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FINE STRUCTURE OF THE INTERFACE MOTION*

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Abstract. We study a non local evolution and define the interface in terms of a local equilibrium condition. We prove that in a diffusive scaling limit the local equilibrium condition propagates in time thus defining an interface evolution which is given by a motion by mean curvature. The analysis extend through all times before the appearance of singularities.

1. Introduction. Interface dynamics describes the evolution of systems after phase segregation. This is a problem of great theoretical and practical importance that originates from the study of alloys subject to rapid cooling from high into low temperatures where only two different stable concentration phases exist. The evolution after the quenching consists of two stages. During the first one the phases separate into clusters and at the end of this stage the system is a mixture of clusters of fairly large size. The second stage describes the evolution of the clusters which is ruled by a motion by mean curvature in systems like those we consider with non conserved order parameter and with two symmetric stable phases.

In our case the order parameter is the magnetization m as ours is a spin system. The equilibrium phases have magnetization $\pm m_\beta$, $m_\beta > 0$, β the inverse temperature (after the cooling). In the second regime local equilibrium has already been established and this state is thus described by magnetization profiles $m(r)$, the connected regions where $m(r) \approx +m_\beta$ are

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the + clusters and those where $m(r) \approx -m_\beta$ are the − clusters. Typically the fraction of volume not occupied by the ± clusters is "very small". The notion of ± clusters is clearly vague as it depends on when we decide that $m(r)$ is close enough to $\pm m_\beta$. Correspondingly vague is the notion of interface which at this point is only defined as the region not occupied by the ± clusters. To have a mathematical theory one considers a scaling limit where the size of the system is increased without bound and the time is correspondingly scaled (in our case the correct procedure is to scale space and time diffusively). While this scaling does not change the fact that the region occupied by the interface is still as "thick" as before, it makes the fraction of volume occupied by the interface vanishingly small. Thus, in the macroscopic units defined so that the size of the system is unchanged under the scaling, the interface has vanishing volume and in the limit it is identified with a regular surface under suitable assumptions of regularity at the initial time. In this frame for a large class of models (including ours), [5], [7], [15], [20], [21], [31], it has been proven that these geometric interfaces remain regular at least for a finite time and evolve by mean curvature. In dimension greater than 2 singularities may develop, however there are generalized definitions of the motion by mean curvature which rules the evolution after the onset of singularities, [3], [8], [9], [22], [23], [29], [30].

It should be kept in mind that the interfaces defined above are geometric objects which do not have an intrinsic physical meaning. They are an artifact of the scaling procedure and of the chosen macroscopic units which allow to identify in the limit a thick region with a surface. In this paper we show that for our systems it is possible to define intrinsically the interface in terms of a local equilibrium property of the magnetization profile, so that interfaces have a direct physical interpretation. In fact, [6], there is a definite magnetization pattern connecting two phases when they coexist at equilibrium and it turns out that the local equilibrium established at the end of the first stage of phase separation ensures not only that most of the volume is at equilibrium occupied by the clusters of the ± phases, but also that the magnetization pattern between clusters is (approximately) the same as when the two phases coexist at equilibrium, [14]. This statement becomes sharper as the size of the clusters increases and exact in the diffusive limit considered earlier. We prove that in such a limit this sharp local equilibrium structure persists through time so that it is possible to define an interface evolution which we show to be the same as that of the geometrical surfaces described earlier, at least until singularities develop. This same fine structure of the interface has been found in the Allen-Cahn equation by De Mottoni and Schatzmann, [20], [21].

The paper is organized as follows. In Section 2 we define our model and state the result, in Section 3 we prove the main theorem and, finally, Section 4 contains the crucial and more delicate estimates that are left over from Section 3.

2. Main definitions and results. In [12], [13] and [14] the Glauber evolution of Ising spin systems with Kac potentials has been introduced to study phase separation and interface dynamics. At the equilibrium these models undergo phase transition in the sense of the van der Waals theory. In the mean field limit the magnetization of the Ising spin system with Glauber dynamics evolves according to the following non local equation:

$$\begin{aligned} &\frac{\partial m}{\partial t}(r,t) = -m(r,t) + \tanh\Big\{\beta(J\star m)(r,t)\Big\}, \qquad (r,t)\in\mathbb{R}^n\times\mathbb{R}_+ \\ &(J\star m)(r,t) \doteq \int dr'\, J(|r-r'|)\, m(r',t), \end{aligned} \tag{2.1}$$

where $\beta > 0$ is the inverse temperature, $J \in C^\infty(\mathbb{R})$ is related to the Kac potential and it is non negative with support on $[0,1]$. We assume also that J is normalized so that

$$\int dr\, J(|r|) = 1. \tag{2.2}$$

Observe that, under the hypothesis (2.2), if $\beta > 1$ there exists a positive number m_β solution of

$$\tanh\{\beta m_\beta\} = m_\beta$$

so that $\pm m_\beta$ are two spatially homogeneous stationary solutions of (2.1) that will be called pure phases. The existence of these two stable stationary solutions corresponds to a phase transition in the Ising spin model which is the microscopic evolution that in the mean field limit gives rise to (2.1). To remind the limit procedure from which the equation has been derived, we will call *mesoscopic evolution* the one described by (2.1).

We study here the problem of interface dynamics which consists on the analysis of the Cauchy problem for (2.1) with an initial datum which has the two pure phases cohexisting: the $+m_\beta$ phase is in a large bounded region and the $-m_\beta$ occupies the outside, the region of separation is called interface. We analyze the evolution of such an initial datum in the limit when the interface becomes very flat and the region very large. This is formalized by introducing a scaling parameter $\lambda \in (0,1]$ (that will eventually go to 0) which gives the correspondence between the meso and macro variables in the following way. We denote by $(r,t) \in \mathbb{R}^n \times \mathbb{R}_+$ the mesoscopic coordinates and by $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}_+$ the macroscopic ones. The relation between them is given in terms of λ:

$$r = \lambda^{-1}\xi, \qquad t = \lambda^{-2}\tau.$$

In [12] it is proven that there exists an unique bounded function which solves the Cauchy problem for the equation (2.1) with initial datum in $L^\infty(\mathbb{R}^n;[-1,1])$. Given any function $u \in L^\infty(\mathbb{R}^n;[-1,1])$, we denote by

$$S_t(u),\ t \in \mathbb{R}_+ \quad \text{the solution of (2.1) with} \quad S_0(u) = u. \tag{2.3}$$

It can be proven that $S_t(u) \in L^\infty(\mathbb{R}^n;[-1,1])$ for any $t \in \mathbb{R}_+$ and it is differentiable in t in the sup-norm, see [12]. In [14] the phase segregation for the Glauber dynamics has been proven to be characterized by the development of interfaces, which are smooth surfaces in $\mathbb{R}^n$ that separate clusters where the magnetization has value $\pm m_\beta$. The location of the interfaces is random with a known distribution and the system after phase separation is in local equilibrium also at the interfaces. This local equilibrium is characterized in terms of a particular function, called "instanton", which is a standing wave connecting the two pure phases in the one dimensional case. Since the interfaces after phase separation are almost flat, the instanton gives a planary symmetric stationary solution of (2.1). This will be essentially our initial datum for (2.1). We then give the following definitions and results.

2.1. The instanton. ([11], [16], [17]). Let $\tilde{J}(|x|)$, $x \in \mathbb{R}$, be the following C^∞ function:

$$\tilde{J}(|x|) = \int_{\mathbb{R}^{n-1}} dy\, J\Big(\big|x^2 + y^2\big|^{1/2}\Big). \tag{2.4}$$

There exists (see Section 3 of [16] and Section 2 of [17]) a unique antisymmetric strictly increasing function $\bar{m}(x)$, $x \in \mathbb{R}$, (hereafter called instanton) that verifies

$$\bar{m}(x) = \tanh\{\beta(\tilde{J} \star m)(x)\}, \qquad x \in \mathbb{R} \tag{2.5}$$

$$\lim_{x\to\pm\infty} \bar{m}(x) = \pm m_\beta. \tag{2.6}$$

Furthermore, there are positive constants C_1, C_2, C_3 and C_4 so that (see also [11])

$$|\bar{m}(x) - \mathrm{sign}(x)m_\beta| \le C_1 e^{-\alpha|x|}, \tag{2.7}$$

$$0 < \bar{m}'(x) \le C_2 e^{-\alpha|x|}, \quad |\bar{m}''(x)| \le C_3\bar{m}'(x), \quad |\bar{m}'''(x)| \le C_4\bar{m}'(x) \tag{2.8}$$

where $\bar{m}'$, $\bar{m}''$ and $\bar{m}'''$ are, respectively, the first, second and third derivatives of $\bar{m}$ while α is the positive number such that

$$\beta(1-m_\beta^2)\int_{\mathbb{R}} dx\, \tilde{J}(|x|)\, e^{-\alpha x} = 1. \tag{2.9}$$

We next introduce the *macroscopic evolution* by giving in 2.3 below the definition of the (classical) motion by mean curvature. Before that we need some more definitions and notation.

2.2. Regular surfaces. We say that a surface Σ in $\mathbb{R}^n$ is regular if it is the embedding of a smooth compact manifold of dimension $(n-1)$. Then Σ is the boundary of an open bounded set, called the interior of Σ and denoted by $I(\Sigma)$. The set $\mathbb{R}^n \setminus (\Sigma \cup I(\Sigma))$, called the exterior of Σ, is denoted by $O(\Sigma)$. We define the signed distance $d(\xi,\Sigma)$ of a point $\xi \in \mathbb{R}^n$ from Σ by setting

$$d(\xi,\Sigma) \doteq \begin{cases} \mathrm{dist}(\xi,\Sigma) & \text{if } \xi \in I(\Sigma) \\ 0 & \text{if } \xi \in \Sigma \\ -\mathrm{dist}(\xi,\Sigma) & \text{if } \xi \in O(\Sigma) \end{cases} \tag{2.10}$$

where

$$\mathrm{dist}(\xi,\Sigma) \doteq \inf_{\zeta\in\Sigma} \mathrm{dist}(\xi,\zeta).$$

Let $\xi \in \Sigma$. We denote by $\kappa_i(\xi,\Sigma)$, $i = 1, ..., n-1$, the principal curvatures of Σ at ξ, labelled in order of increasing value. We denote by $\underline{\kappa}(\xi,\Sigma)$ the vector in $\mathbb{R}^{n-1}$ whose components are the curvatures $\kappa_i(\xi,\Sigma)$ and we set

$$\kappa(\xi,\Sigma) \doteq \sum_{i=1}^{n-1} \kappa_i(\xi,\Sigma). \tag{2.11}$$

Thus, $\kappa(\xi,\Sigma)$ is $(n-1)$ times the mean curvature of Σ at $\xi \in \Sigma$. When there is no risk of confusion we shorthand $\kappa_i(\xi,\Sigma) = \kappa_i(\xi)$, $\underline{\kappa}(\xi,\Sigma) = \underline{\kappa}(\xi)$, $\kappa(\xi,\Sigma) = \kappa(\xi)$.

Given any $a > 0$ we define

$$U_a(\Sigma) \doteq \{ \xi \in \mathbb{R}^n : |d(\xi, \Sigma)| > a \}, \quad W_a(\Sigma) \doteq \{ \xi \in \mathbb{R}^n : |d(\xi, \Sigma)| \le a \}. \tag{2.12}$$

If Σ is regular, then for any a small enough and any point $\xi \in W_a(\Sigma)$ there exists a unique projection $s(\xi, \Sigma) \in \Sigma$ such that

$$\mathrm{dist}(s(\xi, \Sigma), \xi) = |d(\xi, \Sigma)|. \tag{2.13}$$

Given $\lambda \in (0, 1]$ and any set A in the macroscopic space, we denote by $\lambda^{-1}A$ the image of this set in the mesoscopic space. Clearly

$$I(\lambda^{-1}\Sigma) = \lambda^{-1}I(\Sigma), \quad O(\lambda^{-1}\Sigma) = \lambda^{-1}O(\Sigma), \quad U_a(\lambda^{-1}\Sigma) = \lambda^{-1}U_{\lambda a}(\Sigma),$$
$$W_a(\lambda^{-1}\Sigma) = \lambda^{-1}W_{\lambda a}(\Sigma); \qquad d(r, \lambda^{-1}\Sigma) = \lambda^{-1}d(\xi, \Sigma) \quad (\xi = \lambda r).$$

For any $\xi_0 \in \Sigma$, we define a frame in the mesoscopic space as follows. We fix the origin in $r_0 = \lambda^{-1}\xi_0$, x axis directed along the normal to $\lambda^{-1}\Sigma$ at r_0 pointing toward its interior, and the other ones, y_i, $i = 1, ..., n-1$, along the principal axes of curvature. So any point $r \in \mathbb{R}^n$ has coordinates (x, y), $x \in \mathbb{R}$, $y = (y_1, ..., y_{n-1}) \in \mathbb{R}^{n-1}$. Hereafter we will refer to it as the *local frame* with origin in $r_0 = \lambda^{-1}\xi_0$. Finally, for any $a > 0$, we will denote by $B_a(r_0)$ the open ball of radius a centered in the origin r_0.

2.3. Motion by mean curvature. We say that the surface Σ_τ in $\mathbb{R}^n$ evolves according to the classical motion by mean curvature with parameter $\theta > 0$ in the time interval $[0, \tau_0]$ if the following holds. There is a smooth $(n-1)$ dimensional compact manifold S_0 and a smooth map $\xi : [0, \tau_0] \times S_0 \to \mathbb{R}^n$ such that, for any $\tau \in [0, \tau_0]$, $\xi(\tau, \cdot)$ is an embedding of S_0 in $\mathbb{R}^n$,

$$\Sigma_\tau = \{\xi = \xi(\tau, \eta) | \eta \in S_0\}$$

and

$$\frac{d\xi}{d\tau} = \theta\kappa\nu \tag{2.14}$$

where ν is the unit vector normal to Σ_τ at ξ and pointing toward the interior of Σ_τ, while $\kappa = \kappa(\xi, \Sigma_\tau)$ is $(n-1)$ times the mean curvature of Σ_τ at ξ.

There is a local existence and uniqueness theorem regarding the classical motion by mean curvature which follows from general results on parabolic equations, see [23] and [28]. It is known that if $n > 2$ singularities may develop after a finite time, while for $n = 2$ the only singularity which may arise is the disappearence of a cluster, see [24] and [25]. More recent results describe what happens after the appearence of singularities yielding global existence theorems for the evolution, see [3], [8], [9], [23] and references therein. Our results only cover the classical case.

2.4. The initial value problem. Let Σ_0 be a regular surface. For each $\lambda \in (0,1]$ we define

$$m_0^{(\lambda)}(r) = \bar{m}(d(r, \lambda^{-1}\Sigma_0)). \tag{2.15}$$

We denote by $m(r,t)$ the solution of the Cauchy problem (2.1) with initial datum (2.15) and by $m^\lambda(\xi,\tau)$ its macroscopic version, that is

$$m(r,t) \doteq S_t(m_0^{(\lambda)})(r), \qquad m^\lambda(\xi,\tau) \doteq S_{\lambda^{-2}\tau}(m_0^{(\lambda)})(\lambda^{-1}\xi). \tag{2.16}$$

We are now able to state our result.

Theorem 2.5. *There exist a positive number θ, given in (2.42) below, such that the following holds. For any regular surface Σ_0, let $m(r,t)$ and $m^{(\lambda)}(\xi,\tau)$ as in definition (2.16), let Σ_τ be the classical motion by mean curvature with parameter θ starting from Σ_0 and τ_s be its first singularity time. Then, for any $\tau_0 < \tau_s$ there is $\lambda_0 \in (0,1]$ and a positive constant C, depending on τ_0, Σ_0, the inverse temperature β and the interaction J, such that, for any $\lambda \in (0,\lambda_0]$ the following holds.*

i)

$$\sup_{t \le \lambda^{-2}\tau_0} \|m(\cdot,t) - \bar{m}(d(\cdot, \lambda^{-1}\Sigma_{\lambda^2 t}))\|_\infty \le C\lambda|\log\lambda|^{20}. \tag{2.17}$$

ii) *Let*

$$\Sigma_{\lambda,\tau} \doteq \{\, \xi \in \mathbb{R}^n \,:\, m^{(\lambda)}(\xi,\tau) = 0 \,\}. \tag{2.18}$$

Then

$$\Sigma_{\lambda,\tau} \subseteq \{\xi \in \mathbb{R}^n : dist(\xi, \Sigma_\tau) \le C\lambda^2|\log\lambda|^{20}\} \;\forall\, \tau \in [2\lambda^2|\log\lambda|^2, \tau_0] \tag{2.19}$$

$$\Sigma_\tau \subseteq \{\xi \in \mathbb{R}^n : dist(\xi, \Sigma_{\lambda,\tau}) \le C\lambda^2|\log\lambda|^{20}\} \;\forall\, \tau \in [2\lambda^2|\log\lambda|^2, \tau_0]. \tag{2.20}$$

From estimate (2.17) it follows that the instanton-like structure of the magnetization pattern at the interface is persistent, at least until times when the surface is regular. That is, the local equilibrium property at the interface propagates, to leading orders in λ, for times proportional to λ^{-2} (even thought the interface during this time moves by distances proportional to

λ^{-1}). On the other hand, if we look at the Cauchy problem $m^\lambda(\xi,\tau)$ as an approximation of the evolution of an interface by its mean curvature, the results (2.19) and (2.20) give a quasi-optimal error estimate of order $\mathcal{O}(\lambda^2|\log\lambda|^{20})$, which is valid before the onset of singularities. From our analysis it follows that in principle we could controll the error at all order, but we did not make the effort to do it, since the proof of the previous result is already rather involved.

The fact that $m^{(\lambda)}(\xi,\tau)$ verifies

$$\lim_{\lambda\to 0} m^{(\lambda)}(\xi,\tau) = \pm m_\beta \tag{2.21}$$

according whether $\xi \in I(\Sigma_\tau)$ or $\xi \in O(\Sigma_\tau)$ respectively, has already been proven in [15] for all times τ for which the interface Σ_τ is regular. The result has been extended to all times in [5] for $n = 2$ and in [30] for any dimension, provided the motion by curvature is defined in a generalized sense. These results however do not establish the location of the interface in mesoscopic variables, $r = \lambda^{-1}\xi$, and indeed from (2.21) the fact that the interface might have fattened cannot be excluded.

The strategy of proof of Theorem 2.5 is the same as that used in [15], based essentially on two ingredients: a good control of the *linear evolution*, see below, and the construction of appropriate barrier functions to control the non linear term.

We therefore recall below the properties of the linear evolution that will be extensively used in the sequel and that we state using the same notation as in [15].

2.6. The linear evolution. ([15]) Given any unit vector $\nu \in \mathbb{R}^n$ we call linear evolution the one obtained by linearizing (2.1) around the instanton in the direction of ν. More precisely, we define the operator L acting on $L^\infty(\mathbb{R}^n)$ as follows

$$Lu(r) \doteq -u(r) + \big(1 - \bar m(r\cdot\nu)^2\big)\beta(J\star u)(r), \qquad u \in L^\infty(\mathbb{R}^n). \tag{2.22}$$

Under the following mapping

$$u(r) = \bar m'(r\cdot\nu)\psi(r) \tag{2.23}$$

the operator L is mapped into a new one, $\mathcal{L}$, defined by

$$\mathcal{L}\psi(r) = \int dr' K(r,r')[\psi(r') - \psi(r)] \tag{2.24}$$

$$K(r,r') = \big(1 - \bar m(r\cdot\nu)^2\big)\beta J(|r-r'|)\frac{\bar m'(r'\cdot\nu)}{\bar m'(r\cdot\nu)}. \tag{2.25}$$

Observe that $K(r,r') \geq 0$ and, since

$$L\bar{m}'(r\cdot\nu) = 0, \tag{2.26}$$

then

$$\int dr'\, K(r,r') = 1 \qquad \forall\, r \in \mathbb{R}^n$$

so that $\mathcal{L}$ is a Markov generator. We set $x = r\cdot\nu$ and we fix coordinates y in the plane $\{r\cdot\nu = 0\}$, so that any point $r \in \mathbb{R}^n$ has coordinates $(x,y) \in \mathbb{R}\times\mathbb{R}^{n-1}$. The Markov process associated to $\mathcal{L}$ can be canonically realized on $D(\mathbb{R}^n;\mathbb{R}_+)$, the Skorohod space of cadlag trajectories (continuous from the right and with left limits). Given $t \geq 0$, we denotes by $r_t = (X_t, Y_t)$, $Y_t = (Y_{1,t}, ..., Y_{n-1,t})$ the coordinate mapping on such a space and by $\mathbb{E}_r[u(r_t)]$, u any measurable function on $\mathbb{R}^n$, the expectation with respect to the Markov process starting from r. The process has an invariant measure $\mu(dx)dy$, where

$$\mu(dx) = N\frac{\bar{m}'(x)^2}{1-\bar{m}(x)^2}dx, \qquad N^{-1} = \int_{\mathbb{R}} dx\, \frac{\bar{m}'(x)^2}{1-\bar{m}(x)^2}. \tag{2.27}$$

We also define the operator $L^{(1)}$ acting on $L^\infty(\mathbb{R})$ by (see (2.4) for the definition of $\tilde{J}$ below)

$$L^{(1)}\phi(x) = -\phi(x) + [1-\bar{m}(x)^2]\beta(\tilde{J}\star\phi)(x), \qquad \phi \in L^\infty(\mathbb{R}) \tag{2.28}$$

and the Markov generator corresponding to the map $\phi(x) = \bar{m}'(x)\psi(x)$ is

$$\mathcal{L}^{(1)}\psi(x) = \int dx' K^{(1)}(x,x')[\psi(x') - \psi(x)] \tag{2.29}$$

$$K^{(1)}(x,x') = \big(1-\bar{m}(x)^2\big)\beta\tilde{J}(|x-x'|)\frac{\bar{m}'(x')}{\bar{m}'(x)}. \tag{2.30}$$

The invariant measure for the Markov process with generator $\mathcal{L}^{(1)}$ is $\mu(dx)$.

Observe that, given any unit vector $\nu \in \mathbb{R}^n$ and any functions $u, \psi \in L^\infty(\mathbb{R}^n)$ that depends only on $x = r\cdot\nu$, we have

$$Lu(r) = L^{(1)}\hat{u}(x),\ u(r) = \hat{u}(x); \quad \mathcal{L}\psi(r) = \mathcal{L}^{(1)}\hat{\psi}(x),\ \psi(r) = \hat{\psi}(x) \tag{2.31}$$

so that the marginal on $x = r\cdot\nu$ of the Markov process in $\mathbb{R}^n$ with generator $\mathcal{L}$ is a realization of the Markov process in $\mathbb{R}$ with generator $\mathcal{L}^{(1)}$. In the sequel we denote by $\mathbb{E}_x^{(1)}[\phi(X_t)]$, ϕ any measurable function on $\mathbb{R}$, the expectation with respect to the Markov process with generator $\mathcal{L}^{(1)}$ starting from $x \in \mathbb{R}$.

Finally we denote by $g_t(r,r')$, $r,r' \in \mathbb{R}^n$ and by $g_t^{(1)}(x,x')$, $x,x' \in \mathbb{R}$ the Green functions associated to L and $L^{(1)}$ respectively and we set

$$(g_t \star u)(r) \doteq \int_{\mathbb{R}^n} dr' g_t(r,r')u(r'), \quad (g_t^{(1)} \star \phi)(x) \doteq \int_{\mathbb{R}} dx'\, g_t^{(1)}(x,x')\phi(x').$$

Observe that for any $t \in \mathbb{R}_+$ and any function f

$$\frac{1}{\bar{m}'(x)}\big(g_t \star (\bar{m}' f)\big)(r) = \mathbb{E}_r\big[f(r_t)\big]. \tag{2.32}$$

From the analysis done in [11], essentially based on the Perron-Frobenius Theorem, the next Theorem follows.

Theorem 2.7 ([11]). *There are $a, c > 0$ and $b \in (0,\alpha)$, α as in (2.9), so that, for any $\phi \in L^\infty(\mathbb{R})$,*

$$\|g_t^{(1)} \star \phi - C_\phi \bar{m}'\|_\infty \le c e^{-at}\|\phi - C_\phi \bar{m}'\|_\infty, \tag{2.33}$$

where

$$C_\phi = \int \mu(dx)\, \phi(x)\, \bar{m}'(x)^{-1}. \tag{2.34}$$

Furthermore, there is $C > 0$ such that, for any function $\phi \in L^\infty(\mathbb{R})$,

$$\Big|\mathbb{E}_x^{(1)}\big[\phi(X_t)\big] - \mu(\phi)\Big| \le C\|\phi\|_\infty \Big[\, \mathbf{1}_{|x|>t} + e^{-b[t-|x|]}\, \mathbf{1}_{|x|\le t}\Big], \tag{2.35}$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A and

$$\mu(\phi) \doteq \int \mu(dx)\, \phi(x). \tag{2.36}$$

In order to give the value of the constant θ of Theorem 2.5 we study the solution of the Cauchy problem (2.1) with initial datum (2.15) in a small neighbourhood of the interface, by analyzing its linear evolution around the planar instanton. Before doing this we mention that, according to [31], the value of θ is related to the mobility of the interface and to the surface tension through an "Einstein Relation" and this has been proven for this model in [4].

Let Σ_0 be a regular surface. We fix $\xi_0 \in \Sigma_0$, $r_0 = \lambda^{-1}\xi_0$, and consider the local frame with origin in r_0. In a small neighbourhood of r_0 the surface $\lambda^{-1}\Sigma_0$ can be represented as the graph of a function $x = x^\star(y)$ which, to the first order in λ, is given by

$$\lambda\omega_1(y, \underline{\kappa}) = \frac{\lambda}{2}\sum_{i=1}^{n-1} \kappa_i(\xi_0) y_i^2, \qquad \underline{\kappa} = \underline{\kappa}(\xi_0). \tag{2.37}$$

By expanding the initial datum (2.15) we get that $m_0^{(\lambda)}(r) = \psi(r) + O(\lambda^2)$, where

$$\psi(r) = \bar{m}(x) - \lambda\omega_1(y, \underline{\kappa})\bar{m}'(x). \tag{2.38}$$

Evolving $\psi(r)$ with the linear evolution and using the Markov process introduced above we get the function $\psi(r,t)$ given by

$$\psi(r,t) = \bar{m}(x) - \lambda\bar{m}'(x)\mathbb{E}_r\left[\omega_1(y, \underline{\kappa})\right]. \tag{2.39}$$

Since r_0 is a generic point we can consider only the value of the solution along the normal to $\lambda^{-1}\Sigma_0$ in r_0, that is for points $(r,t) = ((x,0),t)$. Then it is easy to see (see §6 in [15]) that

$$-\mathbb{E}_{(x,0)}\left[\omega_1(Y_t, \underline{\kappa})\right] = \int_0^t ds\, \mathbb{E}_x^{(1)}\left[f(X_s)\right] \tag{2.40}$$

with

$$f(x) = \tag{2.41}$$
$$-\left(1 - \bar{m}(x)^2\right)\beta \int_{\mathbb{R}} dx' \int_{\mathbb{R}^{n-1}} dz\, J\left(|(x'-x)^2 + z^2|^{1/2}\right)\frac{\bar{m}'(x')}{\bar{m}'(x)}\frac{1}{2}\sum_{i=1}^{n-1}\kappa_i(\xi_0) z_i^2.$$

We define

$$\theta = \int \mu(dx)\left(1 - \bar{m}(x)^2\right)\beta \int_{\mathbb{R}} dx' \int_{\mathbb{R}^{n-1}} dz\, J\left(|(x'-x)^2 + z^2|^{1/2}\right)\frac{\bar{m}'(x')}{\bar{m}'(x)}\frac{z_1^2}{2}, \tag{2.42}$$

then

$$\int \mu(dx)\, f(x) = -\kappa(\xi_0)\theta. \tag{2.43}$$

From (2.35) it follows that there is C so that for all x and t

$$\Big| - \lambda \mathbb{E}_{(x,0)} \left[\omega_1(Y_t, \underline{\kappa})\right] + \theta\kappa(\xi_0)\lambda t\Big| \le C(1+|x|)|\underline{\kappa}(\xi_0)|\lambda. \tag{2.44}$$

We consider now the linear solution (2.39) for a time T which diverges with λ but is macroscopically very small, e.g. $T = |\log\lambda|^2$. From (2.44) we get, to leading orders in λ, $m((x,0),T) \approx \bar{m}(x - \theta\kappa(\xi_0)\lambda T)$, which means that, on the macroscopic scale, the interface has moved locally with velocity $\theta\kappa(\xi_0)$ along the normal.

We finally recall two basic properties of the evolution (2.1) that will be often used in the sequel.

2.8 The Barrier Lemma. ([12]). There are V and $\bar{C}$ positive so that if $u(r,t)$ and $v(r,t)$ solve (2.1), $\|u(\cdot,0)\|_\infty \le 1$, $\|v(\cdot,0)\|_\infty \le 1$ and, for some $T > 0$, $u(r,0) = v(r,0)$ for all $|r| \le VT$, then

$$\Big|u(0,T) - v(0,T)\Big| \le \bar{C}e^{-T} \tag{2.45}$$

2.9. The Comparison Theorem. ([12]). Let $u(r,t)$ and $v(r,t)$ be two solutions of (2.1) for $t \ge 0$, such that $u(r,0) \ge v(r,0)$ for all $r \in \mathbb{R}^n$. Then $u(r,t) \ge v(r,t)$ for all $(r,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

We conclude this section with a few bibliographical remarks. For papers on Ising spin systems with Kac potentials and Glauber dynamics we refer to [12], [13] and [14]. There is another stochastic spin dynamics where phase separation phenomena and interface dynamics has been analyzed, this is the Glauber and Kawasaki process introduced in [10], see also [18], to model reaction diffusion equations. For this spin system the phase separation has been studied in one dimension in [19] and in higher dimensions in [26] and [27]. Results on convergence to the motion by mean curvature for the Glauber and Kawasaki process has been obtained in [2] and [29]. In particular the analysis in [2] is in the same spirit of our paper, in fact it allows to characterize the magnetization pattern also at the interface and it uses the local equilibrium result proven in [20] and [21] for the Allen-Cahn equation. Beside [20] and [21] there are many others very interesting papers on the derivation of the motion by mean curvature starting from the Allen-Cahn equation, concerning both the classical motion, [1], [7], and the generalized motion, [22] (we do not pretend here to have given the complete list of references and we apologize for that).

3. Proof of Theorem 2.5. Hereafter the parameter θ of the motion by mean curvature is given by the expression in (2.42). Moreover V denotes the constant in the Barrier Lemma. Other parameters that will be used in the sequel are (to simplify notation we omit their dependence on λ):

$$T \doteq \chi |\log \lambda|^2, \qquad h = \lambda^2 |\log \lambda|^{20}, \tag{3.1}$$

where $\chi \in [1,2]$. Finally the sets U_a and W_a are defined in subsection 2.2.

In this Section we prove Theorem 2.5. The proof is based on an iteration scheme which consists in the construction of functions that are above and below the solution at the times jT, j integer and T as in (3.1) above. These barrier functions are related to a biased motion by mean curvature that we define in Definition 3.3 below. The arguments that we will use are different according whether we are giving estimates close or far from the interface: the Barrier Lemma will allow us to match together the two estimates. In the generic step of the iteration we consider the evolution in the time interval $[jT,(j+1)T]$. Far from the interface the estimates are easy consequences of the attractiviness of the two pure phases $\pm m_\beta$. Close to the interface we use that the equation (2.1) is well appoximated by the equation linearized around the instanton. We compute the evolution up to the order λ^2 and, since T is "sufficiently small", we can control the remainder in the sup-norm so that we are able to absorbe it in suitable barrier functions defined by means of the h-biased motion by mean curvature. As we shall see the first time interval $[0,T]$ does not give the generic step of the iteration. However we start from the analysis of this first step hoping that this will give an explanation of our strategy of proof. Therefore in subsection 3.6 below we explain how to control the λ and λ^2 order of the expansion assuming that the remainder is of order λ^3. The argument will be made rigorous in Proposition 3.9 below.

First of all we need some definitions and statements that refer to properties of regular surfaces and their evolution by mean curvature. These properties are quite standard, we therefore only sketch some of the proofs, suggesting to the reader to skip this part at the first reading and to go directly to subsection 3.6.

Definition 3.1. For any $\underline{\kappa} = \{\underline{\kappa}_i, i = 1, \dots n\} \in \mathbb{R}^{n-1}$ and $\underline{c} = \{c_{i,j,k} : i,j,k = 1,...,n-1\} \in \mathbb{R}^{3(n-1)}$ we define the following functions of $r = (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$:

$$\omega_1(y,\underline{\kappa}) \doteq -\frac{1}{2}\sum_{i=1}^{n-1} \underline{\kappa}_i y_i^2 \tag{3.2}$$

$$\hat{\omega}_2(r,\underline{\kappa},\underline{c}) \doteq -\frac{x}{2}\sum_{i=1}^{n-1}\underline{\kappa}_i^2 y_i^2 - \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k \tag{3.3}$$

$$\omega_2(r,\underline{\kappa}) \doteq \frac{1}{8}\left[\frac{\bar{m}''(x)}{\bar{m}'(x)}\left(\sum_{i=1}^{n-1}\underline{\kappa}_i y_i^2\right)^2\right] - \frac{x}{2}\sum_{i=1}^{n-1}\underline{\kappa}_i^2 y_i^2 \tag{3.4}$$

$$\bar{\omega}_2(r,\underline{\kappa},\underline{c}) \doteq \omega_2(r,\underline{\kappa}) - \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k. \tag{3.5}$$

Lemma 3.2. *Let Σ be a regular surface (recall the definitions given in subsection 2.2). Let T be as in (3.1) and $A > 0$. There is $\lambda_0 \in (0,1]$ such that for any $\lambda \in (0,\lambda_0]$ the following holds. Given any point $\xi_0 \in \Sigma$ consider the local frame with origin in $r_0 = \lambda^{-1}\xi_0$. Then there exist functions $\underline{c}(\xi_0) = \underline{c}(\xi_0,\Sigma) = \{c_{i,j,k}(\xi_0,\Sigma) : i,j,k = 1,\dots,n-1\}$ and $\hat{\mathcal{R}}_\lambda(r,\xi_0) = \hat{\mathcal{R}}_\lambda(r,\xi_0,\Sigma)$ such that for any $r \in \mathcal{B}_{AT}(r_0)$, $r = (x,y)$ the following holds.*

$$d(r,\lambda^{-1}\Sigma) = x + \lambda\omega_1(y,\underline{\kappa}(\xi_0)) + \lambda^2\hat{\omega}_2(r,\underline{\kappa}(\xi_0),\underline{c}(\xi_0)) + \lambda^3\hat{\mathcal{R}}_\lambda(r,\xi_0) \tag{3.6}$$

with $\underline{\kappa}(\xi_0) = \underline{\kappa}(\xi_0,\Sigma)$ as defined in subsection 2.2. Furthermore there are constants a_1 and a_2 such that

$$\sup_{\xi_0\in\Sigma}\ \sup_{i,j,k} |c_{i,j,k}(\xi_0)| \le a_1 \tag{3.7}$$

$$\sup_{\xi_0\in\Sigma}\ \sup_{r\in\mathcal{B}_{AT}(r_0)} |\hat{\mathcal{R}}_\lambda(r,\xi_0)| \le a_2 T^4. \tag{3.8}$$

Proof. For λ small enough, in the ball $\mathcal{B}_{AT}(r_0)$ the surface $\lambda^{-1}\Sigma$ can be represented as the graph of a function $x = x^\star(y)$ such that

$$x^\star(y) = -\lambda\omega_1(y,\underline{\kappa}(\xi_0)) + \lambda^2\sum_{i,j,k=1}^{n-1} c_{i,j,k}(\xi_0)\, y_i y_j y_k + \lambda^3 X_\lambda^\star(r,\xi_0), \tag{3.9}$$

where $c_{i,j,k}(\xi_0)$ and $X_\lambda^\star(r,\xi_0)$ are suitable smooth functions such that (3.7) holds and, for some positive constant a_3,

$$\sup_{\xi_0\in\Sigma}\ \sup_{r\in\mathcal{B}_{AT}(r_0)} |X_\lambda^\star(r,\xi_0)| \le a_3 T^4. \tag{3.10}$$

Moreover, for λ small enough and any $r \in \mathcal{B}_{AT}(r_0)$ there is a unique projection $s(\lambda r,\Sigma) \in \Sigma$ such that

$$d(r,\lambda^{-1}\Sigma) = \pm\text{dist}(r,\bar{r}), \qquad \bar{r} \doteq \lambda^{-1}s(\lambda r,\Sigma) \in \lambda^{-1}\Sigma \cap \mathcal{B}_{AT}(r_0) \tag{3.11}$$

with the + (−) sign according whether $r \in I(\lambda^{-1}\Sigma)$ ($r \in O(\lambda^{-1}\Sigma)$) respectively. Let $\bar{r} = (\bar{x}, \bar{y})$. From the definition of $s(\lambda r, \Sigma)$, $(\bar{x}, \bar{y})$ solves

$$\bar{x} = x^\star(\bar{y}), \qquad (x - \bar{x})\nabla_y x^\star(\bar{y}) = y - \bar{y}. \tag{3.12}$$

Then, from (3.11) and (3.12) we get

$$d(r, \lambda^{-1}\Sigma) = (x - \bar{x})\sqrt{1 + |\nabla_y x^\star(\bar{y})|^2}. \tag{3.13}$$

From the first equality in (3.12) and from (3.9) it follows that $x^\star(\bar{y}) = O(\lambda)$. This fact and (3.9) imply that $\nabla_{y_i} x^\star(\bar{y}) = \lambda\kappa_i(\xi_0)y_i + O(\lambda^2)$. Therefore, there is a positive constant a_4 so that for any $i = 1, ..., n-1$,

$$\bar{y}_i = y_i + \lambda\kappa_i(\xi_0)xy_i + \lambda^2 Y_i^{(\lambda)}(r, \xi_0), \qquad \sup_{\xi_0 \in \Sigma} \sup_{r \in \mathcal{B}_{AT}(r_0)} |Y^{(\lambda)}(r, \xi_0)| \le a_4 T^2. \tag{3.14}$$

Putting this expansion in the first equality in (3.12) we get that there is $Q_\lambda(r, \xi_0)$ so that

$$\begin{aligned} x - \bar{x} = x + \lambda\omega_1(y, \underline{\kappa}(\xi_0)) - \lambda^2 x \sum_{i=1}^{n-1} \kappa_i(\xi_0)^2 y_i^2 \\ - \lambda^2 \sum_{i,j,k=1}^{n-1} c_{i,j,k}(\xi_0)\, y_i y_j y_k + \lambda^3 Q_\lambda(r, \xi_0) \end{aligned} \tag{3.15}$$

and for some positive constant a_5

$$\sup_{\xi_0 \in \Sigma} \sup_{r \in \mathcal{B}_{AT}(r_0)} |Q_\lambda(r, \xi_0)| \le a_5 T^4.$$

Using (3.14) and (3.15), from (3.9) we then get that there are $P_i^{(\lambda)}(r, \xi_0)$, $i = 1, ..., n-1$ so that

$$\left(\nabla_y x^\star(\bar{y})\right)_i = -\lambda\kappa_i(\xi_0)y_i + \lambda^2 P_i^{(\lambda)}(r, \xi_0), \qquad i = 1, ..., n-1 \tag{3.16}$$

and, for some positive constant a_6,

$$\sup_{\xi_0 \in \Sigma} \sup_{r \in \mathcal{B}_{AT}(r_0)} |P^{(\lambda)}(r, \xi_0)| \le a_6 T^2.$$

Using (3.15) and (3.16) to expand (3.13) in powers of λ up to the second order one easily gets (3.6). $\square$

Definition 3.3. The biased motion by mean curvature. The surface $\Sigma_\tau^{(h)}$ evolves according to the classical h-biased motion by mean curvature with parameter $\theta > 0$ and forcing term $h \in \mathbb{R}$ in the time interval $[0, \tau_0]$ if the following holds. There is a smooth $(n-1)$ dimensional compact manifold S_0 and a smooth map $\xi : [0, \tau_0] \times S_0 \to \mathbb{R}^d$ such that, for any $\tau \in [0, \tau_0]$, $\xi(\tau, \cdot)$ is an embedding of S_0 in $\mathbb{R}^n$,

$$\Sigma_\tau^{(h)} = \{\xi = \xi(\tau, \eta) \,|\, \eta \in S_0\} \tag{3.17}$$

and

$$\frac{d\xi}{d\tau} = (\theta\kappa - h)\nu, \tag{3.18}$$

where ν is the unit vector normal to $\Sigma_\tau^{(h)}$ at ξ and pointing toward the interior of $\Sigma_\tau^{(h)}$, while $\kappa = \kappa(\xi, \Sigma^{(h)})$ is $(n-1)$ times the mean curvature of $\Sigma_\tau^{(h)}$ at ξ.

The following Theorem that we state without proof, is based on classical results on parabolic equations and can be proven by reasoning as in [28].

Theorem 3.4. *Let Σ_τ, be the classical motion by mean curvature as defined in subsection 2.3 with parameter θ. Let τ_s be the first singularity time and let $\xi = \xi(\tau, \xi_0)$, $\xi_0 \in S_0$ be the corresponding parameterization. Let $A > 0$, and $\tau_0 \in (0, \tau_s)$ and let h and T be as in (3.1). Then there is $\lambda_0 \in (0, 1]$ such that for any $\lambda \in (0, \lambda_0]$ the following holds.*

1) *There exist the $(\pm h)$-biased motion by curvature flows $\Sigma_\tau^{(\pm h)}$, $\tau \in [0, \tau_0]$, in the sense of Definition 3.3, starting at $\tau = 0$ from Σ_0 and with parametrizations $\xi_\lambda^\pm(\tau, \eta)$, $\eta \in S_0$. Furthermore there is c, that depends only on τ_0 and Σ_0 so that, for any $\tau \in [0, \tau_0]$ and $\eta \in S_0$,*

$$\left|\xi_\lambda^\pm(\tau, \eta) - \xi(\tau, \eta)\right| \le ch. \tag{3.19}$$

2) *For any $\tau \in [0, \tau_0]$ and $\xi \in W_{A\lambda T}(\Sigma_\tau)$ (or $\xi \in W_{A\lambda T}(\Sigma_\tau^{(\pm h)})$), the projections $s(\xi, \Sigma_\tau)$ (respectively $s(\xi, \Sigma_\tau^{(\pm h)})$) are well defined.*

3) *The curvatures $\kappa_i(s(\xi, \Sigma_\tau^{(\pm h)}), \Sigma_\tau^{(\pm h)})$, as functions on the compact set*

$$\hat{W}_\lambda \doteq \bigcup_{\tau \in [0, \tau_0]} W_{A\lambda T}(\Sigma_\tau^{(\pm h)}) \times \{\tau\}$$

are smooth and then uniformly bounded with their derivatives on $\hat{W}_\lambda$.

Lemma 3.5. *In the same hypothesis of Theorem 3.4, define*

$$d_\pm = d_\pm(r,t) \doteq d\big(r, \lambda^{-1}\Sigma^{(\pm h)}_{\lambda^2 t}\big), \qquad (r,t) \in \mathbb{R}^n \times [0, \lambda^{-2}\tau_0] \tag{3.20}$$

and the vector functions $\underline{\kappa}_\pm = \underline{\kappa}_\pm(r,t) \in \mathbb{R}^{n-1}$, $r \in W_{AT}\big(\lambda^{-1}\Sigma^{(\pm h)}_{\lambda^2 t}\big)$, $t \in [0, \lambda^{-2}\tau_0]$, *such that*

$$(\underline{\kappa}_\pm)_i(r,t) \doteq \kappa_i\big(s\big(\lambda r, \Sigma^{(\pm h)}_{\lambda^2 t}\big), \Sigma^{(\pm h)}_{\lambda^2 t}\big). \tag{3.21}$$

The functions $\underline{\kappa}_\pm$ *are well defined and smooth. Furthermore there is* $\bar{c} > 0$ *so that for any non negative integer* $j \le T^{-1}\lambda^{-2}\tau_0$ *the following holds.*

1) *For any* $\xi_0 \in \Sigma^{(\pm h)}_{\lambda^2 jT}$ *and any* r *such that* $|r - \lambda^{-1}\xi_0| \le AT$,

$$\big|\underline{\kappa}_\pm(r, jT) - \underline{\kappa}(\xi_0, \Sigma^{(\pm h)}_{\lambda^2 jT})\big| \le \bar{c}\lambda^2 T^2. \tag{3.22}$$

2) *For any* $\xi_0 \in \Sigma_{\lambda^2 jT}$, *consider the local frame with origin in* $r_0 = \lambda^{-1}\xi_0$. *Then, for any point* r *that in this frame has coordinates equal to* $(x, 0)$ *with* $|x| \le AT$,

$$\Big|d\Big((x,0), \lambda^{-1}\Sigma_{\lambda^2(j+1)T}\Big) - x + \theta\kappa(\xi_0)\lambda T\Big| \le \bar{c}\lambda^3 T^2. \tag{3.23}$$

3) *For any* $\xi_0 \in \Sigma^{(\pm h)}_{\lambda^2 jT}$, *consider the local frame with origin in* $r_0 = \lambda^{-1}\xi_0$. *Then, for any point* r *that in this frame has coordinates equal to* $(x, 0)$ *with* $|x| \le AT$,

$$\Big|d\Big((x,0), \lambda^{-1}\Sigma^{(\pm h)}_{\lambda^2(j+1)T}\Big) - x + (\theta\kappa(\xi_0) \mp h)\lambda T\Big| \le \bar{c}\lambda^3 T^2, \tag{3.24}$$

$$\Big|\underline{\kappa}_\pm((x,0), (j+1)T) - \underline{\kappa}(\xi_0, \Sigma^{(\pm h)}_{\lambda^2 jT})\Big| \le \bar{c}\lambda^2 T. \tag{3.25}$$

Proof. From Theorem 3.4 the definition (3.21) is well posed and the functions $\underline{\kappa}_\pm(r,t)$ are smooth functions of the macroscopic variables $(\xi, \tau) = (\lambda r, \lambda^2 t)$. Then (3.22) and (3.25) follow immediately. Finally, the bounds (3.23) and (3.24) follow easily from the definition of the classical mean curvature motion (without and with bias respectively). We omit the details. ☐

We have now all the ingredients to prove Theorem 2.5. As mentioned earlier we now present informally the expansion that we use close to the interface and in the time interval $[0, T]$.

3.6. Evolution in the time interval $[0, T]$. Let $\xi_0 \in \Sigma_0$ and $r_0 = \lambda^{-1}\xi_0$. We consider the evolution of the initial datum $m_0^{(\lambda)}$ given in (2.15) in the time interval $[0, T]$ and in a neighbourhood of r_0 (e.g. $r \in \mathcal{B}_{AT}(r_0)$ for some constant $A > 0$). More precisely we work in the local frame with origin in r_0 and, since the choice of r_0 is arbitrary, we compute the solution $m(r, T)$ for $r = (x, 0)$ and $|x| \le AT$.

We write $\bar{m}(d(r, \lambda^{-1}\Sigma_0))$ in the local frame. From Definition 3.1 and Lemma 3.2 for any $r = (x, y)$, $r \in \mathcal{B}_{AT}(r_0)$ we get

$$\begin{aligned}\bar{m}(d(r, \lambda^{-1}\Sigma_0)) = \bar{m}(x) + \lambda\omega_1(y, \underline{\kappa}(\xi_0))\bar{m}'(x) \\ + \lambda^2\bar{\omega}_2(r, \underline{\kappa}(\xi_0), \underline{c}(\xi_0))\bar{m}'(x) + \lambda^3 U_0^{(\lambda)}(r),\end{aligned} \tag{3.26}$$

where $U_0^{(\lambda)}(r)$ is defined so that (3.26) holds.

We define

$$u^{(\lambda)}(r, t) = m(r, t) - \bar{m}(x), \tag{3.27}$$

where $m(r, t)$ solves (2.1) with initial datum (3.26). Observe that since the expansion (3.26) holds only in $\mathcal{B}_{AT}(r_0)$, the function $m(r, t)$ so defined is not the solution to (2.1) with initial datum $m_0^{(\lambda)}$ which is what we want to compute. However from the barrier lemma we know that the two functions are exponentially close to each other at time T and in $\mathcal{B}_{AT/2}(r_0)$. With this in mind we analyze below $u^{(\lambda)}(r, t)$.

Since $\bar{m}(x)$ satisfies (2.5) and recalling the definition (2.22), the following function

$$\begin{aligned}\mathcal{R}_\lambda(r, t) &\doteq \tanh\{\beta(J \star (\bar{m} + u^{(\lambda)}))(r, t)\} - \bar{m}(x) \\ &\quad - Lu^{(\lambda)}(r, t) - \frac{\Phi(x)}{2}(J \star u^{(\lambda)})^2(r, t), \qquad (3.28)\\ \Phi(x) &\doteq -2\beta^2\bar{m}(x)[1 - \bar{m}(x)^2] \qquad (3.29)\end{aligned}$$

is of the order $(u^{(\lambda)})^3$. We then get that the evolution equation for $u^{(\lambda)}$ can be written as follows:

$$\partial_t u^{(\lambda)} = Lu^{(\lambda)} + \frac{\Phi}{2}\big(J \star u^{(\lambda)}\big)^2 + \mathcal{R}_\lambda. \tag{3.30}$$

We write the solution of (3.30) in the following way,

$$u^{(\lambda)}(r, t) = \lambda\Omega_1(r, t)\bar{m}'(x) + \lambda^2\Omega_2(r, t)\bar{m}'(x) + \lambda^3 U^{(\lambda)}(r, t) \tag{3.31}$$

with (see (3.26))

$$\Omega_1(r,0) = \omega_1(y, \underline{\kappa}(\xi_0)), \quad \Omega_2(r,0) = \bar{\omega}_2(r, \underline{\kappa}(\xi_0), \underline{c}(\xi_0)), \quad U^{(\lambda)}(r,0) = U_0^{(\lambda)}(r) \tag{3.32}$$

and we want to compute $u^{(\lambda)}(r,T)$ for points r that have coordinates $(x,0)$.

By (3.31) and (3.30), after identifying the various powers in λ, we get (recall the definitions in subsection 2.6)

$$\partial_t \Omega_1 = \frac{1}{\bar{m}'} L\bar{m}'\Omega_1 = \mathcal{L}\Omega_1 \tag{3.33}$$

$$\bar{m}'\partial_t\Omega_2 = L\Omega_2 + \frac{\Phi}{2}\big(J \star (\bar{m}'\Omega_1)\big)^2 \tag{3.34}$$

$$\partial_t U^{(\lambda)} = LU^{(\lambda)} + \lambda^{-3}\mathcal{C}_\lambda(U^{(\lambda)}), \tag{3.35}$$

where

$$\begin{aligned}\mathcal{C}_\lambda(U^{(\lambda)}) &= \tanh\{\beta\big(J \star (\bar{m} + u^{(\lambda)})\big)\} - \bar{m} - Lu^{(\lambda)} \\ &\quad - \frac{\Phi}{2}\big[(J \star u^{(\lambda)})^2 - \big(J \star (\lambda\bar{m}'\Omega_1)\big)^2\big].\end{aligned} \tag{3.36}$$

Using the notation of subsection 2.6 from (3.33) it follows that

$$\Omega_1(r,t) = \mathbb{E}_r\Big[\Omega_1(r_t,0)\Big] = \mathbb{E}_r\Big[\omega_1(Y_t, \underline{\kappa}(\xi_0))\Big] \tag{3.37}$$

and noticing that since ω_1 is the same function defined in (2.37), from (2.44) we get that, to leading order in T, $\Omega_1((x,0),T)$ is equal to $\theta\kappa(\xi_0)T$. With this in mind we write

$$\Omega_1(r,T) = -\theta\kappa(\xi_0)T + \mathcal{F}_T(r, \underline{\kappa}(\xi_0)), \tag{3.38}$$

where

$$\mathcal{F}_T(r, \underline{\kappa}(\xi_0)) = \mathbb{E}_r\Big[\theta\kappa(\xi_0)T + \omega_1(Y_T, \underline{\kappa}(\xi_0))\Big]. \tag{3.39}$$

We now solve (3.34) with the same strategy. First we notice that

$$L^{(1)}\bar{m}''(x) + \Phi(x)\big(\tilde{J} \star \bar{m}'\big)^2(x) = 0 \tag{3.40}$$

and then we subtract to (3.34) the left hand side of (3.40) multiplied by $(\theta\kappa(\xi_0)T)^2/2$, we then get

$$\bar{m}'\partial_t\Omega_2 = L\Omega_2 - \frac{1}{2}(\theta\kappa(\xi_0)T)^2 L^{(1)}\bar{m}'' + \bar{m}'\hat{\mathcal{G}}, \tag{3.41}$$

where (see (3.37))

$$\hat{\mathcal{G}}(r,t) = \frac{\Phi(x)}{2\bar{m}'(x)}\Big[\big(J \star \bar{m}'\mathbb{E}.\big[\omega_1(Y_t;\underline{\kappa}(\xi_0)]\big]\big)^2(r) - (\theta\kappa(\xi_0)T)^2\big(\tilde{J}\star\bar{m}'\big)^2(x)\Big]. \tag{3.42}$$

Therefore, see subsection 2.6 for notation,

$$\begin{aligned}\bar{m}'(x)\Omega_2(r,t) = \big(g_t \star \Omega_2(\cdot,0)\big)(r) - \frac{1}{2}(\theta\kappa(\xi_0)T)^2\int_0^t ds\,\big(g_s^{(1)}\star(L^{(1)}\bar{m}'')\big)(x)\\ + \int_0^t ds \int dr' g_{t-s}(r,r')\bar{m}'(x')\hat{\mathcal{G}}(r',s).\end{aligned} \tag{3.43}$$

Noticing that

$$-\int_0^t ds\big(g_s^{(1)}\star(L^{(1)}\bar{m}'')\big)(x) = -\int_0^t ds\frac{d}{ds}(g_s^{(1)}\star\bar{m}'')(x) = \bar{m}''(x) - (g_t^{(1)}\star\bar{m}'')(x) \tag{3.44}$$

from (3.43), (3.44) and observing that $\Omega_2(r,0) = \bar{\omega}_2(r,\kappa(\xi_0),\underline{c}(\xi_0))$, we get

$$\begin{aligned}\Omega_2(r,T)\bar{m}'(x) = \frac{1}{2}(\theta\kappa(\xi_0)T)^2\bar{m}''(x) + \mathcal{G}_T(r,\underline{\kappa}(\xi_0),\underline{c}(\xi_0))\bar{m}'(x)\\ - \frac{1}{2}(\theta\kappa(\xi_0)T)^2(g_T^{(1)}\star\bar{m}'')(x),\end{aligned} \tag{3.45}$$

where (recall (2.32))

$$\mathcal{G}_T(r,\underline{\kappa}(\xi_0),\underline{c}(\xi_0)) = \mathbb{E}_r\big[\bar{\omega}_2(r_T,\underline{\kappa}(\xi_0),\underline{c}(\xi_0))\big] + \int_0^T ds\,\mathbb{E}_r\big[\hat{\mathcal{G}}(r_{T-s},s)\big]. \tag{3.46}$$

Observe that since $C_{\bar{m}''} = 0$ from (2.33) we get

$$\big\|g_T^{(1)}\star\bar{m}''\big\|_\infty \le ce^{-aT}. \tag{3.47}$$

We define $\bar{R}_\lambda(x,T)$ in the following way

$$\begin{aligned}\lambda^3\bar{R}_\lambda(x,T) = \bar{m}(x) - \theta\kappa(\xi_0)\lambda T\bar{m}'(x)\\ + \frac{1}{2}\big(\theta\kappa(\xi_0)\lambda T\big)^2\bar{m}''(x) - \bar{m}\big(x - \theta\kappa(\xi_0)\lambda T\big),\end{aligned}$$

then from (3.31), (3.38) and (3.45) we get

$$m(r,T) = \bar{m}\big(x - \theta\kappa(\xi_0)\lambda T\big) + \lambda \mathcal{F}_T(r, \underline{\kappa}(\xi_0))\bar{m}'(x) + \lambda^2 \mathcal{G}_T(r, \underline{\kappa}(\xi_0), \underline{c}(\xi_0))\bar{m}'(x) + \lambda^3 \hat{U}^{(\lambda)}(r,T), \tag{3.48}$$

where

$$\hat{U}^{(\lambda)}(r,T) = U^{(\lambda)}(r,T) + \bar{R}_\lambda(x,T) - \frac{1}{2}(\theta\kappa(\xi_0)T)^2(g_T^{(1)} \star \bar{m}'')(x).$$

Observe that (3.48) is written in the local frame with origin in $r_0 = \lambda^{-1}\xi_0$; the evolution of the point ξ_0 under the motion by mean curvature is given, at first order in λ, by $\theta\kappa(\xi_0)\lambda^2 T$. On the other hand, from (3.23) and (3.48) it follows that $m((x,0),T)$, at first order in λT, is given by $\bar{m}\big(d(r, \lambda^{-1}\Sigma_{\lambda^2 T})\big)$, which is then what we want. To control the other terms in (3.48) (since (3.24) holds) we use the h-biased motion by curvature as we are going to explain. In Lemma 3.12 below we prove that there are positive constants $\bar{c}_1$, c_1', $\bar{c}_2$ and c_2' so that

$$\sup_x \bar{m}'(x)\Big|\mathbb{E}_x^{(1)}\big[\mathcal{F}_T((X_T,0), \underline{\kappa}(\xi_0))\big]\Big| \le \bar{c}_1 e^{-c_1' T} \tag{3.49}$$

and

$$\sup_x \bar{m}'(x)\Big|\mathbb{E}_x^{(1)}\big[\mathcal{G}_T((X_T,0), \underline{\kappa}(\xi_0), \underline{c}(\xi_0))\big]\Big| \le \bar{c}_2 e^{-c_2' T}. \tag{3.50}$$

From (3.49) and (3.50) it follows that only after another time step T the terms of order λ and λ^2 in (3.48) will be negligible. The idea is then to define, for $r = (x,0)$,

$$M^{\pm}(r,T) = \bar{m}\big(d(r, \lambda^{-1}\Sigma_{\lambda^2 T}^{(\pm h)})\big) + \lambda \mathcal{F}_T(r, \underline{\kappa}(\xi_0))\bar{m}'(x) + \lambda^2 \mathcal{G}_T(r, \underline{\kappa}(\xi_0), \underline{c}(\xi_0))\bar{m}'(x) \tag{3.51}$$

and prove that

$$M^-(r,T) \le m(r,T) \le M^+(r,T), \qquad r = (x,0). \tag{3.52}$$

Let us consider the upper bound part (the argument for the other bound is analogous) observing that from (3.24), for $r = (x,0)$,

$$\bar{m}\big(d(r, \lambda^{-1}\Sigma_{\lambda^2 T}^{(+h)})\big) \ge \bar{m}(x - \theta\kappa(\xi_0)\lambda T) + h\lambda T \bar{m}'(x) - \bar{c}\lambda^3 T^2 \bar{m}'(x).$$

Therefore, $M^+(r,T) \geq m(r,T)$ if

$$hT \geq \lambda^2 \left[\bar{c}T^2 + \frac{1}{\bar{m}'(x)} \hat{U}^{(\lambda)}(r,T) \right]. \tag{3.53}$$

In order to locate the interface in mesoscopic coordinates with an accuracy of order $\lambda |\log \lambda|^p$ for some $p > 0$, we need $h \sim \lambda^2 |\log \lambda|^p$. From (3.53) it follows that this can be done provided the second term on the right hand side can be bounded as a power of T uniformly in r and λ. Here we face another reason why this argument cannot be used in the whole space. In fact, even if we have a good estimate of $\|\hat{U}^{(\lambda)}\|_\infty$, for x very large (i.e., far away from the interface) the factor $\bar{m}'(x)^{-1}$ is exponentially large, (see (2.8)).

The above discussion is the motivation for the following definitions.

The separation between the regions close and far from the interface will be given in terms of the following number,

$$R_\lambda = R_0 |\log \lambda|, \qquad \text{with } R_0 \text{ such that } 3 + \frac{1}{10} < \alpha R_0 < 3 + \frac{1}{5}, \tag{3.54}$$

where α is defined via eq. (2.9).

Definition 3.7. The functions $\mathcal{F}_T$ and $\mathcal{G}_T$.

Let T as in (3.1) and recall the Definition 3.1. For any $\underline{\kappa} \in \mathbb{R}^{n-1}$, $\underline{c} = \{c_{i,j,k} : i,j,k = 1,...,n-1\} \in \mathbb{R}^{3(n-1)}$ and $r = (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ we define

$$\mathcal{F}_T(r, \underline{\kappa}) \doteq \theta \kappa T + \mathbb{E}_r \big[\omega_1(Y_T, \underline{\kappa}) \big] \tag{3.55}$$

$$\mathcal{G}_T(r, \underline{\kappa}, \underline{c}) \doteq \mathbb{E}_r \big[\bar{\omega}_2(r_T, \underline{\kappa}, \underline{c}) \big] + \int_0^T ds \, \mathbb{E}_r \big[\hat{\mathcal{G}}(r_{T-s}, s) \big], \tag{3.56}$$

where

$$\kappa \doteq \sum_{i=1}^{n-1} \underline{\kappa}_i$$

and (see (3.29) for the definition of Φ)

$$\hat{\mathcal{G}}(r,t) \doteq \frac{\Phi(x)}{2\bar{m}'(x)} \left[\big(J \star \bar{m}' \mathbb{E}_{\cdot} \big[\omega_1(Y_t, \underline{\kappa}) \big] \big)^2 (r) - (\theta \kappa T)^2 \big(\tilde{J} \star \bar{m}' \big)^2 (x) \right]. \tag{3.57}$$

Finally we denote

$$F_T(x, \underline{\kappa}) = \mathcal{F}_T((x,0), \underline{\kappa}) \qquad G_T(x, \underline{\kappa}) = \mathcal{G}_T((x,0), \underline{\kappa}, \underline{c}). \tag{3.58}$$

Remark. Notice that, since the marginals $\{Y_{i,T}\}_{i=1}^{n-1}$ of the Markov process starting from $(x, 0)$ have symmetric distribution,

$$\mathbb{E}_{(x,0)}\Big[\sum_{i,j,k=1}^{n-1} c_{i,j,k} Y_{i,T} Y_{j,T} Y_{k,T}\Big] = 0 \tag{3.59}$$

so that (see definitions (3.4) and (3.5))

$$\mathbb{E}_{(x,0)}\big[\bar{\omega}_2(r_T, \underline{\kappa}, \underline{c})\big] = \mathbb{E}_{(x,0)}\big[\omega_2(r_T, \underline{\kappa})\big]$$

and then $G_T(x, \underline{\kappa})$ actually does not depend on $\underline{c}$.

Definition 3.8. Upper and lower bounds at time T. With the same notation of Definition 3.7, recalling the definitions (3.20) and (3.21), we define, for any λ small enough,

$$\hat{m}^{\pm}(r) \doteq \begin{cases} \hat{m}_0^{\pm}(r) & \text{if } |d_{\pm}(r,T)| \le R_\lambda \\ \operatorname{sign}(d_{\pm}(r,T)) m_\beta \pm \lambda^{3+1/10} & \text{if } |d_{\pm}(r,T)| > R_\lambda, \end{cases} \tag{3.60}$$

where

$$\begin{aligned} \hat{m}_0^{\pm}(r) &\doteq \bar{m}\big(d_{\pm}(r,T)\big) + \lambda F_T\big(d_{\pm}(r,T), \underline{\kappa}_{\pm}(r,T)\big)\bar{m}'\big(d_{\pm}(r,T)\big) \\ &\quad + \lambda^2 G_T\big(d_{\pm}(r,T), \underline{\kappa}_{\pm}(r,T)\big)\bar{m}'\big(d_{\pm}(r,T)\big). \end{aligned} \tag{3.61}$$

In the sequel we will prove the following Proposition.

Proposition 3.9. *There is $\lambda_0 \in (0, 1]$, depending on Σ_0, β and J, so that for any $\lambda \in (0, \lambda_0]$ and any $r \in \mathbb{R}^n$, (see (2.3) for notation)*

$$\hat{m}^-(r) \le S_T(m_0^{(\lambda)})(r) \le \hat{m}^+(r). \tag{3.62}$$

From the Comparison Theorem we are then reduced to bound from above (below) $S_t(\hat{m}^+)$ (respectively $S_t(\hat{m}^-)$). Since in the definition of $\hat{m}^{\pm}$ there is the term $\lambda F_T \bar{m}'$ of order λ, in the evolution of $\hat{m}^{\pm}$ this term will give a contribution of order λ^2 at time T (that will be called H_T in the sequel). In this discussion we are using the fact that the linear evolution of $\lambda F_T \bar{m}'$ and $\lambda^2 G_T \bar{m}'$ give a negligible contribution up to the second order in λ because of (3.49) and (3.50). With this in mind we give the following Definition.

Definition 3.10. The function H_T. With the same notation of Definition 3.7 we set

$$\Omega_1(r,t) = \mathbb{E}_r\Big[\omega_1(Y_t,\underline{\kappa}) + F_T(X_t,\underline{\kappa})\Big]. \tag{3.63}$$

We then define

$$H_T(x,\underline{\kappa}) \doteq H_T^{(0)}(x,\underline{\kappa}) + H_T^{(1)}((x,0),\underline{\kappa}) + \mathcal{H}_T((x,0),\underline{\kappa}), \tag{3.64}$$

where, for $x \in \mathbb{R}$ and $r = (x,y) \in \mathbb{R}\times\mathbb{R}^{n-1}$,

$$H_T^{(0)}(x,\underline{\kappa}) \doteq \theta\kappa T \frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}' F_T(\cdot,\underline{\kappa})\Big)(x) \tag{3.65}$$

$$H_T^{(1)}(r,\underline{\kappa}) \doteq \mathbb{E}_r\left[\frac{1}{\bar{m}'(X_T)}\frac{d}{dx}\Big(\bar{m}' F_T(\cdot,\underline{\kappa})\Big)(X_T)\omega_1(Y_T,\underline{\kappa})\right] \tag{3.66}$$

$$\begin{aligned}\mathcal{H}_T(r,\kappa) \doteq \frac{1}{2}\int_0^T ds\, \Bigg\{&\mathbb{E}_r\left[\frac{\Phi}{\bar{m}'}(X_{T-s})\big(J\star(\bar{m}'\mathbb{E}_\cdot[\Omega_1(r,s)])\big)^2(r_{T-s})\right] \\ &- \mathbb{E}_r\left[\frac{\Phi}{\bar{m}'}(X_{T-s})\big(J\star(\bar{m}'\mathbb{E}_\cdot[\omega_1(Y_s,\underline{\kappa})])\big)^2(r_{T-s})\right]\Bigg\}.\end{aligned} \tag{3.67}$$

Finally we define

$$Q_1^{(T)}(x,\underline{\kappa})) \doteq F_T(x,\underline{\kappa}), \qquad Q_2^{(T)}(x,\underline{\kappa}) \doteq G_T(x,\underline{\kappa}) + H_T(x,\underline{\kappa}). \tag{3.68}$$

Definition 3.11. Upper and lower bounds at times jT. With the same notation of Definitions 3.7 and 3.10, recalling the definitions (3.20) and (3.21), we define, for any λ small enough,

$$m^\pm(r,t) \doteq \begin{cases} \bar{m}(d_\pm) + \lambda Q_1^{(T)}(d_\pm,\underline{\kappa}_\pm)\bar{m}'(d_\pm) + \lambda^2 Q_2^{(T)}(d_\pm,\underline{\kappa}_\pm)\bar{m}'(d_\pm) \\ \qquad \text{if } |d_\pm| \le R_\lambda \\ \operatorname{sign}(d_\pm)m_\beta \pm \lambda^{3+1/10} \qquad \text{if } |d_\pm| > R_\lambda. \end{cases} \tag{3.69}$$

In the next Lemma we give the necessary estimates on the functions H_T, F_T and G_T. One of the key ingredients in the proof of Lemma 3.12 below (that will be given in the next section) is that H_T and G_T are odd functions of x.

Lemma 3.12. *There is $\lambda_0 \in (0,1]$ such that, for any $\lambda \in (0,\lambda_0]$, the following holds.*

1) *For any $\underline{\kappa} \in \mathbb{R}^{n-1}$, $F_T(\cdot,\underline{\kappa}) \in C^\infty(\mathbb{R})$ and depends linearly on $\underline{\kappa}$. Moreover there is a positive constant C_1 such that, for any $x \in \mathbb{R}$ and $\underline{\kappa} \in \mathbb{R}^{n-1}$,*

$$|F_T(x,\underline{\kappa})| \le C_1(1+|\underline{\kappa}|^2)(1+|x|), \tag{3.70}$$

$$\left|\frac{\partial F_T}{\partial x}(x,\underline{\kappa})\right| \le C_1(1+|\underline{\kappa}|^2)(1+|x|), \tag{3.71}$$

$$\left|\frac{\partial^2 F_T}{\partial x^2}(x,\underline{\kappa})\right| \le C_1(1+|\underline{\kappa}|^2)(1+|x|). \tag{3.72}$$

Furthermore,

$$\int \mu(dx)\, F_T(x,\underline{\kappa}) = 0 \qquad \forall \underline{\kappa} \in \mathbb{R}^{n-1}. \tag{3.73}$$

Finally, there are constants $\bar{c}_1$ and c_1' so that

$$\left\|\bar{m}'\mathbb{E}^{(1)}_{\cdot}\left[F_T(X_T,\underline{\kappa})\right]\right\|_\infty \le \bar{c}_1 e^{-c_1' T}. \tag{3.74}$$

2) *For any $\underline{\kappa} \in \mathbb{R}^{n-1}$, $G_T(\cdot,\underline{\kappa})$, $H_T(\cdot,\underline{\kappa}) \in C^\infty(\mathbb{R})$ and are polynomial of degree 2 in $\underline{\kappa}$. Moreover, there is a positive constant C_2 such that, for any $x \in \mathbb{R}$ and $\underline{\kappa} \in \mathbb{R}^{n-1}$,*

$$|G_T(x,\underline{\kappa})| + |H_T(x,\underline{\kappa})| \le C_2(1+|\underline{\kappa}|^2)(1+|x|)(|\log T|^2 T + |x|^5)T \tag{3.75}$$

$$\left|\frac{\partial G_T}{\partial x}(x,\underline{\kappa})\right| + \left|\frac{\partial H_T}{\partial x}(x,\underline{\kappa})\right| \le C_2(1+|\underline{\kappa}|^2)(1+|x|)(|\log T|^2 T + |x|^5)T. \tag{3.76}$$

Furthermore,

$$\int \mu(dx)\, G_T(x,\underline{\kappa}) = 0, \qquad \int \mu(dx)\, H_T(x,\underline{\kappa}) = 0 \qquad \forall \underline{\kappa} \in \mathbb{R}^{n-1}. \tag{3.77}$$

Finally, there are constants $\bar{c}_2$ and c_2' so that

$$\left\|\bar{m}'\mathbb{E}^{(1)}_{\cdot}\left[G_T(X_T,\underline{\kappa})\right]\right\|_\infty + \left\|\bar{m}'\mathbb{E}^{(1)}_{\cdot}\left[H_T(X_T,\underline{\kappa})\right]\right\|_\infty \le \bar{c}_2 e^{-c_2' T}. \tag{3.78}$$

Actually some of the previous estimates can be improved, but these are sufficient for our purposes.

We now prove the following Proposition that will be the main statement from which Theorem 2.5 (whose proof will be given at the end of this section) will easily follow.

Proposition 3.13. *With the same notation of Definitions* 3.8 *and* 3.11, *there is* $\lambda_0 \in (0,1]$, *depending on* τ_0, Σ_0, β *and* J, *so that the following estimates hold.*

1) *For any* $\lambda \in (0, \lambda_0]$,

$$S_T(\hat{m}^+)(r) \le m^+(r, 2T) \tag{3.79}$$

and

$$m^-(r, 2T) \le S_T(\hat{m}^-)(r). \tag{3.80}$$

2) *For any* $\lambda \in (0, \lambda_0]$, $j \in \mathbb{N}_+$ *such that* $(j+1)T \le \lambda^{-2}\tau_0$ *and* $r \in \mathbb{R}^n$,

$$S_T\big(m^+(\cdot, jT)\big)(r) \le m^+(r, (j+1)T) \tag{3.81}$$

and

$$m^-(r, (j+1)T) \le S_T\big(m^-(\cdot, jT)\big)(r). \tag{3.82}$$

Proof. We start with the proof of (3.81). We fix j as in the statement of the Proposition and we shorthand $\Sigma^{(j)} \doteq \Sigma^{(+h)}_{j\lambda^2 T}$. We consider the sets $U_{2VT}(\lambda^{-1}\Sigma^{(j)})$ and $W_{3VT}(\lambda^{-1}\Sigma^{(j)})$ (see subsection 2.2) and we prove (3.81) separately in these two regions.

Estimate away from the interface. We fix a point $r \in \lambda^{-1}I(\Sigma^{(j)}) \cap U_{2VT}(\lambda^{-1}\Sigma^{(j)})$ for which we want to prove (3.81). For any $\bar{r}$ such that $|r - \bar{r}| \le VT$, from the triangular inequality we have that $d(\bar{r}, \lambda^{-1}\Sigma^{(j)}) \ge VT$, therefore, since for λ small enough $VT > R_\lambda$, we have that $m^+(\bar{r}, jT) = m_\beta + \lambda^{3+1/10}$. Then, from the Barrier Lemma it follows that, for λ small enough,

$$\big|S_T\big(m^+(\cdot, jT)\big)(r) - v(T)\big| \le \bar{C}e^{-T} \le \frac{1}{2}\lambda^{3+1/10}, \tag{3.83}$$

where $v(t) \doteq S_t(m_\beta + \lambda^{3+1/10})$ solves the homogenous equation

$$\frac{dv(t)}{dt} = -v(t) + \tanh\{\beta v(t)\}, \qquad v(0) = m_\beta + \lambda^{3+1/10}. \tag{3.84}$$

It is easy to see that there are constants c and c' so that

$$|v(t) - m_\beta| \le c' e^{-ct}|v(0) - m_\beta|.$$

Therefore, for λ small enough, $|v(T) - m_\beta| \le \lambda^{3+1/10}/2$ so that

$$S_T\big(m^+(\cdot, jT)\big)(r) \le m_\beta + \lambda^{3+1/10}. \tag{3.85}$$

Observe now that from (3.19) the displacement of the h-biased motion by mean curvature in the time T is of order λT. Then, for some $c'' > 0$, $d_+(r,(j+1)T) \geq d(r, \lambda^{-1}\Sigma^{(j)}) - c''\lambda T \geq 2VT - c''\lambda T$ so that, for λ small enough, $d_+(r,(j+1)T) \geq R_\lambda$ and therefore $m^+(r,(j+1)T) = m_\beta + \lambda^{3+1/10}$. Then (3.81) for $r \in \lambda^{-1}I(\Sigma^{(j)}) \cap U_{2VT}(\lambda^{-1}\Sigma^{(j)})$ follows from (3.85). The proof for $r \in \lambda^{-1}O(\Sigma^{(j)}) \cap U_{2VT}(\lambda^{-1}\Sigma^{(j)})$ is analogous and it is omitted.

Estimate close to the interface. We fix a point $\xi_0 \in \Sigma^{(j)}$ and we consider the local frame with origin in $r_0 = \lambda^{-1}\xi_0$. We consider the points r that in this frame have coordinates $(x,0)$ with $|x| \leq 3VT$ and we prove that

$$S_T\big(m^+(\cdot,jT)\big)\big((x,0)\big) \leq m^+((x,0),(j+1)T) \qquad \text{for any } x : |x| \leq 3VT. \tag{3.86}$$

Since ξ_0 is an arbitrary point in $\Sigma^{(j)}$, (3.81) in the region $W_{3VT}(\lambda^{-1}\Sigma^{(j)})$ follows from (3.86).

In the sequel the local frame is fixed, and we denote by $\mathcal{B}_a$ the ball of radious a centered in the origin.

Estimate of $m^+(r,jT)$. We call

$$\bar{R}_\lambda \doteq R_\lambda - 1$$

and we prove that there are $\lambda_0 \in (0,1]$ and a positive constant C so that the following holds. We define

$$n(r,0) \doteq \bar{m}(x) + u_0^{(\lambda)}(r), \tag{3.87}$$

where

$$\begin{aligned} u_0^{(\lambda)}(r) = &\Big[\lambda\Big(\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\leq 4VT} + Q_1^{(T)}(x,\underline{\kappa})\Big) \\ &+ \lambda^2\Big(\bar{\omega}_2(r,\underline{\kappa},\underline{c})\mathbf{1}_{|y|\leq 4VT} + Q_2^{(T)}(x,\underline{\kappa})\Big) \\ &+ \lambda^2\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\leq 4VT}\frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x) + C\lambda^3T^9\Big]\bar{m}'(x)\mathbf{1}_{|x|\leq\bar{R}_\lambda} \\ &+ \Big[\mathrm{sign}(x)m_\beta - \bar{m}(x) + \lambda^{3+1/10}\Big]\mathbf{1}_{|x|>\bar{R}_\lambda}. \end{aligned} \tag{3.88}$$

We prove that there is λ_0 so that if $\lambda \in (0,\lambda_0]$ then

$$m^+(r,jT) \leq n(r,0) \qquad \text{for all } r \in \mathcal{B}_{4VT}. \tag{3.89}$$

To prove (3.89) we use Lemma 3.2 with $\Sigma = \Sigma^{(j)}$ and $A = 4V$. Let $\underline{\kappa} = \underline{\kappa}(\xi_0, \Sigma^{(+h)}_{\lambda^2 jT})$ and $\underline{c} = \underline{c}(\xi_0, \Sigma^{(+h)}_{\lambda^2 jT})$ as in this lemma. For any $r = (x,y) \in \mathcal{B}_{4VT}$ we define

$$A_\lambda(x,y) = \bar{m}(d_+(r,jT)) - \bar{m}(x) - \big[\lambda\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT} + \lambda^2\bar{\omega}_2(r,\underline{\kappa},\underline{c})\mathbf{1}_{|y|\le 4VT}\big]\bar{m}'(x). \tag{3.90}$$

From Lemma 3.2 and (2.8) we get that there is a constant $c_1 > 0$ so that

$$\sup_{r\in\mathcal{B}_{4VT}} \big|A_\lambda(x,y)\big| \le c_1\lambda^3 T^6 \bar{m}'(x). \tag{3.91}$$

We next bound the term with $Q_1^{(T)}$. Using again Lemma 3.2 and (2.8) togheter with (3.22), (3.70), (3.71) and (3.72) we get that there is a positive constant c_2 so that, for any $r \in \mathcal{B}_{4VT}$,

$$\Big|\bar{m}'(d_+(r,jT))Q_1^{(T)}(d_+(r,jT),\underline{\kappa}_+(r,jT)) - \bar{m}'(x)Q_1^{(T)}(x,\underline{\kappa}) - \lambda\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)\Big| \le c_2\lambda^2T^5\bar{m}'(x). \tag{3.92}$$

Analogously we bound the term with $Q_2^{(T)}$ and using (2.8), (3.6), (3.22), (3.75) and (3.76) we get that there is a positive constant c_3 so that, for any $r \in \mathcal{B}_{4VT}$,

$$\Big|\bar{m}'(d_+(r,jT))Q_2^{(T)}(d_+(r,jT),\underline{\kappa}(r,jT)) - \bar{m}'(x)Q_2^{(T)}(x,\underline{\kappa})\Big| \le c_3\lambda T^9\bar{m}'(x). \tag{3.93}$$

Notice that from Lemmas 3.2 and 3.5 all the constants which appear in the previous estimates can be chosen independently on $\xi_0 \in \Sigma^{(j)}$ and j for all j so that $(j+1)T \le \lambda^{-2}\tau_0$.

From (3.90), (3.91), (3.92) and (3.93) we conclude that there is $c_4 > 0$ such that, for any $r \in \mathcal{B}_{4VT}$,

$$\Big|\bar{m}(d_+(r,jT)) + \lambda Q_1^{(T)}(d_+(r,jT),\underline{\kappa}_+(r,jT))\bar{m}'(d_+(r,jT)) + \lambda^2 Q_2^{(T)}(d_+(r,jT),\underline{\kappa}_+(r,jT))\bar{m}'(d_+(r,jT)) - \bar{m}(x) - \lambda Q_1^{(T)}(x,\underline{\kappa})\bar{m}'(x) - \lambda^2 Q_2^{(T)}(x,\underline{\kappa})\bar{m}'(x) - \Big[\lambda\omega_1(y,\underline{\kappa}) + \lambda^2\bar{\omega}_2(r,\underline{\kappa},\underline{c})\Big]\mathbf{1}_{|y|\le 4VT}\bar{m}'(x) - \lambda^2\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)\Big| \le c_4\lambda^3T^9\bar{m}'(x). \tag{3.94}$$

We now prove that for $C = c_4$ in (3.88) the inequality (3.89) holds if λ is small enough. We analyze two different cases:

a) Let $r = (x, y) \in \mathcal{B}_{4VT}$ such that $|x| \le \bar{R}_\lambda$. Then from (3.6) it follows that $|d_+(r, jT)| \le |x| + c_5 \lambda T^2$ for some constant $c_5 > 0$, so that, for λ small enough, $|d_+(r, jT)| \le R_\lambda$. By the definition (3.69) and the bound (3.94) the inequality $m^+(r, jT) \le n(r, 0)$ follows immediately.

b) Let $r = (x, y) \in \mathcal{B}_{4VT}$ such that $|x| > \bar{R}_\lambda$. Then from (3.6) it follows that $|d_+(r, jT)| \ge |x| - c_6 \lambda T^2 \ge R_\lambda - 1 - c_6 \lambda T^2$ for some $c_6 > 0$. If $|d_+(r, jT)| > R_\lambda$ by the definitions (3.69), (3.87) and (3.88), $m^+(r, jT) = n(r, 0)$. If $R_\lambda - 1 - c_6 \lambda T^2 \le |d_+(r, jT)| \le R_\lambda$, using, (2.7), (2.8) and the bounds (3.70) and (3.75) we have, for some constant $c_7 > 0$ and any λ small enough,

$$\begin{aligned} m^+(r, jT) &\le \text{sign}(d_+(r, jT)) m_\beta + C_1 \lambda^{\alpha R_0} + c_7 \lambda^{1 + \alpha R_0} T \\ &\le \text{sign}(d_+(r, jT)) m_\beta + \lambda^{3 + 1/10} = n(r, 0) \end{aligned}$$

(recall that $\alpha R_0 > 3 + 1/10$).

Evolution of $m^+(r, jT)$. We define

$$\tilde{m}(r, 0) \doteq m^+(r, jT) \mathbf{1}_{|r| \le 4VT} + n(r, 0) \mathbf{1}_{|r| > 4VT}. \tag{3.95}$$

By (3.89) $\tilde{m}(r, 0) \le n(r, 0)$ and $\tilde{m}(r, 0) = m^+(r, jT)$ for $r \in \mathcal{B}_{4VT}$. Using the Barrier Lemma and the Comparison Theorem we conclude that:

$$S_T\big(m^+(\cdot, jT)\big)(r) \le n(r, T) + \bar{C} e^{-T} \qquad \text{for any } r \in \mathcal{B}_{3VT}, \tag{3.96}$$

where $n(r, t) \doteq S_t\big(n(\cdot, 0)\big)(r)$. We are now going to estimate from above $n(r, T)$ for $r = (x, 0)$ and $|x| \le 3VT$.

Let

$$u(r, t) \doteq n(r, t) - \bar{m}(x). \tag{3.97}$$

Then $u(r, t)$ solves the equation (3.30) with initial datum $u_0^{(\lambda)}(r)$ defined in (3.88). In analogy with (3.26) we write

$$u_0^{(\lambda)}(r) = \lambda \Omega_1(r, 0) \bar{m}'(x) + \lambda^2 \Omega_2(r, 0) \bar{m}'(x) + U_0^{(\lambda)}(r) \tag{3.98}$$

with

$$\Omega_1(r, 0) \doteq \omega_1(y, \underline{\kappa}) + Q_1^{(T)}(x, \underline{\kappa}), \tag{3.99}$$

$$\Omega_2(r,0) \doteq \bar{\omega}_2(r,\underline{\kappa},\underline{c}) + Q_2^{(T)}(x,\underline{\kappa}) + \omega_1(y,\underline{\kappa})\frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x), \tag{3.100}$$

$$\begin{aligned} U_0^{(\lambda)}(r) &\doteq C\lambda^3T^9\bar{m}'(x)\mathbf{1}_{|x|\le \bar{R}_\lambda} + \psi_\lambda(r)\bar{m}'(x) \\ &+ \big[\mathrm{sign}(x)m_\beta - \bar{m}(x) + \lambda^{3+1/10}\big]\mathbf{1}_{|x|>\bar{R}_\lambda} \\ &- \Big[\lambda\big(\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT} \\ &+ Q_1^{(T)}(x,\underline{\kappa})\big) + \lambda^2\big(\bar{\omega}_2(y,\underline{\kappa},\underline{c})\mathbf{1}_{|y|\le 4VT} + Q_2^{(T)}(x,\underline{\kappa})\big) \\ &+ \omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT}\frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)\Big]\bar{m}'(x)\mathbf{1}_{|x|>\bar{R}_\lambda}, \end{aligned} \tag{3.101}$$

where

$$\begin{aligned} \psi_\lambda(r) &\doteq -\lambda\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|>4VT} - \lambda^2\bar{\omega}_2(r,\underline{\kappa},\underline{c})\mathbf{1}_{|y|>4VT} \\ &- \lambda^2\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|>4VT}\frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x). \end{aligned} \tag{3.102}$$

In analogy with equations (3.31)–(3.35) we write

$$u(r,t) = \lambda\Omega_1(r,t)\bar{m}'(x) + \lambda^2\Omega_2(r,t)\bar{m}'(x) + U^{(\lambda)}(r,t) \tag{3.103}$$

with Ω_i, $i=1,2$ solutions of

$$\partial_t\Omega_1 = \mathcal{L}\Omega_1 \tag{3.104}$$

$$\bar{m}'\partial_t\Omega_2 = L\Omega_2 + \frac{\Phi}{2}\big(J\star(\bar{m}'\Omega_1)\big)^2 \tag{3.105}$$

$$\partial_t U^{(\lambda)} = LU^{(\lambda)} + \mathcal{C}_\lambda(U^{(\lambda)}), \tag{3.106}$$

where $\mathcal{C}_\lambda$ is defined in (3.36).

We compute $u(r,t)$ for $r=(x,0)$ and $|x|\le 3VT$ and we estimate the various terms separately.

Estimate of $\Omega_1\big((x,0),T\big)$. Using the notation of subsection 2.6 from (3.99) and (3.104) it follows that

$$\Omega_1\big((x,0),T\big) = \mathbb{E}_{(x,0)}\big[\omega_1(Y_T,\underline{\kappa})\big] + \mathbb{E}_x^{(1)}\big[Q_1^{(T)}(X_T,\underline{\kappa})\big]. \tag{3.107}$$

Since $Q_1^{(T)} = F_T$, from (3.74) we have that the second term on the right hand side of (3.107) is negligible small. For the first we use that

$$\mathbb{E}_{(x,0)}\big[\omega_1(Y_T,\underline{\kappa})\big] = -\theta\kappa T + Q_1^{(T)}(x,\underline{\kappa}). \tag{3.108}$$

Therefore,

$$\big|\bar{m}'(x)\Omega_1\big((x,0),T\big) + \theta\kappa T\bar{m}'(x) - \bar{m}'(x)Q_1^{(T)}(x,\underline{\kappa})\big| \le \bar{c}_1 e^{-c_1' T}. \tag{3.109}$$

Estimate of $\Omega_2\big((x,0),T\big)$. We use the same argument as in (3.41)–(3.46), we then write

$$\begin{aligned}\bar{m}'\partial_t\Omega_2 &= L\Omega_2 - \frac{1}{2}(\theta\kappa T)^2 L^{(1)}\bar{m}'' \\ &+ \frac{\Phi}{2}\Big[\big(J\star(m'\Omega_1)\big)^2 - \big(J\star(\bar{m}'\mathbb{E}_{\cdot}[\omega_1(Y_t,\underline{\kappa})])\big)^2\Big] + \hat{\mathcal{G}}\bar{m}',\end{aligned} \tag{3.110}$$

where $\hat{\mathcal{G}}$ is defined in (3.57). We solve (3.110) using (3.44) and (3.59), from the definitions (3.56), (3.66) and (3.67) we then get

$$\begin{aligned}\Omega_2(r,T) &= \frac{1}{2}(\theta\kappa T)^2\frac{\bar{m}''(x)}{\bar{m}'(x)} + \big[\mathcal{G}_T(r,\underline{\kappa},\underline{c}) + \mathcal{H}_T(r,\underline{\kappa})\big] \\ &- \frac{1}{2}(\theta\kappa T)^2\frac{(g_T^{(1)}\star\bar{m}'')(x)}{\bar{m}'(x)} + \mathbb{E}_r\big[Q_2^{(T)}(X_T,\underline{\kappa})\big] + H_1^{(T)}(r,\underline{\kappa}).\end{aligned} \tag{3.111}$$

Using (3.47), (3.78) and the definition of $Q_2^{(T)}$, we get that there are c_8 and c_8' so that

$$\begin{aligned}&\big|\bar{m}'(x)\Omega_2((x,0),T) - \frac{1}{2}(\theta\kappa T)^2\bar{m}''(x) - Q_2^{(T)}(x,\underline{\kappa})\bar{m}'(x) - H_T^{(0)}(x,\underline{\kappa})\bar{m}'(x)\big| \\ &\quad\le c_8 e^{-c_8' T}.\end{aligned} \tag{3.112}$$

Estimate of $U^{(\lambda)}\big((x,0),T\big)$. We rewrite (3.106) in integral form in the following way:

$$U^{(\lambda)}(r,t) = \big(g_t\star U_0^{(\lambda)}\big)(r) + \int_0^t ds\,\big(g_{t-s}\star B_s[u_0^{(\lambda)}]\big)(r) + \int_0^t ds\,\big(g_{t-s}\star R_s\big)(r), \tag{3.113}$$

where

$$B_s[u_0^{(\lambda)}](r) \doteq \Phi(x)\big[\big(J\star(g_s\star u_0^{(\lambda)})\big)^2(r) - \big(J\star(\bar m'\mathbb{E}.[\lambda\Omega_1(r_s,0)])\big)^2(r)\big] \tag{3.114}$$

and

$$\begin{aligned} R_s(r) \doteq \tanh\big\{\beta\big(J\star(\bar m+u)\big)(r,s)\big\} - \bar m(x) - Lu(r,s) \\ - \frac{\Phi(x)}{2}\big[(J\star u)^2(r,s) - (J\star(g_s\star u_0^{(\lambda)}))^2(r)\big]. \end{aligned} \tag{3.115}$$

We estimate the three terms on the right hand side of (3.113) separately starting from the first one.

Bound on $\big(g_T\star U_0^{(\lambda)}\big)\big((x,0)\big)$. Since $\alpha R_0 > 3+1/10$, from (2.7) we get

$$\big|\mathrm{sign}(x)m_\beta - \bar m(x)\big|\mathbf{1}_{|x|>\bar R_\lambda} \le c_9\lambda^{\alpha R_0}\mathbf{1}_{|x|>\bar R_\lambda} \le \lambda^{3+1/10}\mathbf{1}_{|x|>\bar R_\lambda}, \tag{3.116}$$

and from (2.8) we get that for any $p>0$ there is c_p so that

$$\sup_{|x|>R_\lambda}(1+|x|^p)\bar m'(x) \le c_p\lambda^{\alpha R_0}R_\lambda^p. \tag{3.117}$$

From (3.117), (3.70), (3.71) and (3.75) it follows that there is $c_9>0$ so that, for any λ small enough (recall that $R_\lambda\sim\sqrt{T}$ as $\lambda\to 0$),

$$\begin{aligned} \lambda\big|\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT} + Q_1^{(T)}(x,\underline{\kappa})\big|\bar m'(x)\mathbf{1}_{|x|>\bar R_\lambda} \\ \le c_9\lambda^{1+\alpha R_0}T^2\mathbf{1}_{|x|>\bar R_\lambda} \le \lambda^{3+1/10}\mathbf{1}_{|x|>\bar R_\lambda} \end{aligned} \tag{3.118}$$

and

$$\begin{aligned} &\lambda^2\bigg|\bar\omega_2(r,\underline{\kappa},\underline{c})\mathbf{1}_{|y|\le 4VT} + Q_2^{(T)}(x,\underline{\kappa}) \\ &+\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT}\frac{1}{\bar m'(x)}\frac{d}{dx}\Big(\bar m'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)\bigg|\bar m'(x)\mathbf{1}_{|x|>\bar R_\lambda} \\ &\le c_9\lambda^{2+\alpha R_0}T^4\mathbf{1}_{|x|>\bar R_\lambda} \le \lambda^{3+1/10}\mathbf{1}_{|x|>\bar R_\lambda}. \end{aligned} \tag{3.119}$$

Therefore, for small λ,

$$U_0^{(\lambda)}(r) \le C\lambda^3T^9\bar m'(x)\mathbf{1}_{|x|\le\bar R_\lambda} + \psi_\lambda(r)\bar m'(x) + 3\lambda^{3+1/10}\mathbf{1}_{|x|>\bar R_\lambda}. \tag{3.120}$$

In Lemma 4.3 (see Section 4) we will prove that there is a constant $c_{10} > 0$ so that

$$\left|\left(g_T \star (m'\psi_\lambda)\right)\left((x,0)\right)\right| = \bar{m}'(x)\left|\mathbb{E}_{(x,0)}[\psi_\lambda(r_T)]\right| \le c_{10} e^{-T} \tag{3.121}$$

and in [15] it is proven that, for some constant $c_{11} > 0$

$$\left|\left(g_T \star \mathbf{1}_{|\cdot|>\bar{R}_\lambda}\right)\left((x,0)\right)\right| \le c_{11}\lambda^{\alpha R_0}, \qquad \text{for all } |x| \le \bar{R}_\lambda. \tag{3.122}$$

By (3.120), (3.121) and (3.122), recalling that $L\bar{m}'(x) = 0$, we finally get, for some $c_{12} > 0$,

$$g_T \star U_0^{(\lambda)}\left((x,0)\right) \le C\lambda^3 T^9 \bar{m}'(x) + c_{12}\lambda^{3+1/10+\alpha R_0}. \tag{3.123}$$

Bound on $1/2\int_0^T ds\, \left(g_{T-s} \star B_s[u_0^{(\lambda)}]\right)\left((x,0)\right)$. We write

$$u_0^{(\lambda)}(r) = \lambda\Omega_1(r,0)\bar{m}'(x) + \bar{\psi}_\lambda(y)\bar{m}'(x) + I_\lambda(r) \tag{3.124}$$

with

$$\bar{\psi}_\lambda(y) \doteq -\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|>4VT} \tag{3.125}$$

and then

$$I_\lambda(r) = u_0^{(\lambda)}(r) - \lambda\Omega_1(r,0)\bar{m}'(x) - \bar{\psi}_\lambda(y)\bar{m}'(x). \tag{3.126}$$

By arguing as made to obtain (3.120) we easily get

$$|I_\lambda(r)| \le c_{13}\lambda^2 T^4 \bar{m}'(x) + c_{14}\lambda^{3+1/10}\mathbf{1}_{|x|>\bar{R}_\lambda} \tag{3.127}$$

for some positive constants c_{13} and c_{14}.

Recalling the definition (3.114) of $B_s[u_0^{(\lambda)}](r)$ and using (3.124) we get

$$\frac{1}{2}\int_0^T ds\, \left(g_{T-s} \star B_s[u_0^{(\lambda)}]\right)\left((x,0)\right) = \sum_{i=0}^{4} K_\lambda^{(i)}(x), \tag{3.128}$$

where

$$K_\lambda^{(0)}(x) \doteq \frac{1}{2}\int_0^T ds\, \left(g_{T-s} \star \left[\Phi\left(J \star (g_s \star I_\lambda)\right)^2\right]\right)\left((x,0)\right),$$

$$K_\lambda^{(1)}(x) \doteq \lambda\int_0^T ds\, \left(g_{T-s} \star \left[\Phi\left(J \star (g_s \star I_\lambda)\right)\right.\right.$$
$$\left.\left.\times \left(J \star (g_s \star (\bar{m}'\Omega_1(\cdot,0)))\right)\right]\right)\left((x,0)\right),$$

$$
\begin{aligned}
K_\lambda^{(2)}(x) &\doteq \frac{\lambda^2}{2}\int_0^T ds\,\big(g_{T-s}\star\big[\Phi\big(J\star(g_s\star(\bar m'\bar\psi_\lambda))\big)^2\big]\big)((x,0)),\\
K_\lambda^{(3)}(x) &\doteq \lambda\int_0^T ds\,\big(g_{T-s}\star\big[\Phi\big(J\star(g_s\star I_\lambda)\big)\big(J\star(g_s\star(\bar m'\bar\psi_\lambda))\big)\big]\big)((x,0)),\\
K_\lambda^{(4)}(x) &\doteq \lambda\int_0^T ds\,\big(g_{T-s}\star\big[\Phi\big(J\star(g_s\star(\bar m'\bar\psi_\lambda))\big)\\
&\quad\times\big(J\star(g_s\star(\bar m'\Omega_1(\cdot,0)))\big)\big]\big)((x,0)).
\end{aligned}
\tag{3.129}
$$

In Section 4 we will prove that, for some constant $c_{15} > 0$,

$$
\begin{aligned}
|K_\lambda^{(0)}(x)| &\le c_{15}\big(\lambda^4 T^{11}\bar m'(x) + \lambda^{6+1/5}T\big),\\
|K_\lambda^{(1)}(x)| &\le c_{15}\lambda^3 T^7(1+|x|)\bar m'(x),\\
\sum_{i=2}^4 |K_\lambda^{(i)}(x)| &\le c_{15}e^{-T}T.
\end{aligned}
\tag{3.130}
$$

Then there is a constant $c_{16} > 0$ such that, for $|x| \le 3VT$,

$$
\frac{1}{2}\int_0^T ds\,\big(g_{T-s}\star B_s[u_0^{(\lambda)}]\big)((x,0)) \le c_{16}\big(\lambda^3 T^8\bar m'(x) + \lambda^{6+1/5}T\big). \tag{3.131}
$$

Bound on $1/2\int_0^T ds\,\big(g_{T-s}\star R_s\big)((x,0))$**.** We need an upper bound for $u(r,t)$ when $t\in[0,T]$. We observe that $u(r,t)$ solves the integral equation

$$
u(r,t) = \big(g_t\star u_0^{(\lambda)}\big)(r) + \int_0^t ds\,\big(g_{t-s}\star\Psi_s\big)(r), \tag{3.132}
$$

where

$$
\Psi_s(r) \doteq \tanh\{\beta\big(J\star(\bar m+u)\big)(r,s)\} - \bar m(x) - Lu(r,s). \tag{3.133}
$$

Moreover, by the previous estimates, there are positive constants b_1 and b_2 so that

$$
|u_0^{(\lambda)}(r)| \le b_1\lambda T^2\bar m'(x) + b_2\lambda^{3+1/10}. \tag{3.134}
$$

In [15] it is proven that there is $c_0 >$ so that

$$
\sup_{t\ge 0}\sup_{r\in\mathbb{R}^n}\big|(g_t\star 1)(r)\big| \le c_0. \tag{3.135}
$$

Let $c = 1 + b_1 \vee (b_2 c_0)$ and

$$t_c \doteq \inf\{\, t \geq 0 \, : \, |u(r,t)| > c(\lambda T^2 \bar{m}'(x) + \lambda^{3+1/10}) \;\; \forall\, r \in \mathbb{R}^n \,\}. \tag{3.136}$$

We are going to prove that for any λ small enough $t_c > T$. Observe that if λ is small enough, $\|u(\cdot,s)\|_\infty \leq 1$ for all $s \leq t_c$. Then, expanding Ψ_s up to the second order in $u(\cdot,s)$, we get that there is a positive constant $b > 0$ such that, for any λ small enough and $s \leq t_c$,

$$|\Psi_s(r)| \leq b(J \star u)^2(r,s). \tag{3.137}$$

Then, by the positivity of the kernel g_{t-s}, (3.132), (3.134), (3.135) and (3.137) we get

$$|u(r,t)| \leq b_1 \lambda T^2 \bar{m}'(x) + b_2 c_0 \lambda^{3+1/10} + b \int_0^t ds\, \big(g_{t-s} \star (J \star u)^2\big)(r,s) \tag{3.138}$$

for any λ small enough and any $t \leq t_c$. But it is easy to prove that, for some positive constant $c^\star$,

$$|\tilde{J} \star \bar{m}'(x)| \leq c^\star \bar{m}'(x). \tag{3.139}$$

From (3.139) and the definition of t_c, we easily obtain that, for λ small enough,

$$|u(r,t)| \leq f_t^{(\lambda)}(x) \qquad \forall\, t \in [0, t_c], \tag{3.140}$$

where $f_t^{(\lambda)}(x)$ is a linear function of t of the form

$$f_t^{(\lambda)}(x) = \big[b_1 \lambda T^2 + b_3 \lambda^2 T^4 t\big] \bar{m}'(x) + \big[b_2 c_0 \lambda^{3+1/10} + b_4 \lambda^{6+1/5} t\big] \tag{3.141}$$

for suitable $b_3, b_4 > 0$. Let us suppose that $T > t_c$. Then $f_{t_c}^{(\lambda)}(x) \leq f_T^{(\lambda)}(x) < |u_{t_c}(r)|$, for any λ small enough, which contradicts (3.140) by continuity of $u(r,t)$. Then, for any λ small enough, $T \leq t_c$ so that

$$|u(r,t)| \leq c(\lambda T^2 \bar{m}'(x) + \lambda^{3+1/10}) \qquad \forall\, r \in \mathbb{R}^n, \quad \forall\, t \in [0,T]. \tag{3.142}$$

Since $\|u(\cdot,s)\|_\infty \leq 1$ for $s \leq T$, by (3.115) we get, for some $c_{17} > 0$,

$$|R_s(r)| \leq c_{17} \Big[\big| (J \star u)^2(r,s) - \big(J \star (g_s \star u_0^{(\lambda)})\big)^2(r) \big| + \big| (J \star u)^3(r,s) \big| \Big]. \tag{3.143}$$

We rewrite

$$
\begin{aligned}
\big(J\star u\big)^2(r,s) - \big(J\star(g_s\star u_0^{(\lambda)})\big)^2(r) &= \big(J\star(u(\cdot,s)-g_s\star u_0^{(\lambda)})\big)^2(r) \\
&+ 2\big(J\star(u(\cdot,s)-g_s\star u_0^{(\lambda)})\big)(r)\big(J\star(g_s\star u_0^{(\lambda)})\big)(r).
\end{aligned}
\tag{3.144}
$$

By (3.139), (3.142) and the integral equation (3.132) there is a positive constants c_{18} such that, for any $s\le T$,

$$
\begin{aligned}
&\big|(J\star u)(r,s)\big| \le c_{18}(\lambda T^2\bar m'(x)+\lambda^{3+1/10}),\\
&\big|\big(J\star(g_s\star u_0^{(\lambda)})\big)(r)\big| \le c_{18}(\lambda T^2\bar m'(x)+\lambda^{3+1/10}),\\
&\big|(u(\cdot,s)-g_s\star u_0^{(\lambda)})(r)\big| \le c_{18}(\lambda^2T^5\bar m'(x)+\lambda^{6+1/5})
\end{aligned}
\tag{3.145}
$$

and then, by (3.143) and (3.145), there is $c_{19}>0$ so that

$$
|R_s(r)| \le c_{19}(\lambda^3T^7\bar m'(x)+\lambda^{9+3/10}) \qquad \forall\, s\in[0,T]
$$

and by (3.135), we finally get

$$
\frac12\int_0^T ds\,\big(g_{T-s}\star R_s\big)\big((x,0)\big) \le c_0c_{19}(\lambda^3T^8\bar m'(x)+\lambda^{9+3/10}T) \tag{3.146}
$$

Upper bound for $S_T\big(m^+(\cdot,jT)\big)\big((x,0)\big)$. Collecting all the previuos estimates, we have proven that

$$
\begin{aligned}
S_T(m^+(\cdot,jT))(x,0) &\le \bar m(x) - \theta\kappa\lambda T\bar m'(x) + \frac{(\theta\kappa\lambda T)^2}{2}\bar m''(x)\\
&+ \Big[\lambda Q_1^{(T)}(x,\underline{\kappa}) + \lambda^2\big(Q_2^{(T)}(x,\underline{\kappa}) - H_T^{(0)}(x,\underline{\kappa})\big)\Big]\bar m'(x)\\
&+ \lambda\bar c_1 e^{-c_1'T} + \lambda^2 c_8 e^{-c_8'T} + C\lambda^3T^9\bar m'(x) + c_{12}\lambda^{3+1/10+\alpha R_0}\\
&+ c_{16}(\lambda^3T^8\bar m'(x)+\lambda^{6+1/5}T) + c_0c_{19}(\lambda^3T^8\bar m'(x)+\lambda^{9+3/10}T) + \bar C e^{-T}.
\end{aligned}
\tag{3.147}
$$

Using (3.1) and the fact that $\alpha R_0 > 3+1/10$, we get that there are constants M_1 and M_2 so that, for λ small enough,

$$
S_T\big(m^+(\cdot,jT)\big)\big((x,0)\big) \le M_\lambda(x) \qquad \forall\,|x|\le 3VT, \tag{3.148}
$$

where

$$M_\lambda(x) = \bar{m}(x) - \theta\kappa\lambda T\bar{m}'(x) + \frac{(\theta\kappa\lambda T)^2}{2}\bar{m}''(x)$$
$$+ \Big[\lambda Q_1^{(T)}(x,\underline{\kappa}) + \lambda^2\big(Q_2^{(T)}(x,\underline{\kappa}) - H_T^{(0)}(x,\underline{\kappa})\big)\Big]\bar{m}'(x)$$
$$+ M_1\lambda^3T^9\bar{m}'(x) + M_2\lambda^{6+1/5}T. \tag{3.149}$$

Lower bound for $m^+((x,0),(j+1)T)$. Let

$$d_1 = d_1(x) \doteq d((x,0),\lambda^{-1}\Sigma^{(j+1)})$$

and

$$\underline{\kappa}_1 = \underline{\kappa}_1(x) \doteq \underline{\kappa}_+((x,0),(j+1)T).$$

From (3.24), for any λ small enough and $|x| \le 3VT$,

$$|d_1 - x + (\theta\kappa - h)\lambda T| \le \bar{c}\lambda^3T^2. \tag{3.150}$$

From (3.150) it follows that, for λ small enough,

$$\begin{cases} |x| \ge R_\lambda + 1 & \Rightarrow |d_1| \ge R_\lambda \\ |x| \le R_\lambda - 1 & \Rightarrow |d_1| \le R_\lambda. \end{cases} \tag{3.151}$$

Moreover, from (3.25) and arguments similar to the previous ones, one easily proves that there is a positive constant c_{20} so that, for $|x| \le R_\lambda + 1$,

$$\Big|\bar{m}(d_1) + \lambda Q_1^{(T)}(d_1,\underline{\kappa}_1)\bar{m}'(d_1) + \lambda^2Q_2^{(T)}(d_1,\underline{\kappa}_1)\bar{m}'(d_1) - \bar{m}(x)$$
$$- \lambda Q_1^{(T)}(x,\underline{\kappa})\bar{m}'(x) - \lambda^2Q_2^{(T)}(x,\underline{\kappa})\bar{m}'(x) + (\theta\kappa - h)\lambda T\bar{m}'(x)$$
$$- \frac{(\theta\kappa\lambda T)^2}{2}\bar{m}''(x) + \theta\kappa\lambda^2T\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)\Big| \le c_{20}\lambda^3T^5\bar{m}'(x). \tag{3.152}$$

Recalling the definition of $H_T^{(0)}$, from (3.152) and the fact that $\lim_{\lambda\to 0} \lambda^{-(3+1/10)}\,\bar{m}'(R_\lambda - 1) = 0$, we easily get, for λ small enough,

$$m^+((x,0),(j+1)T) \ge N_\lambda(x), \tag{3.153}$$

where

$$N_\lambda(x) \doteq \Big[\bar{m}(x) - (\theta\kappa - h)\lambda T \bar{m}'(x) + \frac{(\theta\kappa\lambda T)^2}{2}\bar{m}''(x)$$
$$+ \Big[\lambda Q_1^{(T)}(x,\underline{\kappa}) + \lambda^2\big(Q_2^{(T)}(x,\underline{\kappa}) - H_T^{(0)}(x,\underline{\kappa})\big) - c_{20}\lambda^3 T^5\Big]\bar{m}'(x)\Big]\mathbf{1}_{|x|\le R_\lambda+1}$$
$$+ \Big[\operatorname{sign}(x) m_\beta + \lambda^{3+1/10}\Big]\mathbf{1}_{|x|>R_\lambda+1}. \tag{3.154}$$

We now prove that

$$M_\lambda(x) \le N_\lambda(x) \qquad \forall\, |x| \le 3VT. \tag{3.155}$$

We prove (3.155) separately in the two different cases:

i) $|x| \le R_\lambda + 1$. Comparing the forms of the functions $M_\lambda(x)$ and $N_\lambda(x)$ for these values of x, and using the fact that $\sup_{|x|\le R_\lambda+1} m'(x)^{-1} \le \lambda^{-\alpha R_0}$ we get that (3.155) is implied by

$$h\lambda T \ge c_{20}\lambda^3 T^5 + M_1 \lambda^3 T^9 + M_2 \lambda^{6+1/5-\alpha R_0} \tag{3.156}$$

which is true, for small λ, because $h \ge \lambda^2 T^{10}$ and R_0 is such that $\alpha R_0 < 3 + 1/5$.

ii) $R_\lambda + 1 < |x| \le 3VT$. Now we need that for λ small enough,

$$M_\lambda(x) \le \operatorname{sign}(x) m_\beta + \lambda^{3+1/10}. \tag{3.157}$$

The inequality (3.157) can be easily proven using the estimates (3.70) and (3.75) together with the bound (3.117) and recalling that R_0 is such that $\alpha R_0 > 3 + 1/10$.

We have thus concluded the proof of (3.81). The proof of (3.82) is completely analogous and it is omitted.

The proof of (3.79) and (3.80) in W_{3VT} is analogous to the previuos one with a different second order shape correction in the initial datum, but it does not change the form of the bound M_λ in (3.148). We omit the details, the Proposition is therefore proven. □

Proof of Proposition 3.9. The proof is very similar to the one of the previous Proposition. Again one proves the estimates separately into the two regions $U_{2VT}(\lambda^{-1}\Sigma^{(0)})$ and $W_{3VT}(\lambda^{-1}\Sigma^{(0)})$. We just make some remarks.

The proof in U_{2VT} follows easily comparing the solution in $r \in U_{2VT}$ with the one starting from the function

$$\tilde{m}(r') \doteq \bar{m}(d(r, \lambda^{-1}\Sigma_0))\mathbf{1}_{\mathcal{B}_{VT}(r)}(r') + \bar{m}(\operatorname{sign}(d(r, \lambda^{-1}\Sigma_0))VT)\mathbf{1}_{\mathbb{R}^n \setminus \mathcal{B}_{VT}(r)}(r'). \tag{3.158}$$

The proof in W_{3VT} is exactly the same as the one for (3.81) but without the shape corrections $Q_i^{(T)}$, $i = 1, 2$; the definition (3.60) is such that the first and second order powers in λ cancel in this case, (see also the subsection 3.6). □

Proof of Theorem 2.5. For any choice of $\chi \in [1,2]$ let us consider the time grid

$$\mathcal{T}_\chi \doteq \{\, t = jT \,:\, j \in \mathbb{N}_+,\, T = \chi|\log\lambda|^2,\, 2T \le t \le \lambda^{-2}\tau_0 \,\}.$$

From Propositions 3.9, 3.13 and the Comparison Theorem it follows that there is $\lambda_1 \in (0,1]$ such that, for any $\lambda \in (0, \lambda_1]$ and $\chi \in [1,2]$ the solution m of the initial value problem (see subsection 2.4) verifies

$$m^-(r,t) \le m(r,t) \le m^+(r,t) \qquad \forall\, r \in \mathbb{R}^n \quad \forall\, t \in \mathcal{T}_\chi, \tag{3.159}$$

where $m^\pm(r,t)$ are defined as in (3.69) with $T = \chi|\log\lambda|^2$.

Observe that, from (3.19) and recalling the definition (3.1) of h, it follows that

$$|d(\xi, \Sigma_\tau^{(\pm h)}) - d(\xi, \Sigma_\tau)| \le c\lambda^2|\log\lambda|^{20} \qquad \forall\, \xi \in \mathbb{R}^n. \tag{3.160}$$

We also have

$$d(\xi, \Sigma_\tau^{(-h)}) \le d(\xi, \Sigma_\tau) \le d(\xi, \Sigma_\tau^{(+h)}) \qquad \forall\, \xi \in \mathbb{R}^n. \tag{3.161}$$

Therefore, from the definition of the barrier functions $m^\pm$, Lemma 3.12 and (3.159), we get that there is a positive constant C_1 so that

$$\sup_{\chi \in [1,2]} \sup_{t \in \mathcal{T}_\chi} \|m(\cdot,t) - \bar{m}(d(\cdot, \lambda^{-1}\Sigma_{\lambda^2 t}))\|_\infty \le C_1\lambda|\log\lambda|^{20},$$

that is,

$$\sup_{2T \le t \le \lambda^{-2}\tau_0} \|m(\cdot,t) - \bar{m}(d(\cdot, \lambda^{-1}\Sigma_{\lambda^2 t}))\|_\infty \le C_1\lambda|\log\lambda|^{20}. \tag{3.162}$$

To prove (2.17) we are left with the bound in the time interval $[0, 2T]$. By studying the evolution in the time interval $[0, 2T]$ as done in the proof of Lemma 3.9, one easily gets, for some constant $C' > 0$,

$$\sup_{0 \le t \le 2T} \|m(\cdot, t) - \bar{m}(d(\cdot, \lambda^{-1}\Sigma_0))\|_\infty \le C'\lambda|\log\lambda|^2. \tag{3.163}$$

On the other hand, for some constant $C'' > 0$,

$$\sup_{0 \le t \le 2T} \|\bar{m}(d(\cdot, \lambda^{-1}\Sigma_{\lambda^2 t})) - \bar{m}(d(\cdot, \lambda^{-1}\Sigma_0))\|_\infty \le C''\lambda|\log\lambda|^2. \tag{3.164}$$

From (3.162), (3.163) and (3.164) we get that there is $\lambda_2 \in (0, \lambda_1]$ so that, for any $\lambda \in (0, \lambda_2]$,

$$\sup_{0 \le t \le \lambda^{-2}\tau_0} \|m(\cdot, t) - \bar{m}(d(\cdot, \lambda^{-1}\Sigma_{\lambda^2 t}))\|_\infty \le C_1\lambda|\log\lambda|^{20}. \tag{3.165}$$

We prove (2.19) and (2.20) separately in the time grid $\lambda^2\mathcal{T}_\chi \doteq \{\tau = \lambda^2 t : t \in \mathcal{T}_\chi\}$ for any choice of $\chi \in [1, 2]$.

We denote by $m_\pm^{(\lambda)}(\xi, \tau) \doteq m^\pm(\lambda^{-1}\xi, \lambda^{-2}\tau)$ and we observe that from (3.159), for any $\lambda \in (0, \lambda_1]$ and $\chi \in [1, 2]$,

$$m_-^{(\lambda)}(\xi, \tau) \le m^{(\lambda)}(\xi, \tau) \le m_+^{(\lambda)}(\xi, \tau) \qquad \forall \xi \in \mathbb{R}^n \quad \forall \tau \in \lambda^2\mathcal{T}_\chi. \tag{3.166}$$

Let $\xi \in \Sigma_{\lambda,\tau}$. By (3.166) and definition (2.18),

$$m_-^{(\lambda)}(\xi, \tau) \le 0 \le m_+^{(\lambda)}(\xi, \tau). \tag{3.167}$$

But, by (3.160), for some positive constant c_1,

$$|d(\xi', \Sigma_\tau^{(-h)}) - d(\xi', \Sigma_\tau^{(+h)})| \le c_1\lambda^2|\log\lambda|^{20} \qquad \forall \xi' \in \mathbb{R}^n. \tag{3.168}$$

From the definition of $m^\pm$, Lemma 3.12 and (3.167), it follows that there are $\lambda_3 \in (0, \lambda_1]$ and $c_2 > 0$ so that, for any $\lambda \in (0, \lambda_3]$,

$$\bar{m}(\lambda^{-1}d(\xi, \Sigma_\tau^{(-h)})) - c_2\lambda \le 0 \le \bar{m}(\lambda^{-1}d(\xi, \Sigma_\tau^{(+h)})) + c_2\lambda. \tag{3.169}$$

(3.168) and (3.169) imply that there is $\lambda_4 \in (0, \lambda_3]$ such that, for any $\lambda \in (0, \lambda_4]$,

$$\lambda^{-1}|d(\xi, \Sigma_\tau^{(\pm h)})| \le C_2\lambda|\log\lambda|^{20} \tag{3.170}$$

for some $C_2 > 0$ and then from (3.161) we get $|d(\xi, \Sigma_\tau)| \le C_2\lambda^2 |\log\lambda|^{20}$.

Let $\xi \in \Sigma_\tau$, i.e., $d(\xi, \Sigma_\tau) = 0$. By (3.161) and (3.168) it follows that

$$d(\xi, \Sigma_\tau^{(-h)}) \le 0 \le d(\xi, \Sigma_\tau^{(+h)}), \qquad |d(\xi, \Sigma_\tau^{(\pm h)})| \le c_1\lambda^2 |\log\lambda|^{20}. \quad (3.171)$$

If we define $\Sigma_{\lambda,\tau}^{\pm} \doteq \{\, \xi' \in \mathbb{R}^n \,:\, m_{\pm}^{(\lambda)}(\xi', \tau) = 0 \,\}$, (3.171) implies that there is $\lambda_5 \in (0,1]$ such that, for any $\lambda \in (0, \lambda_5]$,

$$|d(\xi, \Sigma_{\lambda,\tau}^{\pm})| \le c_4 \lambda^2 |\log\lambda|^{20} \qquad (3.172)$$

for some positive constant c_4. Call $\xi_\pm$ two points in $\Sigma_{\lambda,\tau}^{\pm}$ such that $|\xi - \xi_\pm| \le c_4\lambda^2|\log\lambda|^{20}$. By continuity there exists a point $\xi_0 \in \Sigma_{\lambda,\tau}$ of the form $\xi_0 = q\xi_- + (1-q)\xi_+$ for some $q \in (0,1)$. Then, for some positive constant C_3, it is $|\xi - \xi_0| \le C_3\lambda^2|\log\lambda|^{20}$ and so $|d(\xi, \Sigma_{\lambda,\tau})| \le C_3\lambda^2|\log\lambda|^{20}$. Collecting togheter the previous results we get Theorem 2.5 with $\lambda_0 = \min\{\lambda_2, \lambda_4, \lambda_5\}$ and $C = \min\{C_1, C_2, C_3\}$. □

4. Proofs. In this section we prove Lemma 3.12 and the bounds (3.121) and (3.130). In the following lemmas we give the keys estimates that will be used in the sequel.

Lemma 4.1. *For any $p_1, p_2 \in \mathbb{N}_+$ there is a constant $C = C(p_1, p_2) > 0$ such that for all $t > 1$,*

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p_1} Y_{2,t}^{2p_2}\right] \le C(1 + t + |x|)t^{p_1+p_2-1} \qquad (4.1)$$

and, if ϕ is a bounded odd function on $\mathbb{R}$,

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p_1} Y_{2,t}^{2p_2}\phi(X_t)\right] \le \begin{cases} C(1 + |\log t|^2 + |x|)\|\phi\|_\infty & \text{if } p_1 + p_2 = 1 \\ C(1+|x|)(1 + |\log t|^2\sqrt{t} + |x|^5)\|\phi\|_\infty & \text{if } p_1 + p_2 = 2. \end{cases} \qquad (4.2)$$

Proof. Observe that the bounds (4.1) and (4.2) are true also for small t, but we will use them for t large.

If $p_1 + p_2 = 1$, arguing as in (2.40), we get

$$\mathbb{E}_{(x,0)}\left[Y_{i,t}^2\right] = \int_0^t ds\, \mathbb{E}_x^{(1)}[f_1(X_s)], \qquad i = 1, 2, \qquad (4.3)$$

where f_1 is is the bounded function defined by

$$f_1(x) \doteq \int_{\mathbb{R}} dx' \int_{\mathbb{R}^{n-1}} dz\, K((x,0),(x',z))z_1^2. \tag{4.4}$$

Then from (2.35) we get

$$\mathbb{E}_{(x,0)}\left[Y_{i,t}^2\right] \le C(1+t+|x|) \tag{4.5}$$

for some positive constant C. The estimate (4.1) for $p_1+p_2>1$ is obtained by induction. We then prove (4.1) for $p_1+p_2=n_0$ assuming it is true for $p_1+p_2<n_0$. Using the definition of the Markov process via its generator $\mathcal{L}$ given in (2.24) we get

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p_1}Y_{2,t}^{2p_2}\right] = \int_0^t ds\, \mathbb{E}_{(x,0)}\Big[\int dr'\, K(r_s,r')\left[(y_1')^{2p_1}(y_2')^{2p_2} - Y_{1,s}^{2p_1}Y_{2,s}^{2p_2}\right]\Big]. \tag{4.6}$$

We write $(y_i')^{2p_i}$, $i=1,2$ in the following way

$$(y_i')^{2p_i} = \sum_{n=0}^{2p_i} \binom{2p_i}{n} (y_i' - Y_{i,s})^n Y_{i,s}^{2p_i-n}.$$

We take in (4.6) the conditional expectation to the σ-algebra of the events till time s and we get

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p_1}Y_{2,t}^{2p_2}\right] = \tag{4.7}$$
$$\sum_{n_1=0}^{2p_1} \sum_{\substack{n_2=0 \\ n_1+n_2<2n_0}}^{2p_2} \binom{2p_1}{n_1}\binom{2p_2}{n_2} \int_0^t ds\mathbb{E}_{(x,0)}\left[Y_{1,s}^{2p_1-n_1}Y_{2,s}^{2p_2-n_2}F_{n_1,n_2}(X_s)\right],$$

where

$$F_{n_1,n_2}(X_s) = \mathbb{E}_{r_s}\Big[\int dr'\, K(r_s,r')[(y_1'-Y_{1,s})^{n_1}(y_2'-Y_{2,s})^{n_2}\Big]. \tag{4.8}$$

In the last equality we have used the fact that since the kernel $K(r,r')$ is a function of $|r-r'|$ then the right hand side of (4.8) is a function only of X_s.

By symmetry we also have that

$$\mathbb{E}_{(x,0)}\left[Y_{1,s}^{2p_1-n_1}Y_{2,s}^{2p_2-n_2}F_{n_1,n_2}(X_s)\right] = 0,$$
$$\text{if } 2p_1-n_1 \text{ or/and } 2p_2-n_2 \text{ are odd numbers.} \tag{4.9}$$

Finally since $F_{n_1,n_2}(x)$ defined in (4.8) is a bounded function from (4.7), (4.9) and the inductive hypothesis we get

$$
\begin{aligned}
\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p_1}Y_{2,t}^{2p_2}\right] &= \|F\|_\infty \hat{c} \sum_{n_1=0}^{p_1} \sum_{\substack{n_2=0\\ n_1+n_2<n_0}}^{p_2} \int_0^t ds \mathbb{E}_{(x,0)}\left[Y_{1,s}^{2n_1}Y_{2,s}^{2n_2}\right] \\
&\le ct(1+t+|x|) \sum_{n_1=0}^{p_1} \sum_{\substack{n_2=0\\ n_1+n_2<n_0}}^{p_2} t^{n_1+n_2-1} \\
&\le Ct(1+t+|x|)t^{n_0-2}.
\end{aligned}
$$

We now prove the bound (4.2). We start with the proof for $p_1 = 1$ and $p_2 = 0$. Let $s = |\log t|^2$, taking the conditional expectation to the σ-algebra of the events till time $t-s$ we get

$$
\begin{aligned}
\mathbb{E}_{(x,0)}\left[Y_{1,t}^2\phi(X_t)\right] &= \mathbb{E}_{(x,0)}\left[\mathbb{E}_{r_{t-s}}\left[Y_{1,s}^2\phi(X_s)\right]\right] \\
&= \mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2\mathbb{E}^{(1)}_{X_{t-s}}\left[\phi(X_s)\right]\right] + \mathbb{E}^{(1)}_x\left[\mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2\phi(X_s)\right]\right] \\
&\doteq E_1(x,t) + E_2(x,t).
\end{aligned} \tag{4.10}
$$

In (4.10) the second equality is obtained by writing (with some abuse of notation) $Y_{1,s}^2 = (Y_{1,s} - Y_{1,t-s})^2 + Y_{1,t-s}^2 + 2Y_{1,t-s}(Y_{1,s} - Y_{1,t-s})$ and using the fact that $Y_{1,s}$ has symmetric distribution. By (4.1),

$$
|E_2(x,t)| \le C(1 + s + \mathbb{E}^{(1)}_x[|X_{t-s}|])\|\phi\|_\infty. \tag{4.11}
$$

To bound $\mathbb{E}^{(1)}_x[|X_{t-s}|]$ we write

$$
\mathbb{E}^{(1)}_x[|X_{t-s}|] = \int_0^{t-s} d\tau\, \mathbb{E}^{(1)}_x\left[\int dx'\, K^{(1)}(X_\tau, x')(|x'| - |X_\tau|)\right]. \tag{4.12}
$$

Using (2.35) and the equality

$$
\int \mu(dx) \int dx'\, K^{(1)}(x,x')(|x'| - |x|) = 0, \tag{4.13}
$$

from (4.12) it follows that, for some constant $c_1 > 0$,

$$
\mathbb{E}^{(1)}_x[|X_{t-s}|] \le c_1|x| \tag{4.14}
$$

so that

$$|E_2(x,t)| \le C(1+s+c_1|x|)\|\phi\|_\infty. \tag{4.15}$$

To bound $E_1(x,t)$ we use again (2.35). Since ϕ is an odd function, $\mu(\phi)=0$ therefore there is a constant $c_2>0$ so that

$$|E_1(x,t)| \le c_2\|\phi\|_\infty \mathbb{E}_{(x,0)}\Big[Y_{1,t-s}^2\Big(e^{-b(s-|X_{t-s}|)}\mathbf{1}_{|X_{t-s}|<s}+\mathbf{1}_{|X_{t-s}|\ge s}\Big)\Big], \tag{4.16}$$

where $b\in(0,\alpha)$. Using the Cauchy-Schwartz inequality we get

$$|E_1(x,t)| \le c_2\|\phi\|_\infty\sqrt{\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^4\right]}\left(\sqrt{\mathbb{E}_x^{(1)}\left[e^{-2b(s-|X_{t-s}|)}\mathbf{1}_{|X_{t-s}|<s}\right]}\right.$$
$$\left.+\sqrt{\mathbb{E}_x^{(1)}\left[\mathbf{1}_{|X_{t-s}|\ge s}\right]}\right). \tag{4.17}$$

Using the fact that $b\in(0,\alpha)$, (2.8) and (2.27) one easily proves that, for some $\hat c>0$,

$$\int\mu(dx)\,e^{-2b(s-|x|)}\mathbf{1}_{|x|<s}\le \hat c e^{-2bs}, \qquad \int\mu(dx)\,\mathbf{1}_{|x|\ge s}\le \hat c e^{-2\alpha s}. \tag{4.18}$$

By (2.35) and (4.1) for $p_1=2$, $p_2=0$, from (4.17) and (4.18) there is a constant $c_3>0$ so that

$$|E_1(x,t)| \le c_3\|\phi\|_\infty\sqrt{(1+t-s+|x|)(t-s)}$$
$$\times\sqrt{e^{-2bs}+e^{-b(t-s-|x|)}\mathbf{1}_{|x|<t-s}+\mathbf{1}_{|x|\ge t-s}}\ . \tag{4.19}$$

We finally note that, for some constant $c_4>0$,

$$(t-s)\mathbf{1}_{|x|>t-s}\le|x|, \qquad (t-s)^2\mathbf{1}_{|x|>t-s}\le x^2,$$
$$(t-s)^2e^{-b(t-s-|x|)}\mathbf{1}_{|x|<t-s}.\le c_4(1+x^2). \tag{4.20}$$

From (4.19) and (4.20) we get, for some constant $c_5>0$,

$$|E_1(x,t)|\le c_5(1+|x|)\|\phi\|_\infty. \tag{4.21}$$

From (4.15) and (4.21) the bound (4.2) for $p_1 = 1$ and $p_2 = 0$ follows. We now give the proof for $p_1 = 2$ and $p_2 = 0$. We restrict to the case $t > e$ (so $s > 1$) and write

$$\begin{aligned}\mathbb{E}_{(x,0)}\left[Y_{1,t}^4\phi(X_t)\right] &= \mathbb{E}_{(x,0)}\left[\mathbb{E}_{r_{t-s}}\left[Y_{1,s}^4\phi(X_s)\right]\right]\\ &= \mathbb{E}_{(x,0)}\left[Y_{1,t-s}^4\mathbb{E}^{(1)}_{X_{t-s}}\left[\phi(X_s)\right]\right] + 6\,\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2\mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2\phi(X_s)\right]\right]\\ &\quad + \mathbb{E}_{(x,0)}\left[\mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^4\phi(X_s)\right]\right]\\ &\doteq I_1(x,t) + I_2(x,t) + I_3(x,t). \end{aligned} \tag{4.22}$$

In (4.22) the equality (4.9) has been used.

We treat $I_1(x,t)$ and $I_3(x,t)$ as done before for $E_1(x,t)$ and $E_2(x,t)$ and we get, for some positive constant c_6,

$$|I_1(x,t)| \le c_6(1+|x|^2)\|\phi\|_\infty, \qquad |I_3(x,t)| \le c_6(1+s+|x|)s\|\phi\|_\infty. \tag{4.23}$$

To estimate $I_2(x,t)$ we decompose it as follows:

$$I_2(x,t) = I_2^{(1)}(x,t) + I_2^{(2)}(x,t), \tag{4.24}$$

where

$$\begin{aligned}I_2^{(1)}(x,t) &\doteq 6\,\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2\mathbf{1}_{|X_{t-s}|\le\sqrt{t}}\mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2\phi(X_s)\right]\right],\\ I_2^{(2)}(x,t) &\doteq 6\,\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2\mathbf{1}_{|X_{t-s}|>\sqrt{t}}\mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2\phi(X_s)\right]\right].\end{aligned} \tag{4.25}$$

We notice that

$$I_2^{(1)}(x,t) = 6\,\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 G(X_{t-s})\right], \tag{4.26}$$

where

$$G(x) \doteq \mathbf{1}_{|x|\le\sqrt{t}}\mathbb{E}_{(x,0)}\left[Y_{1,s}^2\phi(X_s)\right]] \tag{4.27}$$

$G(x)$ is an odd function so that from (4.2) for $p_1 + p_2 = 1$ we get

$$|G(x)| \le C(1+|\log s|^2+\sqrt{t})\|\phi\|_\infty. \tag{4.28}$$

We can then apply (4.1) for $p_1 + p_2 = 1$ and obtain

$$|I_2^{(1)}(x,t)| \le C(1+|\log(t-s)|^2+|x|)\|G\|_\infty \tag{4.29}$$

and so, by (4.28) there is $c_7 > 0$ so that

$$|I_2^{(1)}(x,t)| \le c_7(1 + |\log t|^2 + |x|)(1 + \sqrt{t})\|\phi\|_\infty. \tag{4.30}$$

Using again (4.2) with $p_1 + p_2 = 1$ there is $c_8 > 0$ so that

$$|I_2^{(2)}(x,t)| \le c_8\|\phi\|_\infty \mathbb{E}_{(x,0)}\big[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|>\sqrt{t}}(1 + |X_{t-s}|)\big] \tag{4.31}$$

(in (4.31) we have used the fact that, since $|X_{t-s}| > \sqrt{t}$, the term $|\log t|^2$, which appears applying (4.2), can be bounded by a constant times $|X_{t-s}|$). Using now the expression of the Markov semigroup, see [15], and the properties of the rate transition function we get, for some constant $c_9 > 0$,

$$\begin{aligned}
&\mathbb{E}_{(x,0)}\big[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|>\sqrt{t}}(1 + |X_{t-s}|)\big] \\
&\quad = e^{-(t-s)} \sum_{n\ge 1} \frac{(t-s)^n}{n!} \int dr'\, K^n((x,0),r')(y')^2(1 + |x'|)\mathbf{1}_{|x'|>\sqrt{t}} \\
&\quad \le c_9 e^{-(t-s)} \sum_{n \ge \max\{1;\sqrt{t}-|x|\}} \frac{(t-s)^n}{n!} n^2(1 + n + |x|).
\end{aligned} \tag{4.32}$$

It is not difficult to check that, from the bound in (4.32), there is a positive constant c_{10} such that

$$\mathbb{E}_{(x,0)}\big[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|>\sqrt{t}}(1 + |X_{t-s}|)\big] \le c_{10}(1 + x^6). \tag{4.33}$$

From (4.23), (4.30), (4.31) and (4.33) the bound (4.2) for $p_1 = 2$, $p_2 = 0$ follows immediately. The proof for $p_1 = 1$ and $p_2 = 1$ is similar and it is omitted. □

Lemma 4.2. *Let f be a C^∞, $\mathbb{E}_x^{(1)}[\cdot]$-integrable function on $\mathbb{R}$. For any $p,q \in \mathbb{N}_+$ define*

$$\Psi_{p,q}(x,t) \doteq \mathbb{E}_{(x,0)}\big[Y_{1,t}^{2p} Y_{2,t}^{2q} f(X_t)\big]. \tag{4.34}$$

Then $\Psi_{p,q} \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ and, for any $N \ge 0$, there is a positive constant C_N such that, for any $t \ge 0$,

$$\left|\frac{\partial^N \Psi_{p,q}}{\partial x^N}(x,t)\right| \le C_N \sup_{s\le t} \sup_{p'\le p,\, q'\le q} \big|\Psi_{p',q'}(x,s)\big|. \tag{4.35}$$

Proof. Using the form of the generator $\mathcal{L}$ of the Markov semigroup in term of the rate transition function $K(r, r')$ (see (2.24)) and observing that, for any $N \geq 1$,

$$\frac{\partial^N}{\partial x^N} \left(y_1^{2p} y_2^{2q} f(x) \right) \Big|_{(x,0)} = 0$$

it is easy to verify the following identity

$$\frac{\partial^N \Psi_{p,q}}{\partial x^N}(x,t) = \int_0^t ds\, e^{-(t-s)} \int dr' \, \frac{\partial^N K}{\partial x^N}((x,0), r') \mathbb{E}_{(x',y')} \left[Y_{1,s}^{2p} Y_{2,s}^{2q} f(X_s) \right]. \tag{4.36}$$

By using identities similar to (4.40), it is easy to verify also that there exists a positive constant $C^\star(p,q)$ such that

$$\sup_{|y'| \leq 1} \mathbb{E}_{(x',y')} \left[Y_{1,s}^{2p} Y_{2,s}^{2q} f(X_s) \right] \leq C^\star(p,q) \sup_{p' \leq p,\, q' \leq q} \left| \Psi_{p',q'}(x', s) \right|. \tag{4.37}$$

Recalling finally that $K((x,0),r')$ is a smooth function, identically zero for $|(x-x')^2 + (y')^2| \geq 1$, the Lemma follows immediately from (4.36) and (4.37). □

Proof of Lemma 3.12. First of all we rewrite the functions F_T, G_T and H_T in a more convenient way. For any $\underline{\kappa} \in \mathbb{R}^{n-1}$ we set

$$\kappa \doteq \sum_{i=1}^{n-1} \underline{\kappa}_i, \quad \bar{\kappa}^2 \doteq \sum_{i=1}^{n-1} \underline{\kappa}_i^2, \quad \hat{\kappa} \doteq \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} \underline{\kappa}_i \underline{\kappa}_j. \tag{4.38}$$

We observe now that the marginals $\{(X_t, Y_{i,t})\}_{i=1}^{n-1}$ of the Markov process starting from $(x,0)$ are equally distributed. Then

$$F_T(x, \underline{\kappa}) = \kappa \mathbb{E}_{(x,0)} \left[\theta T - \frac{Y_{1,T}^2}{2} \right]. \tag{4.39}$$

We rewrite the functions G_T and H_T by using also the following property of the process r_t: if $\phi(r) = \psi(x) y_i^2$, then

$$\mathbb{E}_r[\phi(r_t)] = \mathbb{E}_{(x,0)}[\phi(r_t)] + y_i^2 \mathbb{E}_{(x,0)}[\psi(X_t)]. \tag{4.40}$$

Using the above observations, after some long but not difficult computations, we get

$$G_T(x, \underline{\kappa}) = \sum_{i=0}^{9} G_T^{(i)}(x, \underline{\kappa}), \qquad H_T(x, \kappa) = \sum_{i=0}^{5} H_T^{(i)}(x, \kappa), \tag{4.41}$$

where

$$
\begin{aligned}
G_T^{(0)}(x,\underline{\kappa}) &\doteq \frac{\bar{\kappa}^2}{8}\mathbb{E}_{(x,0)}[A(X_T)Y_{1,T}^4],\\
G_T^{(1)}(x,\underline{\kappa}) &\doteq \frac{\hat{\kappa}}{8}\mathbb{E}_{(x,0)}[A(X_T)Y_{1,T}^2Y_{2,T}^2],\\
G_T^{(2)}(x,\underline{\kappa}) &\doteq -\frac{\bar{\kappa}^2}{2}\mathbb{E}_{(x,0)}[X_TY_{1,T}^2],\\
G_T^{(3)}(x,\underline{\kappa}) &\doteq \frac{\kappa^2}{8}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_s)(\hat{J}\star\bar{m}')^2(X_s)\big],\\
G_T^{(4)}(x,\underline{\kappa}) &\doteq \frac{\bar{\kappa}^2}{8}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_s)(\tilde{J}\star\bar{m}')^2(X_s)Y_{1,s}^4\big],\\
G_T^{(5)}(x,\underline{\kappa}) &\doteq \frac{\hat{\kappa}}{8}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_s)(\tilde{J}\star\bar{m}')^2(X_s)Y_{1,s}^2Y_{2,s}^2\big],\\
G_T^{(6)}(x,\underline{\kappa}) &\doteq \frac{\kappa^2}{4}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_s)(\hat{J}\star\bar{m}')(X_s)(\tilde{J}\star\bar{m}')(X_s)Y_{1,s}^2\big],\\
G_T^{(7)}(x,\underline{\kappa}) &\doteq \frac{\kappa^2}{4}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_{T-s})(\hat{J}\star\bar{m}')(X_{T-s})\\
&\qquad\times\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[Y_{1,s}^2])\big)(X_{T-s})\big],\\
G_T^{(8)}(x,\underline{\kappa}) &\doteq \frac{\kappa^2}{4}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_{T-s})(\tilde{J}\star\bar{m}')(X_{T-s})\\
&\qquad\times\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[Y_{1,s}^2])\big)(X_{T-s})Y_{1,T-s}^2\big],\\
G_T^{(9)}(x,\underline{\kappa}) &\doteq \frac{\kappa^2}{8}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[Y_{1,s}^2])\big)^2(X_{T-s})\big]\\
&\quad-\frac{(\theta\kappa T)^2}{2}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_s)(\tilde{J}\star\bar{m}')^2(X_s)\big]. \qquad (4.42)
\end{aligned}
$$

The functions $H_T^{(0)}$ and $H_T^{(1)}$ are defined in (3.65) and (3.66) respectively, and

$$
\begin{aligned}
H_T^{(2)}(x,\underline{\kappa}) &\doteq \frac{1}{2}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)^2(X_{T-s})\big]\\
H_T^{(3)}(x,\underline{\kappa}) &\doteq -\frac{1}{2}\int_0^T ds\,\mathbb{E}_{(x,0)}\big[B(X_{T-s})(\hat{J}\star\bar{m}')(X_{T-s})\\
&\quad\times\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)(X_{T-s})\big]
\end{aligned}
$$

$$H_T^{(4)}(x,\underline{\kappa}) \doteq -\frac{1}{2}\int_0^T ds\, \mathbb{E}_{(x,0)}\big[B(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[Y_{1,s}^2])\big)(X_{T-s})$$
$$\times\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)(X_{T-s})\big]$$
$$H_T^{(5)}(x,\underline{\kappa}) \doteq -\frac{1}{2}\int_0^T ds\, \mathbb{E}_{(x,0)}\big[B(X_{T-s})(\tilde{J}\star\bar{m}')(X_{T-s})$$
$$\times\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)(X_{T-s})\big] \tag{4.43}$$

with

$$A(x) \doteq \frac{\bar{m}''(x)}{\bar{m}'(x)}, \quad B(x) \doteq \frac{\Phi(x)}{\bar{m}'(x)}, \quad \hat{J}(x) \doteq \int_{\mathbb{R}^{n-1}} dy\, J\Big(|x^2+y^2|^{1/2}\Big)\, y_1^2. \tag{4.44}$$

We call $F_t(x,\underline{\kappa})$, $G_t^{(i)}(x,\underline{\kappa})$, $i=0,\dots,9$, and $H_t^{(j)}(x,\underline{\kappa})$, $j=0,\dots,5$, the functions as defined in (4.39), (4.42), (4.43) but computed for $t\in[0,T]$ instead of T. By (2.42) and (4.4) it is $\mu(f_1)=2\theta$. Then, using (2.35) and (4.3) we get, for some $C_1>0$,

$$|F_t(x,\underline{\kappa})| \le C_1(1+|\underline{\kappa}|^2)(1+|x|). \tag{4.45}$$

By (4.2) we get, for some $C_2>0$,

$$|G_t^{(i)}(x,\underline{\kappa})| \le \begin{cases} C_2(1+|\underline{\kappa}|^2)(1+|x|)(1+|\log t|^2\sqrt{t}+|x|^5) & \text{if } i=0,1\\ C_2(1+|\underline{\kappa}|^2)(1+|x|)(1+|\log t|^2\sqrt{t}+|x|^5)t & \text{if } i=4,5\\ C_2(1+|\underline{\kappa}|^2)(1+|\log t|^2+|x|)t & \text{if } i=6. \end{cases} \tag{4.46}$$

where we used the fact that $A(x)$, $B(x)(\tilde{J}\star\bar{m}')(x)$ and $B(x)(\hat{J}\star\bar{m}')(x)$ are bounded odd functions. By reasoning as in the estimate of $I_2(x,t)$, see (4.24)–(4.33), we get, for some $C_3>0$,

$$|G_t^{(2)}(x,\underline{\kappa})| \le C_3(1+|\underline{\kappa}|^2)(1+|x|)(1+|\log t|^2\sqrt{t}+|x|^5). \tag{4.47}$$

Since $\mu(B(\tilde{J}\star\bar{m}')^2)=0$ by (2.35) there is $C_4>0$,

$$|G_t^{(3)}(x,\underline{\kappa})| \le C_4(1+|\underline{\kappa}|^2)(1+|x|). \tag{4.48}$$

Using (2.35) and (4.1) there is a constant C_5 so that

$$|G_t^{(7)}(x,\underline{\kappa})| \le C_5(1+|\underline{\kappa}|^2)\int_0^t ds\,(1+s)\Big(e^{-b(t-s-|x|)}\mathbf{1}_{t-s>|x|}+\mathbf{1}_{t-s\le|x|}\Big)$$
$$\le C_5(1+|\underline{\kappa}|^2)(1+|x|)t. \tag{4.49}$$

From (4.1) and (4.2), it is easy to prove that

$$|G_t^{(8)}(x,\underline{\kappa})| \le C_6(1+|\underline{\kappa}|^2)(1+|\log t|^2+|x|)t^2 \tag{4.50}$$

for some positive constant C_6. $G_t^{(9)}(x,\underline{\kappa})$ can be bounded as $G_t^{(7)}(x,\underline{\kappa})$ and one gets

$$|G_t^{(9)}(x,\underline{\kappa})| \le C_7(1+|\underline{\kappa}|^2)(1+|x|)t$$

for some $C_7 > 0$ (note that this term grows as t and not as t^2 since, by the definition of θ, in (4.42) there is an exact cancellation of the leading term in t). Clearly there is $C_8 > 0$ so that

$$|H_t^{(0)}(x,\underline{\kappa})| \le C_8(1+|\underline{\kappa}|^2)(1+|x|)t. \tag{4.51}$$

By (4.2), for some $C_9 > 0$,

$$|H_t^{(1)}(x,\underline{\kappa})| \le C_9(1+|\underline{\kappa}|^2)(1+|\log t|^2+|x|). \tag{4.52}$$

Using (2.35), for some $C_{10} > 0$,

$$|H_t^{(j)}(x,\underline{\kappa})| \le C_{10}(1+|\underline{\kappa}|^2)(1+|x|), \qquad j=2,3,5 \tag{4.53}$$

and finally, by arguing as in (4.49), we get

$$|H_t^{(4)}(x,\underline{\kappa})| \le C_{11}(1+|\underline{\kappa}|^2)(1+|x|)t \tag{4.54}$$

for some $C_{11} > 0$.

The proof of Lemma 3.12 follows easily from all the previous bounds and Lemma 4.2. □

Proof of (3.121) **and** (3.130). We need the following Lemma.

Lemma 4.3. *Let* $r = (x,y) \in \mathbb{R}\times\mathbb{R}^{n-1}$, $\psi(r) = \phi(r)\mathbf{1}_{|y|>4VT}$ *with* $|\phi(r)| \le c\big((1+|x|)y^2+|y|^4\big)$ *for some positive constant c. Then there is a constant* $C > 0$ *such that*

$$\big|\big(g_T \star (\bar{m}'\psi)\big)\big((x,0)\big)\big| \le Ce^{-T} \tag{4.55}$$

and, for any bounded function W *on* $\mathbb{R}^n$,

$$\Big|\int_0^T ds\,\big(g_{T-s}\star[WJ\star(g_s\star(\bar{m}'\psi))]\big)\big((x,0)\big)\Big| \le C\|W\|_\infty Te^{-T}. \tag{4.56}$$

Proof. We use again the explicit expression of the Markov semigroup. Recalling the hypothesis on $\phi(r)$ we get, for some positive constant $\hat{C} > 0$,

$$\begin{aligned} &\left|\left(g_T \star (\bar{m}'\psi)\right)\left((x,0)\right)\right| = \bar{m}'(x)\left|\mathbb{E}_{(x,0)}[\psi(r_T)]\right| \\ &\le c\, e^{-T}\bar{m}'(x)\sum_{n\ge 1}\frac{T^n}{n!}\int dr'\, K^n((x,0),r')\left((1+|x'|)(y')^2+|y'|^4\right)\mathbf{1}_{|y'|>4VT} \\ &\le \hat{C}e^{-T}\sum_{n>4VT}\frac{T^n}{n!}n^4 \le \hat{C}e^{-T}. \end{aligned} \tag{4.57}$$

Analogously we estimate, for some $\tilde{C} > 0$,

$$\begin{aligned} &\left|\int_0^T ds\left(g_{T-s}\star\left[WJ\star\left(g_s\star(\bar{m}'\psi)\right)\right]\right)\left((x,0)\right)\right| \\ &\le c\|W\|_\infty \bar{m}'(x)e^{-T}\int_0^T ds\sum_{n,q\ge 1}\frac{(T-s)^n s^q}{n!q!}\int dr\,dr'\,dr''\,K^n((x,0),r) \\ &\times J(r-r')\bar{m}'(x')K^q(r',r'')\left((1+|x''|)(y'')^2+|y''|^4\right)\mathbf{1}_{|y''|>4VT} \\ &\le \tilde{C}\|W\|_\infty e^{-T}\int_0^T ds\sum_{N\ge 4VT}\sum_{q\le N}\frac{(T-s)^{N-q}s^q}{(N-q)!q!}N^4 \\ &= \tilde{C}\|W\|_\infty e^{-T}T\sum_{N\ge 4VT}\frac{N^4T^N}{N!} \le \tilde{C}Te^{-T}\|W\|_\infty. \end{aligned} \tag{4.58}$$

The Lemma is proved. □

The estimate (4.55) with $\psi = \psi_\lambda$ gives (3.121). The estimate (4.56) with $\psi = \bar{\psi}_\lambda$ proves the last bound in (3.130). We are left with the bounds on $K_\lambda^{(i)}(x)$ for $i = 0, 1$. These follows easily from the estimate (3.127) and the nice properties of the kernel $g_t(r,r')$ (see (3.135)). We omit the details. □

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