to an almost stationary state which represents a front connecting the two stable phases, $m = \pm 1$. This front is very close to the "instanton" $\bar{m}_{x_0}(x) = \bar{m}(x - x_0)$ restricted to the finite interval (In the literature it is usual to call instanton the space derivative $\bar{m}'$ of $\bar{m}$). As already mentioned the front which has been formed in $\mathscr{T}_{\varepsilon,\kappa}$ is not truly stationary, in fact it moves but extremely slowly, with speed $\approx e^{-c\ell}$, c a positive "slowly varying" factor, ℓ the distance of the center from the boundary of $\mathscr{T}_{\varepsilon,\kappa}$. During this motion the front keeps almost the same shape.

If we take into account the stochastic term, the picture initially does not change much: except for small deviations we still have a short relaxation time and the formation of a profile very close to a front. However, under the action of the noise, the front moves at times dramatically shorter than in the deterministic case. At times $t_\varepsilon \doteq t\varepsilon^{-a}$, $0 < a < 1/3$, $t > 0$, the shape is still the same but the center x_0 has moved by $\approx \sqrt{\varepsilon t_\varepsilon}$ and on this scale it is a Brownian motion.[3] At times $t_\varepsilon = \doteq t\varepsilon^{-1}$, $t > 0$, the displacement is now finite and the motion of the center converges as $\varepsilon \to 0^+$ to a Brownian motion b_t, as shown by Brassesco *et al.*[5] when $\kappa = 1$. The motion is still Brownian also at times $t_\varepsilon \doteq t\varepsilon^{-1-\gamma}$, $t > 0$, $\gamma > 0$ small enough, as shown by Funaki,[16] in a somewhat different setup. At much longer times the picture may in principle change, for instance the system could pick up some drift, as it happens when the potential V is nonsymmetric, as shown by Brassesco and Buttà.[4] In this paper we prove that in the symmetric case there is no drift for times that grow as any inverse power of ε: roughly speaking we prove that the process m_t is close to $\bar{m}_{x_0 + \sqrt{\varepsilon}\, b_t}$ (in the sense that the sup norm of the difference vanishes as $\varepsilon \to 0^+$) with b_t a Brownian motion with diffusion $D = 3/4$. Convergence is proved by suitably scaling space and times and before "extinction", i.e., when one of the two phases disappears and only one remains. Many problems are left out of our analysis as for instance a precise estimate of the extinction time. We have only estimates on the first time τ when one of the two phases shrinks to cover $\mathscr{T}_{\varepsilon,\kappa}$ except an interval of size proportional to ε^{-1} close to one of the endpoints of $\mathscr{T}_{\varepsilon,\kappa}$. We show that as $\varepsilon \to 0^+$, τ has the same law as a stopping time defined in terms of b_t. We could improve (with a lot of work) the result till intervals of the order of $c \log \varepsilon^{-1}$, with $c > 0$ large enough. But for a critical value of c the Fusco–Hale drift should become dominant with the minority phase shrinking deterministically still extinction. Since the time (after τ) when all this happens should be significantly smaller than τ itself, τ would become a good estimate for the extinction time. The one phase regime, however, is not yet the true equilibrium, as, at much longer times, tunnelling phenomena become important, see Faris and Lasinio,(11) and Cassandro *et al.*(7) for the analysis of these aspects in finite volumes and, respectively, in intervals that grow logarithmically with ε^{-1}.

The convergence to a one-dimensional process described by a simple Brownian motion holds in a much stronger sense than one might suspect from this presentation. In fact by extending the work of Mueller,(17), to the present case (his stable point being replaced by our one-dimensional manifold of equilibria, i.e., the translates of the instanton) we construct, in the limit as $\varepsilon \to 0^+$, a coupling of two processes starting from two different data, m and m', where the time of success of the coupling (i.e., when m_t and m'_t become almost everywhere equal) has the same law as the first encounter of two independent Brownian motions in $d = 1$ that start from the centers of the instantons associated to m and m'. We refer to the next section for the precise statements. In Section 2, we prove some properties of the deterministic equation. In Section 3, we extend the results to the case with noise. In Section 4, we prove convergence to the Brownian motion and in Section 5 we construct the coupling which proves the loss of memory of the initial datum. A brief outline of the main ideas of the proof is given in Section 3 after the proof of Theorem 1.

1.1. Definitions and Main Results

We consider the Ginzburg–Landau one-dimensional stochastic partial differential equation

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 m}{\partial x^2} - [m^3 - m] + \sqrt{\varepsilon}\,\alpha, \quad t \geqslant 0; \qquad x \in \mathscr{T}_{\varepsilon,\kappa} \doteq [-\varepsilon^{-\kappa}, \varepsilon^{-\kappa}]$$
$$m(x, 0) = m_0(x) \tag{1.2}$$

with Neumann boundary conditions (N.b.c.). The noise $\alpha=\alpha(x,t)$ is a standard space-time white noise (see Walsh[19] for a precise definition) and ε is a small positive parameter that will eventually go to zero. Let us denote by $H_t^{(\varepsilon)}$ the Green operator for the heat equation with N.b.c. in $\mathcal{T}_{\varepsilon,\kappa}$, and by $H_t^{(\varepsilon)}(x,y)$ the corresponding kernel. The standard way to give a precise meaning to (1.2) is to consider the integral equation that results of formally applying $H_t^{(\varepsilon)}$ on both sides of (1.2). For a given continuous function m_0 defined in $\mathcal{T}_{\varepsilon,\kappa}$ and satisfying N.b.c., we then get

$$m_t=H_t^{(\varepsilon)}m_0-\int_0^t ds\, H_{t-s}^{(\varepsilon)}(m_s^3-m_s)+\sqrt{\varepsilon}\, Z_t, \qquad t\geqslant 0, \quad w\in\mathcal{T}_{\varepsilon,\kappa} \tag{1.3}$$

where $Z_t(x)$ is the Gaussian process defined by the stochastic integral in the sense of Walsh.[19]

$$Z_t(x)\doteq\int_0^t ds\int_{\mathcal{T}_{\varepsilon,\kappa}} dy\, \alpha(s,y)\, H_{t-s}^{(\varepsilon)}(x,y) \tag{1.4}$$

The process $Z_t(x)$ can be seen to be continuous in both variables [Walsh[19]], and its follows as in Faris Lasinio[11] that (1.3) has a unique continuous solution m_t, which we will call the Ginzburg–Landau process.

Since we are interested in studying the evolution of m_t when the initial datum is close to an instanton, and to use the stability of the instanton under the dynamics in the whole line, it will be convenient in the sequel to consider the integrals in $\mathbb{R}$ instead of $\mathcal{T}_{\varepsilon,\kappa}$ as in Ref. 5. To this end, given a continuous function f defined in $\mathcal{T}_{\varepsilon,\kappa}$, call $\check{f}$ it extension to $\mathbb{R}$ given by successive reflections around $(2n+1)\varepsilon^{-\kappa}$, $n\in\mathbb{Z}$, and define the space of functions so obtained

$$C_{\varepsilon,\kappa}(\mathbb{R})=\{f\colon f\in C^0(\mathbb{R}), f \text{ is invariant by reflections around the points } (2n+1)\,\varepsilon^{-\kappa}, n\in\mathbb{Z}\}$$

Consider then

$$m_t=H_t m_0-\int_0^t ds\, H_{t-s}(m_s^3-m_s)+\sqrt{\varepsilon}\, Z_t, \qquad t\geqslant 0, \quad x\in\mathbb{R}, \tag{1.5}$$

where H_t is the heat operator in $\mathbb{R}$, and the last term Z_t is really the extension to $\mathbb{R}$ of the process Z_t defined by (1.4). As it follows for instance from Doering,[10] Eq. (1.5) has a unique continuous solution m_t if the initial datum m_0 is bounded and continuous. We set

$$m_t=\colon T_t(m_0; \sqrt{\varepsilon}\, Z) \qquad t\geqslant 0 \tag{1.6}$$

for the solution m_t of (1.5). As proved by Brassesco *et al.*[5] in Prop. 2.3, given m_0 defined in $\mathcal{T}_{\varepsilon,\kappa}$ and satisfying N.b.c., m_t is a solution of (1.3) if and only if its extension to $\mathbb{R}$, $\check{m}_t$ solves (1.5), with initial condition and noise being the respective extensions of m_0 and Z_t. In case $m_0 \in C_{\varepsilon,\kappa}(\mathbb{R})$, by an abuse of notation we will also refer to $T_t(m_0; \sqrt{\varepsilon}\, Z)$ as the Ginzburg–Landau process (in $\mathcal{T}_{\varepsilon,\kappa}$ with N.b.c.). We thus have two ways of representing the same process, one in $\mathcal{T}_{\varepsilon,\kappa}$ with $H_t^{(\varepsilon)}$ and the other in $\mathbb{R}$ with H_t and we will switch from one to the other according to which one is more convenient in the specific application.

We leave for a while the discussion on the Ginzburg–Landau process turning to the deterministic case, i.e., $\varepsilon = 0$, and the Eq. (1.2) considered in $\mathbb{R}$; we denote by $T_t(m) \doteq T_t(m, 0)$ the corresponding flow. Since $\bar{m}(x) \doteq \tanh(x)$ verifies the identity

$$\frac{1}{2}\frac{d^2\bar{m}}{dx^2} = V'(\bar{m}); \qquad V'(m) \doteq \frac{dV(m)}{dm}, \qquad V(m) \doteq -\frac{1}{2}m^2 + \frac{1}{4}m^4, \qquad m \in \mathbb{R} \tag{1.7}$$

it is a stationary point of the flow, namely $\bar{m} = T_t(\bar{m})$ for any $t \geqslant 0$. The function $\bar{m}(x)$ is what we call "instanton" and 0 is "its center". Its translate by $x_0 \in \mathbb{R}$,

$$\bar{m}_{x_0}(x) \doteq \bar{m}(x - x_0), \qquad x \in \mathbb{R} \tag{1.8}$$

is also stationary and will be called "instanton centered at x_0" (thus $\bar{m} \equiv \bar{m}_0$), Fife and McLeod,[12,13] have proved that the manifold

$$\mathcal{M} \doteq \{\bar{m}_{x_0}, x_0 \in \mathbb{R}\} \subset C^0(\mathbb{R}) \tag{1.9}$$

is locally attractive under the deterministic flow $T_t(\cdot)$. More precisely, let $\|\cdot\|$ denote the sup norm in $\mathbb{R}$, and, for $\delta \geqslant 0$, define

$$\mathcal{M}_\delta \doteq \{m \in C^0(\mathbb{R}): \operatorname{dist}(m, \mathcal{M}) \doteq \inf_{x_0 \in \mathbb{R}} \|m - \bar{m}_{x_0}\| \leqslant \delta\} \tag{1.10}$$

Then, there exists $\delta^* \geqslant 0$ and a real valued function $\zeta(m)$ defined on $\mathcal{M}_{\delta^*}$, such that

$$\lim_{t \to +\infty} T_t(m) = \bar{m}_{\zeta(m)}, \qquad \text{for all} \quad m \in \mathcal{M}_{\delta^*} \tag{1.11}$$

in sup norm and exponentially fast. Thus $\mathcal{M}_{\delta^*}$ is foliated by the submanifolds (transversal to $\mathcal{M}$):

$$\mathcal{V}_{x_0} \doteq \{m \in \mathcal{M}_{\delta^*}: \zeta(m) = x_0\} \tag{1.12}$$

which are space translated of each other. $\zeta(m)$ will be called the "true center" of $m \in \mathcal{M}_{\delta^*}$. In practice it will be more convenient to work with an approximate center, the "linear center", but for expository reasons in this section we refer to the true center.

When we restrict to $\mathcal{T}_{\varepsilon,\kappa}$ we evidently loose the notion of instanton and the reader may ask why to consider $\mathcal{T}_{\varepsilon,\kappa}$ instead of the whole $\mathbb{R}$. The reason is technical, the price would be having to deal with an unbounded process. Essentially the same problems arise in intervals which grow exponentially with ε^{-1}, hence the restriction to $\mathcal{T}_{\varepsilon,\kappa}$. With the choice of the Neumann boundary conditions in $\mathcal{T}_{\varepsilon,\kappa}$ we have the advantage of recovering to some extent the instanton structure present in $\mathbb{R}$, as proved by Carr and Pego,[6] and Fusco and Hale,[15] and explained in the introduction, see also the introduction in Brassesco *et al.*[5] The choice of Neumann boundary conditions is important here, periodic b.c., for instance, would give rise to two fronts while Dirichlet b.c. would force the unstable $m = 0$ phase at the boundary with extra complications that our choice of N.b.c. avoids.

Our analysis will extensively exploit the stability properties of the instantons and in this respect we will take great advantage of the representation (1.5) where the only memory of the boundary conditions is in the "small perturbation" $\sqrt{\varepsilon}\, Z_t$ and in the initial state. The Eq. (1.5) is thus well suited for a perturbative analysis of data close to instantons. However even if $m \in C_{\varepsilon,\kappa}(\mathbb{R})$ is very close to an instanton in $\mathcal{T}_{\varepsilon,\kappa}$, it is not close to any instanton in the sup norm in $\mathbb{R}$. We overcome this point by using barrier lemmas that allow to modify the function away from $\mathcal{T}_{\varepsilon,\kappa}$ without changing too much evolution in $\mathcal{T}_{\varepsilon,\kappa}$. The modified function can then be taken in a neighborhood of $\mathcal{M}$ and we can adapt the results of Fife and McLeod about the convergence to an instanton. The noise will counteract this trend by preventing the orbit from getting too close to the instanton and the process will live in a small neighborhood of $\mathcal{M}$, whose size depends on the strength of the noise and vanishes as $\varepsilon \to 0^+$, see Theorem 1 later. At the same time, however, these small deviations caused by the noise have the important effect of changing the center of the deterministic evolution. Their cumulative effect will be in the end responsible for the Brownian motion on $\mathcal{M}$ that describes the limit process, see Theorem 2 later.

As explained before, the first step was to modify the functions in $C_{\varepsilon,\kappa}(\mathbb{R})$, outside $\mathcal{T}_{\varepsilon,\kappa}$ so that they are in a small neighborhood of an instanton. With this in mind, given $m \in C^0(\mathbb{R})$, we define $m^{\varepsilon,\kappa} \doteq m$ when $m \notin C_{\varepsilon,\kappa}(\mathbb{R})$ setting in the other case

$$m^{\varepsilon,\kappa}(x) \doteq \begin{cases} m(x) & \text{for} \quad |x| \leqslant \varepsilon^{-\kappa} \\ m(\pm\varepsilon^{-\kappa}) & \text{for} \quad x \geqslant \varepsilon^{-\kappa}, \text{ respectively } x \leqslant -\varepsilon^{-\kappa} \end{cases} \tag{1.13}$$

We then define

$$\zeta^{\varepsilon,\kappa}(m) \doteq \begin{cases} x_0 & \text{if } m^{\varepsilon,\kappa} \in \mathscr{V}_{x_0} \\ 2\varepsilon^{-\kappa} & \text{otherwise} \end{cases} \tag{1.14}$$

Given m_t a solution of (1.5) we set

$$\zeta_t \doteq \zeta^{\varepsilon,\kappa}(m_t) \tag{1.15}$$

and for any $0 < \ell < 1$ we define the stopping time

$$\tau_\varepsilon(\kappa, \ell) \doteq \inf\{t \geqslant 0: |\zeta^{\varepsilon,\kappa}(m_t)| \geqslant \varepsilon^{-\kappa} - \ell\varepsilon^{-1}\} \tag{1.16}$$

and the stopped process $\{\hat{m}_t\}_{t \geqslant 0}$,

$$\hat{m}_t \doteq m_{t \wedge \tau_\varepsilon}, \qquad \hat{\zeta}_t \doteq \zeta_{t \wedge \tau_\varepsilon}, \qquad \tau_\varepsilon \doteq \tau_\varepsilon(\kappa, \ell)$$

where $a \wedge b$ stands for the minimum between a and b.

Finally we denote by P^ε the probability on the space where the noise $\sqrt{\varepsilon}\,\alpha$, and consequently all the processes we consider, are constructed. E^ε denotes the corresponding expectation.

Theorem 1. There is $\delta \in (0, \delta^*/2]$, δ^* as in (1.11), and for any $\kappa \geqslant 1$, $h > 0$, $a \in (0, 1/4)$ and $\ell \in (0, 1)$ there are $c > 0$ and $p > 0$ so that the following holds. Let $m_0 \in \mathscr{M}_\delta$ and $|\zeta^{\varepsilon,\kappa}(m_0)| < \varepsilon^{-\kappa} - \ell\varepsilon^{-1}$, and m_t as in (1.6). Then

$$\begin{aligned} &P^\varepsilon((m_t)^{\varepsilon,\kappa} \in \mathscr{M}_{2\delta}\ \forall t \leqslant (\log\varepsilon)^2, (m_t)^{\varepsilon,\kappa} \in \mathscr{M}_{\varepsilon^{1/2-a}}\ \forall t \in [(\log\varepsilon)^2, \varepsilon^{-h} \wedge \tau_\varepsilon]) \\ &\quad \geqslant 1 - ce^{-\varepsilon^{-p}} \end{aligned} \tag{1.17}$$

Our next result states that before being stopped the process $\hat{\zeta}_t$ converges as $\varepsilon \to 0^+$ to a Brownian motion with diffusion coefficient $D = 3/4$.

Theorem 2 (Convergence to Brownian Motion). In the same contest as in Theorem 1, given any $R_\pm \in [-\infty, +\infty]$ we suppose that the initial data m_0 is such that

$$\lim_{\varepsilon \to 0^+} \varepsilon^{h/2}[\pm(\varepsilon^{-\kappa} - \ell\varepsilon^{-1}) - \zeta^{\varepsilon,\kappa}(m_0)] = R_\pm$$

Then for any $t^* > 0$ the process $\varepsilon^{h/2}(\hat{\zeta}_{\varepsilon^{-h-1}t} - \zeta^{\varepsilon,\kappa}(m_0))$ converges weakly on $C([0, t^*])$ to a Brownian motion with diffusion coefficient D, starting from 0 and stopped at $R_\pm$.

Theorem 1 thus states that the Ginzburg–Landau process is locally attracted by the manifold $\mathscr{M}$ (when the center is sufficiently far from the endpoints of $\mathcal{T}_{\varepsilon,\kappa}$) and Theorem 2 that it performs a Brownian motion on $\mathcal{M}$. We also have sharper results, see Proposition 4 and Corollary 1, on the small deviations of the process transversally to $\mathcal{M}$.

Our last result refers to couplings:

Theorem 3 (Couplings). Let m_0 and m'_0 be in $C_{\varepsilon,\kappa}$, let both verify the assumptions of Theorem 2 with respectively $R_\pm$ and $R'_\pm$ and, calling $x_0 \doteq \zeta(m_0)$, $x'_0 \doteq \zeta(m'_0)$, suppose

$$\lim_{\varepsilon \to 0^+} \varepsilon^{h/2}(x_0 - x'_0) =: r^*$$

Then there is a realization on the same space of the two Ginzburg–Landau processes m_t and m'_t that start from m_0 and, respectively m'_0 so that, if

$$\sigma = \inf\{t: m_t \equiv m'_t\} \tag{1.18}$$

then $\varepsilon^{h+1}\sigma$ converges in law to the distribution of a variable S defined as follows. Let b_t and b'_t be independent Brownians with diffusion D starting from 0 and stopped respectively at $R_\pm$ and $R'_\pm$. Then S is the first time when $b'_t - b_t = r^*$, provided S occurs before any of the Brownians is stopped, in that case $S = +\infty$.

2. THE DETERMINISTIC FLOW

As proved by Fife and McLeod, $T_t(m)$, $m \in \mathcal{M}_{\delta^*}$, is attracted by $\mathcal{M}$, see (1.11), so that it is eventually very close to an instanton. (This will also be true with large probability when the noise is present, if $\varepsilon > 0$ is small enough.) The flow T_t is thus well approximated (after a relaxation time) by its linearization around an instanton. It is then possible, see Theorem 5 later, to estimate the true center $\zeta(m)$ in terms of the (easier to handle) "linear center $\xi(m)$", defined as the center of the instanton that attracts the linearized flow, a condition equivalent to that in Definition 1. These are the main issues discussed in this section.

If $m \in C^0(\mathbb{R})$ then $m(x,t) \doteq T_t(m)(x)$, $t > 0$, is in $C^2(\mathbb{R})$, it is differentiable with respect to t, see Fife and McLeod,[12] and it satisfies the Ginzburg–Landau equation

$$\frac{\partial m}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 m}{\partial x^2}(x,t) + m(x,t) - m(x,t)^3 \tag{2.1}$$

The instantons $\bar{m}_{x_0}$, $x_0 \in \mathbb{R}$, are stationary solutions of (2.1). The linearized flow around $\bar{m}_{x_0}$ is the linear semigroup g_{t, x_0} on $C^0(\mathbb{R})$ whose generator L_{x_0} acts on $f \in C^2(\mathbb{R})$ as

$$L_{x_0} f(x) = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x) + [1 - 3\bar{m}_{x_0}(x)^2]\, f(x) \tag{2.2}$$

Denote by $\bar{m}'_{x_0}(x)$ the derivative w.r.t. x of $\bar{m}_{x_0}(x)$. From (1.7), $L_{x_0}\bar{m}'_{x_0} = 0$, (for any $x_0 \in \mathbb{R}$). Denoting by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\mathbb{R})$ we set for any $x_0 \in \mathbb{R}$,

$$\tilde{m}'_{x_0} \doteq \frac{\sqrt{3}}{2} \bar{m}'_{x_0}, \qquad \langle \tilde{m}'_{x_0}, \tilde{m}'_{x_0} \rangle = 1 \tag{2.3}$$

so 0 is an eigenvalue of L_{x_0}, and $\tilde{m}'_{x_0}$ is the corresponding unitary eigenvector in $L^2(\mathbb{R})$. The operator L_{x_0} has a spectral gap:

Theorem 4. There are α and c positive so that for any $f \in C^0(\mathbb{R})$ and $x_0 \in \mathbb{R}$

$$\|g_{t, x_0}[f - \langle \tilde{m}'_{x_0}, f \rangle\, \tilde{m}'_{x_0}]\| \leqslant c e^{-\alpha t}\, \|f - \langle \tilde{m}'_{x_0}, f \rangle\, \tilde{m}'_{x_0}\| \tag{2.4}$$

A proof of Theorem 4 in a L^2 setting may be found in Ref. 12, the proof with sup-norms is similar to that in Section 4 of Ref. 8 and it is omitted.

We will exploit Theorem 4 by observing that $m(x, t) \doteq T_t(m)(x)$ solves for $t > 0$ the equation:

$$\frac{\partial(m - \bar{m}_{x_0})}{\partial t} = L_{x_0}(m - \bar{m}_{x_0}) - 3\bar{m}_{x_0}(m - \bar{m}_{x_0})^2 - (m - \bar{m}_{x_0})^3 \tag{2.5}$$

We next define the "linear center" of a function $f \in C^0(\mathbb{R})$. This notion was introduced by Brassesco *et al.*,[5] where it was called simply "center".

Definition 1. The point $x_0 \in \mathbb{R}$ is a linear center of $m \in C^0(\mathbb{R})$ if

$$\langle \bar{m}'_{x_0}, m - \bar{m}_{x_0} \rangle = 0 \tag{2.6}$$

Existence and uniqueness of the linear center are stated in the next lemma. The proof, being essentially the same as that of Brassesco *et al.*,[5] [Prop. 3.2] is omitted:

Lemma 1. There is $\delta_0>0$ so that any $m\in\mathcal{M}_{\delta_0}$ has a unique linear center $\xi(m)$. Moreover there is $c_0>0$ so that if $m\in C^0(\mathbb{R})$, $y_0\in\mathbb{R}$ and

$$\|m-\bar{m}_{y_0}\| =: \delta \leqslant \delta_0 \tag{2.7}$$

then the linear center x_0 of m is such that

$$|x_0-y_0|\leqslant c_0\delta; \qquad \left|x_0-\left[y_0-\frac{3}{4}\langle \bar{m}'_{y_0}, (m-\bar{m}_{y_0})\rangle\right]\right|\leqslant c_0\delta^2 \tag{2.8}$$

Let m and $\tilde{m}$ be in $\mathcal{M}_{\delta_0}$, x_0 and $\tilde{x}_0$, their respective linear centers and $\|m-\tilde{m}\|\leqslant\delta_0$. Then

$$|x_0-\tilde{x}_0|\leqslant\frac{c_0}{2}\,|\langle \bar{m}'_{x_0}, m-\tilde{m}\rangle|\leqslant c_0\,\|m-\tilde{m}\| \tag{2.9}$$

In the sequel we take (for notational simplicity) $\delta_0<1$ and $\delta_0<\delta^*$, δ^* as in (1.11), so that if $m\in\mathcal{M}_{\delta_0}$ both $\xi(m)$ and $\zeta(m)$ are well defined. Theorem 5 is the main result in this section.

Theorem 5. There are $\beta>0$, $c>0$ and $\delta_1\in(0,\delta_0]$, so that for any $m\in\mathcal{M}_{\delta_1}$

$$|\xi(m)-\zeta(m)|\leqslant c[\,\|m-\bar{m}_{\xi(m)}\|\log\{\|m-\bar{m}_{\xi(m)}\|\}\,]^2 \tag{2.10}$$

Moreover $T_t(m)\in\mathcal{M}_{\delta_0}$ for all $t\geqslant 0$ and, setting $\xi_t\doteq\xi(T_t(m))$,

$$\begin{aligned}\|T_t(m)-\bar{m}_{\zeta(m)}\|&\leqslant c\,\|m-\bar{m}_{\xi(m)}\|\,e^{-\beta t},\\ \|T_t(m)-\bar{m}_{\xi_t}\|&\leqslant c\,\|m-\bar{m}_{\xi(m)}\|\,e^{-\beta t}\end{aligned} \tag{2.11}$$

$$\begin{aligned}\sup_{t\geqslant 0}|\xi_t-\xi(m)|&\leqslant c\,\|m-\bar{m}_{\xi(m)}\|,\\ \sup_{t\geqslant 0}\|T_t(m)-\bar{m}_{\xi(m)}\|&\leqslant c\,\|m-\bar{m}_{\xi(m)}\|\end{aligned} \tag{2.12}$$

The relevant parameter in Theorem 5 is thus $\|m-\bar{m}_{\xi(m)}\|$, which controls the difference between the linear and the true center, see (2.10), and the convergence of $T_t(m)$ to the instanton, see the first relation in (2.11). We start by proving that the first relation in (2.11) follows from the second one and (2.10), we will then complete the proof of Theorem 5 after a preliminary lemma, Lemma 2.

Proof of the first inequality in (2.11). Since $\bar{m}' \leqslant 1$,

$$\|\bar{m}_{x_0} - \bar{m}_{y_0}\| \leqslant |x_0 - y_0|$$

hence, by the second inequality in (2.11),

$$\|T_t(m) - \bar{m}_{\zeta(m)}\| \leqslant |\zeta(m) - \xi_t| + c\,\|m - \bar{m}_{\xi(m)}\|\, e^{-\beta t}$$

Recalling that $\zeta(m) = \zeta(T_t(m))$ and $\xi_t = \xi(T_t(m))$, by (2.10),

$$|\zeta(m) - \xi_t| = |\zeta(T_t(m)) - \xi(T_t(m))|$$
$$\leqslant c[\,\|T_t(m) - \bar{m}_{\xi(T_t(m))}\| \log\{\|m - \bar{m}_{\xi(T_t(m))}\|\}\,]^2$$

which using again the second inequality in (2.11) completes the proof of the first one. □

Lemma 2. There is $c_1 > 0$ so that for any $\lambda \in (0, 1]$, $x_0 \in \mathbb{R}$ and $v \in C(\mathbb{R})$ such that $\|v\| \leqslant 1$ and $\langle \bar{m}'_{x_0}, v\rangle = 0$ the following holds. Let

$$m \doteq \bar{m}_{x_0} + \lambda v, \qquad t_\lambda \doteq \max\{1, (\log \lambda)^2\} \tag{2.13}$$

and c, α as in Theorem 4. Then for all $0 \leqslant t \leqslant t_\lambda$

$$|T_t(m) - \bar{m}_{x_0}| \leqslant ce^{-\alpha t}\lambda + c_1 \lambda^2 t \tag{2.14}$$

Proof. Let $u_t \doteq T_t(m) - \bar{m}_{x_0}$, $u_0 \doteq \lambda v$. By (2.5) and Theorem 4 there is $c_2 > 0$ so that

$$\|u_t\| \leqslant ce^{-\alpha t}\lambda + c_2 \int_0^t ds(\|u_s\|^2 + \|u_s\|^3) \tag{2.15}$$

Call T the first time such that $\|u_T\| = 2c\lambda$, and suppose that $T \leqslant t_\lambda$. Then by (2.15)

$$2c\lambda \leqslant c\lambda + c_2 t_\lambda([2c\lambda]^2 + [2c\lambda]^3)$$

which does not hold when λ is small enough, say $\lambda \leqslant \lambda_0$, $\lambda_0 \in (0, 1]$.

But (2.14) follows from (2.15) when $\lambda \in (\lambda_0, 1]$, since $\|u_s\|$ is bounded for all $s \geqslant 0$, by the maximum principle. For $\lambda \leqslant \lambda_0$, we have seen that $T > t_\lambda$ so $\|u_t\| < 2c\lambda$ for all $t \leqslant t_\lambda$. With this bound on the right-hand side of (2.15) we obtain (2.14), completing the proof of the Lemma. □

Proof of Theorem 5. We use the same notation as in Lemmas 1 and 2. We take $\delta_1 > 0$ (other conditions on δ_1 will be specified later) so that for any $\lambda \in (0, \delta_1]$

$$\sup_{t \leqslant t_\lambda} [ce^{-\alpha t}\lambda + c_1 \lambda^2 t] \leqslant \delta_0, \qquad \delta_0 \text{ as in Lemma 1} \tag{2.16}$$

Let $m \in M_{\delta_1}$ and call $x_0 = \xi(m)$. If $m = \bar{m}_{x_0}$, then the theorem follows since $\bar{m}_{x_0}$ is stationary for T_t. By writing

$$m = \bar{m}_{x_0} + \frac{m - \bar{m}_{x_0}}{\|m - \bar{m}_{x_0}\|} \|m - \bar{m}_{x_0}\|$$

we can apply Lemma 2 with $\lambda = \|m - \bar{m}_{x_0}\|$ and, by Lemma 1, (2.14) and (2.16), the linear center $\xi(T_t(m))$ is well defined for $t \leqslant t_\lambda$, and, setting $m^* \doteq T_{t_\lambda}(m)$ there is a constant $c_2 > 0$ so that,

$$|\xi(m^*) - x_0| \leqslant c_0 \|m^* - \bar{m}_{x_0}\| \leqslant c_2[\lambda \log \lambda]^2 \tag{2.17}$$

We will prove that there is a constant $c_3 > 0$ so that

$$|\xi(m^*) - \zeta(m^*)| \leqslant c_3 \|m^* - \bar{m}_{\xi(m^*)}\| \tag{2.18}$$

Proof of (2.10). Since $\zeta(m) = \zeta(T_t(m))$, $\zeta(m) = \zeta(m^*)$. Then by (2.18) and (2.17)

$$|\zeta(m) - x_0| \leqslant c_2[\lambda \log \lambda]^2 + c_3 \|m^* - \bar{m}_{\xi(m^*)}\|$$

Recalling that $\bar{m}' \leqslant 1$ and using the inequalities in (2.17)

$$\begin{aligned} \|m^* - \bar{m}_{\xi(m^*)}\| &\leqslant \|m^* - \bar{m}_{x_0}\| + \|\bar{m}_{x_0} - \bar{m}_{\xi(m^*)}\| \\ &\leqslant (1 + c_0) \|m^* - \bar{m}_{x_0}\| \\ &\leqslant \frac{(1 + c_0)\, c_2}{c_0} [\lambda \log \lambda]^2 \end{aligned} \tag{2.19}$$

(2.10) is proved, conditioned on the validity of (2.18).

Proof of (2.18). We further specify δ_1 by requiring that for all $\lambda \in (0, \delta_1]$

$$(1 + c_0)[ce^{-\alpha t_\lambda}\lambda + c_1 \lambda^2 t_\lambda] \leqslant \delta_1 \tag{2.20}$$

By (2.19) and (2.14), (2.20) implies

$$\|m^* - \bar{m}_{x^*}\| \leqslant \delta_1, \qquad \text{where} \quad x^* \doteq \xi(m^*) \tag{2.21}$$

thus enforcing the initial conditions also at time t_λ.

We define τ so that $\tau \leqslant t_{\lambda_0}$ (where λ_0 is defined later) and also that

$$e^{-\alpha\tau} c(1 + c_0) \leqslant \tfrac{1}{4} \tag{2.22}$$

We require that δ_1 is such that for all $\lambda \in (0, \delta_1]$

$$(1 + c_1)[ce^{-\alpha\tau}\lambda + c_1 \lambda^2 \tau] \leqslant \frac{\lambda}{2} \tag{2.23}$$

Let $m^{(0)} \doteq m^*$, $x^{(0)} \doteq x^*$, $m^{(1)} \doteq T_\tau(m^{(0)})$, $x^{(1)} \doteq \xi(m^{(1)})$

$$\lambda_0 \doteq (1 + c_0)[ce^{-\alpha t}\lambda + c_1 \lambda^2 t_\lambda], \qquad \lambda \doteq \|m - \bar{m}_{x_0}\|, \qquad \|m^* - \bar{m}_{x^{(0)}}\| \leqslant \lambda_0$$

and

$$\lambda_1 \doteq (1 + c_0)[ce^{-\alpha\tau}\lambda_0 + c_1 \lambda_0^2 \tau]$$

By iteration we then define for $n > 1$

$$m^{(n)} \doteq T_{n\tau}(m^{(0)}), \qquad c^{(n)} \doteq \xi(m^{(n)}), \qquad \lambda_n \doteq (1 + c_0)[ce^{-\alpha\tau}\lambda_{n-1} + c_1 \lambda_{n-1}^2 \tau]$$

We have from (2.23)

$$\|T_{n\tau}(m^{(0)}) - \bar{m}_{x^{(n)}}\| \leqslant \lambda_n \leqslant 2^{-n}\lambda_0$$

and since $|x^{(n)} - x^{(n-1)}| \leqslant c_0 \, \|T_\tau(m^{(n-1)}) - \bar{m}_{x^{(n-1)}}\|$ by (2.14)

$$|x^{(n)} - x^{(0)}| \leqslant \sum_{i=1}^{n} \lambda_i \leqslant 2\lambda_0$$

Since $x^{(n)} \to \zeta(m^{(0)})$ as $n \to +\infty$ this proves (2.18) (recall that $m^* = m^{(0)}$).

Proof of (2.12) and of the second inequality in (2.11). The second inequality in (2.11) has been previously proved at the times $n\tau$. For $t \in (n\tau, (n+1)\tau)$ we have:

$$\begin{aligned} |T_t(m) - \bar{m}_{\xi_t}\| &\leqslant \|T_t(m) - \bar{m}_{x^{(n)}}\| + \|\bar{m}_{x^{(n)}} - \bar{m}_{\xi_t}\| \\ &\leqslant \|T_t(m) - \bar{m}_{x^{(n)}}\| + |x^{(n)} - \xi_t| \\ &\leqslant (1 + c_0) \, \|T_t(m) - \bar{m}_{x^{(n)}}\| \end{aligned}$$

In the last inequality we have used Lemma 1. The second inequality in (2.11) then follows from Lemma 2.

To prove (2.12) we write

$$|\xi_t - \xi(m)| \leqslant |\xi_t - \zeta(m)| + |\zeta(m) - \xi(m)| \tag{2.24}$$

The last term is bounded using (2.10). For the other one observe that by (2.9) for any $t \geqslant 0$

$$\begin{aligned} \|\xi_t - \zeta(m)\| &= \|\xi(\bar{m}_{\xi_t}) - \xi(\bar{m}_{\zeta(m)})\| \\ &\leqslant c_0 \|\bar{m}_{\xi_t} - \bar{m}_{\zeta(m)}\| \\ &\leqslant c_0 \|\bar{m}_{\xi_t} - T_t(m)\| + c_0 \|\bar{m}_{\zeta(m)} - T_t(m)\| \end{aligned} \tag{2.25}$$

which by (2.11) proves the first inequality in (2.12). The second one follows from the first one and (2.11), and the proof of Theorem 5 is finished. □

3. THE STOCHASTIC FLOW

In this section we prove Theorem 1 and some of the key estimates that will be used in Sections 4 and 5 to prove the other theorems of Section 1. We start with Proposition 1 where we derive the basic bounds on the Gaussian process Z_t. The proposition is proved in Ref. 5 for $\kappa = 1$. Its extension to $\kappa > 1$ is not difficult and for completeness we report it in the Appendix.

Proposition 1. Let $\kappa \geqslant 1$, $\varepsilon > 0$ Z_t as defined in (1.4), and $T_t(m, \sqrt{\varepsilon}\, Z)$ the solution of (1.5) starting from m. Then there are positive constants b_0 and b_1 such that, if we set $t_\varepsilon \doteq (\log \varepsilon)^2$ and

$$\mathscr{B}_{p,\varepsilon} \doteq \{ \sup_{0 \leqslant t \leqslant t_\varepsilon} \|\sqrt{\varepsilon}\, Z_t\| \leqslant \varepsilon^{1/2 - p} \} \tag{3.1}$$

the, for any $p > 0$ and $\varepsilon > 0$,

$$P^\varepsilon(\mathscr{B}_{p,\varepsilon}) \geqslant 1 - b_0 e^{-b_1 |\log \varepsilon|^{-1} \varepsilon^{-2p}} \tag{3.2}$$

and for all $m \in C^0(\mathbb{R})$ with $\|m\| \leqslant 1 + 10^{-2}$, any $S \geqslant t_\varepsilon$ and any $\varepsilon > 0$ small enough,

$$P^\varepsilon(\sup_{0 \leqslant t \leqslant S} \|T_t(m; \sqrt{\varepsilon}\, Z)\| \leqslant 2, \|T_S(m; \sqrt{\varepsilon}\, Z)\| \leqslant 1 + 10^{-2}) \geqslant 2 - S b_0 e^{-b_1 \varepsilon^{-1}} \tag{3.3}$$

Note that (3.3) implies that, for any $N>0$, the process remains uniformly bounded up to times ε^{-N} with probability close to 1 exponentially in ε^{-1}.

Let $m \in C^0(\mathbb{R})$, set $m_t \doteq T_t(m; \sqrt{\varepsilon}\, Z)$, $t \geqslant 0$, then, for any $x_0 \in \mathbb{R}$, $u_t \doteq m_t - \bar{m}_{x_0}$ solved the following integral version of the Ginzburg–Landau stochastic equation (also considered in Ref. 5)

$$u_t = g_{t, x_0} u_0 - \int_0^t ds\, g_{t-s, x_0}(3\bar{m}_{x_0} u_s^2 + u_s^3) + \sqrt{\varepsilon}\, \hat{Z}_{t, x_0} \tag{3.4}$$

The operator g_{t, x_0} was defined in the beginning of Section 2 and

$$\hat{Z}_{t, x_0} \doteq Z_t - \int_0^t ds\, g_{t-s, x_0}[(3\bar{m}_{x_0}^2 - 1)\, Z_s] \tag{3.5}$$

$\hat{Z}_{t, x_0}$ is also given by the stochastic integral (see Ref. 5)

$$\hat{Z}_{t, x_0} = \int_0^t ds \int_{\mathcal{T}_{\varepsilon, \kappa}} dy\, g_{t-s, x_0}^{(\varepsilon)}(x, y)\, \alpha(s, y) \tag{3.6}$$

where

$$g_{t-s, x_0}^{(\varepsilon)}(x, y) \doteq \sum_{j \in \mathbb{Z}} (g_{t-s, x_0}(x, y + 4j\varepsilon^{-\kappa}) + g_{t-s, x_0}(x, 4j\varepsilon^{-\kappa} + 2\varepsilon^{-\kappa} - y)) \tag{3.7}$$

and $g_{t, x_0}(\cdot, \cdot)$ stands for the kernel corresponding to the operator g_{t, x_0}. An estimate analogous to that given by Proposition 1 follows for $\hat{Z}_{t, x_0}$:

Proposition 2. Let $\hat{Z}_{t, x_0}$ as before. Then there are positive constant b_0 and b_1 such that, if we set $t_\varepsilon \doteq (\log \varepsilon)^2$ and

$$\mathcal{B}_{p, \varepsilon, x_0} \doteq \{ \sup_{0 \leqslant t \leqslant t_\varepsilon} \| \sqrt{\varepsilon}\, \hat{Z}_{t, x_0} \| \leqslant \varepsilon^{1/2 - p} \}$$

then, for any $p>0$ and $\varepsilon>0$,

$$P^\varepsilon(\mathcal{B}_{p, \varepsilon, x_0}) \geqslant 1 - b_0 e^{-b_1 \varepsilon^{-p}} \tag{3.8}$$

Proof. It is a consequence of (3.2), (3.5) and (A.30). □

Recall from Lemma 1 that $\xi(m)$ is the linear center of m (see (2.6)), and that there is δ_0 so that if $m \in \mathcal{M}_{\delta_0}$ then $\xi(m)$ is uniquely defined. In analogy with (1.14), given any $\kappa \geqslant 1$, we set

$$\xi^{\varepsilon,\kappa}(m) \doteq \begin{cases} \xi(m^{\varepsilon,\kappa}) & \text{if } \ m^{\varepsilon,\kappa} \in \mathcal{M}_{\delta_0} \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

and, given any $\ell \in (0, 1)$ and $\delta \in (0, \delta_0]$,

$$M^{\varepsilon}_{\kappa,\ell,\delta} \doteq \{ m \in C^0(\mathbb{R}) : m^{\varepsilon,\kappa} \in \mathcal{M}_\delta, \ |\xi^{\varepsilon,\kappa}(m)| \leqslant \varepsilon^{-\kappa} - \ell\varepsilon^{-1} \} \tag{3.10}$$

By (2.10), Theorem 1 follows from the analogous statement with the true center replaced by the linear center. Theorem 1 will then be a consequence of the following Proposition.

Proposition 3. There is $\delta_2 \in (0, \delta_1]$ (δ_1 as in Theorem 5) so that the following holds. Let $a \in (0, 1/2)$, $\ell \in (0, 1)$, $\kappa \geqslant 1$, $\delta \in (0, \delta_2]$ and $p \in (0, a/2)$. Then there are positive constants b_0, b_1 and c so that for any $\varepsilon > 0$ small enough and for any $m \in M^{\varepsilon}_{\kappa,\ell,\delta}$, if $m_t \doteq T_t(m; \sqrt{\varepsilon}\, Z)$, $\ell' = \ell - \varepsilon c(\delta \vee \varepsilon^{1/2-a})$, and $t_\varepsilon = (\log \varepsilon)^2$,

$$P^{\varepsilon}(m_t \in M^{\varepsilon}_{\kappa,\ell',c\delta} \text{ for all } t \leqslant t_\varepsilon, m_{t_\varepsilon} \in M^{\varepsilon}_{\kappa,\ell,\varepsilon^{1/2-a}}) \geqslant 1 - b_0 e^{-b_1 \varepsilon^{-p}} \tag{3.11}$$

Proof. Let m be as in the statement and consider first the case $m \in \mathcal{M}_{\delta_1}$. We study $m_t \doteq T_t(m; \sqrt{\varepsilon}\, Z)$ as a perturbation of $m^0_t \doteq T_t(m)$. Let

$$\begin{aligned} &x_0 = \xi(m), \qquad u^0_t \doteq m^0_t - \bar{m}_{x_0}, \qquad D_t \doteq \|m_t - m^0_t\|, \\ &\xi_t \doteq \xi(m_t), \qquad \xi^0_t \doteq \xi(m^0_t) \end{aligned} \tag{3.12}$$

By Theorem 5 there is a constant $c = c(\delta_1)$ so that for all $t \geqslant 0$

$$\|m^0_t - \bar{m}_{\xi^0_t}\| \leqslant c e^{-\beta t}, \qquad |\xi^0_t - x_0| \leqslant c \tag{3.13}$$

Next, we write the integral equation for $m_t - m^0_t$ in terms of the operator g_{t,x_0}. Recalling that $V'(m_t) - V'(m^0_t) = (m_t - m^0_t)[3(m^0_t)^2 - 1 + 3\bar{m}^2_{x_0} - 3\bar{m}^2_{x_0}] + 3m^0_t(m_t - m^0_t)^2 + (m_t - m^0_t)^3$, we obtain

$$\begin{aligned} m_t - m^0_t = {} & -\int_0^t ds\, g_{t-s,x_0}[(m_s - m^0_s)\, 3(m^0_s - \bar{m}_{x_0})(m^0_s + \bar{m}_{x_0}) \\ & + 3m^0_s(m_s - m^0_s)^2 + (m_s - m^0_s)^3] + \varepsilon^{1/2} \hat{Z}_{t,x_0} \end{aligned}$$

where $\hat{Z}_{t,x_0}$ was defined in (3.5). By (2.12) $\|m_t^0 - \bar{m}_{x_0}\| \leqslant \|u_0^0\|$. Then, by (3.12) and (A.30) we get that in $\mathscr{B}_{p,\varepsilon,x_0}$ (which, by (3.8), has probability greater than $1 - b_0 e^{-b_1 \varepsilon^{-p}}$),

$$D_t \leqslant c_3 \int_0^t ds\{\|u_0^0\| D_s + D_s^2 + D_s^3\} + \varepsilon^{1/2 - p} \tag{3.14}$$

for some positive constant c_3. By (3.13) and (3.3), $D_t \leqslant 4$ for all $t \leqslant \varepsilon^{-2}$ with probability exponentially (in ε^{-p}) close to 1. By (3.14) and the Gronwall's Lemma, in $\mathscr{B}_{p,\varepsilon,x_0}$ intersected with the set in (3.3), we have

$$D_t \leqslant e^{c_3(\|u_0^0\| + 100)t} \varepsilon^{1/2 - p}$$

Calling $t^* \doteq p(c_3(\|u_0^0\| + 100))^{-1} |\log \varepsilon|$, for any $\varepsilon > 0$ small enough we have that

$$\sup_{t \leqslant t^*} D_t \leqslant \varepsilon^{1/2 - p} \tag{3.15}$$

By (3.13) there is $b \in (0, 1/2 - 2p)$, so that calling $y_0 \doteq \xi(m_{t^*}^0)$, for any $\varepsilon > 0$ small enough,

$$\|m_{t^*}^0 - \bar{m}_{y_0}\| \leqslant \varepsilon^b \tag{3.16}$$

Then

$$\|m_{t^*} - \bar{m}_{y_0}\| \leqslant \varepsilon^{1/2 - 2p} + \varepsilon^b \leqslant 2\varepsilon^b \tag{3.17}$$

Hence for $\varepsilon > 0$ small enough the linear center $x^* \doteq \xi(m_{t^*})$ of m_{t^*} exists and by Lemma 1 $|x^* - y_0| \leqslant c_0 2\varepsilon^b$. Then

$$|x^* - x_0| \leqslant |x^* - y_0| + |y_0 - x_0| \leqslant c + c_0 2\varepsilon^b \leqslant: c' \tag{3.18}$$

where c is the constant in the second inequality in (3.13). Thus by (3.17) and (3.18) $m_{t^*} \in M^{\varepsilon}_{\kappa, \ell_1, \delta}$, $\ell_1 = \ell - \varepsilon c'$ and $\delta \doteq 2\varepsilon^b$.

Moreover, recalling that $\bar{m}' \leqslant 1$,

$$\|m_{t^*} - \bar{m}_{x^*}\| \leqslant \|m_{t^*} - \bar{m}_{y_0}\| + \|\bar{m}_{y_0} - \bar{m}_{x^*}\| \leqslant (1 + c_0)\, 2\varepsilon^b \tag{3.19}$$

We next consider $m_t \doteq T_t(m_{t^*}; \sqrt{\varepsilon}\, Z)$. We call $m_t^0 \doteq T_t(m_{t^*})$ and $D_t \doteq \|m_t - m_t^0\|$. Again in $\mathscr{B}_{p,\varepsilon,x_0}$, (3.14) holds in this setup, but now, by (3.19), $\|u_0^0\| \leqslant (1 + c_0)\, 2\varepsilon^b$. Hence, by considering $\sigma = \inf\{t \geqslant 0\colon D_t \geqslant c\varepsilon^{1/2 - p}\}$, it follows from (3.14) that $\sigma \geqslant t_\varepsilon$ for some convenient $c < \infty$ so that $D - t \leqslant c\varepsilon^{1/2 - p}$ for all $t \leqslant t_\varepsilon$.

By the same argument used earlier we then complete the proof of the Proposition under the additional assumption that $m \in \mathcal{M}_{\delta_1}$ and consider next the case $m \in M^{\varepsilon}_{\kappa, \ell, \delta} \cap C_{\varepsilon, \kappa}(\mathbb{R})$ with $\delta \in (0, \delta_1/(1+c_0)]$, c_0 as in Lemma 1. Then $m^{\varepsilon, \kappa} \in \mathcal{M}_\delta$ and $|\xi^{\varepsilon, \kappa}(m)| \leqslant \varepsilon^{-\kappa} - \ell\varepsilon^{-1}$ and hence (3.11) holds for $T_t(m^{\varepsilon, \kappa}; \sqrt{\varepsilon}\, Z)$. Let $\mathscr{B}_{p, \varepsilon}$ be the set in (3.1). Then there are constants c' and V positive so that setting $L_\varepsilon \doteq \varepsilon^{-\kappa} - Vt_\varepsilon$

$$\sup_{0 \leqslant t \leqslant t_\varepsilon} \sup_{|x| \leqslant L_\varepsilon} |T_t(m; \sqrt{\varepsilon}\, Z) - T_t(m^{\varepsilon, \kappa}; \sqrt{\varepsilon}\, Z)| \leqslant \hat{c} e^{-t_\varepsilon} \tag{3.20}$$

(3.20) is proved in Proposition 5.3 of Ref. 5 (Barrier Lemma) for $\kappa = 1$, but the proof is also valid for $\kappa > 1$. Let next $x \in [L_\varepsilon, \varepsilon^{-\kappa}]$ (the proof for $x \in [-\varepsilon^{-\kappa}, -L_\varepsilon]$, is similar). Since $m^{\varepsilon, \kappa} \in \mathcal{M}_\delta$ with $|\xi(m^{\varepsilon, \kappa})| \leqslant \varepsilon^{-\kappa} - \ell\varepsilon^{-1}$, using (2.8) and recalling that $\bar{m}(x) = \tanh(x)$, we have, for any $\varepsilon > 0$ small enough,

$$\sup_{|x - \varepsilon^{-\kappa}| \leqslant 3Vt_\varepsilon} |m^{\varepsilon, \kappa}(x) - 1| \leqslant (1 + c_0)\, \delta + 2e^{-(\ell\varepsilon^{-1} - Vt_\varepsilon)}$$

Then, since $m(x) = m^{\varepsilon, \kappa}(x)$ for any $x \in [L_\varepsilon, \varepsilon^{-\kappa}]$ and $\delta < \delta_1/(1 + c_0)$ with $\delta_1 < 1$, there is a constant $c \in (0, 1)$ so that, for any $\varepsilon > 0$ small enough,

$$\sup_{|x - \varepsilon^{-\kappa}| \leqslant 2Vt_\varepsilon} |m(x) - 1| \leqslant c \tag{3.21}$$

Recalling that $m \in C_{\varepsilon, \kappa}(\mathbb{R})$, we define $\hat{m} \in C(\mathbb{R})$ as

$$\hat{m}(x) \doteq \begin{cases} m(x) & \text{if} \quad |x - \varepsilon^{-\kappa}| \leqslant 2Vt_\varepsilon \\ m(\varepsilon^{-\kappa} - 2Vt_\varepsilon) & \text{if} \quad x \leqslant \varepsilon^{-\kappa} - 2Vt_\varepsilon \\ m(\varepsilon^{-\kappa} + 2Vt_\varepsilon) & \text{if} \quad x \geqslant \varepsilon^{-\kappa} + 2Vt_\varepsilon \end{cases} \tag{3.22}$$

Using again the Barrier Lemma there is $c > 0$ so that in $\mathscr{B}_{p, \varepsilon}$

$$\sup_{0 \leqslant t \leqslant t_\varepsilon} \sup_{L_\varepsilon < x \leqslant \varepsilon^{-\kappa}} |T_t(m; \sqrt{\varepsilon}\, Z) - T_t(\hat{m}; \sqrt{\varepsilon}\, Z)| \leqslant ce^{-t_\varepsilon} \tag{3.23}$$

Since $m(\cdot) \equiv 1$ is stable (see the proof of Lemma A.2 in the Appendix) there is a constant $c > 0$ so that in $\mathscr{B}_{p, \varepsilon}$ for any $t \in [0, t_\varepsilon]$

$$\| T_t(\hat{m}; \sqrt{\varepsilon}\, Z) - 1 \| \leqslant c[e^{-t} + \varepsilon^{1/2 - p}] \tag{3.24}$$

By (3.20), (3.23), and (3.24) there is $c > 0$ so that $(T_t(m; \sqrt{\varepsilon}\, Z))^{\varepsilon, \kappa}$ is in $\mathcal{M}_{c\delta}$ for all $t \leqslant t_\varepsilon$ and in $\mathcal{M}_{c\varepsilon^{1/2 - a}}$ at time t_ε.

It remains to control the position of the center. From (3.20), (3.11), and (3.24), there is a constant $c>0$ so that

$$\|(T_t(m;\sqrt{\varepsilon}\,Z))^{\varepsilon,\kappa}-T_t(m^{\varepsilon,\kappa};\sqrt{\varepsilon}\,Z)\|\leqslant c\delta \tag{3.25}$$

By choosing $\delta_2\in(0,\delta_1/(1+c_0)]$ so small that $c\delta_2\leqslant\delta_0$, δ_0 as in Lemma 1 we conclude that, for any $\delta\in(0,\delta_2]$,

$$|\xi((T_t(m;\sqrt{\varepsilon}\,Z))^{\varepsilon,\kappa})-\xi(T_t(m_{\varepsilon,\kappa};\sqrt{\varepsilon}\,Z))|\leqslant c_0c\delta \tag{3.26}$$

Since $|\xi(T_t(m^{\varepsilon,\kappa};\sqrt{\varepsilon}\,Z))|\leqslant\varepsilon^{-\kappa}-\varepsilon^{-1}\ell'$ similar conclusion (with ℓ' replaced by $\ell'-\varepsilon(c_0c\delta)$) holds for $\xi((T_t(m;\sqrt{\varepsilon}\,Z))^{\varepsilon,\kappa})$, and Proposition 3 is proved. □

Proof of Theorem 1. By iterating (3.11) and recalling that by (2.10) the true and the linear centers are close we conclude the proof of Theorem 1. □

3.1. Main Ideas of the Proofs

The proof of Theorem 1 is a (simple) perturbative argument that relates $T_t(m;\sqrt{\varepsilon}\,Z)$ and $T_t(m)$; convergence to a Brownian motion is a quite different game, its proof much more delicate.

When times are scaled as $\varepsilon^{-1}\tau$, τ in a compact of $\mathbb{R}$, the argument used in Ref. 5 applies. It is based on the bound

$$\|T_{t_\varepsilon}(m;\sqrt{\varepsilon}\,Z)-T_{t_\varepsilon}(\bar m_{\xi(m)};\sqrt{\varepsilon}\,Z)\|\leqslant c\varepsilon^{1-2a} \tag{3.27}$$

which holds if $m\in M^{\varepsilon}_{\kappa,\ell,\varepsilon^{1/2-a}}$ and $c>0$ a suitable constant. The bound follows easily from the integral representation (3.4) using the estimates of Section 2. The idea then is to split the time into intervals of length t_ε (a different value of t_ε is actually used in Ref. 5) and to replace at the beginning of each interval the true process by the one which starts from the instanton with the same linear center. One can see that if a is small the sum of all errors, bounded using (3.27), vanishes when $\varepsilon\to 0^+$ so that we can consider a process that at each interval starts from an instanton. Then except for the (negligible) influence of the boundaries, the increments of the linear center are mutually independent and convergence to a Brownian motion is easily proved.

If times are proportional to ε^{-h}, h large, the sum of the errors is no longer negligible and this approach fails even though the bound (3.27) is optimal. We find a way out exploiting the fact that we can prove a much better bound (that goes even like $c_n \varepsilon^n$ for any given n) provided we construct the two processes in a more refined way than taking just the same noise in the whole interval $[0, t_\varepsilon]$. But the most important point is that we only compare the processes modulo translations. We can then conclude that after a time delay t_ε the two processes, suitably shifted, are in law very close to each other. The successive increments of the linear centers are independent of the shift (except for the influence of the boundary, controlled by using the Barrier Lemma) so that they are in law very close to each other.

The crucial bound involving $c_n \varepsilon^n$ is proved later, see Proposition 4 and Corollary 1; its application to the convergence to a Brownian motion in Section 4.

To investigate the process modulo translation, i.e., the deviations transversal to $\mathcal{M}$ (neglecting the localization along $\mathcal{M}$), it is convenient to introduce a function $D_\varepsilon(m, m^*)$ which is a substitute for a distance between m and m^*.

Definition 2. Let $f \in C(\mathbb{R})$, $\varepsilon > 0$, x and $y \in \mathbb{R}$, we set

$$\|f\|_\varepsilon \doteq \sup_{|x| \leqslant \varepsilon^{-1/10}} |f(x)|; \qquad \tau_y f(x) \doteq f(x+y) \qquad \|f\|_{\varepsilon, x} \doteq \|\tau_x f\|_\varepsilon \tag{3.28}$$

For m and m^* in $C(\mathbb{R})$, we then define $D_\varepsilon(m, m^*)$ by

$$D_\varepsilon(m, m^*) \doteq \begin{cases} \inf_{r \in \mathbb{R}} \{ |r| \vee \|\tau_{x_0} m - \tau_{x_0^* + r} m^*\|_\varepsilon \} \\ \quad \text{if} \quad m^{\varepsilon, \kappa}, (m^*)^{\varepsilon, \kappa} \in \mathcal{M}_{\varepsilon^{1/2-a}} \\ 1 \quad \text{otherwise} \end{cases} \tag{3.29}$$

$x_0 \doteq \xi^{\varepsilon, \kappa}(m)$, $x_0^* \doteq \xi^{\varepsilon, \kappa}(m^*)$ (see (3.9) for notation) and $a \vee b$ stands for the maximum between a and b.

In general $D_\varepsilon(m, m^*) \neq D_\varepsilon(m^*, m)$. By its definition for any $\delta \in (0, 1)$, $D_\varepsilon(m, m^*) \leqslant \delta$ if and only if $m^{\varepsilon, \kappa}$, $(m^*)^{\varepsilon, \kappa} \in \mathcal{M}_{\varepsilon^{1/2-a}}$ and there is η such that $|\eta - (x_0^* - x_0)| \leqslant \delta$ and $\|m - \tau_\eta m^*\|_{\varepsilon, x_0} \leqslant \delta$.

We next prove a contraction property of the evolution with respect to D_ε.

Proposition 4. Let $\kappa \geqslant 1$, $\ell \in (0, 1)$, $a \in (0, 1/2)$, $n \geqslant 1$, $b > 0$ and $\gamma \in (0, 1/2 - a)$. Then there are $p > 0$ and $c > 0$ such that for all $0 < \varepsilon < 1$ the following holds. For all pairs m and $m^* \in C(\mathbb{R})$ so that $D_\varepsilon(m, m^*) \leqslant \varepsilon^b$, we can construct the processes m_t and m_t^*, solutions of (1.1) with initial data m and m^* respectively in the same probability space and such that, if we set $t_\varepsilon = (\log \varepsilon)^2$, then

$$P^\varepsilon(D_\varepsilon(m_{t_\varepsilon/n}, m^*_{t_\varepsilon/n}) \leqslant \varepsilon^{b+\gamma}) \geqslant 1 - ce^{-\varepsilon^{-p}} \tag{3.30}$$

Proof. Notion: for simplicity we consider the case $n=1$. Let $x_0 \doteq \xi^{\varepsilon,\kappa}(m)$, $x_0^* \doteq \xi^{\varepsilon,\kappa}(m^*)$, $\Delta \doteq x_0^* - x_0$, $\lambda \doteq \varepsilon^{1/2-a}$, $\delta \doteq \varepsilon^b$,

$$v(x) \doteq [m(x) - \bar{m}_{x_0}(x)], \qquad u(x) \doteq [m^*(x) - \bar{m}_{x_0^*}(x)] \tag{3.31}$$

By assumption there is η, $|\eta - \Delta| \leqslant \delta$, so that (recall that $\bar{m}' \leqslant 1$)

$$\|v - \tau_\eta u\|_{\varepsilon, x_0} \leqslant 2\delta \tag{3.32}$$

We divide the proof into several steps.

Step 1. *Construction of the coupling.* Starting from a white noise process α, consider the processes Z_t and $\hat{Z}_{t,x_0}$ as defined in (1.4) and (3.5), (or (3.6)) respectively.

Next, take a second white noise $\bar{\alpha}$ independent of α and set

$$\begin{aligned} Z_t^*(x) \doteq \int_0^t ds \int dz\, \mathbf{1}_{\{|z+\Delta-x_0^*| \leqslant 4\varepsilon^{-1/10};\ z+\Delta \in \mathscr{T}_{\varepsilon,\kappa}\}} H_{t-s}^{(\varepsilon)}(x, z+\Delta)\, \alpha(s,z) \\ + \int_0^t ds \int dy\, \mathbf{1}_{\{|y-x_0^*| > 4\varepsilon^{-1/10};\ y \in \mathscr{T}_{\varepsilon,\kappa}\}} H_{t-s}^{(\varepsilon)}\, \bar{\alpha}(s,y) \end{aligned} \tag{3.33}$$

($\mathbf{1}_A$ denotes the characteristic function of the set A). It is easy to check (by comparing covariances) that the process Z_t^* and Z_t have the same law. Using them, we construct the Ginzburg–Landau processes by setting

$$m_t \doteq T_t(m; \sqrt{\varepsilon}\, Z), \qquad m_t^* \doteq T_t(m^*; \sqrt{\varepsilon}\, Z^*) \tag{3.34}$$

Define

$$v_t \doteq m_t - \bar{m}_{x_0}, \qquad v_0 \doteq v, \qquad u_t \doteq m_t^* - \bar{m}_{x_0^*}, \qquad u_0 \doteq u \tag{3.35}$$

We also call $v_t^{(\varepsilon,\kappa)} \doteq (m_t)^{\varepsilon,\kappa} - \bar{m}_{x_0}$ and $u_t^{(\varepsilon,\kappa)} \doteq (m_t^*)^{\varepsilon,\kappa} - \bar{m}_{x_0^*}$.

Step 2. *The good sets.* Let $p \in (0, a/2)$, $\varepsilon > 0$, $c > 0$ and

$$\begin{aligned} \mathscr{B}_{\varepsilon,p}^{(1)} \doteq \{ \sup_{0 \leqslant t \leqslant t_\varepsilon} \{ \|\hat{Z}_{t,x_0}\| + \|\hat{Z}_{t,\bar{x}_0^*}^*\| \} \leqslant \varepsilon^{-2p}, \\ \sup_{0 \leqslant t \leqslant t_\varepsilon} \{ \|v_t^{(\varepsilon,\kappa)}\| + \|u_t^{(\varepsilon,\kappa)}\| \} \leqslant c\varepsilon^{1/2-a} \} \end{aligned} \tag{3.36}$$

$$\mathscr{B}_\varepsilon^{(2)} \doteq \{ \sup_{0 \leqslant t \leqslant t_\varepsilon} \|Z_t - \tau_\Delta Z_t^*\|_{x_0,\varepsilon} < e^{-\varepsilon^{-1/100}} \}, \tag{3.37}$$

$$\mathscr{B}_\varepsilon^{(3)} \doteq \{ \sup_{0 \leqslant t \leqslant t_\varepsilon} \|\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}_{t,x_0^*}^*\|_{x_0,\varepsilon} < e^{-\varepsilon^{-1/100}} \} \tag{3.38}$$

where $\hat{Z}_{t,x_0}$ is given by (3.5), and $\hat{Z}^*_{t,x_0^*}$ is given by (3.5), with Z^*_t (given by (3.33)) in the place of Z_t and x_0^* in the place of x_0. That is,

$$\hat{Z}^*_{t,x_0^*} \doteq Z^*_t - \int_0^t ds\, g_{t-s,x_0^*}[(3\bar{m}^2_{x_0^*} - 1)\, Z^*_s] \tag{3.39}$$

We will prove that there is $c > 0$ so that

$$P^\varepsilon(\hat{\mathscr{B}}_{\varepsilon,p}) \geq 1 - ce^{-\varepsilon^{-p}}, \qquad \hat{\mathscr{B}}_{\varepsilon,p} \doteq \mathscr{B}^{(1)}_{\varepsilon,p} \cap \mathscr{B}^{(2)}_\varepsilon \cap \mathscr{B}^{(3)}_\varepsilon \tag{3.40}$$

Since $D_\varepsilon(m, m^*) \leq \varepsilon^b < 1$, both m and m^* are in $\mathscr{M}_{\varepsilon^{1/2-a}}$. Then by (3.8) and the proof of Proposition 3 there is a constant c so that

$$P^\varepsilon(\mathscr{B}^{(1)}_{\varepsilon,p}) \geq 1 - ce^{-\varepsilon^{-p}} \tag{3.41}$$

A similar bound holds for the probability of $\mathscr{B}^{(i)}_\varepsilon$, $i = 2, 3$. (See Lemma A.5 of the Appendix and recall that $p < 1/2$).

Step 3. *Bounds close to the center.* Let $\gamma \in (0, 1/2 - a)$ and $b > 0$, $\bar{\gamma} \in (\gamma, 1/2 - a)$ and M a positive integer such that $b + \bar{\gamma} < M(1/2 - a)$. It follows that there exists ε_0 so that for $\varepsilon < \varepsilon_0$

$$[t_\varepsilon \varepsilon^{1/2-a}]^M \leq \delta \varepsilon^{\bar{\gamma}} \tag{3.42}$$

We will prove that there are c and p positive so that for $\varepsilon < \varepsilon_0$, in the set $\hat{\mathscr{B}}_{\varepsilon,p}$,

$$\sup_{|x - x_0| \leq \varepsilon^{-1/10} - M\varepsilon^{-1/20}} |v_{t_\varepsilon}(x) - \tau_\Delta u_{t_\varepsilon}(x)| \leq c\delta\varepsilon^{\bar{\gamma}} \tag{3.43}$$

Proof of (3.43). We set

$$d_t(x) \doteq v_t(x) - \tau_\Delta u_t(x) \tag{3.44}$$

For x, y and h in $\mathbb{R}$ and $t > 0$

$$g_{t,x_0+h}(x+h, y+h) = g_{t,x_0}(x, y), \qquad \bar{m}_{x_0}(x-h) = \bar{m}_{x_0+h}(x) \tag{3.45}$$

hence, for any function $f \in C^0(\mathbb{R})$,

$$\int dy\, g_{t,x_0^*}(x+\Delta, y)\, \bar{m}_{x_0^*}(y)\, f(y) = \int dy\, g_{t,x_0}(x, y)\, \bar{m}_{x_0}(y)\, \tau_\Delta f(y) \tag{3.46}$$

Writing the integral equation (3.4) for v_t and u_t (the former with g_{t, x_0}, the latter with g_{t, x_0^*}) we get

$$d_t = [g_{t, x_0} v - \tau_\Delta g_{t, x_0^*} u] + \int_0^t ds (g_{t-s, x_0} A_s d_s) + \sqrt{\varepsilon}[\hat{Z}_{t, x_0} - \tau_\Delta \hat{Z}^*_{t, x_0^*}] \tag{3.47}$$

where

$$A_s \doteq -3\bar{m}_{x_0}[v_s + \tau_\Delta u_s] - [v_s^2 + (\tau_\Delta u_s)^2 + v_s \tau_\Delta u_s] \tag{3.48}$$

For any $k = 0, \ldots, M$ we define

$$B_k \doteq \{y : |y - x_0| \leqslant \varepsilon^{-1/10} - k\varepsilon^{-1/20}\} \tag{3.49}$$

$$D_{k,t} \doteq \sup_{x \in B_k} |d_t(x)| \tag{3.50}$$

We next prove that there are $c_1 > 0$, $c_2 > 0$ and for any n there is $c'_n > 0$ so that in $\hat{\mathscr{B}}_{\varepsilon, p}$

$$D_{k,t} \leqslant c_1 \delta \left[1 + \frac{\varepsilon^{1/2 - a}}{\sqrt{t}}\right] + c_2 \int_0^t ds\, \varepsilon^{1/2 - a} D_{k-1, s} + c'_n \varepsilon^n \tag{3.51}$$

To prove (3.51) we first notice that by (3.36) and (3.40) there is $c > 0$ so that in $\hat{\mathscr{B}}_{\varepsilon, p}$

$$\sup_{t \leqslant t_\varepsilon} \|A_t\|_{\varepsilon, x_0} \leqslant c\varepsilon^{1/2 - a} \tag{3.52}$$

Furthermore from (3.38) and (3.40) it follows that for any n there is c_n so that

$$\sup_{t \leqslant t_\varepsilon} \sqrt{\varepsilon}\, \|\hat{Z}_{t, x_0} - \tau_\Delta \hat{Z}^*_{t, x_0^*}\|_{x_0, \varepsilon} \leqslant c_n \varepsilon^n \tag{3.53}$$

By (3.45),

$$\begin{aligned}(g_{t, x_0} v)(x) &- (\tau_\Delta g_{t, x_0^*} u)(x) \\ &= [g_{t, x_0}(v - \tau_\eta u)] + \int dy\, \tau_\eta u(y)[g_{t, x_0}(x, y) - g_{t, x_0}(x, y - \Delta + \eta)]\end{aligned} \tag{3.54}$$

We then use (3.32), that $\|u\| \leqslant 2\varepsilon^{1/2-a}$ and (A.31) to conclude that for any $k=0,..., M$

$$\sup_{x \in B_k} |g_{t, x_0} v(x) - \tau_\Delta g_{t, x_0^*} u(x)| \leqslant c_1 \delta \left[1 + \frac{\varepsilon^{1/2-a}}{\sqrt{t}} \right] \tag{3.55}$$

Furthermore by Theorem 4 and the Appendix, see (A.30) and (A.32), for any n there is $c_n > 0$ so that for all $k=0,..., M$

$$\sup_{t \geqslant 0} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} dy\, g_{t, x_0}(x, y) \leqslant c_0, \qquad \sup_{t > 0} \sup_{x \in B_k} \int_{\mathbb{R} \setminus B_{k-1}} dy\, g_{t, x_0}(x, y) \leqslant c_n \varepsilon^n \tag{3.56}$$

so (3.51) follows from (3.47), (3.52), (3.53), (3.55), and (3.56).

By Theorem 4

$$\|g_{t_\varepsilon, x_0} v\| \leqslant c e^{-\alpha t}, \qquad \|g_{t_\varepsilon, x_0^*} u\| \leqslant c e^{-\alpha t_\varepsilon} \tag{3.57}$$

By (3.47), (3.57), and (3.52) we have that, in the set $\hat{\mathscr{B}}_{\varepsilon, p}$

$$D_{M, t_\varepsilon} \leqslant 2ce^{-\alpha t_\varepsilon} + c_2 \int_0^{t_\varepsilon} ds\, \varepsilon^{1/2-a} D_{M-1, s} + c_n \varepsilon^n$$

Therefore by iterating (3.51) we get

$$D_{M, t_\varepsilon} \leqslant 2ce^{-\alpha t_\varepsilon} + \sum_{k=1}^{M} [c_2 \varepsilon^{1/2-a}]^k \int_0^{t_\varepsilon} ds_1 \cdots \int_0^{s_{k-1}} ds_k\, c_1 \delta \left(1 + \frac{\varepsilon^{1/2-a}}{\sqrt{s_k}} \right)$$
$$+ [c_2 \varepsilon^{1/2-a}]^{M-1} \int_0^{t_\varepsilon} ds_1 \cdots \int_0^{s_{M-1}} ds_M\, D_{0, s_M} + c_n \varepsilon^n$$

By the choice of M and since $D_{0, s_M} \leqslant \varepsilon^{1/2-a}$ we conclude the proof of Step 3.

Step 4. *Bounds away from the center.* We prove that there are c, p and ε_0 positive so that if $\varepsilon > \varepsilon_0$ and the processes m_t and m_t^* are in $\hat{\mathscr{B}}_{\varepsilon, p}$, then

$$\sup_{\varepsilon^{-1/10} - M\varepsilon^{-1/20} \leqslant |x - x_0| \leqslant 2\varepsilon^{-1/10}} |v_t(x) - \tau_\Delta u_t(x)| \leqslant c e^{-t\lambda}, \qquad \text{for all} \quad t \leqslant t_\varepsilon \tag{3.58}$$

We set

$$r_{\varepsilon, M} = \varepsilon^{-1/10} - M\varepsilon^{-1/20}$$

and consider the case $x-x_0\in[r_{\varepsilon,M}, 2\varepsilon^{-1/10}]$. The analysis of the other interval involved in the sup in (3.58) is similar and omitted. Given $V>0$ we define

$$m^+(x)=\begin{cases} m(x) & \text{for} \quad x-x_0\in(r_{\varepsilon,M}-Vt_\varepsilon, 2^{\varepsilon}-1/10+Vt_\varepsilon) \\ 1 & \text{for} \quad x-x_0\in\{(-\infty, r_{\varepsilon,M}-2Vt_\varepsilon) \\ & \qquad\qquad \cup(2\varepsilon^{-1/10}+2Vt_\varepsilon, +\infty)\} \end{cases} \tag{3.59}$$

and complete the definition of m^+ in the missing intervals by linear interpolation. $\tilde{m}^+$ is defined similarly with m replaced by m^* and x_0 by x_0^*.

We set

$$m_t^+=T_t(m^+, \sqrt{\varepsilon}\, Z), \qquad \tilde{m}_t^+=T_t(\tilde{m}^+, \sqrt{\varepsilon}\, Z^*)$$

We choose V in (3.59) as the parameter entering in the Barrier Lemma (Ref. 5, Prop. 5.3). Then there is $c>0$ so that in $\hat{\mathscr{B}}_{\varepsilon,p}$ for all $t\leqslant t_\varepsilon$,

$$\sup_{x-x_0\in[r_{\varepsilon,M}, 2\varepsilon^{-1/10}]} |m_t(x)-m_t^+(x)|\leqslant ce^{-t} \tag{3.60}$$

The same bound holds for $m_t^*(x)-\tilde{m}_t^+(x)$ when x_0 is replaced by x_0^*. We define

$$v_t^+=m_t^+-1, \qquad u_t^+=\tilde{m}_t^+-1 \tag{3.61}$$

It is not difficult to prove the following *a priori* bound for v_t^+ and u_t^+: for $p<a$ there is $c>0$ so that

$$\|v_t^+\|\leqslant 2\lambda c, \qquad \|u_t^+\|\leqslant 2\lambda c, \qquad \text{for all} \quad t\leqslant t_\varepsilon \tag{3.62}$$

We consider next the versions of v_t^+ and u_t^+ given by the corresponding solutions of the equations:

$$v_t^+=e^{-2t}H_t v_0^+ +\int_0^t ds\, e^{-2(t-s)}H_{t-s}(-3(v_s^+)^2-(v_s^+)^3)+\sqrt{\varepsilon}\, V_t$$

$$u_t^+=e^{-2t}H_t u_0^+ +\int_0^t ds\, e^{-2(t-s)}H_{t-s}(-3(u_s^+)^2-(u_s^+)^3)+\sqrt{\varepsilon}\, V_t^*$$

where

$$V_t(x) \doteq \int_0^t ds \int_{\mathscr{T}_{\varepsilon,\kappa}} dy\, e^{-2(t-s)} H^{(\varepsilon)}_{t-s}(x, y)\, \alpha(s, y)$$

$$V^*_t(x) \doteq \int_0^t ds \int dz\, \mathbf{1}_{\{|z+\Delta-x^*_0| \leqslant 4\varepsilon^{-1/10};\, z+\Delta \in \mathscr{T}_{\varepsilon,\kappa}\}} e^{-t-s} H^{(\varepsilon)}_{t-s}(x, z+\Delta)\, \alpha(s, z)$$
$$+ \int_0^t ds \int dy\, \mathbf{1}_{\{|y-s^*_0| > 4\varepsilon^{-1/10};\, y \in \mathscr{T}_{\varepsilon,\kappa}\}} e^{-(t-s)} H^{(\varepsilon)}_{t-s}(x, y)\, \tilde{\alpha}(s, y) \quad (3.63)$$

for α and $\tilde{\alpha}$ as in Step 1. We call

$$d^+_t \doteq \sup_{x-x_0 \in (r_{\varepsilon,M},\, 2\varepsilon^{-1/10})} (v^+_t - \tau_\Delta u^+_t)$$

From (3.62), we get that there is a constant $c' > 0$ such that

$$d^+_t \leqslant e^{-2t} 2\lambda c + c'\lambda \int_0^t ds\, e^{-2(t-s)} d^+_s + \sqrt{\varepsilon}\, \| V_t - \tau_\Delta V^*_t \|_{\varepsilon, x_0} \quad (3.64)$$

But $V_t - \tau_\Delta V^*_t$ is a Gaussian process, and it is not difficult to see that the proof of (3.40) is still valid for a set like (3.37) with $\| V_t - \tau_\Delta V^*_t \|_{\varepsilon, x_0}$ instead of $\| Z_t - \tau_\delta Z^*_t \|_{\varepsilon, x_0}$. Then by (3.64) there is $c > 0$ so that

$$d^+_t \leqslant c e^{-2t} \lambda, \qquad \text{for all} \quad t \leqslant t_\varepsilon \quad (3.65)$$

Since

$$v_t - \tau_\Delta u_t = (v^+_t - \tau_\Delta u^+_t) + (m_t - m^+_t) - \tau_\Delta (m^*_t - \tilde{m}^+_t)$$

from (3.65) and (3.60), we get that

$$\sup_{x-x_0 \in (r_{\varepsilon,M},\, 2\varepsilon^{-1/20})} |v_t(x) - \tau_\Delta u_t(x)| \leqslant c(e^{-2t}\lambda + e^{-t}), \qquad \text{for all} \quad t \leqslant t_\varepsilon \quad (3.66)$$

Step 5. *Conclusion.* By (3.43) and (3.58) there are c and ε_0 so that if $\varepsilon < \varepsilon_0$ and the processes m_t and $m^*_t \in \hat{\mathscr{B}}_{\varepsilon,p}$, then

$$\sup_{|x-x_0| \leqslant 2\varepsilon^{-1/20}} |m_{t_\varepsilon}(x) - \tau_\Delta m^*_{t_\varepsilon}(x)|$$
$$= \sup_{|x-x_0| \leqslant 2\varepsilon^{-1/10}} |v_{t_\varepsilon}(x) - \tau_\Delta u_{t_\varepsilon}(x)| \leqslant \delta \varepsilon^{\bar{\gamma}} \quad (3.67)$$

We then observe that from Lemma 1 it follows that in $\hat{\mathscr{B}}_{\varepsilon, p}$,

$$|\xi^{\varepsilon,\kappa}(m_{t_\varepsilon}) - x_0| \leqslant c\varepsilon^{1/2-a}$$

so that from (3.67) it follows that

$$\|m_{t_\varepsilon} - \tau_\Delta m^*_{t_\varepsilon}\|_{\varepsilon,\, \xi(m^{\varepsilon,\kappa}_{t_\varepsilon})} \leqslant \delta\varepsilon^{\bar{\gamma}} \tag{3.68}$$

By Lemma 1

$$|\xi^{\varepsilon,\kappa}(m^*_{t_\varepsilon}) - \xi^{\varepsilon,\kappa}(m_{t_\varepsilon})| \leqslant c_0\, |\langle \bar{m}'_{\xi^{\varepsilon,\kappa}(m_{t_\varepsilon})}, (m_{t_\varepsilon})^{\varepsilon,\kappa} - \tau_\Delta (m^*_{t_\varepsilon})^{\varepsilon,\kappa}\rangle| \tag{3.69}$$

By (3.68) and the exponential decay at infinity of $\bar{m}'_{\xi(m_{t_\varepsilon})}$ there is $c > 0$ so that

$$|\xi^{\varepsilon,\kappa}(m^*_{t_\varepsilon}) - \xi^{\varepsilon,\kappa}(m_{t_\varepsilon}) - \Delta| \leqslant c\, \delta\varepsilon^{\bar{\gamma}} \tag{3.70}$$

Since $\bar{\gamma} > \gamma$, we have thus proven that in $\hat{\mathscr{B}}_{\varepsilon, p}$ with p small enough,

$$\|m_{t_\varepsilon} - \tau_\Delta m^*_{t_\varepsilon}\|_{\varepsilon,\, \xi^{\varepsilon,\kappa}(m_{t_\varepsilon})} \leqslant \delta\varepsilon^{\gamma}, \qquad |\xi^{\varepsilon,\kappa}(m^*_{t_\varepsilon}) - \xi^{\varepsilon,\kappa}(m_{t_\varepsilon}) - \Delta| \leqslant \delta\varepsilon^{\gamma} \tag{3.71}$$

Proposition 4 is proved. □

As a corollary of the previous proposition we have a fast decay transversally to $\mathscr{M}$, as we are going to see. We define

$$\psi\colon C(\mathbb{R}) \to C(\mathbb{R}) \qquad \text{such that} \quad \psi(m)(x) \doteq m^{\varepsilon,\kappa}(x + \xi^{\varepsilon,\kappa}(m)) \tag{3.72}$$

Note that we omit in the expression of ψ the explicit dependence on ε and κ.

Corollary 1. Let $\kappa \geqslant 1$, $\ell \in (0, 1)$, $a \in (0, 1/2)$, $m, m^* \in M^{\varepsilon}_{\kappa, \ell, \varepsilon^{1/2-a}}$. Then we can construct m_t and m^*_t, solutions of (1.5) with initial datum m_0 and m^* respectively, in the same probability space (but with different noises), and such that, for any positive integer N there are $c > 0$, and $p > 0$ so that

$$P^{\varepsilon}(\|\psi - \psi^*\|_{\varepsilon} \leqslant \varepsilon^N) \geqslant 1 - ce^{-\varepsilon^{-p}} \tag{3.73}$$

where we set

$$\psi \doteq \psi(m_{t_\varepsilon}) \qquad \psi^* \doteq \psi(m^*_{t_\varepsilon})$$

Proof. Let $\gamma \in (0, 1/2-a)$, $n>1$, m_t and m_t^* constructed as in Proposition 4. By iterating (3.30) n times we get (for a constant c possibly different from that in (3.30))

$$P^\varepsilon(D_\varepsilon(m_{t_\varepsilon}, m^*_{t_\varepsilon}) \leqslant \varepsilon^{1/2-a}\varepsilon^{n\gamma}) \geqslant 1 - ce^{-\varepsilon^{-p}} \tag{3.74}$$

Then, by the comment after Definition 2, there is η such that (recall $\varepsilon^{1/2-a}<1$)

$$P^\varepsilon(|\eta - \Delta| \leqslant \varepsilon^{n\gamma}, \|m_{t_\varepsilon} - \tau_\eta m^*_{t_\varepsilon}\|_{\varepsilon, x_0} \leqslant \varepsilon^{n\gamma}) \geqslant 1 - ce^{-\varepsilon^{-p}} \tag{3.75}$$

where $\Delta = x_0^* - x_0$, with $x_0 = \xi(m_{t_\varepsilon}^{\varepsilon,\kappa})$ and $x_0^* = \xi((m^*_{t_\varepsilon})^{\varepsilon,\kappa})$. We note that

$$\begin{aligned}\|\psi - \psi^*\|_\varepsilon &= \|m_{t_\varepsilon} - \tau_\Delta m^*_{t_\varepsilon}\|_{\varepsilon, x_0} \\ &\leqslant \|m_{t_\varepsilon} - \tau_\eta m^*_{t_\varepsilon}\|_{\varepsilon, x_0} + \|\tau_{\eta-\Delta} m^*_{t_\varepsilon} - m^*_{t_\varepsilon}\|_{\varepsilon, x_0^*}\end{aligned} \tag{3.76}$$

In the Appendix, see Lemma A.3, we prove the following property of the Ginzburg–Landau process. For any $\delta>0$, $\alpha \in (0, 1/2)$ there is a constant $c>0$ so that for any ε small enough

$$P^\varepsilon\left(\sup_{|x|,\,|y| \leqslant \varepsilon^{-\kappa}} \sup_{\substack{x \neq y \\ |x-y| \leqslant 1}} \frac{|m_{t_\varepsilon}(x) - m_{t_\varepsilon}(y)|}{|x-y|^\alpha} > \delta\right) \leqslant e^{-c\delta^2} \tag{3.77}$$

Applying the previous inequality to m_t^*, we obtain

$$\begin{aligned}&P^\varepsilon\left(\sup_{|x|,\,|y| \leqslant \varepsilon^{-\kappa}} \sup_{\substack{x \neq y \\ |x-y| \leqslant |\eta-\Delta|}} \frac{|m^*_{t_\varepsilon}(x) - m^*_{t_\varepsilon}(y)|}{|x-y|^{1/4}} > 1\right) \\ &\quad\leqslant P^\varepsilon\left(\sup_{|x|,\,|y| \leqslant \varepsilon^{-\kappa}} \sup_{\substack{x \neq y \\ |x-y| \leqslant 1}} \frac{|m^*_{t_\varepsilon}(x) - m^*_{t_\varepsilon}(y)|}{|x-y|^{3/8}} > |\eta-\Delta|^{-1/2}\right) \\ &\quad\leqslant e^{-c\,|\eta-\Delta|^{-1/4}}\end{aligned} \tag{3.78}$$

By (3.75), (3.76), and (3.78), and choosing n such that $\varepsilon^{n\gamma/4} < \varepsilon^N$ we then derive (3.73). Corollary 1 is proved. □

Remark. Corollary 1 proves that two processes that start from different data become almost equal modulo translations, with probability going to 1 as $\varepsilon \to 0^+$ as fast as in (3.73). We improve this result in the next proposition in the sense that we have a similar statement without translations. The price we pay is twofold. The processes must start from data with linear centers close to each other and, more importantly, the rate of convergence is not as fast as before. In particular it is not fast enough for what needed in Section 4, where we prove Theorem 2. In that case we use Corollary 1 as we can reduce the analysis to events invariant under translations. This is no longer possible when proving Theorem 3, where however we need only convergence in probability, without bounds on the rate of convergence: for this Proposition 5 next will suffice.

Proposition 5. Let $a \in (0, 1/2)$, $b \geq 1-a$, $\gamma \in (0, 1/2-a)$, m and $\tilde{m}$ both in $M^{\varepsilon}_{\kappa, \ell, \varepsilon^{1/2-a}}$ and

$$\|m-\tilde{m}\| \leq \varepsilon^b, \qquad |\xi^{\varepsilon,\kappa}(m)-\xi^{\varepsilon,\kappa}(\tilde{m})| \leq \varepsilon^b \tag{3.79}$$

Then we can construct the Ginzburg–Landau processes $\{m_t\}_{t\geq 0}$ and $\{\tilde{m}_t\}_{t\geq 0}$ starting respectively from m and $\tilde{m}$ in the same probability space and so that

$$\lim_{\varepsilon\to 0^+} P^{\varepsilon}(\|(m_{t_\varepsilon})^{\varepsilon,\kappa}-(\tilde{m}_{t_\varepsilon})^{\varepsilon\kappa}\| \leq \varepsilon^{b+\gamma}, |\xi^{\varepsilon,\kappa}(m_{t_\varepsilon})-\xi^{\varepsilon,\kappa}(\tilde{m}_{t_\varepsilon})| \leq \varepsilon^{b+\gamma})=1 \tag{3.80}$$

Proof. We set $\delta \doteq \varepsilon^b$ and $\lambda \doteq \varepsilon^{1/2-a}$.

We consider first the case when m and $\tilde{m}$ are not in $C_{\varepsilon,\kappa}(\mathbb{R})$. Let $x_0 \doteq \xi(m)$, $\tilde{x}_0 \doteq \xi(\tilde{m})$ and $t^*_\varepsilon \doteq \varepsilon^q t_\varepsilon$, $q>0$ will be specified later. Let $\{e^{(\varepsilon)}_j\}_{j\geq 1}$ be an orthonormal basis of $L^2(\mathscr{T}_{\varepsilon,\kappa})$, such that $e^{(\varepsilon)}_1=\sqrt{D_\varepsilon}\,\bar{m}'_{x_0}$ on $\mathscr{T}_{\varepsilon,\kappa}$, D_ε the normalization constant, ($D_\varepsilon \to D=3/4$ as $\varepsilon\to 0^+$). Set

$$\langle f, g\rangle_{\varepsilon,\kappa} \doteq \int_{\mathscr{T}_{\varepsilon,\kappa}} dx\, f(x)\, g(x), \qquad f, g \in L^2(\mathscr{T}_{\varepsilon,\kappa}) \tag{3.81}$$

Let $\{b_j(t)\}_{j\geq 0}$ be a family of standard independent Brownian motions, and consider the Gaussian process

$$\hat{Z}^{(1)}_{t,x_0}(x) \doteq \int_0^t db_1(s) \int_{\mathscr{T}_{\varepsilon,\kappa}} dy\, g^{(\varepsilon)}_{t-s,x_0}(x,y)\, e^{(\varepsilon)}_1(y) + R_t(x) \tag{3.82}$$

where

$$R_t(x) \doteq \sum_{j\geq 2} \int_0^t db_j(s) \int_{\mathscr{T}_{\varepsilon,\kappa}} dy\, g^{(\varepsilon)}_{t-s,x_0}(x,y)\, e^{(\varepsilon)}_j(y) \tag{3.83}$$

and $g^{(\varepsilon)}_{t-s,x_0}$ was defined in (3.7). By comparing covariances, it is easy to check that $\hat{Z}^{(1)}_{t,x_0}$ has the same law of the process $\hat{Z}_{t,x_0}$ defined in (3.6). We will construct another Gaussian process $\hat{Z}^{(2)}_{t,x_0}$ with the same law. Consider, for y_0 that will be conveniently chosen later, the process

$$\tilde{b}_1(t) \doteq \begin{cases} b_0(t) & \text{if} \quad t \in [0,\tau] \\ b_1(t) - y_0 & \text{if} \quad t \geqslant \tau \end{cases} \tag{3.84}$$

where

$$\tau \doteq \inf\{t \geqslant 0: [b_1(t) - b_0(t)] = y_0\} \tag{3.85}$$

The process $\tilde{b}_1(t)$ is a Brownian motion, independent of $\{b_j(t)\}_{j \geqslant 2}$. Finally, let

$$\hat{Z}^{(2)}_{t,x_0}(x) \doteq \int_0^t d\tilde{b}_1(s) \int_{\mathscr{T}_{\varepsilon,\kappa}} dy\, g^{(\varepsilon)}_{t-s,x_0}(x,y)\, e^{(\varepsilon)}_1(y) + R_t(x) \tag{3.86}$$

Write the integral equations (3.4) for m_t and $\tilde{m}_t$ as in the statement of the proposition, using the Gaussian processes $\hat{Z}^{(1)}_{t,x_0}$ and $\hat{Z}^{(2)}_{t,x_0}$ respectively. Then,

$$\begin{aligned} m_\tau - \tilde{m}_\tau = {} & g_{\tau,x_0}(m - \tilde{m}) - \int_0^\tau ds\, g_{\tau-s,x_0}(A_{x_0}[v_s] - A_{x_0}[u_s]) \\ & + \sqrt{\varepsilon} \int_0^\tau d(b_1 - b_0)(s) \int_{\mathscr{T}_{\varepsilon,\kappa}} dy\, g^{(\varepsilon)}_{t-s,x_0}(x,y)\, e^{(\varepsilon)}_1(y) \end{aligned} \tag{3.87}$$

where $A_{x_0}[f] \doteq 3\bar{m}_{x_0} f^2 + f^3$, $v_t \doteq m_t - \bar{m}_{x_0}$, $u_t \doteq \tilde{m}_t - \bar{m}_{x_0}$. We multiply both sides by $\bar{m}'_{x_0}$ and integrate over $\mathbb{R}$. We get, in $\{\tau \leqslant t^*_\varepsilon\}$:

$$\begin{aligned} \langle \bar{m}'_{x_0}, (m_\tau - \tilde{m}_\tau)\rangle = {} & -\langle \bar{m}'_{x_0}, \tilde{m}\rangle - \int_0^\tau ds\, \langle \bar{m}'_{x_0}, (A_{x_0}[v_s] - A_{x_0}[u_x])\rangle \\ & + \sqrt{\varepsilon} \int_0^\tau d(b_1 - b_0)(s) \langle a_\varepsilon, e^{(\varepsilon)}_1\rangle_{\varepsilon,\kappa} \end{aligned} \tag{3.88}$$

where

$$\begin{aligned} a_\varepsilon(y) & \doteq \langle g^{(\varepsilon)}_{\tau-s,x_0}(\cdot, y), \bar{m}'_{x_0}\rangle \\ & = \sum_{j \in \mathbb{Z}} (\bar{m}'_{x_0}(x, y + 4j\varepsilon^{-\kappa}) + \bar{m}'_{x_0}(x, 4j\varepsilon^{-\kappa} + 2\varepsilon^{-\kappa} - y)) \end{aligned}$$

Choosing now $y_0 = \langle \bar{m}'_{x_0}, \tilde{m} \rangle (\sqrt{\varepsilon}\, \langle a_\varepsilon, e_1^{(\varepsilon)} \rangle_{\varepsilon,\kappa})^{-1}$, from the definition of τ we then get

$$\langle \bar{m}'_{x_0}, m_\tau - \tilde{m}_\tau \rangle = -\int_0^\tau ds\, \langle \bar{m}'_{x_0}, A_{x_0}[v_s] - A_{x_0}[u_s] \rangle \tag{3.89}$$

By standard results on Brownian motions and since $\lim_{\varepsilon \to 0^+} \langle a_\varepsilon, e_1^{(\varepsilon)} \rangle_{\varepsilon,\kappa} = \sqrt{D}$, there is $c > 0$ so that

$$P^\varepsilon(\tau \leqslant t_\varepsilon^*) \geqslant 1 - c\, \frac{|\langle \bar{m}'_{x_0}, \tilde{m} \rangle|}{\sqrt{\varepsilon t_\varepsilon^*}} \geqslant 1 - c\varepsilon^{1-a-1/2-q/2}$$

By choosing $q < 1 - 2a$,

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(\tau \leqslant t_\varepsilon^*) = 1 \tag{3.90}$$

The set

$$\mathscr{G} \doteq \{\sup_{t \leqslant t_\varepsilon} (\|v_t\| + \|u_t\|) \leqslant 2\delta\}$$

has, by Propositions 3, probability that goes to 1 as $\varepsilon \to 0^+$. There is $c_2 > 0$ so that in $\mathscr{G}$

$$\sup_{0 \leqslant t \leqslant t_\varepsilon^*} \|m_t - \tilde{m}_t\| \leqslant c_2 \delta \tag{3.91}$$

Let $\bar{\gamma} \in (\gamma, 1/2 - a)$, then in $\{\tau \leqslant t_\varepsilon^*\} \cap \mathscr{G}$ and for all $\varepsilon > 0$ small enough

$$|\langle \bar{m}'_{x_0}, m_\tau - \tilde{m}_\tau \rangle| \leqslant \delta \varepsilon^{\bar{\gamma}} \tag{3.92}$$

By Lemma 1 there is $c_3 > 0$ so that in the same set

$$|\xi(m_\tau) - \xi(\tilde{m}_\tau)| \leqslant c_3 \delta \varepsilon^{\bar{\gamma}} \tag{3.93}$$

Finally, using the integral Eq. (3.4) in the time interval $[\tau, t_\varepsilon^*]$, by (3.91) and (3.92), there is $c_4 > 0$ so that in $\{\tau \leqslant t_\varepsilon^*\} \cap \mathscr{G}$

$$|\xi(m_{t_\varepsilon^*}) - \xi(\tilde{m}_{t_\varepsilon^*})| \leqslant c_4 \delta \varepsilon^{\bar{\gamma}} \tag{3.94}$$

We next consider the time interval $[t_\varepsilon^*, t_\varepsilon]$. Let $x_\varepsilon^* \doteq \xi(m_{t_\varepsilon^*})$. We set, for any $t \in [0, \bar{t}_\varepsilon]$, $\bar{t}_\varepsilon \doteq (1 - \varepsilon^q)\, t_\varepsilon$,

$$v_t^* \doteq m_{t_\varepsilon^* + t} - \bar{m}_{x_\varepsilon^*}, \qquad u_t^* \doteq \tilde{m}_{t_\varepsilon^* + t} - \bar{m}_{x_\varepsilon^*} \tag{3.95}$$

We write (3.4) for v_t and u_t relatively to $\tilde{m}_{x_\varepsilon^*}$. Setting $\Delta_* = x_\varepsilon^* - \xi(\tilde{m}_{t_\varepsilon^*})$,

$$\begin{aligned} g_{\bar{t}_\varepsilon, x_\varepsilon^*}(v_0^* - u_0^*) &= g_{\bar{t}_\varepsilon, x_\varepsilon^*}(m_{t_\varepsilon^*} - \tilde{m}_{t_\varepsilon^*}) \\ &= g_{\bar{t}_\varepsilon, x_\varepsilon^*}(m_{t_\varepsilon^*} - \tau_{\Delta_*}\tilde{m}_{t_\varepsilon^*}) \\ &\quad + \int dy \, [\, g_{\bar{t}_\varepsilon, x_\varepsilon^*}(x, y - \Delta_*) - g_{\bar{t}_\varepsilon, x_\varepsilon^*}(x, y)] \, \tilde{m}_{t_\varepsilon^*}(y) \end{aligned} \tag{3.96}$$

Then Theorem 4 (3.94) and (A.31) imply that there are c_5 and c_6 so that

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(\|g_{\bar{t}_\varepsilon, x_\varepsilon^*}(v_0^* - u_0^*)\| \leqslant c_5 e^{-\alpha \bar{t}_\varepsilon} + c_6 \delta \varepsilon^{\bar{\gamma}}) = 1 \tag{3.97}$$

Using (3.97) and the integral equations for v_t^* and u_t^*, we get, for some constant c_7,

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(\|v_{\bar{t}_\varepsilon}^* - u_{\bar{t}_\varepsilon}^*\| \leqslant c_7 \delta \varepsilon^{\bar{\gamma}}) = 1 \tag{3.98}$$

By (3.98), using Lemma 1, we have also, for some constant c_8,

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(|\xi(m_{t_\varepsilon}) - \xi(\tilde{m}_{t_\varepsilon})| \leqslant c_8 \delta \varepsilon^{\bar{\gamma}}) = 1 \tag{3.99}$$

and so, since $\bar{\gamma} > \gamma$, by (3.98) and (3.99) we finally obtain

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(\|m_{t_\varepsilon} - \tilde{m}_{t_\varepsilon}\| \leqslant \delta \varepsilon^\gamma, \, |\xi(m_{t_\varepsilon}) - \xi(\tilde{m}_{t_\varepsilon})| \leqslant \delta \varepsilon^\gamma) = 1 \tag{3.100}$$

This proves the proposition for $m, \tilde{m} \notin C_{\varepsilon,\kappa}(\mathbb{R})$. By using conveniently the Barrier Lemma, one easily extends the result to the general case. We omit the details. □

When proving Theorem 3 we will consider the case when at some time T we have two data, m_T and $\tilde{m}_T$, both in $C_{\varepsilon,\kappa}(\mathbb{R})$ and in $M^\varepsilon_{\kappa,\ell,\varepsilon^{1/2-a}}$, and such that $\xi^{\varepsilon,\kappa}(m_T) = \xi^{\varepsilon,\kappa}(\tilde{m}_T)$, see the end of Section 5. This case is not directly covered by Proposition 5, but we will see in the following lemma that if we construct with the same noise the two processes then after a time t_ε they will verify with great probability the conditions of Proposition 5.

Lemma 3. Let m and $\tilde{m}$ both in $M^\varepsilon_{\kappa,\ell,\varepsilon^{1/2-a}}$, $a \in (0, 1/4)$, with $x_0 \doteq \xi^{\varepsilon,\kappa}(m) = \xi^{\varepsilon,\kappa}(\tilde{m})$ and let $p \in (0, a)$. Then for any $\omega \in (2a, 1/2)$ and any ε small enough, in $\mathscr{B}_{p,\varepsilon,x_0}$ the following estimate holds:

$$\|T_{t_\varepsilon}(m, \sqrt{\varepsilon}\, Z) - T_{t_\varepsilon}(\tilde{m}, \sqrt{\varepsilon}\, Z)\| \leqslant \varepsilon^{1-\omega}$$

Proof. Writing the integral equation (3.4) around $\bar{m}_{x_0}$ for the two processes one obtains an equation for the difference like (3.47) but with the same center and noises. Then the estimate follows easily. □

We conclude the section with two lemmas consequence of general properties of the Ginzburg–Landau process. For any $a \in (0, 1/2)$ and any $\phi \in \mathscr{M}_{\varepsilon^{1/2-a}}$, let us denote by E^{ε}_{ϕ} the expectation with respect to the Ginzburg–Landau process starting from ϕ. We indicate with m_t the coordinate map on $C(\mathbb{R}_+; C(\mathbb{R}))$ and let $\xi_t = \xi^{\varepsilon, \kappa}(m_t)$. We also recall that $\psi(m)$ is defined in (3.72).

Lemma 4. For any t, $s \geqslant 0$,

$$E^{\varepsilon}_{\bar{m}}[E^{\varepsilon}_{\psi(m_s)}[\xi_t]] = 0 \tag{3.101}$$

Proof. Consider the symmetry transformation $\mathscr{R}$: $C(\mathbb{R}) \to C(\mathbb{R})$ defined by $(\mathscr{R}\phi)(x) \doteq -\phi(-x)$. We want first to prove that starting from $\bar{m}$ the laws of $\psi(m_s)$ and $\mathscr{R}(\psi(m_s))$ are identical. In fact for any $m \in \mathscr{M}_{\varepsilon^{1/2-a}}$, we have that $T_t(m, \sqrt{\varepsilon}\, \underline{Z}) = \mathscr{R}T_t(\mathscr{R}m, \sqrt{\varepsilon}\, \mathscr{R}\underline{Z})$ for all $t \geqslant 0$. Since $\bar{m} = \mathscr{R}\bar{m}$, for any bounded functional F on $C(\mathbb{R}_+; C(\mathbb{R}))$, $E^{\varepsilon}_{\bar{m}}[f] = E^{\varepsilon}_{\bar{m}}[\mathscr{R}F]$. By choosing $F = f(\psi(m_s))$ with f any bounded continuous function in $\mathbb{R}$ we prove that claimed, that is that the law of $\psi(m_s)$ and $\mathscr{R}(\psi(m_s))$ are the same. Therefore

$$E^{\varepsilon}_{\bar{m}}[E^{\varepsilon}_{\psi(m_s)}[\xi]] = \tfrac{1}{2} E^{\varepsilon}_{\bar{m}}[E^{\varepsilon}_{\psi(m_s)}[\xi] + E^{\varepsilon}_{\mathscr{R}\psi(m_s)}[\xi]] \tag{3.102}$$

On the other hand, by symmetry, if $\xi(m) = 0$ then $E^{\varepsilon}_{m}[\xi_t] = -E^{\varepsilon}_{\mathscr{R}m}[\xi_t]$. The lemma is proved. □

Lemma 5. Let $a \in (0, 1/4)$ and $m \in \mathscr{M}_{\varepsilon^{1/2-a}}$ such that $\xi(m) = 0$. Then

$$\lim_{\varepsilon \to 0^+} E^{\varepsilon}_{m}[(\varepsilon t_{\varepsilon})^{-1} \xi^2_{t_\varepsilon}] = D, \qquad D = 3/4 \tag{3.103}$$

Proof. Given $p \in (0, a)$ let $\mathscr{D}_p$ be the nice set where $\sup_{0 \leqslant t \leqslant t_\varepsilon} \|\sqrt{\varepsilon}\, \hat{Z}_{t,0}\| \leqslant \varepsilon^{1/2-p}$. By (2.8), for small ε, in this set $\bar{m} + \sqrt{\varepsilon}\, \hat{Z}_{t_\varepsilon,0}$ has a unique linear center ξ^Z_t and furthermore

$$|\xi^Z_{t_\varepsilon} + \tfrac{3}{4} \langle \bar{m}' + \sqrt{\varepsilon}\, \hat{Z}_{t_\varepsilon,0} \rangle| \leqslant c_0(\varepsilon^{1/2-p})^2 \tag{3.104}$$

On the other hand, looking at the integral version (3.4) of the Ginzburg–Landau equation, since m is orthogonal to $\bar{m}'$, one easily obtains that, in $\mathscr{D}_p$,

$$\|m_{t_\varepsilon} - (\bar{m} + \sqrt{\varepsilon}\, \hat{Z}_{t_\varepsilon,0})\| \leqslant t_\varepsilon \varepsilon^{1/2-a} \tag{3.105}$$

and then, by Lemma 1,

$$|\xi_{t_\varepsilon} - \xi^Z_{t_\varepsilon}| \leqslant c_0 t_\varepsilon \varepsilon^{1-2a} \tag{3.106}$$

In the proof of Theorem 6 we will use the following consequence of (3.106); in $\mathscr{D}_p$ one has

$$|\xi_{t_\varepsilon} - \bar{\xi}_{t_\varepsilon}| \leqslant c_0 t_\varepsilon \varepsilon^{1-2a} \tag{3.107}$$

where $\bar{\xi}_{t_\varepsilon} = \xi(T_{t_\varepsilon}(\bar{m}, \sqrt{\varepsilon}\, Z))$.

Since $P^\varepsilon(\mathscr{D}_p)$ converge to 1 as $\varepsilon \to 0^+$ faster than any power of ε, from (3.104) and (3.106),

$$\lim_{\varepsilon \to 0^+} E^\varepsilon_m[(\varepsilon t_\varepsilon)^{-1} \xi^2_{t_\varepsilon}] = \lim_{\varepsilon \to 0^+} E^\varepsilon_m[(\varepsilon t_\varepsilon)^{-1} |\tfrac{3}{4} \langle \bar{m}', \sqrt{\varepsilon}\, \hat{Z}_{t_\varepsilon, 0}\rangle|^2] \tag{3.108}$$

But one easily computes

$$E^\varepsilon_m[(\varepsilon t_\varepsilon)^{-1} |\langle \bar{m}', \sqrt{\varepsilon}\, \hat{Z}_{t_\varepsilon, 0}\rangle|^2] = |\langle \bar{m}', e^{(\varepsilon)}_1\rangle|^2 \tag{3.109}$$

where $e^{(\varepsilon)}_1$ is defined in the proof of Proposition 5 (here it is considered as an element of $C_{\varepsilon,\kappa}(\mathbb{R})$). The Lemma follows from (3.108) and (3.109) since $\lim_{\varepsilon \to 0^+} \langle \bar{m}', e^{(\varepsilon)}_1\rangle = 1/\sqrt{D} = \sqrt{4/3}$. □

4. CONVERGENCE TO A BROWNIAN MOTION

In this section we prove that the linear center $\xi(m_t)$, suitably normalized, converges to a Brownian motion.

Let $a \in (0, 1/2)$, $\kappa \geqslant 1$, $\varepsilon > 0$, and

$$\mathscr{X} \doteq \{\psi \in C(\mathbb{R}) \colon \xi^{\varepsilon,\kappa}(\psi) = 0\}; \qquad \mathscr{X}_{\varepsilon, a} \doteq \mathscr{X} \cap \mathscr{M}_{\varepsilon^{1/2-a}} \tag{4.1}$$

As in the previous section, set $t_\varepsilon = (\log \varepsilon)^2$. We consider an auxiliary Markov chain $(x_n, \psi_n)_{n \in \mathbb{N}}$ with state space $\mathbb{R} \times \mathscr{X}$. We denote by $\mathbb{P}^\varepsilon_{(x_n, \psi_n)}$ its transition probabilities, given by:

$$\mathbb{P}^\varepsilon_{(x_n, \psi_n)}((x_{n+1}, \psi_{n+1}) = (x_n, \psi_n)) = 1 \qquad \text{if} \quad \psi_n \notin \mathscr{X}_{\varepsilon, a}$$

If instead $\psi_n \in \mathscr{X}_{\varepsilon, a}$ we define,

$$m_t \doteq T_t(\psi_n; \sqrt{\varepsilon}\, Z) \qquad \theta \doteq \xi^{\varepsilon,\kappa}(m_{t_\varepsilon})$$

Then, for any given B and A Borel sets in $\mathbb{R}$ and $C^0(\mathbb{R})$ respectively

$$\mathbb{P}^{\varepsilon}_{(x_n, \psi_n)}((x_{n+1}, \psi_{n+1}) \in B \times A) = P^{\varepsilon}(x_n + \theta \in B, \tau_{\theta} m_{t_{\varepsilon}} \in A)$$

We next introduce some stopping times: for $r > 0$ we set

$$t^0(r) \doteq \inf\{n \in \mathbb{N}: |x_n| \geqslant r\} \tag{4.2}$$

and for $\varepsilon > 0$ and $a \in (0, 1/2)$

$$t_{\varepsilon, a} \doteq \inf\{n \in \mathbb{N}: \psi_n \notin \mathscr{X}_{\varepsilon, a}\} \tag{4.3}$$

Finally, the stopping time $s_{\varepsilon, \kappa}(\zeta)$, $\varepsilon > 0$, $\kappa \geqslant 1$, $\zeta > 0$ is defined on $(\mathbb{R} \times X_{\varepsilon, a})^{\mathbb{N}} \times C(\mathbb{R}_+, \mathbb{R})$, i.e. the product of the Markov chain and the Ginzburg–Landau process:

$$s_{\varepsilon, \kappa}(\zeta) \doteq \inf\{n \in \mathbb{N}: |x_n - \xi^{\varepsilon, \kappa}(m_{nt_{\varepsilon}})| + \|\psi_n(x) - \tau_{x_n} m_{nt_{\varepsilon}}\|_{\varepsilon} \geqslant \zeta\} \tag{4.4}$$

The seminorm $\|\cdot\|_{\varepsilon}$ is defined in (3.28). In these definitions the stopping times are set equal to $+\infty$ if the sets on the right hand side are empty.

In the next proposition we indeed construct the original Ginzburg–Landau process and the auxiliary Markov chain in the same probability space, and prove lower bounds on $s_{\varepsilon, \kappa}(\zeta)$ thus showing that the two processes are close to each other.

Proposition 6. Let $\ell \in (0, 1)$, $a \in (0, 1/2)$, $\kappa \geqslant 1$, $h > 0$, $q > 0$. Then there is $c > 0$ so that the following holds. Let $\varepsilon > 0$, $m \in C^0(\mathbb{R})$ with $m^{\varepsilon, \kappa} \in \mathscr{M}_{\varepsilon^{1/2 - a}}$,

$$x_0 \doteq \xi(m^{\varepsilon, \kappa}), \qquad \psi_0(x) \doteq m^{\varepsilon, \kappa}(x + x_0), \qquad |x_0| < \varepsilon^{-\kappa} - \ell \varepsilon^{-1} \tag{4.5}$$

Then we can construct the Ginzburg–Landau process m_t (that starts from m) and the Markov chain (that starts from (x_0, ψ_0)) in the same probability space so that

$$P^{\varepsilon}(s_{\varepsilon, \kappa}(\varepsilon^q) \geqslant t^0(\varepsilon^{-\kappa} - \ell \varepsilon^{-1}) \wedge \varepsilon^{-h};\, t_{\varepsilon, a} \geqslant \varepsilon^{-h}) \geqslant 1 - c\varepsilon^q \tag{4.6}$$

Proof. The proof follows by applying iteratively Corollary 1 together with (2.9) and Proposition 4. □

We next study the Markov chain (x_n, ψ_n) and prove convergence to a Brownian motion. We set $z_0 \doteq 0$ and for $n \geqslant 1$

$$z_n \doteq \frac{x_n - x_{n-1}}{\sqrt{\varepsilon t_{\varepsilon}}} \tag{4.7}$$

We then define

$$Z_n \doteq \sum_{i=0}^{n} z_i = \frac{x_n - x_0}{\sqrt{\varepsilon t_\varepsilon}} \tag{4.8}$$

and given N (eventually we let $N = N(\varepsilon)$ and $N(\varepsilon) \to +\infty$)

$$X(t) \doteq \frac{1}{\sqrt{N}} Z_n \qquad t = n/N \tag{4.9}$$

We finally extend $X(t)$ to $t \in \mathbb{R}_+$ by linear interpolation.

Theorem 6. Let $h > 0$, $x_0 \in \mathbb{R}$, $a \in (0, 1/4)$ and $\psi_0 \in \mathcal{X}_{\varepsilon, a}$. Let $\mathbb{P}^\varepsilon$ be the law on $C(\mathbb{R}_+)$ of the process $X(t)$ induced via (4.9) with $N \doteq [\varepsilon^{-h}]$ by the Markov chain that starts from (x_0, ψ_0). Then $\mathbb{P}^\varepsilon$ converges weakly on the compacts to P as $\varepsilon \to 0^+$, where P is the law of a Brownian motion starting from 0 with diffusion equal to 3/4.

Proof. Without loss of generality we may restrict to $t \in [0, 1]$. Tightness on $C([0, 1])$ follows from the existence of $c > 0$ for which

$$\mathbb{E}^\varepsilon(\sup_{n \leqslant N} N_n^2) \leqslant cN \tag{4.10}$$

$$\mathbb{E}^\varepsilon([Z_{n_3} - Z_{n_2}]^2 [Z_{n_2} - Z_{n_1}]^2) \leqslant c(n_3 - n_1)^2, \quad \text{for all} \quad 1 \leqslant n_1 < n_2 < n_3 \leqslant N \tag{4.11}$$

see Billingsley.[2]

We first prove (4.10) and (4.11), then a martingale relation for the limit laws that will identify the law P of the theorem.

We call $\mathcal{F}_n$, $n \in \mathbb{N}$, the σ-algebra generated by the coordinates (x_i, ψ_i), $0 \leqslant i \leqslant n$, of the Markov chain and denote by $\mathbb{E}_n^\varepsilon$, $n \in \mathbb{N}$, the conditional expectation given $\mathcal{F}_n$ (sometimes we write more explicitly $\mathbb{E}^\varepsilon_{(x_n, \psi_n)}$). We set

$$\gamma_{1,n} \doteq \mathbb{E}_n^\varepsilon(z_{n+1}), \qquad \Gamma^*_{1,n} \doteq \mathbb{E}_n^\varepsilon(\gamma_{1,n+1}) \qquad \text{for} \quad n \geqslant 0 \quad \text{and} \quad \gamma^*_{1,-1} \doteq \gamma_{1,0} \tag{4.12}$$

We then have for $n \geqslant 1$

$$Z_n = \Gamma^*_{1,n-2} + M^*_{n-1} + M_n \tag{4.13}$$

where

$$\Gamma^*_{1,n} \doteq \sum_{i=-1}^{n} \Gamma^*_{1,i}, \qquad n \geqslant -1 \tag{4.14}$$

$$M^*_n \doteq \sum_{i=1}^{n} [\gamma_{1,i} - \gamma^*_{1,i-1}], \qquad n \geqslant 1, \qquad M^*_0 = 0 \tag{4.15}$$

$$M_n \doteq \sum_{i=1}^{n} [z_i - \gamma_{1,i-1}], \qquad n \geqslant 1, \qquad M_0 \doteq 0 \tag{4.16}$$

Observe that M^*_n and M_n are $\mathscr{F}_n$-martingales. The usual semimartingale representation for Z_n in terms of the compensators $\gamma_{1,i}$ and M_n is not useful in the present context: the time delay in the definition of the compensators $\gamma^*_{1,i}$ allows in fact to exploit the relaxation properties of the Ginzburg–Landau process stated in Corollary 1.

The semimartingale representation of M^2_n is

$$M^2 = \Gamma_{2,n-1} + N_n, \qquad n \geqslant 1 \tag{4.17}$$

$$\Gamma_{2,n} \doteq \sum_{i=0}^{n} \gamma_{2,i}, \qquad \gamma_{2,n} \doteq \mathbb{E}^{\varepsilon}_n((z_{n+1} - \gamma_{1,n})^2), \qquad n \geqslant 0 \tag{4.18}$$

where we set $N_0 = 0$ and for $n \geqslant 1$

$$N_n \doteq 2 \sum_{1 \leqslant j < i \leqslant n} [z_i - \gamma_{1,i-1}][z_j - \gamma_{1,j-1}] + \sum_{i=1}^{n} [(z_i - \gamma_{1,i-1})^2 - \gamma_{2,i-1}] \tag{4.19}$$

is a $\mathscr{F}_n$ martingale. For $(M^*_n)^2$ we have

$$(M^*_n)^2 = \Gamma^*_{2,n-1} + N'_n + N''_n \tag{4.20}$$

$$\Gamma^*_{2,n} \doteq \sum_{i=0}^{n} \gamma^*_{2,i}, \quad \gamma^*_{2,n} \doteq \mathbb{E}^{\varepsilon}_n((\gamma_{1,n+1} - \gamma^*_{1,n})^2), \qquad n \geqslant 0 \tag{4.21}$$

$$N'_n \doteq \sum_{i=0}^{n-1} [(\gamma_{1,i+1} - \gamma^*_{1,i})^2 - \gamma^*_{2,i}], \qquad n \geqslant 1, \; N'_0 = 0 \tag{4.22}$$

$$N''_n \doteq 2 \sum_{1 \leqslant j < i \leqslant n} [\gamma_{1,i} - \gamma^*_{1,i-1}][\gamma_{1,j} - \gamma^*_{1,j-1}], \qquad n > 1, \qquad N''_1 = N''_0 = 0 \tag{4.23}$$

where N'_n and N''_n are $\mathscr{F}_n$-martingales.

Proof of tightness. By (4.13)

$$\tfrac{1}{4}\,\mathbb{E}^{\varepsilon}(\sup_{n\leqslant N} Z_n^2)\leqslant \mathbb{E}^{\varepsilon}(\sup_{n\leqslant N}(M_{n-1}^*)^2)+\mathbb{E}^{\varepsilon}(\sup_{n\leqslant N}(M_n)^2)+2N^2\sup_{n\leqslant N}((\gamma_{1,n}^*)^2)$$

Using Doob's inequality we then get

$$\tfrac{1}{4}\,\mathbb{E}^{\varepsilon}(\sup_{n\leqslant N} Z_n^2)\leqslant \sup_{n\leqslant N}\mathbb{E}^{\varepsilon}(2N^2(\gamma_{1,n}^*)^2+4N\gamma_{2,n}+4N\gamma_{2,n}^*) \tag{4.24}$$

We set $N\equiv N(\varepsilon)\doteq[\varepsilon^{-h}]$, then we obtain

$$\mathbb{E}^{\varepsilon}(\sup_{n\leqslant N(\varepsilon)} Z_n^2)\leqslant 16N(\varepsilon)\sup_{n\leqslant N(\varepsilon)}\mathbb{E}^{\varepsilon}(N(\varepsilon)(\gamma_{1,n}^*)^2+\gamma_{2,n}+\gamma_{2,n}^*) \tag{4.25}$$

Since the chain is stopped once it is not in $\mathscr{X}_{\varepsilon,a}$ it follows that if $\psi_n\notin\mathscr{X}_{\varepsilon,a}$ then $\gamma_{1,n}^*=\gamma_{2,n}=\gamma_{2,n}^*=0$ so that (4.10) holds if there is $c>0$ so that

$$\sup_{\psi\in\mathscr{X}_{\varepsilon,a}}\mathbb{E}_{(0,\psi)}^{\varepsilon}(N(\varepsilon)(\gamma_{1,0}^*)^2+\gamma_{2,0}+\gamma_{2,0}^*)\leqslant c$$

We will prove next the following stronger inequalities that will be needed later:

$$\sup_{\psi\in\mathscr{X}_{\varepsilon,a}}\mathbb{E}_{(0,\psi)}^{\varepsilon}(N(\varepsilon)^2(\gamma_{1,0}^*)^2)\leqslant c$$

$$\sup_{\psi\in\mathscr{X}_{\varepsilon,a}}\mathbb{E}_{(0,\psi)}^{\varepsilon}(\gamma_{2,0})\leqslant c$$

$$\lim_{\varepsilon\to 0}\sup_{\psi\in\mathscr{X}_{\varepsilon,a}}\mathbb{E}_{(0,\psi)}^{\varepsilon}(\gamma_{2,0}^*)=0 \tag{4.26}$$

We will prove (4.26) later.

To prove (4.11) we use the same argument after conditioning on $\mathscr{F}_{n_2}$. We call (x_{n_2},ψ_{n_2}) the state of the chain at time n_2, $n\doteq n_3-n_2$ and $\bar{n}\doteq n_2-n_1$. Then

$$\begin{aligned}&\mathbb{E}^{\varepsilon}([Z_{n_3}-2_{n_2}]^2\,[Z_{n_2}-Z_{n_1}])\\ &\quad\leqslant\sup_{(x_{n_2},\psi_{n_2})}\mathbb{E}_{n_2}^{\varepsilon}([Z_{n_3}-Z_{n_2}]^2)\,\mathbb{E}^{\varepsilon}([Z_{n_2}-Z_{n_1}]^2)\\ &\quad\leqslant\sup_{\psi\in\mathscr{X}_{\varepsilon,a}}\mathbb{E}_{(0,\psi)}^{\varepsilon}(Z_n^2)\sup_{\psi\in\mathscr{X}_{\varepsilon,a}}\mathbb{E}_{(0,\psi)}^{\varepsilon}(Z_{\bar{n}}^2)\end{aligned} \tag{4.27}$$

and using (4.26) we have, for a suitable constant $c' > 0$,

$$\begin{aligned}
&\sup_{\psi\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi)}(Z_n^2) \sup_{\psi\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi)}(Z_{\bar n}^2)\\
&\quad\leqslant 16^2 n\bar n \sup_{\psi\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi)}(n(\gamma^*_{1,0})^2+\gamma_{2,0}+\gamma^*_{2,0})\\
&\qquad\times \sup_{\psi\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi)}(\bar n(\gamma^*_{1,0})^2+\gamma_{2,0}+\gamma^*_{2,0})\\
&\quad\leqslant c'(n_3-n_1)^2
\end{aligned}\tag{4.28}$$

having used (4.26). Equation (4.11) and tightness are proved provided that (4.26) holds.

Let $\mathbb{P}$ be a limit law on $C([0,1])$ of $\{\mathbb{P}^{\varepsilon}\}_{\varepsilon>0}$. By Levy's characterization theorem it will be sufficient to prove that the coordinate process $X(t)$ in $C([0,1])$ is a square integral $\mathbb{P}$-martingale and that $X(t)^2-3t/4$ is also a $\mathbb{P}$-martingale.

By (4.13) and (4.21)–(4.23) and Doob's inequality we get

$$\begin{aligned}
&\frac{1}{N(\varepsilon)} \sup_{\psi_0\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi_0)}\Big(\sup_{n\leqslant N(\varepsilon)} (Z_n-M_n)^2\Big)\\
&\quad\leqslant \frac{1}{N(\varepsilon)} \sup_{\psi_0\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi_0)}(N(\varepsilon)^2\gamma^*_{1,0}) + \frac{1}{N(\varepsilon)} \sup_{\psi_0\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi_0)}(\Gamma^*_{2,N(\varepsilon)})
\end{aligned}\tag{4.29}$$

Observe that

$$\frac{1}{N(\varepsilon)} \sup_{\psi_0\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi_0)}(\Gamma^*_{2,N(\varepsilon)}) \leqslant \sup_{\psi\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi)}(\gamma^*_{2,0}) \tag{4.30}$$

so that from (4.29), (4.30) and the first and third inequality in (4.26) we get

$$\lim_{\varepsilon\to 0^+} \frac{1}{N(\varepsilon)} \sup_{\psi_0\in\mathscr{X}_{\varepsilon,a}} \mathbb{E}^{\varepsilon}_{(0,\psi_0)}\Big(\sup_{n\leqslant N(\varepsilon)} (Z_n-M_n)^2\Big)=0 \tag{4.31}$$

which proves that $X(t)$ is a $\mathbb{P}$-martingale.

By (4.17)

$$\Lambda_n \doteq \frac{1}{N(\varepsilon)}(Z_n^2-\Gamma_{2,n-1}) \tag{4.32}$$

differs from a $\mathbb{P}^{\varepsilon}$-martingale by the term $[Z_n^2 - M_n^2]$ which by (4.29) vanishes in L^1 as $\varepsilon \to 0^+$. Thus the proof that $X(t)^2 - 3t/4$ is a $\mathbb{P}$-martingale follows from

$$\lim_{\varepsilon \to 0^+} \sup_{\psi_0 \in \mathscr{X}_{\varepsilon, a}} \mathbb{E}^{\varepsilon}_{(0, \psi_0)} \left(\sup_{n \leqslant n(\varepsilon)} \frac{1}{N(\varepsilon)} \left| \Gamma_{2, n-1} - \frac{3}{4} \right| \right) = 0 \tag{4.33}$$

which, by Proposition 6, is implied by

$$\lim_{\varepsilon \to 0^+} \sup_{\psi \in \mathscr{X}_{\varepsilon, a}} \left| \gamma_{2, 0}(\psi) - \frac{3}{4} \right| = 0 \tag{4.34}$$

Proof of (4.26) and (4.34). First of all we recall the notation $\psi_t \doteq T_t(\psi; \sqrt{\varepsilon}\, Z)$. Let $\psi \in \mathscr{X}_{\varepsilon, a}$ we call $\Psi \doteq \psi_{t_\varepsilon} = T_{t_\varepsilon}(\psi; \sqrt{\varepsilon}\, Z)$ and we denote by χ the center of Ψ, i.e. $\chi \doteq \xi^{\varepsilon, \kappa}(\Psi)$; we finally let $\Phi(x) \doteq \Psi(x + \chi)$. By (4.12)

$$\gamma^*_{1, 0}(\psi) = E^{\varepsilon}_{\psi}(\gamma_{1, 0}(\Psi)) \tag{4.35}$$

By Lemma 4, $\gamma^*_{1, 0}(\bar{m}) = 0$ so that $\gamma^*_{1, 0}(\psi) = \gamma^*_{1, 0}(\psi) - \gamma^*_{1, 0}(\bar{m})$. We apply Corollary 1 so that we can construct the processes starting from ψ and $\bar{m}$ in such a way that for any $q > 0$ there is $c > 0$ so that $\|\psi_{t_\varepsilon} - \bar{\psi}_{t_\varepsilon}\|_\varepsilon \leqslant \varepsilon^q$ with probability larger than $1 - c\varepsilon^q$. ($\bar{\psi}_{t_\varepsilon} = T_{t_\varepsilon}(\bar{m}; \sqrt{\varepsilon}\, Z)$). Thus

$$|\gamma^*_{1, 0}(\psi)| \leqslant c\varepsilon^q + \sup_{\phi, \bar{\phi} \in \mathscr{X}_{\varepsilon, a}} \sup_{\|\phi - \bar{\phi}\|_\varepsilon \leqslant \varepsilon^q} |E^{\varepsilon}_{\phi}(\xi^{\varepsilon, \infty}(\phi_{t_\varepsilon})) - E^{\varepsilon}_{\bar{\phi}}(\xi^{\varepsilon, \kappa}(\bar{\phi}_{t_\varepsilon}))| \tag{4.36}$$

By (3.71) with $\Delta = 0$ and $\delta = \varepsilon^q$ we get $|\gamma^*_{1, 0}(\psi)| \leqslant c\varepsilon^q$ (for a suitable constant $c > 0$). The first inequality in (4.26) is thus proved.

In order to prove the second and third inequalities in (4.26), we first observe that by symmetry $\gamma_{1, 0}(\bar{m}) = 0$, so that by (3.107)

$$|\gamma_{1, 0}(\psi)| \leqslant c t_\varepsilon \varepsilon^{1/2 - 2a} \tag{4.37}$$

which vanishes as $\varepsilon \to 0^+$ by the assumption $a < 1/4$. By (4.18)

$$\gamma_{2, 0}(\psi) = \frac{1}{\varepsilon t_\varepsilon} E^{\varepsilon}_{\psi}(\xi^{\varepsilon, \kappa}(\psi_{t_\varepsilon})^2) - \gamma_{1, 0}(\psi)^2$$

By the previous bound the last term vanishes as $\varepsilon \to 0^+$ (uniformly on $\mathscr{X}_{\varepsilon, a}$) while the first term on the right-hand side converges to 3/4 by Lemma 5. We have thus proved both the second inequality in (4.26) and (4.34). The third inequality in (4.26) follows from (4.21), (4.36), and (4.37).

The proof of the theorem is complete. □

We next relate the convergence results proved for the auxiliary Markov chain to the Ginzburg–Landau process.

Proof of Theorem 2. We use the same notation as in Theorem 6 and Proposition 6. We fix the initial position x_0 in the Markov chain so that $\varepsilon^{h/2}x_0 \doteq r_0$ (which is independent of ε) with $|r_0| < \varepsilon^{-\kappa + h/2}$. We consider the Markov and the Ginzburg–Landau process whose initial state is related to that of the Markov chain as in Proposition 6. We call $T_M(r)$, $r \in \mathbb{R}$, the suffix M standing for Markov, the first time when the coordinate process $X(t)$ (that we here suppose starting from r_0) reaches r. The analogous variable in the Ginzburg–Landau process is denoted by $T_{\mathrm{GL}}(r)$. Let $\ell^* \in (0, 1)$ and let $\ell \in (0, \ell^*)$, call $r_\varepsilon^* \doteq \varepsilon^{h/2}[\varepsilon^{-\kappa} - \ell^* \varepsilon^{-1}]$, so that (at least for $\varepsilon > 0$ small enough) $|r_0| < r_\varepsilon^*$. Then, by Proposition 6 with ℓ as earlier, for any $q > 0$ there is $c > 0$ so that

$$P^\varepsilon(T_M(r_\varepsilon^* - \varepsilon^q) \leqslant T_{\mathrm{GL}}(r_\varepsilon^*) \leqslant T_M(r_\varepsilon^* + \varepsilon^q)) \geqslant 1 - c\varepsilon^q \tag{4.38}$$

Similar statement holds for $-r_\varepsilon^*$.

For any r the law of $T_M(r)$ converges as $\varepsilon \to 0^+$ to the law of the stopping time at $\pm r$ for the limit Brownian motion b_t (starting from r_0) because the stopping time for the limit process is almost surely continuous, see Billingsley.[2] Moreover the probability of $|T_M(r \pm \delta) - T_M(r)| > \zeta\delta$ and ζ positive, vanishes as $\delta \to 0^+$, hence by (4.38) the law of the stopping time at $\varepsilon^{h/2}[\varepsilon^{-\kappa} - \ell^* \varepsilon^{-1}]$ in the Ginzburg–Landau process converges to the law of the stopping time for the limit Brownian motion at

$$\lim_{\varepsilon \to 0^+} \{\varepsilon^{h/2}[\varepsilon^{-\kappa} - \ell\varepsilon^{-1}] - r_0\}$$

This together with Theorem 6 proves Theorem 2. □

5. ASYMPTOTIC COUPLING

In this section we prove Theorem 3. By Theorem 2 and Proposition 5 we only need, as we will explain later, the following theorem.

Theorem 7. Let $m, m^* \in C_{\varepsilon,\kappa}(\mathbb{R})$ (eventually depending on ε) such that $\|m\|$, $\|m^*\| \leqslant 3/2$ and $\|m - m^*\| \leqslant \varepsilon^{2+\kappa}$. Then, we can construct a pair of Ginzburg–Landau processes m_t and m_t^*, starting from m and m^* respectively, in the same probability space, and so that, if

$$\eta \doteq \inf\{t \geqslant 0\colon \|m_t - m_t^*\| = 0\}$$

(η is defined to be infinity if the set above is empty), for any $\alpha < 1$

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(\eta > \varepsilon^\alpha) = 0 \tag{5.1}$$

Proof. The proof uses the coupling and the ideas introduced by Mueller[17] to prove Theorem 1, but since V' is not monotonic an extra argument is needed to conclude (5.1). Recall that in fact, we do not prove, as in Ref. 17 that η is finite with probability 1.

Consider the pair (m_t, m_t^*) introduced in Ref. 17, which satisfies

$$\begin{aligned} \frac{\partial m_t}{\partial t} &= \frac{1}{2}\frac{\partial^2 m_t}{\partial x^2} - V'(m_t) + \sqrt{\varepsilon}\,\alpha_1 \\ \frac{\partial m_t^*}{\partial t} &= \frac{1}{2}\frac{\partial^2 m_t^*}{\partial x^2} - V'(m_t^*) + \sqrt{\varepsilon}[(1-(|m_t - m_t^*| \wedge 1))^{1/2}\alpha_1 \\ &\quad + (|m_t - m_t^*| \wedge 1)^{1/2}\alpha_2] \end{aligned} \tag{5.2}$$

for α_1 and α_2 two independent space-time white noises, and with initial conditions

$$m_0 = m, \qquad m_0^* = m^* \tag{5.3}$$

Consider the case $m \geqslant m^*$. The general case follows from this one as in Ref. 17. If we write the equation for the difference $m_t - m_t^*$ and approximate the coefficients of the noise by Lipschitz functions as in Ref. 17, we can conclude, from Shiga[18] [Thm. 2.3], that $m_t \geqslant m_t^*$ $\forall x \in \mathbb{R}$, $t \geqslant 0$. Let $\mathscr{F}_t$ be the filtration generated by α_1 and α_2 up to time t. Next, integrate (5.2) from 0 to t and over the interval $\mathscr{T}_{\varepsilon,\kappa} = [-\varepsilon^{-\kappa}, \varepsilon^{-\kappa}]$. Set

$$U(t) \doteq \int_{\mathscr{T}_{\varepsilon,\kappa}} dx\,(m_t(x) - m_t^*(x)) \tag{5.4}$$

Proceedings as in Ref. 17, we obtain for U the equation

$$U(t) = U(0) + \int_0^t ds\, U(s) - \int_0^t ds \int_{\mathscr{T}_{\varepsilon,\kappa}} dx\,(m_s(x)^3 - m_s^*(x)^3) + M(t) \tag{5.5}$$

where M_t is a martingale with respect to $\mathscr{F}_t$, with compensator

$$\langle M\rangle(t) = 2\varepsilon \int_0^t ds \int_{\mathscr{T}_{\varepsilon,\kappa}} dx \frac{|m_s(x) - m_s^*(x)| \wedge 1}{1 + (1 - |m_s(x) - m_s^*(x)| \wedge 1)^{1/2}} \tag{5.6}$$

Since

$$\frac{d\langle M\rangle(t)}{dt} \geqslant \varepsilon \int_{\mathcal{T}_{\varepsilon,\kappa}} dx(|m_t(x)-m_t^*(x)| \wedge 1)$$
$$= \varepsilon \int_{\mathcal{T}_{\varepsilon,\kappa}} dx \frac{|m_t(x)-m_t^*(x)|}{|m_t(x)-m_t^*(x)| \vee 1} \tag{5.7}$$

we have that

$$\frac{d\langle M\rangle(t)}{dt} = U(t)\, D(t)$$

for some adapted process $D(t)$ satisfying

$$D(t) \geqslant \frac{\varepsilon}{\|m_t - m_t^*\| \vee 1} \tag{5.8}$$

Take

$$\varphi(t) = \int_0^t ds\, D(s) \tag{5.9}$$

It is not difficult to see that Lemma 3.3 of Ref. 17 also holds in our case and, for each fixed ε, $\varphi(\infty) = \infty$. Then, we can define the time changed process

$$X(t) = U(\varphi^{-1}(t)) \tag{5.10}$$

which satisfies

$$X(t) = U(0) + \int_0^t ds \frac{X(s)}{\varphi'(\varphi^{-1}(s))} + \int_0^t ds\, C(s) + \int_0^t dB(s)\, X^{1/2}(s)$$

for some Brownian motion $B(s)$ and non-positive adapted process $C(s)$. Applying Ito's formula with the function $f(x) = 2x^{1/2}$, we have that, as long as $X(t) \geqslant 0$,

$$Y(t) \doteq 2\sqrt{X(t)} \tag{5.11}$$

satisfies

$$Y(t) = 2\sqrt{U(0)} + \int_0^t ds \left(\frac{2C(s)}{Y(s)} - \frac{1}{2Y(s)} + \frac{Y(s)}{2\varphi'(\varphi^{-1}(s))}\right) + B(t) \tag{5.12}$$

Now, let us prove (5.1). From (5.11) and the definition of the time change, for any positive y

$$P^{\varepsilon}(\eta > y) \leqslant P^{\varepsilon}(\gamma > \varphi(y)) \tag{5.13}$$

where

$$\gamma \doteq \inf\{t \geqslant 0\colon Y(t) = 0\}$$

Now, take $\alpha < 1$ as in the statement. From (5.13) we can write, for any given positive a,

$$\begin{aligned} P^{\varepsilon}(\eta > \varepsilon^{\alpha}) &\leqslant P^{\varepsilon}(\gamma > \varphi(\varepsilon^{\alpha}), \varphi(\varepsilon^{\alpha}) > a) + P^{\varepsilon}(\gamma > \varphi(\varepsilon^{\alpha}), \varphi(\varepsilon^{\alpha}) \leqslant a) \\ &\leqslant P^{\varepsilon}(\gamma > \alpha) + P^{\varepsilon}(\varphi(\varepsilon^{\alpha}) \leqslant a) \end{aligned}$$

Using (5.8) and the *a priori* bound on the sup-norm of m_t and m_t^* (see (3.3)), if we take $a = \varepsilon^{\beta}$, for any $\beta > 1 + \alpha$, it is not difficult to prove that this last probability goes to zero as $\varepsilon \to 0$, and so, to prove (5.1) we only need to show that

$$\lim_{\varepsilon \to 0^+} P^{\varepsilon}(\gamma \leqslant \varepsilon^{\beta}) = 1 \tag{5.14}$$

for some β as earlier. Recall equation (5.11) for Y and consider

$$\tau \doteq \inf\{t\colon B(t) + 2\sqrt{U(0)} = 0 \text{ or } B(t) + 2\sqrt{U(0)} = \varepsilon\,|\log \varepsilon|\}$$

and the set S

$$S \doteq \left\{\varphi'(\varphi^{-1}(s)) \geqslant \frac{\varepsilon}{4}\ \forall s \leqslant \varepsilon^{\beta}\right\}$$

Let

$$E \doteq S \cap \{B(\tau) + 2\sqrt{U(0)} = 0\} \cap \{\tau < \varepsilon^{\beta}\}$$

We shall prove that for all ε small enough

$$E \subset \{\gamma \leqslant \varepsilon^{\beta}\} \tag{5.15}$$

Define the stopping time

$$t_0 = \inf\left\{s\colon \frac{1}{2Y(s)} \leqslant \frac{Y(s)}{2\varphi'(\varphi^{-1}(s))}\right\}$$

Take $\omega \in E$ and suppose by contradiction that, for this ω, $Y(t) > 0$ for all $t \leqslant \tau$. Then, equation (5.12) holds for $Y(t)$ for any $t \leqslant \tau$, and so, for the ω we are considering,

$$Y(t) \leqslant B(t) + 2\sqrt{U(0)} \qquad \forall t \leqslant t_0 \wedge \tau \tag{5.16}$$

If $\tau \leqslant t_0$ the evaluation of the previous expression at τ yields $Y(\tau) \leqslant B(\tau) + 2\sqrt{U(0)} = 0$, which is a contradiction. Then $\tau > t_0$ and since $\omega \in E \supset S$,

$$Y(t_0) = \frac{\varphi'(\varphi^{-1}(t_0))}{Y(t_0)} \geqslant \frac{\varepsilon}{Y(t_0)}$$

which implies

$$Y(t_0) \geqslant \frac{\sqrt{\varepsilon}}{2}$$

and this contradicts (5.16) for small ε, from the definition of τ, which finishes the proof of (5.15). To conclude, we only have to show that we can take $\beta > 1 + \alpha$ such that $P^\varepsilon(E) \to 1$ as $\varepsilon \to 0$. But, if we recall that $U(0) \leqslant 2\varepsilon^2$, we obtain

$$\begin{aligned} P^\varepsilon(B(\tau) + 2\sqrt{U(0)} = 0) &= \frac{\varepsilon\,|\log \varepsilon| - 2\sqrt{U(0)}}{\varepsilon\,|\log \varepsilon|} \\ &\geqslant \frac{\varepsilon\,|\log \varepsilon| - 2\sqrt{2}\,\varepsilon}{\varepsilon\,|\log \varepsilon|} \to 1 \qquad \text{as} \quad \varepsilon \to 0 \end{aligned}$$

Also, taking $\beta < 2$, it follows that

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(\tau \leqslant \varepsilon^\beta) = 1$$

To finish, let us prove

$$\lim_{\varepsilon \to 0^+} P^\varepsilon(S) = 1 \tag{5.17}$$

First recall that by Proposition 1 for any $\beta < 2$

$$\lim_{\varepsilon \to 0^+} P^\varepsilon\left(\frac{1}{\|m_s - m_s^*\| \vee 1} > \frac{1}{4}\, \forall s < \varepsilon^\beta\right) = 1$$

so, from (5.8) and (5.9),

$$\lim_{\varepsilon \to 0^+} P^\varepsilon \left(\varphi(s) > \frac{\varepsilon s}{4} \, \forall s < \varepsilon^\beta \right) = 1$$

and then

$$\lim_{\varepsilon \to 0^+} P^\varepsilon \left(\varphi^{-1}(s) \leqslant \frac{4s}{\varepsilon} \, \forall s < \varepsilon^\beta \right) = 1 \tag{5.18}$$

Recalling $\varphi' = D$, from (5.8) and the Proposition 1

$$\lim_{\varepsilon \to 0^+} P^\varepsilon \left(\varphi'(t) > \frac{\varepsilon}{4} \, \forall t < 4\varepsilon^{\beta-1} \right) = 1 \tag{5.19}$$

for any $\beta \geqslant 0$. Finally, (5.18) and (5.19) imply (5.17), and the theorem is proved. □

Proof of Theorem 3. The coupling is constructed as follows. The two processes m_t and m'_t are independent of each other till the first time T_1 when $\xi^{\varepsilon,\kappa}(m_{T_1} = \xi^{\varepsilon,\kappa}(m'_{T_1})$. Let $a \in (0, 1/4)$, then with probability going to 1 as $\varepsilon \to 0^+$, both $(m_t)^{\varepsilon,\kappa}$ and $(m'_t)^{\varepsilon,\kappa}$ are in $\mathscr{M}_{\varepsilon^{1/2-a}}$, we can thus suppose that such a condition is verified. By Lemma 3 at time $T = T_1 + t_\varepsilon$ with great probability we are in the hypothesis of Proposition 5. We construct the processes in the time interval $[T, T+t_\varepsilon]$ using Proposition 5 with $b = 1-\omega$, $\omega \in (2a, 1/2)$, so that (3.80) is verified and we can suppose that the processes at time $T+t_\varepsilon$ are in the set which appears on its left-hand side. We can thus apply again Proposition 5 with $b = 1-\omega+\bar{\gamma}$ and iterate this procedure $N > 1$ times. Then calling $S \doteq T + Nt_\varepsilon$:

$$\|(m_S)^{\varepsilon,\kappa} - (m'_S)^{\varepsilon,\kappa}\| \leqslant \varepsilon^{1/2-a+N\bar{\gamma}}$$

with probability going to 1 as $\varepsilon \to 0^+$. Since by assumption m and m' are both in $C_{\varepsilon,\kappa}(\mathbb{R})$ the above holds as well for the sup norm (without the cutoff (ε, κ)).

We can then apply Theorem 7 to conclude that if N is large enough there is a coupling before $T+(N+1)\, t_\varepsilon$ with probability going to 1 as $\varepsilon \to 0^+$ (recall that $t_\varepsilon = (\log \varepsilon)^2$). Thus the time of coupling differs from the time of first encounter of the linear centers by a term bounded by $(N+1)\, |\log \varepsilon|^2$. The law of first encounter of the linear centers converges to that of the Brownians to which the linear centers converge, by Theorem 6. As the difference between true and linear centers vanishes in the limit, (see the proof of Theorem 2 and the end of Section 4) we then obtain the proof of Theorem 3. □

APPENDIX

In this appendix we prove Proposition 1, the inequalities (3.40) and (3.77) and some properties of the semigroup, g_{t, x_0} defined at the beginning of Section 2. We start with the proof of Proposition 1, more precisely in lemma A.1 next we prove (3.2).

Lemma A.1. Let $Z_t(x)$ be the process defined in Section 1. There are positive constants b_0 and b_1 so that for all $p > 0$ and $\varepsilon > 0$,

$$P^\varepsilon\big(\sup_{0 \leqslant t \leqslant t_\varepsilon} \|\sqrt{\varepsilon}\, Z_t\| > \varepsilon^{1/2} - p\big) \leqslant b_0 e^{-b_1 |\log \varepsilon|^{-1} \varepsilon^{-2p}} \tag{A.1}$$

Proof. The process Z_t is a Gaussian centered process, with bounded and continuous paths a.e. Define

$$\begin{aligned} D_\varepsilon &\doteq \{(x, t): x \in \mathcal{T}_{\varepsilon, \kappa};\, 0 \leqslant t \leqslant t_\varepsilon\} \\ \||Z|\| &\doteq \sup_{(x, t) \in D_\varepsilon} Z_t(x), \\ \sigma_\varepsilon^2 &\doteq \sup_{(x, t) \in D_\varepsilon} E^\varepsilon[Z_t(x)^2] \end{aligned} \tag{A.2}$$

Using the explicit form of the covariance of Z_t, see for instance Walsh,[20] it is not difficult to prove that there exists a constant C_1, independent of ε such that

$$E^\varepsilon[Z_t(x)^2] \leqslant C_1 t \varepsilon^\kappa + C_1 \sqrt{t} \tag{A.3}$$

what yields

$$\sigma_\varepsilon^2 \leqslant C \sqrt{t_\varepsilon} \tag{A.4}$$

for some C independent of ε. Then, we can apply the following inequality, which follows through a symmetry argument from Adler,[1] [Thm. 2.1]: for any $\lambda > E^\varepsilon \||Z|\|$:

$$P^\varepsilon\big(\sup_{0 \leqslant t \leqslant t_\varepsilon} \|Z_t\| > \lambda\big) \leqslant 4 \exp\left[-\frac{(\lambda - E^\varepsilon \||Z|\|)^2}{2\sigma_\varepsilon^2} \right] \tag{A.5}$$

To give an upper bound to $E^\varepsilon \, |||Z|||$, we use Corollary 4.15 of Ref. 1: there exists a universal constant K such that

$$E^\varepsilon \, |||Z||| \leqslant K \int_0^\infty dr \, \sqrt{\log N_\varepsilon(r)} \tag{A.6}$$

where $N_\varepsilon(r)$ is the minimal number of balls of radius r needed to cover D_ε, with respect to the metric

$$d((x, t), (y, s)) \doteq \sqrt{E^\varepsilon[(Z_t(x) - Z_s(y))^2]} \tag{A.7}$$

It can be proved that there are positive constants k_1 and k_2 such that,

$$E^\varepsilon[(Z_t(x) - Z_t(y))^2] \leqslant k_1 \, |x-y| \qquad \forall x, y \in \mathbb{R} \text{ and } \forall t, s \in \mathbb{R}_+ \tag{A.8}$$

$$E^\varepsilon[(Z_t(x) - Z_s(x))^2] \leqslant k_2 \sqrt{|t-s|} \qquad \forall x, y \in \mathbb{R} \text{ and } \forall t, s, \; \|t\| \leqslant 1, \; \|s\| \leqslant 1 \tag{A.9}$$

(see for example Walsh,[20], Prop. 4.2). From (A.8) and (A.9) it is easy to check that there is a constant c such that

$$N_\varepsilon(r) \leqslant \max\{1, c \, |\log \varepsilon|^2 \, \varepsilon^{-\kappa} r^{-3}\} \tag{A.10}$$

By (A.6) and (A.10) it follows that there is a constant K' such that

$$E^\varepsilon \, |||Z||| \leqslant K' \log \varepsilon^{-(\kappa+1)} \tag{A.11}$$

Using (A.4) and (A.11), from inequality (A.6) with $\lambda = \varepsilon^{-p}$ we finally obtain

$$P^\varepsilon\left(\sup_{0 \leqslant t \leqslant t_\varepsilon} \|Z_t\| > \varepsilon^{-p} \right) \leqslant 4 \exp\left[-\frac{(e^{-p} - K' \log \varepsilon^{-(\kappa+1)})^2}{2C \, |\log \varepsilon|} \right] \tag{A.12}$$

The bound (A.12), which is valid for ε small enough, implies the estimate (3.2) for some constants b_0 and b_1. The estimate can be then extended to any $\varepsilon \in (0, 1]$ simply by modifying conveniently the values of b_0 and b_1.

□

Lemma A.2. Let $m \in C^0(\mathbb{R})$, $\|m\| \leqslant 1 + 10^{-2}$. Then there are constants c_0 and c_1 so that, for any $\varepsilon > 0$ small enough,

$$P^\varepsilon\Big(\sup_{0 \leqslant t \leqslant t_\varepsilon} \|T_t(m; \sqrt{\varepsilon}\, Z)\| \leqslant 2, \; \sup_{t_\varepsilon \leqslant t \leqslant 2t_\varepsilon} \|T_t(m; \sqrt{\varepsilon}\, Z)\| \leqslant 1 + 10^{-2} \Big)$$
$$\geqslant 1 - c_0 e^{-c_1 \varepsilon^{-1}} \tag{A.13}$$

Proof. A Comparison Theorem holds for the stochastic Ginzburg–Landau equation, see Ref. 5, Prop. 5.1. So, if $\|m\| \leqslant 1 + 10^{-2}$, then

$$m_t^- \leqslant T_t(m; \sqrt{\varepsilon}\, Z) \leqslant m_t^+ \qquad P^\varepsilon\text{-a.s.} \tag{A.14}$$

where $m_1^\pm \doteq T_t(\pm(1+10^{-2}); \sqrt{\varepsilon}\, Z)$. It is then sufficient to prove (A.13) with $T_t(m; \sqrt{\varepsilon}\, Z)$ replaced by $m_t^\pm$. We define $6u_\pm(x, t) \doteq m_t^\pm(x) \mp 1$. Then $u_\pm(x, t)$ solve the equations

$$\frac{\partial u_\pm}{\partial t} - \frac{1}{2}\frac{\partial^2 u_\pm}{\partial x^2} + 2u\pm = \mp 3u_\pm^2 - u_\pm^3 + \sqrt{\varepsilon}\, \alpha \tag{A.15}$$

i.e. the integral equations

$$\begin{aligned} u_\pm(x, t) = e^{-2t} H_t u_\pm(\cdot, 0) + \int_0^t ds \int dy\, e^{-2(t-s)} H_{t-s}(x-y) \\ \times [\mp 3u_\pm^2 - u_\pm^3](y, s) + \sqrt{\varepsilon}\, V_t \end{aligned} \tag{A.16}$$

where H_t is the heat kernel (1.5), $e^{-2t} H_t u_\pm(0) = \pm e^{-2t} 10^{-2}$ and

$$V_t(x) \doteq \int_0^t ds\, dy\, \alpha(s, y)\, e^{-2(t-s)} H_{t-s}^\varepsilon(x, y) \tag{A.17}$$

(note that $e^{-2t} H_t^{(\varepsilon)}(x-y)$ is the Green function for the operator $\partial_t - (1/2)\,\partial_x^2 + 2Id$). By arguing as in the proof of Lemma A.1, it is easy to prove that for any $b > 0$ there are constants h_0 and h_1 such that

$$P^\varepsilon\Big(\sup_{0 \leqslant t \leqslant 2t_\varepsilon} \|\sqrt{\varepsilon}\, V_t\| > b\Big) \leqslant h_0 e^{-h_1 \varepsilon^{-1}} \tag{A.18}$$

Let $T \doteq \inf\{t \geqslant 0 : \|u_\pm(\cdot, t)\| \geqslant 2(10^{-2} + b)\}$. We will prove that there exists b and ε_0 such that for all $\varepsilon \leqslant \varepsilon_0$

$$\begin{aligned} P^\varepsilon\left(\sup_{0 \leqslant t \leqslant t_\varepsilon} \|m_t^\pm\| \leqslant 2,\ \|m_{t_\varepsilon}^\pm\| \leqslant 1 + \frac{10^{-2}}{4}\right) \\ \leqslant P^\varepsilon\Big(T \geqslant 2t_\varepsilon, \sup_{0 \leqslant t \leqslant 2t_\varepsilon} \|\sqrt{\varepsilon}\, V_t\| \leqslant b\Big) \end{aligned} \tag{A.19}$$

and

$$P^\varepsilon\Big(T \geqslant 2t_\varepsilon, \sup_{0 \leqslant t \leqslant 2t_\varepsilon} \|\sqrt{\varepsilon}\, V_t\| \leqslant b\Big) = P^\varepsilon\Big(\sup_{0 \leqslant t \leqslant 2t_\varepsilon} \|\sqrt{\varepsilon}\, V_t\| \leqslant b\Big) \tag{A.20}$$

Clearly, by definition of T and $u_{\pm}$, $\sup_{0 \leqslant t \leqslant t_\varepsilon} \|m_t^{\pm}\| \leqslant 2$ is implied by $T \geqslant 2t_\varepsilon$ if b is small enough. Moreover, in the set

$$\{T \geqslant 2t_\varepsilon, \sup_{0 \leqslant t \leqslant 2t_\varepsilon} \|\sqrt{\varepsilon}\, V_t\| \leqslant b\}$$

from (A.16) we get

$$\|u_{\pm}(\cdot, t_\varepsilon)\| \leqslant e^{-2t_\varepsilon} 10^{-2} + 12(10^{-2}+b)^2 + 8(10^{-2}+b)^3 + b \tag{A.21}$$

Now there exists $\varepsilon_0 > 0$ and $b_0 > 0$ such that, for any $\varepsilon \leqslant \varepsilon_0$ and $b \leqslant b_0$, (A.21) implies $\|u_{\pm}(\cdot, t_\varepsilon)\| \leqslant 10^{-2}/4$ and hence $\|m_{t_\varepsilon}^{\pm}\| \leqslant 1 + 10^{-2}/4$, which implies (A.19). To prove (A.20) we note that if $\sup_{0 \leqslant t \leqslant 2t_\varepsilon} \|\sqrt{\varepsilon}\, V_t\| \leqslant b$ and $T \leqslant 2t_\varepsilon$ then

$$2(10^{-2}+b) = \|u_{\pm}(\cdot, T)\| \leqslant 10^{-2} + 12(10^{-2}+b)^2 + 8(10^{-2}+b)^3 + b \tag{A.22}$$

which gives a contradiction if, for example, $b \leqslant 10^{-2}$. Then, for $b = b_0 \wedge 10^{-2}$, both (A.19) and (A.20) holds. To obtain (A.13), repeat the argument for the interval $[t_\varepsilon, 2t_\varepsilon]$, with $T_t(\pm 1 + 10^{-2}/4); \sqrt{\varepsilon}\, Z)$ instead of $T_t(\pm(1+10^{-2}); \sqrt{\varepsilon}\, Z)$, and take $\varepsilon < \varepsilon_0$ and $c_0 = h_0$ and $c_1 = h_1$. □

Proof of Proposition 1. Lemma A.1 proves (3.2). To prove (3.3) we fix $s \geqslant t_\varepsilon$ and iterate the estimate (A.13) k times where $k = s/t_\varepsilon$. □

In the next lemma we prove (3.77).

Lemma A.3. Let $m \in C^0(\mathbb{R})$, $m_t \doteq T_t(m, \sqrt{\varepsilon}\, Z)$. For any $\delta > 0$, $\alpha \in (0, 1/2)$ there is a constant $c > 0$ so that for all ε small enough

$$P^\varepsilon\left(\sup_{|x|, |y| \leqslant \varepsilon^{-\kappa}} \sup_{\substack{x \neq y \\ |x-y| \leqslant 1}} \frac{|m_{t_\varepsilon}(x) - m_{t_\varepsilon}(y)|}{|x-y|^\alpha} > \delta \right) \leqslant e^{-c\delta^2} \tag{A.23}$$

Proof. We first prove an analogous estimate for the Gaussian process $\sqrt{\varepsilon}\, Z$. We use Theorem 2.1 of Ref. 1 applied to the Gaussian process

$$G_\varepsilon(x, y) \doteq \sqrt{\varepsilon}\, \frac{Z_1(x) - 2_1(y)}{|x-y|^\alpha} \tag{A.24}$$

By arguing as in the proof of Lemma A.1 we have, for $\delta > E^\varepsilon[\hat{G}_\varepsilon]$,

$$P^\varepsilon(\sup_{|x|, |y| \leqslant \varepsilon^{-\kappa}} \sup_{\substack{x \neq y \\ |x-y| \leqslant 1}} |G_\varepsilon(x, y)| > \delta) \leqslant 4 \exp\left[-\frac{(\delta - E^\varepsilon[\hat{G}_\varepsilon])^2}{2\sigma_\varepsilon^2} \right] \tag{A.25}$$

where

$$\hat{G}_\varepsilon \doteq \sup_{|x|,\,|y| \leqslant \varepsilon^{-\kappa}} \sup_{\substack{x \neq y \\ |x-y| \leqslant 1}} G_\varepsilon(x, y),$$

$$\sigma_\varepsilon^2 \doteq \sup_{|x|,\,|y| \leqslant \varepsilon^{-\kappa}} \sup_{\substack{x \neq y \\ |x-y| \leqslant 1}} E^\varepsilon[G_\varepsilon(x, y)^2] \tag{A.26}$$

Using Corollary 4.15 of Ref. 1 as in Lemma A.1 one easily proves that $E^\varepsilon[\hat{G}_\varepsilon] \leqslant c_1 \sqrt{\varepsilon}$ for some $c_1 > 0$. From (A.8) with $\alpha < 1/2$, we obtain $\sigma_\varepsilon^2 \leqslant c_4 \varepsilon$ for some constant $c_4 > 0$. Then by (A.25) we recover an estimate like (A.23) for the noise $\sqrt{\varepsilon}\, Z$. To prove (A.23) we use the integral form (1.5) of the Ginzburg–Landau equation and write

$$\begin{aligned} m_{t_\varepsilon}(x) - m_{t_\varepsilon}(y) &= H_1 m_{t_\varepsilon - 1}(x) - H_1 m_{t_\varepsilon - 1}(y) \\ &\quad + \int_0^1 ds \int dy\, [H_{1-s}(x-z) - H_{1-s}(y-z)] \\ &\quad \times [m_{t_\varepsilon - 1 + s} - m^3_{t_\varepsilon - 1 + s}](z) + G_\varepsilon(x, y)\, |x-y|^\alpha \end{aligned} \tag{A.27}$$

We consider the intersection of the sets where $\sup_{t \leqslant t_\varepsilon} \|m_t\| \leqslant 2$ and where the noise satisfies the bound like (A.23) with δ' to be fixed. In this set we have

$$|m_{t_\varepsilon}(x) - m_{t_\varepsilon}(y)| \leqslant 2\, |x-y| + c_5\, |x-y| + \delta'\, |x-y|^\alpha \tag{A.28}$$

for any $x \neq y$ such that $|x|$, $|y| \leqslant \varepsilon^{-\kappa}$ and $|x-y| \leqslant 1$. For $\delta > 2(2 + c_5)$ and $\delta' < \delta/2$, (A.23) follows easily. Changing c we get (A.23) for all $\delta > 0$. □

Lemma A.4. Let $x_0 \in \mathbb{R}$ and let $g_{t, x_0}(x, y)$ be the fundamental solution of the equation $\partial_t u = L_{x_0} u$. Then, there exist positive constants c_0, c_1 and c_2 such that the following holds.

$$g_{t, x_0}(x, y) \geqslant 0 \qquad \text{for any} \quad x, y \in \mathbb{R} \tag{A.29}$$

$$\sup_{t \geqslant 0} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} dy\, g_{t, x_0}(x, y) \leqslant c_0, \tag{A.30}$$

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} dy\, |g_{t,x_0}(x, y+d) - g_{t,x_0}(x, y)| \leqslant c_1 \frac{d}{\sqrt{t}} \tag{A.31}$$

$$\sup_{t\leqslant t_\varepsilon} \sup_{x\in B_2} \int_{\mathbb{R}\setminus B_1} dy\, g_{t,x_0}(x, y) \leqslant c_2 e^{3t_\varepsilon} e^{-(r_1-r_2)^2},$$

$$B_i \doteq \{|x-x_0| \leqslant r_i\}, \qquad r_1 > r_2 \tag{A.32}$$

Proof. From (3.45) we can restrict ourselves to the case $x_0 = 0$.

From the Feynman–Kac formula (A.29) follows. Equation (A.30) follows from Theorem 4. To prove (A.31) we use the following integral equations:

$$g_t(x, y) = H_t(x, y) + \int_0^t ds \int dz\, H_{t-s}(x-z)[1 - 3\bar{m}^2(z)]\, g_s(z, y) \tag{A.33}$$

We define

$$f_t(d) = \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} dy\, |g_t(x, y) - g_t(x, y+d)|$$

From (A.38) we then have

$$f_t(d) \leqslant c_1 \frac{d}{\sqrt{t}} + c_2 \int_0^t ds\, f_s(d) \tag{A.34}$$

From (A.34) the inequality (A.31) follows immediately for $t \in (0, 1]$. For $t > 1$ we use the semigroup property getting that $f_t(d) \leqslant c_0 f_1(d)$ with c_0 as in (A.30). (A.32) is based on the estimate

$$g_t(x, y) \leqslant \frac{e^{2t}}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

which follows easily from the Feynman–Kac formula. □

Lemma A.5. Consider the set $\mathscr{B}_\varepsilon^{(i)}$, $i = 2, 3$, defined in (3.37) and (3.38). Then, there are c_0 and c_1 positive constants such that

$$P^\varepsilon(\mathscr{B}_\varepsilon^{(i)}) \geqslant 1 - c_0 e^{-c_1 \varepsilon^{-1/2}}, \qquad i = 2, 3 \tag{A.35}$$

Proof. Recall equations (3.5) for $\hat{Z}_t$ and (3.39) for $\hat{Z}_t^*$, in terms of Z_t and Z_t^*. Using (3.46), we obtain

$$\begin{aligned}
\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}^*_{t,x_0^*} &= Z_t - \tau_\Delta Z_t^* + \int_0^t ds\, g_{t-s,x_0}(3\bar{m}^2_{x_0^*} - 1)(Z_{t,x_0} - \tau_\Delta Z^*_{t,x_0^*}) \\
&= Z_t - \tau_\Delta Z_t^* + \int_0^t ds\, g_{t-s,x_0} \mathbf{1}_{|y-x_0| \leqslant 2\varepsilon^{-1/10}} \\
&\quad \times (3\bar{m}^2_{x_0^*} - 1)(Z_{t,x_0} - \tau_\Delta Z^*_{t,x_0^*}) \\
&\quad + \int_0^t ds\, g_{t-s,x_0} \mathbf{1}_{|y-x_0| > 2\varepsilon^{-1/10}}(3\bar{m}^2_{x_0^*} - 1)(Z_{t,x_0} - \tau_\Delta Z^*_{t,x_0^*}) \\
&\doteq Z_t - \tau_\Delta Z_t^* + A_1(x,t) + A_2(x,t).
\end{aligned} \tag{A.36}$$

We will prove next that

$$P^\varepsilon\Big(\sup_{\substack{0 \leqslant t \leqslant t_\varepsilon \\ |x-x_0| \leqslant 2\varepsilon^{-1/10}}} |Z_t - \tau_\Delta Z_t^*| > e^{-\varepsilon^{-1/50}} \Big) \leqslant c_0 e^{-c_1 \varepsilon^{-1/2}} \tag{A.37}$$

The bound (A.35) for $i = 2$ follows immediately from this inequality. Moreover, from (A.37) and (A.34), we can estimate

$$\begin{aligned}
&P^\varepsilon\left(\sup_{t \leqslant t_\varepsilon} \|A_1(x,t)\|_{x_0,\varepsilon} > \frac{e^{-1/100}}{3} \right) \\
&\quad \leqslant P^\varepsilon\left(\sup_{\substack{0 \leqslant t \leqslant t_\varepsilon \\ |x-x_0| \leqslant 2\varepsilon^{-1/10}}} |Z_t - \tau_\Delta Z_t^*| > \frac{K e^{-\varepsilon^{-1/100}}}{t_\varepsilon} \right) \\
&\quad \leqslant c_0 e^{-c_1 \varepsilon^{-1/2}}
\end{aligned} \tag{A.38}$$

Also, for A_2 from (A.32), we obtain

$$\begin{aligned}
P^\varepsilon\left(\sup_{t \leqslant t_\varepsilon} \|a_2(x,t)\|_{x_0,\varepsilon} > \frac{e^{-1/100}}{3} \right) &\leqslant P^\varepsilon\Big(\sup_{\substack{0 \leqslant t \leqslant t_\varepsilon \\ x \in \mathscr{T}_{\varepsilon,\kappa}}} |Z_t - \tau_\Delta Z_t^*| > \varepsilon^{-1/2} \Big) \\
&\leqslant b_0 e^{-b_1 \varepsilon^{-1/2}}
\end{aligned}$$

Then, let us prove (A.37), to conclude the proof of the lemma. Recall that

$$|Z_t - \tau_\Delta Z_t^*| \leq \left| \int_0^t ds \int dy \, (\mathbf{1}_{\{y \in \mathcal{T}_{\varepsilon,\kappa}\}} H_{t-s}^{(\varepsilon)}(x, y) - \mathbf{1}_{\{|y-x_0| \leq 4\varepsilon^{-1/10},\, y+\Delta \in \mathcal{T}_{\varepsilon,\kappa}\}} \right.$$

$$\left. \times H_{t-s}^{(\varepsilon)}(x+\Delta, y+\Delta)) \, \alpha(y, s) \right|$$

$$+ \left| \int_0^t ds \int dy \, \mathbf{1}_{\{|y-x_0^*| > 4\varepsilon^{-1/10};\, y \in \mathcal{T}_{\varepsilon,\kappa}\}} H_{t-s}^{(\varepsilon)}(x+\Delta, y) \, \bar{\alpha}(y, s) \right|$$

$$\doteq |I_1(x, t)| + |I_2(x, t)|$$

Both I_1 and I_2 are centered Gaussian processes, for which estimates like (A.8) and (A.9) are valid. Moreover, recalling that

$$H_t^{(\varepsilon)}(x, y) = \sum_{j \in \mathbb{Z}} (H_t(x, y + 4j\varepsilon^{-\kappa}) + H_t(x, 4j\varepsilon^{-\kappa} + 2\varepsilon^{-\kappa} - y)),$$

it is not difficult to prove that

$$(\sigma_i)^2 \doteq \sup_{\substack{0 \leq t \leq t_\varepsilon \\ |x-x_0| \leq 2\varepsilon^{-1/10}}} E(I_i(x, t)^2) \leq e^{-c\varepsilon^{-1/16}}$$

Then, proceeding as in the proof of Lemmas A.1, (A.37) follows. □

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