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A Derivation of the Broadwell Equation*

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Abstract. We consider a stochastic system of particles in a two dimensional lattice and prove that, under a suitable limit (i.e. $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon^2 \to \text{const}$, where N is the number of particles and ε is the mesh of the lattice) the one-particle distribution function converges to a solution of the two-dimensional Broadwell equation for all times for which the solution (of this equation) exists. Propagation of chaos is also proven.

1. Introduction

The dynamics of the molecules of a rarefied gas could, in principle, be described by Newton's equations, but a reduced description, as provided by the Boltzmann equation, turns out in practice to be more appropriate and useful. On the other hand it is known that in a low density regime, mathematically expressed by the Boltzmann–Grad limit (B–G limit), the solutions of the Newton equations approximate those of the Boltzmann equation. This is rigorously proven for short times in [6] and globally in time for an expanding cloud of gas in the vacuum, [5]. Both cases are based on the fact that the average number of collisions experienced by any given particle is finite and small. At the present moment no other results concerning more general situations are available.

In this paper we deal with a discrete velocity kinetic equation, namely the two dimensional Broadwell equation. There are pathologies connected with the finite structure of the velocity space: from one side it has been proven, [7], that such an equation does not hold in the B–G limit for a natural class of approximating Hamiltonian systems; on the other side, the recent existence theory, [4], for the Boltzmann equation does not apply to discrete velocity models, essentially for the same reason.

In the present paper we prove that the two-dimensional Broadwell equation holds in the B–G limit for a sequence of stochastic particle systems. The

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stochasticity is absolutely necessary: it delocalizes the particles and, in this way, the pathologies outlined in [7] are avoided. In fact it is easy to prove (see Sect. 3 below) that the short time analysis presented in [6] does apply without difficulties to our case, yielding a derivation of the Broadwell equation for short times. In this paper we develop a new technique which allows us to prove convergence up to any time in which the solution to the Broadwell equation stays bounded. Indeed solutions which are bounded for arbitrary long times have been constructed in some cases so that our result provides a validity proof which is global in time, at least in these situations. Our approach to the problem uses probabilistic ideas. We basically prove that a sample of our stochastic process deviates from the correct kinetic behavior with small probability, when the B–G limit is approached. To prove this we introduce and evaluate a sort of truncated correlation functions (i.e. the v-functions defined in Sect. 4) yielding powerful probability estimates, which allow us to iterate the short time result up to arbitrary times. We remark that also in this part of the proof the stochasticity plays a fundamental role: the same strategy does not apply, at least directly, to Hamiltonian systems. The techniques developed in this paper apply to more general discrete velocity equations with minor modifications. We finally mention that other discrete velocity kinetic equations have been derived from stochastic systems (see [2, 3]) and that techniques similar to those presented here have been used in [1] to study Reaction–Diffusion equations.

The paper is organized as follows. In Sect. 2 we describe the model and state the main result. Section 3 is devoted to the short time analysis, while in Sects. 4 and 5 we prove the main theorem. The detailed definition of the branching process which describes the evolution of the v-functions is given in Appendix A.

2. The Model and the Results

We consider the unit square $\Lambda = [0, 1]^2$ and its discretization

$$\Lambda_\varepsilon = \{q \equiv (q_x, q_y) \in \Lambda : q_x = n_1 \varepsilon, q_y = n_2 \varepsilon, n_i = 1, \ldots, \varepsilon^{-1}\}, \tag{2.1}$$

where ε^{-1} is a positive integer to be fixed later on. We then consider a system of N (identical) particles whose configurations are denoted by:

$$X = \{x_1, \ldots, x_N\}, \tag{2.2a}$$

$$x_i = (q_i, \sigma_i) \quad q_i \in \Lambda_\varepsilon, \quad \sigma_i \in \Omega, \tag{2.2b}$$

where q_i and σ_i are position and velocity of the i^{th} particle and

$$\Omega = \{(0, 1); (1, 0); (0, -1); (-1, 0)\} \tag{2.3}$$

is the velocity space, consisting of four elements.

We denote by $\Gamma_\varepsilon = \Lambda_\varepsilon \times \Omega$ the phase space of a single particle and, accordingly, Γ_ε^N will denote the phase space of our system.

In the sequel we shall avoid boundary problems by assuming periodicity

$$(0, q_y) = (1, q_y), \quad (q_x, 0) = (q_x, 1), \tag{2.4}$$

which makes Λ a torus.

The dynamics of the system is a stochastic process with values in Γ_ε^N, described in the following way. The free motion of each particle is a Poisson process of intensity ε^{-1} associated to the transition:

$$(q_i, \sigma_i) \to (q_i + \sigma_i \varepsilon, \sigma_i). \tag{2.5}$$

All these Poisson processes are independent.

The interaction among the particles is described by $N(N-1)/2$ independent Poisson processes of intensity one (one for each pair of particles) associated to the transition:

$$(q_i, \sigma_i, q_j, \sigma_j) \to (q_i, \sigma_i^\perp, q_j, \sigma_j^\perp), \tag{2.6}$$

where

$$\sigma^\perp = (-\sigma_2, \sigma_1) \quad \text{if} \quad \sigma = (\sigma_1, \sigma_2), \tag{2.7}$$

and the transition (2.6) takes place only if

$$q_i = q_j \quad \text{and} \quad \sigma_i = -\sigma_j. \tag{2.8}$$

In terms of differential equations, if $\mu_t^\varepsilon = \mu_t^\varepsilon(X)$ denotes the probability of finding the configuration $X = \{x_1, \ldots, x_N\}$ at the time t, then the following evolution equation holds:

$$\frac{d\mu_t^\varepsilon}{dt} = \varepsilon^{-1} L_0 \mu_t^\varepsilon + L_I \mu_t^\varepsilon, \tag{2.9}$$

where, denoting by $|X|$ the cardinality of X, (in this case N)

$$L_0 \mu^\varepsilon(X) = \sum_{i=1}^{|X|} [\mu^\varepsilon(X_i) - \mu^\varepsilon(X)], \tag{2.10}$$

$$L_I \mu^\varepsilon(X) = \frac{1}{2} \sum_{i \neq j = 1}^{|X|} [\mu^\varepsilon(X_{i,j}^\perp) - \mu^\varepsilon(X)] \chi_{i,j}(X). \tag{2.11}$$

We have used the following notation: ($X = \{x_1, \ldots, x_{|X|}\}$),

$$X_i = \{x_1, \ldots, (q_i - \sigma_i \varepsilon, \sigma_i), \ldots, x_{|X|}\}, \tag{2.12}$$

$$X_{i,j}^\perp = \{x_1, \ldots, (q_i, \sigma_i^\perp), \ldots, (q_j, \sigma_j^\perp), \ldots, x_{|X|}\}, \tag{2.13}$$

$$\chi_{i,j} = \begin{cases} 1, & \text{if } q_i = q_j \text{ and } \sigma_i = -\sigma_j \\ 0, & \text{otherwise} \end{cases}. \tag{2.14}$$

We want to investigate the behavior of the system in the limit as $\varepsilon \to 0$, $N \to \infty$, $N\varepsilon^2 \to \lambda > 0$. In such a limit the number of particles per site is expected to be finite so that each particle will undergo a finite number of collisions per unit time. On the other hand the displacement per unit time of each particle is finite since $\varepsilon^{-1} L_0$ converges, at least formally, to the generator of the free stream. Therefore we are in a situation which strongly resembles the Boltzmann–Grad limit for Hamiltonian particle systems where the asymptotic behavior of the one-particle distribution function is described by the Boltzmann equation. In our context the kinetic equation which is expected to be valid in this limit is the two dimensional Broadwell equation, as we shall see below by a simple formal argument.

For a probability measure μ which is symmetric under the exchange of the particles, we introduce the rescaled correlation functions:

$$\rho_j^\varepsilon(x_1,\dots,x_j)=(N\varepsilon^2)^{-j}N(N-1)\cdots(N-j+1)\sum_{x_{j+1}\cdots x_N}\mu^\varepsilon(x_1,\dots,x_j,x_{j+1},\dots,x_N). \tag{2.15}$$

Proceeding as in the derivation of the BBGKY hierarchy for Hamiltonian systems, we obtain:

$$\frac{d}{dt}\rho_j^\varepsilon=(\varepsilon^{-1}L_0+L_I)\rho_j^\varepsilon+\varepsilon^2 NC_{j,j+1}\rho_{j+1}^\varepsilon, \tag{2.16}$$

where

$$\begin{aligned}C_{j,j+1}\rho_{j+1}^\varepsilon(x_1,\dots,x_j)=\sum_{i=1}^{j}\,[&\rho_{j+1}^\varepsilon(x_1,\dots,(q_i,\sigma_i^\perp),\dots,x_j,(q_i,-\sigma_i^\perp))\\&-\rho_{j+1}^\varepsilon(x_1,\dots,(q_i,\sigma_i),\dots,x_j,(q_i,-\sigma_i))].\end{aligned} \tag{2.17}$$

Remarks on the Definition in (2.15). Consider the joint distribution densities:

$$f_j^\varepsilon(x_1,\dots,x_j)=\varepsilon^{-2j}\sum_{x_{j+1}\cdots x_N}\mu^\varepsilon(x_1,\dots,x_j,x_{j+1},\dots,x_N) \tag{2.18}$$

(notice therefore that $\varepsilon^{2j}f_j^\varepsilon(x_1,\dots,x_j)$ is the μ^ε-probability of finding the group of particles $1\cdots j$ in the states $x_1\cdots x_j$). Then the correlation functions are:

$$u_j^\varepsilon(x_1,\dots,x_j)=N(N-1)\cdots(N-j+1)f_j^\varepsilon(x_1,\dots,x_j). \tag{2.19}$$

In the B–G limit the correlation functions diverge. Renormalizing by a factor N^{-j} we then get the rescaled correlation functions:

$$\rho_j^\varepsilon(x_1,\dots,x_j)=N^{-j}u_j^\varepsilon(x_1,\dots,x_j). \tag{2.20}$$

These can also be expressed as follows: if $x_1,\dots,x_j$ are distinct states then

$$\rho_j^\varepsilon(x_1,\dots,x_j)=(N\varepsilon^2)^{-j}E_{\mu^\varepsilon}\left(\prod_{i=1}^{j}\eta(x_i)\right), \tag{2.21a}$$

where E_{μ^ε} denotes the expectation with respect to μ^ε and $\eta(x_i)$ the random variable number of particles in the state x_i. If the states are not distinct the expression changes. For instance if $x_1=\cdots=x_j=x$ then

$$\rho_j^\varepsilon(x,\dots,x)=(N\varepsilon^2)^{-j}E_{\mu^\varepsilon}(\eta(x)\cdots(\eta(x)-j+1)). \tag{2.21b}$$

For the other cases see (4.10) below.

The formal limit of (2.16) when $\varepsilon\to 0$, $N\to\infty$, $N\varepsilon^2\to\lambda$ is the following infinite hierarchy:

$$\frac{\partial}{\partial t}g_j=-\left(\sum_{i=1}^{j}\sigma_i\cdot\frac{\partial}{\partial q_i}\right)g_j+\lambda C_{j,j+1}g_{j+1}. \tag{2.22}$$

The reason why in (2.22) there is not a term corresponding to $L_I\rho_j^\varepsilon$ is due to the fact that the flow generated by $\varepsilon^{-1}L_0$ gives vanishing probability to configurations in which two tagged particles sit on the same site.

We shall call the set of Eqs (2.22) the Broadwell hierarchy because if g denotes any solution of the two-dimensional Broadwell equation:

$$\frac{\partial}{\partial t}g(q,\sigma) = -\sigma\cdot\frac{\partial}{\partial q}g(q,\sigma) + \lambda[g(q,-\sigma^{\perp})g(q,\sigma^{\perp}) - g(q,\sigma)g(q,-\sigma)] \quad (2.23)$$

then the products:

$$g_{j,t}(x_1,\ldots,x_j) = \prod_{i=1}^{j} g_t(x_i) \quad (2.24)$$

solve the Broadwell hierarchy (2.22). Conversely, in case of uniqueness of the solutions of (2.22) and (2.23), any solution of the Broadwell hierarchy which initially factorizes (i.e. satisfies (2.24) at time zero) cannot fail to be of the form (2.24) at all times.

The aim of this paper is to prove rigorously the above limit. We make the following hypothesis at time zero. Consider $g_0 \in C^1(\Gamma)$ a positive initial value for the Broadwell equation (2.23) satisfying the normalization condition $\sum_\sigma \int dq g_0(q,\sigma) = 1$. Let μ_0^ε be the probability measure on Γ_ε^N defined as follows:

$$\mu_0^\varepsilon(x_1,\ldots,x_N) = \prod_{i=1}^{N} \int_{\Delta(q_i)} dq g_0(q,\sigma_i) \quad (x_i = (q_i,\sigma_i)), \quad (2.25)$$

where $\Delta(q_i)$ is the atom of the partition into squares induced by the lattice Λ_ε and whose left low corner is q_i. Let μ_t^ε be the measure evolved according to (2.9), or, what is the same, the distribution of the process $\{x_1,\ldots,x_N\}$ at time t. Consider the associated rescaled correlation functions ρ_t^ε. (In what follows we shall compare functions on Γ_ε^j with functions on $(\Lambda\times\Omega)^j$ by restricting these latter to Γ_ε^j.) We have the following theorem (the main result of this paper), which is actually valid under more general assumptions on the convergence at time 0.

Theorem 2.1. *Let $g \in C^1([0,T]; C^1(\Gamma))$ be a solution of (2.23) with initial datum g_0. Let $g_{j,t}$ be given by (2.24) and μ_0^ε by (2.25). Then:*

$$\lim_{\varepsilon\to 0, N\to\infty, N\varepsilon^2\to\lambda} \|\rho_{j,t}^\varepsilon - g_{j,t}\|_\infty = 0 \quad (2.26)$$

uniformly in $t \in [0,T]$.

3. Short Time Analysis

In this section we prove Theorem 2.1 for T sufficiently small. This result is not surprising at all: we simply exploit Lanford's technique used for deriving the Boltzmann equation for hard spheres, [6]. This short time analysis, however, will turn out to be useful in the sequel so that we shall present it here in some detail.

Theorem 3.1. *Suppose that $g_0 \in C^1(\Gamma)$. Then there exists $T(\lambda, \|g_0\|_\infty)$ such that, for $t \leqq T(\lambda, \|g_0\|_\infty)$,*

$$\|\rho_{j,t}^\varepsilon\|_\infty \leqq (2\|g_0\|_\infty)^j \quad (3.1)$$

and (2.26) holds.

Proof. We write the solution of (2.21) by means of a perturbation series around $V^\varepsilon_{j,t}$ which is the semigroup generated by $\varepsilon^{-1}L_0 + L_I$ and acting on functions of j particles. In the sequel we simply write V^ε_t instead of $V^\varepsilon_{j,t}$, making clear the j-dependence from the context. We have:

$$\rho^\varepsilon_{j,t} = V^\varepsilon_t \rho^\varepsilon_{j,0} + \sum_{n=1}^{N} (N\varepsilon^2)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V^\varepsilon_{t-t_1} C_{j,j+1} V^\varepsilon_{t_1-t_2} C_{j+1,j+2} \cdots V^\varepsilon_{t_n} \rho^\varepsilon_{j+n,0}. \tag{3.2}$$

Since V^ε_t is a contraction semigroup in L_∞, and

$$\| C_{j,j+1} \|_\infty \leqq 2j, \tag{3.3}$$

we conclude:

$$\| \rho^\varepsilon_{j,t} \|_\infty \leqq \| \rho^\varepsilon_{j,0} \|_\infty + \sum_{n=1}^{N-j} \frac{(2tN\varepsilon^2)^n}{n!} j(j+1)\cdots(j+n-1) \| \rho^\varepsilon_{j+n,0} \|_\infty. \tag{3.4}$$

Using the inequality:

$$j(j+1)\cdots(j+n-1) \leqq n! 2^{n+j}, \tag{3.5}$$

we finally obtain:

$$\begin{aligned} \| \rho^\varepsilon_{j,t} \|_\infty &\leqq \| \rho^\varepsilon_{j,0} \|_\infty + 2^j \sum_{n=1}^{\infty} (4tN\varepsilon^2)^n \| \rho^\varepsilon_{j+n,0} \|_\infty \\ &\leqq \| g_0 \|^j_\infty + (2\| g_0 \|_\infty)^j \sum_{n=1}^{\infty} (4tN\varepsilon^2)^n \| g_0 \|^n_\infty. \end{aligned} \tag{3.6}$$

Therefore (3.1) is verified with $t \leqq T(\lambda, \| g_0 \|_\infty) \equiv [12\lambda \| g_0 \|_\infty]^{-1}$.

To prove the convergence we set:

$$\Delta^\varepsilon_j(t) = g_{j,t} - \rho^\varepsilon_{j,t}, \tag{3.7}$$

where $g_{j,t}$ is given by (2.24). It is easy to see that $g_{j,t}$ solves the Broadwell hierarchy (2.22) and by using a perturbative expansion as above, we can also prove uniqueness (in a suitable space).

Denoting by V_t the free stream, i.e. the group generated by $-\sum_{i=1}^{j} \sigma_i \frac{\partial}{\partial q_i}$ and by P^ε_t the semigroup generated by $\varepsilon^{-1}L_0$, we have:

$$\begin{aligned} \Delta^\varepsilon_j(t) &= \int_0^t ds P^\varepsilon_{t-s}(L_I \rho^\varepsilon_j(s)) + (V_t - P^\varepsilon_t) g_{j,0} + P^\varepsilon_t \Delta^\varepsilon_j(0) \\ &- \int_0^t (V_{t-s} - P^\varepsilon_{t-s}) C_{j,j+1} g_{j+1,s} ds + \int_0^t P^\varepsilon_{t-s} C_{j,j+1} \Delta^\varepsilon_{j+1}(s) ds. \end{aligned} \tag{3.8}$$

To estimate the first term on the right-hand side of (3.8) we develop, in some detail, an elementary argument which will be often used in the sequel. We want to estimate $P^\varepsilon_t \chi(x, y)$, where χ is the characteristic function of the event that two particles sit on the same site. We denote by $P^\varepsilon_t(x, y | z, u) = P^\varepsilon_t(x|z) P^\varepsilon_t(y|u)$ the transition probability that two particles, initially in the states z, u, are at time t in the states x, y respectively, when they moved according to the process P^ε_t, (the random flights associated to each particle are independent so that the transition probability factorizes). We have:

$$\begin{aligned} P_t^\varepsilon \chi(x, y) &= \sum_{z,u} P_t^\varepsilon(x, y \mid z, u)\chi(z, u) \\ &\leqq \sup_z P_t^\varepsilon(x \mid z) \sum_u P_t^\varepsilon(y \mid u) \\ &\leqq \sup_{x,z} P_t^\varepsilon(x \mid z). \end{aligned} \tag{3.9}$$

Moreover

$$P_t^\varepsilon(x \mid y) \leqq \sup_k e^{-t/\varepsilon} \frac{(t/\varepsilon)^k}{k!}, \tag{3.10}$$

and recalling the Stirling formula

$$\frac{e^n n^{(n+1/2)}}{n!} \rightarrow \sqrt{\frac{1}{2\pi}}$$

we finally get

$$P_t^\varepsilon \chi(x, y) \leqq \text{const.} \sqrt{\frac{\varepsilon}{t}}. \tag{3.11}$$

Therefore for $t < T(\lambda, \| g_0 \|_\infty)$:

$$P_{t-s}^\varepsilon L_I \rho_{j,s}^\varepsilon \|_\infty \leqq \text{const.}\, j^2 \sqrt{\frac{\varepsilon}{t-s}} (2 \| g_0 \|_\infty)^j. \tag{3.12}$$

Furthermore, for a function $f \in C^1(\Gamma)$,

$$\begin{aligned} |(V_t - P_t^\varepsilon) f(x)| &\leqq \sum_u P_t^\varepsilon(x \mid u) | f(u) - f(x - \sigma t)| \\ &\leqq \text{const.} \sum_u P_t^\varepsilon(x \mid u) | u - x + \sigma t| \\ &\leqq \text{const.} \left(\sum_u P_t^\varepsilon(x \mid u) | u - x + \sigma t|^2 \right)^{1/2} \\ &\leqq \text{const.} \sqrt{\varepsilon t}. \end{aligned} \tag{3.13}$$

The third term on the right-hand side of (3.8) can be easily bounded by

$$\| \Delta_j^\varepsilon(0) \|_\infty \leqq \| g_0 \|_\infty j\varepsilon (\| g_0 \|_\infty)^{j-1}, \tag{3.14}$$

so that we obtain the integral inequality:

$$\| \Delta_j^\varepsilon(t) \|_\infty \leqq c^j(\varepsilon + \sqrt{\varepsilon t}) + 2j \int_0^t ds \| \Delta_{j+1}^\varepsilon(s) \|_\infty. \tag{3.15}$$

From (3.15), since $\| \Delta_j^\varepsilon(s) \| \leqq c^j$ for some c, we can obtain a series expansion which converges for short times. It is now easy to conclude that in the B–G limit, ($\varepsilon \to 0$, $N \to \infty$ and $N\varepsilon^2 \to \lambda$) $\| \Delta_j^\varepsilon(t) \|_\infty \to 0$, uniformly in $t \in [0, T(\lambda, \| g_0 \|_\infty)]$. ■

Remarks. The only difference between the Lanford argument for hard spheres and the present one is that here we can prove a stronger (uniform) convergence at time t, the same as that assumed at time zero. This is due to the stochasticity of the free motion. For Hamiltonian systems it is unavoidable to have some weaker convergence at time t because of the reversibility-irreversibility transition occurring in the B–G limit, (see [6] for details).

A general global existence theorem for the two-dimensional Broadwell equation is still lacking. However for special cases, as for space-homogeneous initial data, or when there is a one-dimensional symmetry, a unique, smooth, global solution can be constructed. As an outcome of our analysis we shall see that, in these cases, the short time analysis carried out in this section can be extended to arbitrary times.

We observe that the knowledge of a uniform bound on the limiting solution, say

$$\| g_{j,t} \|_\infty \leqq c(T)^j, \tag{3.16}$$

valid up to the time T introduced in Theorem 2.1 does not allow us to extend the convergence result up to such a time. In this section we have established that for $t \leqq T(\lambda, \| g_0 \|_\infty)$

$$\| \Delta_j^\varepsilon(t) \|_\infty \leqq c^j \sqrt{\varepsilon t}, \tag{3.17}$$

where the above constant c increases with $\sup_j (\| \rho_{j,t}^\varepsilon \|_\infty)^{1/j}$. It is clear that (3.17) combined with (3.16) is not enough to iterate the short time convergence to reach arbitrary times.

4. Proof of Theorem 2.1

According to (3.17) the difference $|g_{j,t} - \rho_{j,t}^\varepsilon|$ is only bounded by $c^j \sqrt{\varepsilon t}$; if we had, instead, a bound like $c^j (\sqrt{\varepsilon t})^j$ then using (3.16) we could improve the bound for $\rho_{j,t}^\varepsilon$ and iterate the short time analysis past the time $T(\lambda, \| g_0 \|_\infty)$. We notice that if $\rho_{j,t}^\varepsilon$ is going to converge to $g_{j,t}$ then in the limit the $\rho_{j,t}^\varepsilon$'s will factorize as the $g_{j,t}$'s actually do. With this in mind we introduce the functions:

$$v_t(X) = \sum_{Y \subset X} (-1)^{|X/Y|} g_t(X/Y) \rho_t^\varepsilon(Y), \tag{4.1}$$

where X is a configuration with finitely many particles, $\sum_{Y \subset X}$ denotes the sum over all the subsets of X, and $\rho_t^\varepsilon(Y)$ is a shorthand for $\rho_{|Y|,t}^\varepsilon(Y)$. In the sequel we shall denote the restriction of v_t on the configurations X with j particles by $v_{j,t}(X)$. The v-functions reduce then to:

$$v_{j,t}(X) = \prod_{i=1}^{j} (g_t - \rho_{1,t}^\varepsilon)(x_i) \tag{4.2}$$

if the $\rho_{j,t}^\varepsilon$'s strictly factorize. Therefore we may hope that, at any fixed time, the v-functions will be bounded by $c_j \omega(\varepsilon)^j$, where c_j does not depend on ε and $\omega(\varepsilon)$ is infinitesimal in ε. A control on the size of the v-functions gives a measure of the tendency of the system not only to converge to the limiting kinetic behavior, but also to gain the factorization property. This means that the processes associated to each particle are going to become independent according to the so-called *propagation of chaos* hypothesis which is used in the heuristic derivation of the Boltzmann equation. However a bound like the above one, i.e. $c_j \omega(\varepsilon)^j$, is not at all easy to derive even for short times. Following the definition given in (4.1), one can write down an evolution equation for the v-functions (see Appendix A) which involves not only operators which increase the number of particles, (as $C_{j,j+1}$) but also others which decrease it. A naive bound can be obtained by taking sup norms as in Sect. 3, getting an integral inequality for $v_j(t) \equiv \|v_j\|_\infty$ which looks like the following one:

$$v_j(t) \leqq v_j(0) + c\int_0^t ds j v_{j+1}(s) + j^2[v_j(s) + v_{j-1}(s)],$$

$$v_j(0) = \varepsilon^j, \quad j = 0, 1 \ldots, \tag{4.3}$$

where c is some constant. In addition to (4.3) we have the a priori bound $v_j(t) \leqq c^j$ for short times, as follows from (4.1) and the analysis in Sect. 3. If we could improve the a priori bound showing that

$$v_j(t) \leqq (C\varepsilon^\delta)^j, \tag{4.4}$$

then we could easily iterate the argument of Sect. 3 by expressing ρ_t^ε in terms of v_t and g_t by means of (4.1). However the estimate (4.4) is clearly incompatible with (4.3): if we just take the contribution coming from the term v_{j-1} we get a bound like

$$\sum_{k=0}^{j} \frac{[j(j-1)\cdots(j-k+1)]^2}{k!} \varepsilon^{j-k} t^k. \tag{4.5}$$

We could still recover a bound like (4.4) if $t \approx \varepsilon^\beta$, $\beta \in (0, 1)$. However this is not a real gain because we would need an estimate like $\varepsilon^{\delta j}$ with $\delta > \beta$ to iterate the procedure to reach arbitrary finite times. The estimate (4.4) with $t < \varepsilon^\beta$ and $\delta > \beta$ will be obtained (see Proposition 4.1 below) by exploiting more carefully the real structure of the evolution equation for the v-functions. Unfortunately the best estimate that we have been able to derive gives a bound for v_j like $c(j)\varepsilon^{\delta j}$ with $c(j)$ which increases in j faster than exponentially. This new difficulty will be overcome by using probabilistic arguments.

Following these considerations we can outline the strategy of our proof. We fix an arbitrary time T and split the interval $[0, T]$ into intervals of length ε^β, $\beta \in (0, 1)$. On each of these time steps we estimate the v-functions conditioned to a *good* set of initial configurations (see Proposition 4.1 below). As a consequence we can compare the empirical distribution of the process at the end of the time step with the solution of an approximate Broadwell equation (see Proposition 4.2) and prove that their difference is small in probability. This allows us to say that at this final time the configuration is, with some large probability, good as required for iterating the procedure to a new time step. In this way we prove, see Proposition 4.4, that, typically, the process follows pieces of trajectories of the approximate Broadwell dynamics. We also prove that these pieces are uniformly close to the Broadwell dynamics so that the proof will be achieved.

From now on we shall put $N\varepsilon^2 = 1$ for notational simplicity and denote by η the function which associates to any $x \in \Gamma_\varepsilon$ the random variable number of particles $\eta(x)$ in the state x. The knowledge of η uniquely determines a configuration $X = \{x_1 \cdots x_N\}$ of particles modulo their labelling. Let $\xi = \xi(x)$ be any integer valued function on Γ_ε and ρ a symmetric function on the configuration space with finitely many particles. We set

$$\rho(\xi) = \rho_j(x_1 \cdots x_j) \tag{4.6}$$

for any configuration $x_1 \cdots x_j$ realizing ξ, that is:

$$\xi(x) = \sum_{i=1}^{j} \delta(x, x_i), \tag{4.7a}$$

where

$$\delta(x, x_i) = \begin{cases} 1, & \text{if } x = x_i \\ 0, & \text{otherwise} \end{cases} \tag{4.7b}$$

The following functions will be useful in solving combinatorial problems:

$$D(\xi, \eta) = \prod_x D_{\xi(x)}(\eta(x)), \tag{4.8}$$

$$D_k(j) = j(j-1)\cdots(j-k+1). \tag{4.9}$$

Let $\rho(\cdot\,; \mu)$ be the family of rescaled correlation functions relative to the measure μ. Then

$$\rho(\xi; \mu) = E_\mu(D(\xi, \eta)). \tag{4.10}$$

To understand (4.10) suppose μ concentrated on the singleton η. Denote by $x_1 \cdots x_j$ a configuration which realizes ξ and by $P_\mu(x_1 \cdots x_j)$ the probability of finding the first j particles in the states $x_1 \cdots x_j$. Then:

$$P_\mu(x_1 \cdots x_j) = \prod_x \frac{\eta(x)(\eta(x)-1)\cdots(\eta(x)-\xi(x)+1)}{N(N-1)\cdots(N-j+1)}. \tag{4.11}$$

Therefore, according to (2.15) we have:

$$\rho(\xi; \mu) = D(\xi, \eta). \tag{4.12}$$

The general statement (4.10) follows by integrating on η.

For any configuration η, consider $g^\varepsilon(\cdot|\eta)$ the solution of the following initial value problem:

$$\begin{aligned} \frac{\partial}{\partial t} g_t^\varepsilon(q, \sigma|\eta) &= \varepsilon^{-1} L_0 g_t^\varepsilon(q, \sigma|\eta) + g_t^\varepsilon(q, \sigma^\perp|\eta) g_t^\varepsilon(q, -\sigma^\perp|\eta) \\ &\quad - g_t^\varepsilon(q, \sigma|\eta) g_t^\varepsilon(q, -\sigma|\eta) \\ g_0^\varepsilon(q, \sigma|\eta) &= \eta(q, \sigma), \end{aligned} \tag{4.13}$$

which is an approximate version of the Broadwell equation on the lattice Λ_ε. Let $\rho_t^\varepsilon(\cdot|\eta)$ be the rescaled correlation functions of the distribution of the process $\eta(t)$ which starts from η at time zero. Alternatively $\rho_t^\varepsilon(\cdot|\eta)$ is the solution of Eq. (2.16) associated to the initial datum $D(\cdot|\eta)$. We define (compare with (4.1)):

$$v_t(\xi|\eta) = \sum_{\zeta \subset \xi} (-1)^{|\zeta|} g_t^\varepsilon(\zeta|\eta) \rho_t^\varepsilon(\xi - \zeta|\eta), \tag{4.14}$$

where $|\zeta|$ denotes the cardinality of the set $\{\zeta(x), x \in \Gamma_\varepsilon\}$, $\sum_{\zeta \subset \xi}$ means that we sum over all the set of particles ζ contained in the set ξ, i.e.

$$\sum_{\zeta \subset \xi} = \prod_x \sum_{\zeta(x) \subset \xi(x)} = \prod_{x \in \zeta} \sum_{\zeta(x) \leq \xi(x)} \frac{D_{\zeta(x)}(\xi(x))}{\zeta(x)!} \tag{4.15}$$

and

$$g_t^\varepsilon(\zeta|\eta) = \prod_x g_t^\varepsilon(x|\eta)^{\zeta(x)}. \tag{4.16}$$

Denoting by E_η the conditional expectation that at time zero the process is in the state η, we have

$$v_t(\xi|\eta) = E_\eta\left(\prod_x \sum_{\zeta(x)\subset\xi(x)} (-1)^{\zeta(x)} g_t^\varepsilon(x|\eta)^{\zeta(x)} D_{(\xi-\zeta)(x)}(\eta_t)\right). \tag{4.17}$$

Notice that if $\xi(x) = 0, 1$ for all x, i.e. there is no more than one particle per state, then:

$$v_t(\xi|\eta) = E_\eta\left(\prod_x (\eta(x) - g_t^\varepsilon(x|\eta)\right). \tag{4.18}$$

To estimate the v-functions we write, after straightforward but tedious calculations, the following evolution equation:

$$\frac{d}{dt} v_t(\xi|\eta) = \varepsilon^{-1} L_0 v_t(\xi|\eta) + \sum_\zeta a_t(\xi,\zeta) v_t(\zeta|\eta). \tag{4.19}$$

The operators $a_t(\xi,\zeta)$ are nonvanishing only when $|\xi| - 2 \leqq |\zeta| \leqq |\xi| + 1$. We do not give here their explicit expression referring the reader to Appendix A, where (4.19) is derived.

The main technical point of our result is the following proposition whose proof will be given in Sect. 5:

Proposition 4.1. *Let $\beta = \frac{1}{100}$, $\delta = \frac{1}{4}$, $\zeta = \frac{1}{1000}$. Suppose that for all x $\eta(x) \leqq \varepsilon^{-\zeta}$. Then for all $t \leqq \varepsilon^\beta$ we have*

$$|v_t(\xi|\eta)| < c(j)\varepsilon^{-\zeta j}\left(\frac{\varepsilon}{t}\right)^{\delta j}, \tag{4.20}$$

where $c(j)$ is a positive constant depending only on j and $j = |\xi|$.

Remark. The choice of the parameters β, δ, ζ in Proposition 4.1 is obviously rather arbitrary, but good enough for our purposes.

The above proposition allows us to control the deviation of the process η_t from the approximate solution of the Broadwell equation $g_t^\varepsilon(x|\eta)$ by means of the following seminorm:

$$\|f\| = \sup_x \left|\sum_y P_{\varepsilon^{1/4}}^\varepsilon(x|y) f(y)\right|. \tag{4.21}$$

We remark that $P_{\varepsilon^{1/4}}^\varepsilon(x|y)$, $x = (q, \sigma)$, is essentially supported by all $y = (q', \sigma)$ such that q varies in an interval of length $\varepsilon^{-3/8} = (\varepsilon^{-1}\varepsilon^{1/4})^{1/2}$. Therefore (4.21) expresses the supnorm of the one dimensional averages on intervals of size $\varepsilon^{-3/8}$. The introduction of these averages is necessary to smooth out the local fluctuations of η_t. However the seminorm (4.21) will be sufficient to ensure a macroscopic local control since $\frac{3}{8} < 1$.

Proposition 4.2. *Under the same hypotheses of Proposition 4.1, setting $\gamma = \frac{1}{16}$, for* *all $n > 0$ there exists $c(n)$ such that*:

$$\Pr_\eta(\| g_{\varepsilon^\beta}(\cdot|\eta) - \eta_{\varepsilon^\beta} \| > \varepsilon^\gamma) < c(n)\varepsilon^n, \tag{4.22}$$

where $\Pr_\eta$ denotes the law of the process starting from η at time zero.

Proof. By the Chebichev inequality, the left-hand side of (4.22) is bounded by:

$$\varepsilon^{-2j\gamma} E_\eta(\| g_{\varepsilon^\beta}(\cdot|\eta) - \eta_{\varepsilon^\beta} \|^{2j}) \leqq \varepsilon^{-2j\gamma} \sum_x \sum_{y_1 \cdots y_{2j}} P^\varepsilon_{\varepsilon^{1/4}}(x|y_1) \cdots P^\varepsilon_{\varepsilon^{1/4}}(x|y_{2j})$$
$$\cdot E_\eta\left(\prod_{k=1}^{2j} (g_{\varepsilon^\beta}(y_k|\eta) - \eta_{\varepsilon^\beta}(y_k)) \right). \tag{4.23}$$

Consider first the contribution coming from points $y_1 \cdots y_{2j}$ which are mutually distinct. By (4.18) and Proposition 4.1:

$$E_\eta\left[\prod_{k=1}^{2j} (g^\varepsilon_{\varepsilon^\beta}(y_k|\eta) - \eta_{\varepsilon^\beta}(y_k)) \right] = v_{\varepsilon^\beta}(y_1 \cdots y_{2j}|\eta) \leqq c(2j)\varepsilon^{2j(1-\beta)\delta}\varepsilon^{-2\zeta j}, \tag{4.24}$$

hence the sum in (4.23) over all distinct states is bounded by

$$c(2j)\varepsilon^{2j(1-\beta)\delta}\varepsilon^{-2}\varepsilon^{-2\gamma j}\varepsilon^{-2\zeta j}. \tag{4.25}$$

The general case can be recovered by combining the probability estimate that two particles are in the same site, see (3.11), with estimates on the v-functions of lower order. In fact if $y_1 \cdots y_{2j}$ is a configuration with n, $(n \leqq 2j)$ particles on the same state x, then in the product in (4.23) we have a factor:

$$(\bar{g} - \bar{\eta})^n, \tag{4.26}$$

where $\bar{\eta} \equiv \eta_{\varepsilon^\beta}(x)$ and $\bar{g} \equiv g^\varepsilon_{\varepsilon^\beta}(x|\eta)$. The following identity is easily proved by induction:

$$\bar{\eta}^k = \sum_{h=1}^{k} A(k,h) D_h(\bar{\eta}), \tag{4.27}$$

where the $A(k,h)$ are defined by recurrence:

$$A(k,h) = 1 \quad \text{if} \quad k = h \quad \text{or} \quad h = 1,$$
$$A(k,h) = hA(k-1,h) + A(k-1,h-1). \tag{4.28}$$

The explicit form of $A(k,h)$ is not relevant at all, we simply remark that $A(k,h)$ may be bounded by $(2h)^k$. We define:

$$V_k(\bar{\eta}, \bar{g}) = \sum_{s=0}^{k} \frac{k!}{(k-s)!s!} (\bar{g})^{k-s} D_s(\bar{\eta}). \tag{4.29}$$

By inverting (4.29) we have:

$$D_k(\bar{\eta}) = \sum_{s=0}^{k} \frac{k!}{(k-s)!s!} \bar{g}^{k-s} V_s(\bar{\eta}, \bar{g}). \tag{4.30}$$

Using (4.27) we can express $\bar{\eta}^n$ in terms of $V_s(\bar{\eta}, \bar{g})$ to obtain:

$$(\bar{g} - \bar{\eta})^n = \sum_{s=0}^{n} \sum_{h=1}^{s} \sum_{r=0}^{h} \frac{n!}{(n-s)!s!} \frac{h!}{(h-r)!r!} A(s,h)(-1)^s \bar{g}^{k+r-s} V_r(\bar{\eta}, \bar{g}). \tag{4.31}$$

Since

$$v_t(\xi|\eta) = E_\eta\left[\prod_x V_{\xi(x)}(\eta_{\varepsilon^\beta}(x), g^\varepsilon_{\varepsilon^\beta}(x))\right] \tag{4.32}$$

by inserting (4.31) in the expectation in (4.23), we obtain a linear combination of v-functions whose minimum order is given by the number of y_i's having only one particle per site. Hence the contribution to the sum $\sum_{y_1 \cdots y_{2j}}$ relative to the event with exactly k isolated particles may be estimated by

$$c(j)\varepsilon^{-2}\varepsilon^{-2j\gamma}\varepsilon^{(1-\beta)\delta k}\varepsilon^{-\zeta k}\varepsilon^{(2j-k)3/8}\varepsilon^{-(2j-k)\zeta}, \tag{4.33}$$

the first factor $c(j)$ is a suitable constant independent of ε, the second factor comes from the sum over $x \in \Lambda_\varepsilon$, the third one was already present in (4.23), the fourth and fifth terms are consequences of Proposition 4.1 and the sixth arises from estimate (3.11). Finally the last one bounds the g's in the expansion (4.31). This last bound follows from the short time analysis (recall that $\beta > \zeta$) of Sect. 3 which applies as well to the discretized Broadwell equation (4.13). Since:

$$\zeta + \gamma < (1-\beta)\delta, \quad \zeta + \gamma < \tfrac{3}{8}, \tag{4.34}$$

the proof is achieved. ■

In order to iterate the probability estimate given by Proposition 4.2, we introduce a set of trajectories well behaving locally in time, with respect to the approximate Broadwell dynamics.

Definition. We denote by H_n the set of all sequences $\{\eta^{(k)}\}_{k=0,\ldots,n}$ in Γ^N_ε, $(\eta^{(0)} = \eta)$ such that:

$$\eta^{(k)} \leqq \varepsilon^{-\zeta}, \tag{4.35}$$

$$\|\eta - g^\varepsilon_0\| \leqq \varepsilon^\gamma, \tag{4.36}$$

$$\|\eta^{(k)} - g^\varepsilon_{\varepsilon^\beta}(\cdot|\eta^{(k-1)})\| \leqq \varepsilon^\gamma, \quad k = 1, \ldots, n. \tag{4.37}$$

Let T be an arbitrary fixed time, we set m = integer part of $T\varepsilon^{-\beta} + 1$.

The next proposition (combined with (4.37)) shows that the solutions of the Broadwell equation are well approximated by the trajectories in H_m.

Proposition 4.3. *In the same hypotheses of Theorem 2.1, if* $\{\eta^{(k)}\} \in H_m$ *then there exists a constant* $c > 0$ *such that*:

$$|g^\varepsilon_{\varepsilon^\beta}(x|\eta^{(k-1)}) - g_{k\varepsilon^\beta}(x)| \leqq ck\varepsilon^\gamma. \tag{4.38}$$

Proof. We prove Proposition 4.3 by induction so that we assume the estimate (4.38) true up to $k-1$. We set

$$h(x,t) = g^\varepsilon_t(x|\eta^{(k-1)}) - g_{(k-1)\varepsilon^\beta + t}(x), \tag{4.39}$$

$$H(x,t) = g^\varepsilon_t(x|\eta^{(k-1)}) + g_{(k-1)\varepsilon^\beta + t}(x), \tag{4.40}$$

Then

$$h(\cdot,t) = P^\varepsilon_t(\eta^{(k-1)} - g_{(k-1)\varepsilon^\beta})(\cdot) + P^\varepsilon_t - V_t(g_{(k-1)\varepsilon^\beta}(\cdot) + \int_0^t ds P^\varepsilon_{t-s} Q(h(\cdot,s), H(\cdot,s))$$

$$+ \int_0^t ds(P^\varepsilon_{t-s} - V_{t-s})Q(g_{(k-1)\varepsilon^\beta+s}(\cdot)), \tag{4.41}$$

where

$$\begin{aligned} Q(f,g)(q,\sigma) &= \tfrac{1}{2}\{f(q,-\sigma^{\perp})g(q,\sigma^{\perp}) - f(q,\sigma)g(q,-\sigma) \\ &\quad + g(q,-\sigma^{\perp})f(q,\sigma^{\perp}) - g(q,\sigma)f(q,-\sigma)\}, \\ Q(f) &= Q(f,f). \end{aligned} \tag{4.42}$$

We denote by T_i, $i=1,\dots,4$ the four terms in the right-hand side of (4.41). We estimate them separately. For $\varepsilon^{\beta} \geqq t \geqq \varepsilon^{1/4}$,

$$|T_1| \leqq \|\eta^{(k-1)} - g^{\varepsilon}_{\varepsilon^{\beta}}(\cdot|\eta^{(k-2)})\| + \|g^{\varepsilon}_{\varepsilon^{\beta}}(\cdot|\eta^{(k-2)}) - g_{(k-1)\varepsilon^{\beta}}\| \leqq \varepsilon^{\gamma} + (k-1)C\varepsilon^{\gamma}, \tag{4.43}$$

where the first term arises by the assumption that $\{\eta^{(k)}\}\in H_m$ and the second follows by the induction hypothesis. Moreover:

$$|T_2| \leqq C_1\varepsilon^{1/2} \tag{4.44}$$

as follows by (3.13) and the hypothesis of the regularity of the solution g. To estimate the third term we split the time interval into two parts: $s\in[0,\varepsilon^{1/4}]$ and $s\in[\varepsilon^{1/4},t]$. In the first we simply use the short time estimate:

$$Q(h(\cdot,s),H(\cdot,s)) \leqq C_2\varepsilon^{-2\zeta} \tag{4.45}$$

in the second:

$$\begin{aligned} |Q(h(\cdot,s),H(\cdot,s))| &\leqq |Q(h(\cdot,s))| + 2|Q(h(\cdot,s),g_{(k-1)\varepsilon^{\beta}+s})| \\ &\leqq C_3\|h(\cdot,s)\|_{\infty}(1+\|h(\cdot,s)\|_{\infty}), \end{aligned} \tag{4.46}$$

so that

$$|T_3| \leqq C_2\varepsilon^{-2\zeta}\varepsilon^{1/4} + C_3\int_0^t ds\,\chi(s>\varepsilon^{1/4})\|h(\cdot,s)\|_{\infty}(1+\|h(\cdot,s)\|_{\infty}). \tag{4.47}$$

Finally the last term can be bounded in the same way as the second one. In conclusion we have obtained the following integral inequality: for $t\in[\varepsilon^{1/4},\varepsilon^{\beta}]$,

$$\begin{aligned} \|h(\cdot,t)\|_{\infty} &\leqq C_4\varepsilon^{1/2} + C_2\varepsilon^{-2\zeta}\varepsilon^{1/4} + \varepsilon^{\gamma} + (k-1)C\varepsilon^{\gamma} \\ &\quad + C_3\int_0^t ds\,\|h(\cdot,s)\|_{\infty}(1+\|h(\cdot,s)\|_{\infty})\chi(s>\varepsilon^{1/4}). \end{aligned} \tag{4.48}$$

Because $\frac{1}{4} - 2\zeta > \gamma$, we can first conclude that the first time $\bar{s}$ for which $\|h(\cdot,\bar{s})\|_{\infty} \geqq 1$ is larger than ε^{β}; we then solve the linear integral inequality obtained from (4.48) replacing $(1+\|h(\cdot,s)\|_{\infty})$ by 2. Finally we choose c so large that the estimate

$$\|h(\cdot,t)\|_{\infty} < ck\varepsilon^{\gamma} \tag{4.49}$$

holds.

The case $k=1$ can be recovered in a similar fashion so that the proof is achieved. ■

We now prove that, with large probability, the sequence $\eta_{k\varepsilon^{\beta}}$ obtained by observing the process at times $k\varepsilon^{\beta}$ is in H_m. Define the set of trajectories:

$$H_n^* = \{(\eta_t)_{t\geqq 0} | \{\eta_{k\varepsilon^{\beta}}\}_{k=0,\dots,n}\in H_n\}. \tag{4.50}$$

Then

Proposition 4.4. *For any $n > 0$ there exists $c(n)$ such that:*

$$P_{\mu^\varepsilon}(H_m^*) > 1 - c(n)\varepsilon^n, \tag{4.51}$$

where P_{μ^ε} is the law of the process distributed at time zero according to μ^ε.

Proof. Denote by χ_k the characteristic function of the set H_k^* and by η the value of the process at time $(k-1)\varepsilon^\beta$. Then:

$$E_{\mu^\varepsilon}(\chi_k) = E_{\mu^\varepsilon}(\chi_{k-1} \Pr_\eta\{\|\eta_{k\varepsilon^\beta}\|_\infty < \varepsilon^{-\zeta}; \|\eta_{\varepsilon^\beta} - g_{\varepsilon^\beta}^\varepsilon(\cdot|\eta)\| < \varepsilon^\gamma\}). \tag{4.52}$$

If $\eta \leqq \varepsilon^{-\zeta}$ we already know by Proposition 4.2 that:

$$\Pr_\eta\{\|\eta_{\varepsilon^\beta} - g_{\varepsilon^\beta}^\varepsilon(\cdot|\eta)\| > \varepsilon^\gamma\} \leqq c(n)\varepsilon^n \tag{4.53}$$

which controls the second event in (4.52). For the first term we have

$$\Pr_\eta\{\|\eta_{\varepsilon^\beta}\|_\infty > \varepsilon^{-\zeta}\} \leqq \Pr_\eta\{\|g_{\varepsilon^\beta}^\varepsilon(\cdot|\eta)\|_\infty > \tfrac{1}{2}\varepsilon^{-\zeta}\} + \Pr_\eta\{\|\eta_{\varepsilon^\beta} - g_{\varepsilon^\beta}^\varepsilon(\cdot|\eta)\|_\infty > \tfrac{1}{2}\varepsilon^{-\zeta}\}. \tag{4.54}$$

By Proposition 4.3 we know that:

$$\|g_{\varepsilon^\beta}^\varepsilon(\cdot|\eta) - g_{k\varepsilon^\beta}(\cdot)\|_\infty < C(\varepsilon^{-\beta}T + 1)\varepsilon^\gamma \tag{4.55}$$

so that, since $\gamma > \beta$, we have an uniform bound on $\|g_{\varepsilon^\beta}^\varepsilon(\cdot|\eta)\|_\infty$ inherited by the assumed boundedness of the Broadwell solution $g_{k\varepsilon^\beta}$. Hence the first probability appearing on the right-hand side of (4.54) is zero. By the Chebichev inequality the second term on the right-hand side of (4.54) is bounded by:

$$2^{2n}\varepsilon^{2\zeta n}\sum_x E_{\mu^\varepsilon}\{(\eta_{\varepsilon^\beta}(x) - g_{\varepsilon^\beta}^\varepsilon(x|\eta))^{2n}\}. \tag{4.56}$$

Now we expand (4.56) by using (4.31) as we did before. We obtain a linear combination of terms with products of v-functions and powers of $g_{\varepsilon^\beta}^\varepsilon$. Both are known to be bounded uniformly in ε so that, for an arbitrary n:

$$E_{\mu^\varepsilon}(\chi_k) \geqq E_{\mu^\varepsilon}(\chi_{k-1})(1 - c(n)\varepsilon^n). \tag{4.57}$$

The estimate of $E_{\mu^\varepsilon}(\chi_0)$ is obvious by the assumption on μ^ε so that by iteration we conclude the proof. ■

We are now ready to conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. By definition, putting $t = k\varepsilon^\beta$:

$$\rho_t^\varepsilon(\xi) = E(D(\xi, \eta_t)) = E(\chi_{k-1}D(\xi, \eta_t)) + E([1 - \chi_{k-1}]D(\xi, \eta_t)). \tag{4.58}$$

By using the Cauchy–Schwartz inequality, the last term is bounded:

$$\begin{aligned} E([1 - \chi_{k-1}]D(\xi, \eta_t)) &\leqq E([1 - \chi_{k-1}])^{1/2}E(D(\xi, \eta_t))^2)^{1/2} \\ &\leqq C(n)\varepsilon^{n/2}E\left(\prod_x \eta_t(x)^{2\xi(x)}\right)^{1/2} \\ &\leqq C(n)\varepsilon^{n/2}\varepsilon^{-2|\xi|}. \end{aligned} \tag{4.59}$$

Here we have used the conservation of the number of particles for which $\eta_t(x) < \varepsilon^{-2}$. For fixed $|\xi|$ and sufficiently large n the right-hand side of (4.59) vanishes in the limit $\varepsilon \to 0$. Finally:

$$E(\chi_{k-1}D(\xi,\eta_t)) - \prod_x g_t(x)^{\xi(x)}) = E\left(\chi_{k-1}\left[D(\xi,\eta_t) - \prod_x g^\varepsilon_{\varepsilon^\beta}(x|\eta)^{\xi(x)}\right]\right)$$
$$+ E\left(\chi_{k-1}\left[\prod_x g^\varepsilon_{\varepsilon^\beta}(x|\eta)^{\xi(x)} - \prod_x g_t(x)^{\xi(x)}\right]\right), \tag{4.60}$$

where η denotes, as before, a configuration at time $(k-1)\varepsilon^\beta$. By Proposition 4.3 the last term of the right-hand side of (4.60) is also vanishing. Finally the first term in the right-hand side of (4.60) can be expressed in terms of a linear combination of v-functions, (see formula 4.30) for which, by Proposition 4.1, we can conclude that all terms are vanishing.

From this it is easy to prove convergence uniformly in x and t on the compacts. ■

5. Proof of Proposition 4.1

The analysis at the beginning of Sect. 3 applies as well when the initial measure is supported by a single configuration η: for each $\varepsilon > 0$ let the configuration η be such that $\eta(x) \leqq \varepsilon^{-\zeta}$, for all x, and denote by $\rho^\varepsilon_{j,t}(\cdot|\eta)$ the correlation functions when the initial measure is supported by η. By (3.6) (here $\|g_0\|_\infty \leqq \varepsilon^{-\zeta}$)

$$\|\rho^\varepsilon_{j,t}(\cdot|\eta)\|_\infty \leqq 2^j \varepsilon^{-j\zeta} \tag{5.1a}$$

for $t \leqq (12\lambda\varepsilon^{-\zeta})^{-1}$. Hence (5.1a) is satisfied for all times $t \leqq \varepsilon^\beta$, those relevant in the proof of Proposition 4.1 (recall that $\beta = 1/100$ and $\zeta = 1/1000$). Analogously it can be proven that the solution $g^\varepsilon_t(\cdot|\eta)$ of (4.13) is bounded by

$$\|g^\varepsilon_t(\cdot|\eta)\|_\infty \leqq 2\varepsilon^{-\zeta}. \tag{5.1b}$$

From (4.17) and (5.1) we then derive (4.20) for $t \leqq \varepsilon$. For the remaining values of t a more careful analysis is required, as we shall see hereafter.

We recall that P^ε_t is the semigroup generated by $\varepsilon^{-1}L_0$ and denote by $P^\varepsilon_t(X|Y)$ its kernel. By (4.19), denoting by $a_t v_t(\xi|\eta) = \sum_\zeta a_t(\xi,\zeta)v_t(\zeta|\eta)$, we have:

$$v_t(\xi|\eta) = P^\varepsilon_t v_0(\xi|\eta) + \int_0^t ds P^\varepsilon_{t-s} a_s v_s(\xi|\eta)$$
$$= \sum_{n=0}^{m-1} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n P^\varepsilon_{t-t_1} a_{t_1} \cdots P^\varepsilon_{t_{n-1}-t_n} a_{t_n} P^\varepsilon_{t_n} v_0(\xi|\eta) + R(t,m), \tag{5.2a}$$

where

$$R(t,m) = \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m P^\varepsilon_{t-t_1} a_{t_1} \cdots P^\varepsilon_{t_{m-1}-t_m} a_{t_m} P^\varepsilon_{t_m} v_{t_m}(\xi|\eta) \tag{5.2b}$$

and $m = 100j$, $j = |\xi| =$ cardinality of ξ. We recall that

$$a_t = \sum_{q=1}^{12} a^q_t,$$

where a_t^q, $q=1,\dots,12$ is the finite sequence of the operators $C^\pm$, $K_{1,2}^\pm$, $A_{1,2}^\pm$, $L_I^\pm$ defined in Appendix A and given in some order. We shall first estimate:

$$P_{t-t_1}^\varepsilon a_{t_1}^{q_1}\cdots P_{t_{n-1}-t_n}^\varepsilon a_{t_n}^{q_n} P_{t_n}^\varepsilon |v_0(\xi|\eta)|. \tag{5.3}$$

The particles appearing in the expansion (5.3) are at most $j+n$ (this value is achieved when only operators of type C are present). Thus we need to estimate the following transition kernel:

$$P_{t-t_1}^\varepsilon a_{t_1}^{q_1}(i_1,j_1)\cdots P_{t_{n-1}-t_n}^\varepsilon a_{t_n}^{q_n}(i_n,j_n) P_{t_n}^\varepsilon |v_0(\xi|\eta)|, \tag{5.4}$$

where $a_t^q=\sum_{i,j} a_t^q(i,j)$, $a_t^q(i,j)$ is the contribution to the operator a_t^q coming from the pair of particles i and j and finally $a_t^q(i,j)=0$ if $i=j$ for the operators of type A and L_I. For the operators C and K, $a_t^q(i,j)$ is the contribution due to the particle i and $a_t^q(i,j)=0$ if $i\neq j$. Moreover we use the convention that $a_t^q(i,j)=0$ if the particle i and j are not present at time t.

We denote by $\Gamma=\{q_1,\dots,q_n;(i_1j_1),\dots,(i_n,j_n)\}$ the set of all possible choices of operators and pairs of interacting particles. The algebraic structure of the process is entirely given by $\gamma\in\Gamma$. For each γ we have the following estimate:

Proposition 5.1. *For each fixed $\gamma\in\Gamma$, $n<m\equiv 100j$ and $\varepsilon\leqq t\leqq\varepsilon^\beta$,*

$$\left|\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \sum_Y (P_{t-t_1}^\varepsilon a_{t_1}^{q_1}\cdots a_{t_n}^{q_n} P_{t_n}^\varepsilon v_0)(X|Y)\right| \leqq c(j)\varepsilon^{-(j+2n)\zeta}\left(\frac{\varepsilon}{t}\right)^{j/4} t^{n/2}. \tag{5.5}$$

Since $t\leqq\varepsilon^\beta$ the factor $\varepsilon^{-2n\zeta}t^{n/2}\to 0$ as $\varepsilon\to 0$, hence Proposition 5.1 will imply Proposition 4.1, once we prove an analogous statement for the remainder term (5.2b).

Proof. Since v_0 vanishes whenever there is a state occupied by a single particle, we have

$$(5.4)\leqq\sum_Y \varepsilon^{-\zeta|Y|}(P_{t-t_1}^\varepsilon a_{t_1}^{q_1}(i_1,j_1)\cdots a_{t_n}^{q_n}(i_n,j_n)P_{t_n}^\varepsilon)(X|Y)\hat{\chi}(Y), \tag{5.6}$$

where X is the set of states realized by ξ and $\hat{\chi}(Y)$ vanishes unless there are at least two particles per site. Therefore

$$\hat{\chi}(Y)=\sum_{\mathscr{D}}\prod_{i=1}^{|\mathscr{D}|}\hat{\chi}_{\mathscr{D}_i}(Y), \tag{5.7}$$

where $\mathscr{D}=\{\mathscr{D}_1,\dots,\mathscr{D}_s\}$, $|\mathscr{D}|=s$, is a partition of the particle labels in Y into atoms $\mathscr{D}_1,\dots,\mathscr{D}_s$ each one containing at least two elements. Finally $\hat{\chi}_{\mathscr{D}_i}(Y)=0$, unless all the particles of Y with labels in $\mathscr{D}_i$ are on the same site. Since the number of partitions is finite (once j is given) it will be enough to prove the estimate (5.5) for a fixed partition $\mathscr{D}$. Hence from now we shall consider a fixed value of γ and $\mathscr{D}$. We first consider those γ's where the terms a_t^q are only of type A and C, i.e. when there is no operator which leaves invariant the number of particles. Then the transition kernel (5.4) has the following structure:

$$S_{t-\tau_1}\mathscr{C}\cdots\mathscr{C}S_{\tau_{k-1}-\tau_k}\mathscr{C}S_{\tau_k}(X|Y), \tag{5.8}$$

where $\tau_1,\dots,\tau_k$ are the times when a new particle is created and the $\mathscr{C}$'s are the corresponding creation operators. Each operator S is a string whose transition kernel is given by:

$$S_{\tau_{r-1}-\tau_r}(X\,|\,Y) = P^{\varepsilon}_{\tau_{r-1}-s_1^r}\mathscr{A}\cdots\mathscr{A}P^{\varepsilon}_{s^r_{n(r)}-\tau_r}(X\,|\,Y), \tag{5.9}$$

where $\{s_i^r\}_{i=1}^{n(r)}$ are the times when the particles die and the $\mathscr{A}$'s denote the corresponding destruction operators. It is obviously assumed that $S_{\tau_{r-1}-\tau_r} = P^{\varepsilon}_{\tau_{r-1}-\tau_r}$ if no particle is destroyed.

By using (5.1b) we get

$$S_{\tau_{r-1}-\tau_r}(X\,|\,Y) \leqq (4\varepsilon^{-2\zeta})^{n(r)} \sum_{Z_1\cdots Z_{n(r)}} \prod_{i=1}^{n(r)} \chi_i P^{\varepsilon}_{\tau_{r-1}-s_1^r}(X|Z_1)P^{\varepsilon}_{s_1^r-s_2^r}(Z_1^*|Z_2)$$

$$\cdots P^{\varepsilon}_{s^r_{n(r)}-\tau_r}(Z^*_{n(r)}|\,Y), \tag{5.10}$$

where χ_i is the characteristic function of the event that the two particle involved in the interaction must stay at the same site at the time s_i^r. Z_i^* is the set Z_i deprived of one or two particles if A_1 respectively A_2 acted at time s_i^r (one or two particles dying in such a case). Furthermore when the operator A_1^+ acts then the velocity of the particle involved (say i) changes $\sigma_i \to \sigma_i^{\perp}$ according to (A.25). By applying the probability estimate (3.11) we easily obtain:

$$\sum_{Z_1} \chi_1 P^{\varepsilon}_{\tau_{r-1}-s_1^r}(X|Z_1)P^{\varepsilon}_{s_1^r-s_2^r}(Z_1^*|Z_2) \leqq c\sqrt{\frac{\varepsilon}{\tau_{r-1}-s_1^r}}\,Q^{\varepsilon}_{\tau_{r-1}-s_2^r}(X(Z_2)|Z_2), \tag{5.11}$$

where the set $X(Z_2)$ is the set X deprived of the particle (or particles) which died; $Q^{\varepsilon}(X\,|\,Y)$ denotes the transition probability associated to $|X|$ independent Poisson jump processes to which we add the transition $\sigma_i \to \sigma_i^{\perp}$ whenever the operator A_1^+ acts on particle i. Notice that Q^{ε} depends on $\gamma\in\Gamma$.

Iterating the procedure we obtain:

$$S_{\tau_{r-1}-\tau_r}(X\,|\,Y) \leqq (4\varepsilon^{-2\zeta})^{n(r)} Q^{\varepsilon}_{\tau_{r-1}-\tau_r}(X(Y)|\,Y)\sqrt{\frac{\varepsilon}{\tau_{r-1}-s_1^r}}\cdots\sqrt{\frac{\varepsilon}{\tau_{r-1}-s^r_{n(r)}}}, \tag{5.12}$$

where $X(Y)$ denotes the subset of X which survives in the time interval $\tau_{r-1}-\tau_r$. Before using (5.12) we need to estimate the action of the operators C. For simplicity let us first consider the case when all the operators C are of the form C^-. We now denote by $z_1\cdots z_k$ the single particle states of the newly created particles, cf. (5.8), (they are completely specified by $\gamma\in\Gamma$ and by the configurations just before the times of creation). Therefore:

$$(5.8) = \sum_{Z_1\cdots Z_k} S_{t-\tau_1}(X, Z_1)S_{\tau_1-\tau_2}(Z_1\cup z_1, Z_2)$$

$$\cdots S_{\tau_{k-1}-\tau_k}(Z_{k-1}\cup z_{k-1}, Z_k)S_{\tau_k}(Z_k\cup z_k, Y). \tag{5.13}$$

We introduce the notation:

$$(Z)_k, (Z)_0 \tag{5.14}$$

to indicate the set of particles in the configuration Z which are present at time t_k and at time 0, respectively. By using (5.12) we can bound (5.13) by the following quantity:

$$\sum_{Z_1\cdots Z_k} F(t,\tau_1)\cdots F(\tau_k,0)Q^{\varepsilon}_{t-\tau_1}((X)_1|Z_1)Q^{\varepsilon}_{\tau_1-\tau_2}((Z_1\cup z_1)_2|Z_2)\cdots Q^{\varepsilon}_{\tau_k}((Z_k\cup z_k)_0|\,Y), \tag{5.15}$$

where

$$F(\tau_r, \tau_{r-1}) = (4\varepsilon^{-2\zeta})^{-n(r)} \sqrt{\frac{\varepsilon}{\tau_{r-1} - s_1^r}} \cdots \sqrt{\frac{\varepsilon}{\tau_{r-1} - s_{n(r)}^r}} \prod_{i=1}^{|\mathscr{D}|} \hat{\chi}_{\mathscr{D}_i}(Y). \tag{5.16}$$

From now on we shall consider only those Y for which $\prod_{i=1}^{|\mathscr{D}|} \hat{\chi}_{\mathscr{D}_i}(Y) = 1$, recall that the partition $\mathscr{D}$ has been fixed once for all.

There are two possibilities. Either the particle created in the state z_k dies in the time interval $(0, \tau_k)$ and therefore

$$Q^{\varepsilon}_{\tau_k}((Z_k \cup z_k)_0 \,|\, Y) \equiv Q^{\varepsilon}_{\tau_k}((Z_k)_0 \,|\, Y) \tag{5.17}$$

or it survives, so that

$$Q^{\varepsilon}_{\tau_k}((Z_k \cup z_k)_0 \,|\, Y) = Q^{\varepsilon}_{\tau_k}((Z_k)_0 \,|\, Y^k) P^{\varepsilon}_{\tau_k}(z_k \,|\, y_k), \tag{5.18}$$

where y_k is the position in Y of this particle and $Y^k = Y/y_k$. In the first case, (5.17), we sum over Z_k and obtain the same expression as (5.15) with k replaced by $k-1$. In the second case, (5.18), we estimate $P^{\varepsilon}_{\tau_k}(z_k \,|\, y_k)$ by (3.11) so that the expression (5.15) is bounded by:

$$\sqrt{\frac{\varepsilon}{\tau_k}} F(t, \tau_1) \cdots F(\tau_k, 0) \sum_{Y^k} \sum_{Z_1 \cdots Z_k} Q^{\varepsilon}_{t-\tau_1}((X)_1 \,|\, Z_1)$$
$$\cdots Q^{\varepsilon}_{\tau_{k-1}}((Z_{k-1} \cup z_{k-1})_k \,|\, Y^k) \prod_{i=1}^{|\mathscr{D}|} \hat{\chi}_{\mathscr{D}_i^k}(Y^k), \tag{5.19}$$

where

$$\mathscr{D}^k \equiv \{\mathscr{D}_i^k\}_{i=1}^s \tag{5.20}$$

is the partition of the labels of Y^k obtained from $\mathscr{D}$ by eliminating y_k. We notice that $\mathscr{D}^k$ may have an atom (say $\mathscr{D}_i^k$) with a single element (correspondingly $\chi_{\mathscr{D}_i^k} = 1$).

We want to iterate the above procedure but now there is a new possibility, namely that the particle created in z_{k-1} belongs, at time 0, to a cluster with a single element. In this case, as well as in the case when the particle dies, we do not gain the factor $\sqrt{\varepsilon/\tau_{k-1}}$. In all cases summing first over the particles which are not alive at time 0 and then over y_{k-1} (if present) we complete this step of the iteration passing from $k-1$ to $k-2$.

We define $Y^h = Y^{h+1}/y_h$ for $h = 1, \ldots, k-1$, with the convention that $Y^h = Y^{h+1}$ whenever the particle z_h is not present at time 0. Analogously we define $\mathscr{D}^h$ as the partition derived from $\mathscr{D}$ by restricting it to the particles in Y^h. At the end of the iteration we obtain

$$\left[\prod_{i=1}^{h} \sqrt{\frac{\varepsilon}{\sigma_i}}\right] F(t, \tau_1) \cdots F(\tau_k, 0) \sum_{Y^1} Q^{\varepsilon}_t((X)_0 \,|\, Y^1) \prod_{i=1}^{|\mathscr{D}|} \hat{\chi}_{\mathscr{D}_i^1}(Y^1), \tag{5.21}$$

where $\sigma_1, \ldots, \sigma_h$ is the following subset of $\tau_1, \ldots, \tau_k$: $\tau_i \in \{\sigma_1, \ldots, \sigma_h\}$ if and only if z_i is present at time 0 and its label belongs to a cluster in $\mathscr{D}^{i+1}$ consisting of more than one element, $\mathscr{D}^{k+1} \equiv \mathscr{D}$. Moreover Y^1 is the image at time 0 of the particles $(X)_0$ in X which are alive at time 0. Finally $\mathscr{D}^1$ is the partition of Y^1 obtained from $\mathscr{D}$ after subtracting all the new particles alive at time 0. We finally bound

(5.21) by

$$\left[\prod_{i=1}^{h}\sqrt{\frac{\varepsilon}{\sigma_i}}\right]F(t,\tau_1)\cdots F(\tau_k,0)\prod_{i=1}^{|\mathscr{D}|}\left(\frac{\varepsilon}{t}\right)^{1/2(|\mathscr{D}_i^1|-1)}, \tag{5.22}$$

where $|\mathscr{D}_i^1|$ denotes the cardinality of $\mathscr{D}_i^1$ whenever it is larger than 1 and it equals 1 otherwise. We now insert the estimate (5.22) in the left-hand side of (5.5) and perform the time integrations. We get

$$\text{l.h.s. of (5.5)} \leqq c(j)\varepsilon^{-(j+2n)\zeta}\varepsilon^{(h/2)+(1/2)\sum_{r=1}^{k+1}n(r)+(1/2)\sum_{i=1}^{s}(|\mathscr{D}_i^1|-1)}(\sqrt{t})^{n-\sum_{i=1}^{s}(|\mathscr{D}_i^1|-1)}. \tag{5.23}$$

After some thought one can realize that

$$\sum_{i=1}^{s}(|\mathscr{D}_i|-1)=\sum_{i=1}^{s}(|\mathscr{D}_i^1|-1)+h, \tag{5.24a}$$

$$\sum_r n(r)\geqq\frac{M}{2}, \tag{5.24b}$$

where M is the total number of particles which have died. Then

$$\sum_i(|\mathscr{D}_i|-1)+\sum_r n(r)\geqq\frac{|Y|}{2}+\frac{M}{2}=\frac{j+k}{2}. \tag{5.25}$$

From (5.23), since $\varepsilon/t<1$ we obtain as a final estimate:

$$\text{l.h.s. of (5.5)}\leqq c(j)\varepsilon^{-(j+2n)\zeta}\left(\frac{\varepsilon}{t}\right)^{(j+k)/4}(\sqrt{t})^n. \tag{5.26}$$

If also the operators C^+ are present in the expansion (5.8), the same estimate (5.26) can be easily obtained after redefining Q^ε to take into account the transitions $\sigma\to\sigma^\perp$ of the particles which are involved in the interaction.

Exactly the same estimate may be obtained in the case when in the expression (5.9) there are operators as L_I and K. Consider the following string:

$$P_{s_0-s_1}\mathscr{L}P_{s_1-s_2}\mathscr{L}\cdots\mathscr{L}P_{s_{r-1}-s_r}(X,Y), \tag{5.27}$$

where $\{s_i\}_{i=1\ldots r}$ are the times when an operator $\mathscr{L}$ of type L_I or K acts, leaving the number of particles invariant. Removing the characteristic functions of the interaction we bound (5.27) by

$$2\varepsilon^{-r\zeta}\Pi^\gamma_{s_0-s_r}(X,Y), \tag{5.28}$$

where Π^γ is the transition probability of $|X|$ independent Poisson jump processes of intensity ε^{-1} in which the velocity of the particle is changed at time s_i according to the sequence γ. We can replace P by Π^γ and obtain the same estimate (5.26) for the general case. The extra factors $2\varepsilon^{-r\zeta}$ can be controlled by the time integrals yielding a factor $\varepsilon^{\beta r}$. Thus the proof of Proposition 5.1 is achieved. ■

By Proposition 5.1 we can easily bound the sum in the right-hand side of (5.2a) by the desired estimate.

Finally we estimate the remainder. By definition

$$|v_t|(\xi|\eta)\leqq\sum_{k=0}^{j}\frac{j!}{(j-k)!k!}(\|g_t^\varepsilon(\cdot|\eta)\|_\infty)^k(\|\rho^\varepsilon_{j-k,t}(\cdot|\eta)\|_\infty). \tag{5.29}$$

Using (5.1) and recalling that $\beta = 1/100$, we get

$$|R(t, m)| \leq c(j)\frac{t^m}{m!}\varepsilon^{-\zeta(j+m)}$$
$$\leq c(j)\varepsilon^j\varepsilon^{-101\zeta j} \tag{5.30}$$

and, with our choice of the parameters, the proof is achieved. ∎

Appendix A

In this appendix we derive Eq. (4.19) which describes the time evolution of the v-functions. We recall the hierarchy satisfied by the rescaled correlation functions (Eq. 2.16):

$$\frac{d}{dt}\rho^\varepsilon = \varepsilon^{-1}L_0\rho^\varepsilon + L_I\rho^\varepsilon + C\rho^\varepsilon. \tag{A.1}$$

Defining:

$$g^\varepsilon(x_1 \cdots x_j) = \prod_{i=1}^{j} g^\varepsilon(\xi), \tag{A.2}$$

where g^ε satisfies Eq. (4.13), we have for the set $\{g^\varepsilon(x_1 \cdots x_j)\}_{j=1}^\infty$ the following hierarchy of equations:

$$\frac{d}{dt}g^\varepsilon = \varepsilon^{-1}L_0 g^\varepsilon + Cg^\varepsilon. \tag{A.3}$$

According to Definition (4.14), the v-functions relative to the correlations ρ^ε are implicitly defined by the formula:

$$\rho^\varepsilon(X) = \sum_{Y \subset X} g^\varepsilon(Y)v(X/Y)) \equiv (g^\varepsilon * v)(X), \tag{A.4}$$

where $X = (x_1 \cdots x_j)$ denotes any set of states of single particle and the convolution product $*$ is defined by the last equality in (A.4). We first observe that the operator L_0 acts as a derivation for the product $*$:

$$L_0\rho = (L_0 v)*g + v*(L_0 g) \tag{A.5}$$

(from now on we skip the suffix ε for notational simplicity). We now express the collision and the interaction operators, C and L_I respectively, in the set formalism. We have:

$$C = C^+ - C^-, \tag{A.6}$$

$$C^+\rho(X) = \sum_{x \in X} \rho(X_x^\perp \cup x_-^\perp), \tag{A.7}$$

$$C^-\rho(X) = \sum_{x \in X} \rho(X \cup x_-), \tag{A.8}$$

where

$$x_- = (q, -\sigma), \quad x^\perp = (q, \sigma^\perp) \quad \textit{if} \quad x = (q, \sigma), \tag{A.9}$$

and

$$X_x^\perp = (X/x) \cup x^\perp. \tag{A.10}$$

Finally

$$L_I = L_I^+ + L_I^-, \tag{A.11}$$

and

$$L_I^+ \rho(X) = \frac{1}{2} \sum_{(x,y)\in X} \rho(X_{xy}^{\perp})\chi_{x,y}, \tag{A.12}$$

$$L_I^- \rho(X) = \frac{1}{2} \sum_{(x,y)\in X} \rho(X)\chi_{x,y}, \tag{A.13}$$

where

$$X_{xy}^{\perp} = (X/\{x, y\}) \cup \{x^{\perp}, y^{\perp}\} \tag{A.14}$$

and $\chi_{x,y} = \chi_{i,j}$ (cfr. Eq. (2.14)) whenever the particles i and j are in the states x and y. The following is straightforward algebra:

$$\begin{aligned} C^- \rho(X) &= \sum_{x\in X} \sum_{Z \subset X \cup x_-} v(Z) g(X \cup x_-/Z) \\ &= \sum_{T\subset X} \sum_{x\in X} v(T \cup x_-) g(X/T) + \sum_{T\subset X} \sum_{x\in X} v(T) g(X \cup x_-/T) \\ &= (C^- v) * g(X) + \sum_{T\subset X} \sum_{x\in X/T} v(T \cup x_-) g(X/T) + v * (C^- g)(X) \\ &\quad + \sum_{T\subset X} \sum_{x\in T} v(T) g(X \cup x_-/T) \\ &= (C^- v) * g(X) + v * (C^- g)(X) + (K_1^- v) * g(X) + (K_2^- v) * g(X), \end{aligned} \tag{A.15}$$

where

$$K_1^- v(T) = \sum_{x\in T} v(T \cup x_-/x) g(x), \tag{A.16}$$

$$K_2^- v(T) = \sum_{x\in T} v(T) g(x_-). \tag{A.17}$$

By analogous computations:

$$C^+ \rho(X) = (C^+ v) * g(X) + v * (C^+ g)(X) + (K_1^+ v) * g(X) + (K_2^+ v) * g(X), \tag{A.18}$$

where

$$K_1^+ v(T) = \sum_{x\in T} v(T \cup x_-^{\perp}/x) g(x^{\perp}), \tag{A.19}$$

$$K_1^+ v(T) = \sum_{x\in T} v(T \cup x^{\perp}/x) g(x_-^{\perp}), \tag{A.20}$$

Moreover

$$L_I^- \rho(X) = \frac{1}{2} \sum_{T\subset X} \sum_{x,y\in X} v(T) g(X/T) \chi_{x,y}. \tag{A.21}$$

We split the sum $x, y \in X$ in the four sums: $x, y \in T$; $y \in X/T$, $x \in T$; $x \in X/T$, $y \in T$; $x, y \in X/T$ and, after this, we arrive easily at the following identity:

$$L_I^- \rho = L_I^- v * g + A_1^- v * g + A_2^- v * g, \tag{A.22}$$

where

$$A_1^- v(T) = \sum_{x,y\in T} v(T/x) g(x) \chi_{x,y}, \tag{A.23}$$

$$A_2^- v(T) = \frac{1}{2} \sum_{x,y \in T} v(T/x, y) g(x) g(y) \chi_{x,y}. \tag{A.24}$$

Analogously

$$L_I^+ \rho = L_I^+ v * g + A_1^+ v * g + A_2^+ v * g, \tag{A.25}$$

where

$$A_1^+ v(T) = \sum_{x,y \in T} v(T/\{x, y\} \cup x^\perp) g(y^\perp) \chi_{x,y}, \tag{A.26}$$

$$(A_2^- v)(T) = \frac{1}{2} \sum_{x,y \in T} v(T/\{x, y\}) g(x^\perp) g(y^\perp) \chi_{x,y}. \tag{A.27}$$

Defining

$$K_i = K_i^+ - K_i^-, \quad i = 1, 2, \tag{A.28}$$

$$A_i = A_+^i - A_i^-, \quad i = 1, 2, \tag{A.29}$$

we arrive, finally, at the equation:

$$\frac{dv}{dt} = (\varepsilon^{-1} L_0 + L_I + K_1 + K_2 + C + A_1 + A_2) v. \tag{A.30}$$

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