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A Stochastic Particle System Modeling the Carleman Equation

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Two species of Brownian particles on the unit circle are considered; both have diffusion coefficient $\sigma > 0$ but different velocities (drift), 1 for one species and -1 for the other. During the evolution the particles randomly change their velocity: if two particles have the same velocity and are at distance $\leqslant \varepsilon$ (ε being a positive parameter), they both may simultaneously flip their velocity according to a Poisson process of a given intensity. The analogue of the Boltzmann–Grad limit is studied when ε goes to zero and the total number of particles increases like ε^{-1}. In such a limit propagation of chaos and convergence to a limiting kinetic equation are proven globally in time, under suitable assumptions on the initial state. If, furthermore, σ depends on ε and suitably vanishes when ε goes to zero, then the limiting kinetic equation (for the density of the two species of particles) is the Carleman equation.

KEY WORDS: Boltzmann–Grad limit; Carleman equation; stochastic interacting particle systems; propagation of chaos.

1. INTRODUCTION

One of the most important and still unsolved problem in non-equilibrium statistical mechanics is the derivation of kinetic and fluid dynamical equations (Boltzmann, Euler, Navier–Stokes equations) starting from a particle system which evolves according to Newton's laws of motion. Very few rigorous results are known. The Boltzmann equation has been derived from the Newton dynamics either for short times[(1)] or at all times but for a dilute cloud of gas in the vacuum.[(2)] These results are obtained in the

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Dedicated to the memory of Paola Calderoni.

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Boltzmann–Grad limit when each molecule has very few collisions per unit time while it otherwise moves freely. When such a simplifying assumption is not fulfilled the analysis of the hydrodynamic behavior of the fluid becomes so intricate that no mathematically founded result is known and none seems likely in the near future. The hydrodynamic equations, though, have been derived rigorously starting from the Boltzmann equation, at least in some cases, by using Hilbert and Chapman–Enskog expansions.[3]

The situation is more satisfactory, in a sense, if one considers from the beginning simplified particle models such as stochastic interacting particle systems or stochastic cellular automata.[4] The long-time analysis of these models is made easier by the stochastic nature of their evolution, so that ergodiclike properties of the dynamics, hopeless to be proven in mechanical systems, can actually be established. The crucial point is that even after such assumptions on the nature of the evolution, the system keeps some of the features of the original mechanical model, and in particular the collective phenomena responsible for establishing the hydrodynamic behavior of the system are preserved in many cases. This gives a concrete hope of proving for such systems the validity of the hydrodynamic description which seems so far off for the purely mechanical systems.

In the same spirit of looking at simplified versions of the general problem, Kurtz[5] and McKean[6] derived the hydrodynamic equation for the Carleman model. This latter is a caricature of the Boltzmann dynamics. The associated Euler equation is in fact trivial, so that one needs to consider a diffusive scaling in which Kurtz and McKean proved convergence to a nonlinear heat equation using the Chapman–Enskog expansion.

In the present paper we complete, in a sense, their analysis, by deriving the Carleman equation from a particle system, in analogy with the Boltzmann–Grad limit. The particle model has a stochastic evolution to avoid the pathology appearing in the derivation of the Boltzmann equation with descrete velocities, as pointed out by Uchiyama[7] for the Broadwell equations.

In Section 2 we introduce the model and state the main theorem, which is proven in Section 3. Concluding remarks are given in Section 4.

2. THE MODEL AND THE MAIN RESULT

Consider N point particles on the circle, i.e., the interval $[0, 1]$ with periodic boundary conditions. We denote by $\mathbf{x} = (x_1, x_2,..., x_N)$ their positions and by $\mathbf{v} = (v_1,..., v_N)$ their velocities and we assume $v_j = \pm 1$, $j = 1,..., N$. The interaction is described in the following way. A Poisson process of intensity one is introduced for each pair (i, j) of particles. Let $\{t_1,..., t_k,...\}$ be a sequence of times distributed according to this process:

then, if at the time t_k, $v_j=v_i$ and $d(x_i, x_j)\leqslant\varepsilon$ [here $d(x, y)$ denotes the distance on the circle and ε is a positive parameter], the two particles i and j invert their velocities, otherwise they go ahead. The Poisson processes for different pairs of particles are mutually independent.

Suppose now that the system is described at time zero by a distribution whose Lebesgue density is $\mu_0^N(x_1, v_1,..., x_N, v_N)\equiv\mu_0^N(\mathbf{x}, \mathbf{v})$. Then the time-evolved distribution density μ_t^N satisfies

$$D_t\mu_t^N(\mathbf{x}, \mathbf{v})=(G_N^{*\varepsilon}\mu_t^N)(\mathbf{x}, \mathbf{v}) \tag{2.1}$$

where

$$D_t=\partial_t+\sum_{j=1}^N v_j\partial_{x_j} \tag{2.2}$$

and

$$(G_N^{*\varepsilon}\mu_t^N)(\mathbf{x}, \mathbf{v})$$
$$\equiv\tfrac{1}{2}\sum_{i=1}^N\sum_{j=1;j\neq i}^N\chi^\varepsilon(i, j)[\mu_t^N(x_1, v_1,..., x_i, -v_i,..., x_j, -v_j,..., x_N, v_N)-\mu_t^N(\mathbf{x}, \mathbf{v})] \tag{2.3a}$$

$$\chi^\varepsilon(i, j)=1 \qquad \text{if} \quad d(x_i, x_j)\leqslant\varepsilon, \quad v_j=v_i, \qquad \text{and 0 otherwise} \tag{2.3b}$$

We assume that μ_0^N is symmetric under permutations of particles, so that it remains symmetric at all times.

Introducing the marginal distribution densities

$$f_j^\varepsilon(x_1, v_1,..., x_j, v_j; t)=\sum_{v_{j+1};...;v_N}\int dx_{j+1}\cdots dx_N\,\mu_t^N(x_1, v_1,..., x_N, v_N) \tag{2.4}$$

we obtain

$$(D_t-G_j^{*\varepsilon})f_j^\varepsilon=C_{j;j+1}^\varepsilon f_{j+1}^\varepsilon \tag{2.5}$$

where

$$\begin{aligned} &C_{j;j+1}^\varepsilon f_{j+1}^\varepsilon(x_1, v_1,..., x_j, v_j; t)\\ &\quad=(N-j)\sum_{i=1}^j\int dx_{j+1}\,\chi^\varepsilon(i, j+1)\\ &\qquad\times\{f_{j+1}^\varepsilon(x_1, v_1,..., x_i, -v_i,..., x_{j+1}, -v_{j+1}; t)\\ &\qquad-f_{j+1}^\varepsilon(x_1, v_1,..., x_{j+1}, v_{j+1}; t)\} \end{aligned} \tag{2.6}$$

We refer to (2.5) and (2.6) as to the BBGKY hierarchy for our model system, in analogy with the system of equations which describe the evolution of mechanical particle models.

In the formal limit $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon \to 1$, the above set of equations converges, formally, to the following hierarchy of equations:

$$D_t f_j = C_{j,j+1} f_{j+1} \tag{2.7}$$

where

$$\begin{aligned} C_{j,j+1} f_{j+1}&(x_1, v_1,..., x_j, v_j; t) \\ &= \sum_{i=1}^{j} \{f_{j+1}(x_1, v_1,..., x_i, -v_i,..., x_j, v_j, x_i, -v_i; t) \\ &\quad - f_{j+1}(x_1, v_1,..., x_i, v_i,..., x_i, v_i; t)\} \end{aligned} \tag{2.8}$$

The set of equations (2.7) is called the Carleman hierarchy. In fact, if we assume that the distribution densities f_j factorize, i.e.,

$$f_j(x_1, v_1,..., x_j, v_j; t) = \prod_{i=1}^{j} f_1(x_i, v_i; t) \tag{2.9}$$

then the first equation of the set (2.7) reduces to the Carleman equation

$$D_t f_1(x, v; t) = f_1(x, -v; t)^2 - f_1(x, v; t)^2 \tag{2.10}$$

Moreover, it is only a matter of simple algebraic manipulations to prove that, if $f_1(x, v; t)$ satisfies the Carleman equation (2.10), then the lhs of (2.9) satisfies the hierarchy (2.7).

In spite of the fact that (2.5) converges formally to (2.7), one can prove that the solutions to (2.5) *do not* converge to those of (2.7). Such a "paradoxical" feature is also present in the four-velocity Broadwell model, as noticed by Uchiyama.[(7)] In the Carleman case it is a consequence of the fact that two isolated particles, initially at the same point, cannot be separated by the dynamics, so that, in the limit $\varepsilon \to 0$ their motion converges to a synchronous random flight (random change of velocity) rather than to a free flow, as in (2.7).

One might wonder about the relevance of such considerations, since the set of configurations with two particles at the same place has zero Lebesgue measure. Notice, however, that in the rhs of (2.8) two particles are at the same place and with the same velocity: it is the same evolution in a sense which forces us to consider such configurations. More details will be given in Section 4.

To overcome the above intrinsic difficulty, we add a Brownian motion on each particle, i.e., we replace $G_N^{*\varepsilon}$ by

$$G_N^\varepsilon = G_N^{*\varepsilon} + \sigma\Delta, \qquad \sigma > 0 \tag{2.11a}$$

$$\Delta \equiv \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \tag{2.11b}$$

and in order to have the same limiting equation we shall choose $\sigma = \sigma(\varepsilon)$ vanishing with ε slowly enough for the particles to be sufficiently delocalized. In this way we shall avoid the pathology that we discussed previously.

For this new model the hierarchy of equations is

$$(D_t - G_j^\varepsilon) f_j^\varepsilon = C_{j;j+1}^\varepsilon f_{j+1}^\varepsilon \tag{2.12}$$

We study the behavior of the f_j^ε in the limit $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon \to 1$. Our main result is the following theorem.

Theorem 2.1. For each ε let $N(\varepsilon)$ be a positive integer such that $\varepsilon N(\varepsilon) \to 1$ as $\varepsilon \to 0$. Assume further that the density of the initial distribution of N particles is

$$\mu_0^N(x_1, v_1,..., x_N, v_N) = \prod_{k=1}^{N} f(x_k, v_k; 0) \tag{2.13a}$$

$$\sum_v \int_0^1 dx\, f(x, v; 0) = 1 \tag{2.13b}$$

where $f(\cdot; 0)$ is in $C^0([0, 1]^2)$ and $f(0, v; 0) = f(1, v; 0)$, $v = \pm 1$. Then the following two statements hold.

(i) There is a constant c_σ not depending on ε such that for all $j > 0$ and $t \geqslant 0$

$$\|f_j^\varepsilon(\cdot, t)\|_\infty \leqslant \|f_j^\varepsilon(\cdot, 0)\|_\infty \exp(c_\sigma j^2 t) \tag{2.14}$$

where $\|\cdot\|_\infty$ denotes the sup norm. [For this result we do not need the factorization property (2.13).]

(ii) There exists a function $\sigma(\varepsilon)$ such that $\sigma(\varepsilon) > 0$ for each ε and $\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for all $j > 0$

$$\lim_{\varepsilon \to 0} f_j^\varepsilon(x_1, v_1,..., x_j, v_j; t) = \prod_{i=1}^{j} f(x_i, v_i; t) \qquad \text{uniformly on compact sets} \tag{2.15}$$

where $f(\cdot, \cdot; t)$ satisfies the Carleman equation in mild form, i.e.,

$$f(x, v; t) = f(x - vt, v; 0))$$
$$+ \int_0^t ds \, \{f(x + v(t-s), -v; s)^2 - f(x + v(t-s), v; s)^2\} \qquad (2.16)$$

Remark. If $\sigma > 0$ is kept fixed when $\varepsilon \to 0$, then (2.15) holds and the limiting density $f(x, v, t)$ satisfies the following equation:

$$\frac{\partial}{\partial t} f(x, v, t) + v \frac{\partial}{\partial x} f(x, v, t) = [f(x, -v, t)^2 - f(x, v, t)^2] + \sigma \frac{\partial^2}{\partial x^2} f(x, v, t) \qquad (2.17)$$

We shall prove Theorem 2.1 in the next section by studying a perturbative expansion of (2.5) as usual when deriving kinetic equations from microscopic-particle models. We may overcome the typical limitation to short times by exploiting the diffusion part of the evolution. While a diffusion is necessary, as explained above, to get convergence to the "right" Carleman equation, here the diffusion plays a more important role by providing also a technical device to gain *a priori* bounds on the correlation functions. The price we pay is that we can only state the existence of a function $\sigma(\varepsilon)$ but we are not able to make it explicit. As we discuss in Section 4, a natural choice for $\sigma(\varepsilon)$ is σ proportional to ε. We expect the theorem to hold also in this case. Actually, for such a choice of $\sigma(\varepsilon)$ convergence at short times holds, as proven in Section 4.

3. PROOF OF THEOREM 2.1

We start by proving the *a priori* estimate (2.14) on f_j^ε, which is the key ingredient in the proof of Theorem 2.1. We shall often write $f_{j;t}^\varepsilon$ instead of $f_j^\varepsilon(\cdot; t)$.

By (2.12) we have

$$f_{j;t}^\varepsilon = V_{j,t} f_{j,0} + \int_0^t ds \, V_{j,t-s} G_j^{*\varepsilon} f_{j;s}^\varepsilon + \int_0^t ds \, V_{j,t-s} C_{j;j+1}^\varepsilon f_{j+1;s}^\varepsilon \qquad (3.1)$$

where

$$(V_{j,t} f)(x_1, v_1, ..., x_j, v_j) = \int dy_1 \cdots dy_j \prod_{k=1}^{j} H_t(x_k - v_k t, y_k) \, f(y_1, v_1, ..., y_j, v_j) \qquad (3.2)$$

and $H_t(x, y)$ is the Green function associated with the heat equation with periodic boundary conditions in [0, 1].

Throughout this section we shall denote by c [respectively c_σ] any positive numerical constant [respectively any positive constant depending only on σ].

The following properties of H_t are well known for $t \leqslant T$, $T \geqslant 0$ arbitrary but fixed:

$$\int_0^1 dy\, H_t(\cdot, y)\, dy = 1 \tag{3.3a}$$

$$\sup_{x,y} |H_t(x, y)| \leqslant c_\sigma \frac{1}{\sqrt{t}} \tag{3.3b}$$

$$\sup_x \|\partial_x H_t(x, \cdot)\|_1 \leqslant c_\sigma \frac{1}{\sqrt{t}} \tag{3.3c}$$

$$\sup_x \|\partial^2_{x^2} H_t(x, \cdot)\|_1 \leqslant c_\sigma \frac{1}{t} \tag{3.3d}$$

where $\|\cdot\|_1$ denotes the L_1-norm.

We then have the following estimates:

$$\|V_{j,t} f_{j,t}\|_\infty \leqslant \|f_{j,t}\|_\infty, \qquad \forall j \geqslant 1, \quad \forall t \geqslant 0 \tag{3.4}$$

$$\begin{aligned} &\left\| \int_0^t ds\, V_{j,t-s} G_j^{*\varepsilon} f^\varepsilon_{j;s} \right\|_\infty \\ &\quad \leqslant 2 \int_0^t ds\, \|f^\varepsilon_{j;s}\|_\infty \sum_{i,k;i\neq k}^{j} \int dy_1 \cdots dy_j \prod_{h=1}^{j} H_{t-s}(x_h - v_h(t-s), y_h)\, \chi^\varepsilon(i, k) \\ &\quad \leqslant j^2 \varepsilon c_\sigma \int_0^t ds \frac{\|f^\varepsilon_{j;s}\|_\infty}{(t-s)^{1/2}} \end{aligned} \tag{3.5}$$

Therefore, since $j\varepsilon \leqslant c$, the lhs of (3.5) is also bounded by

$$c_\sigma j \int_0^t ds \frac{\|f^\varepsilon_{j;s}\|_\infty}{(t-s)^{1/2}} \tag{3.6}$$

The next is the crucial estimate: we use the fact that the kernel H delocalizes the position of the particles to get an *a priori* bound on the last term in the lhs of (3.1) in terms of $\|f^\varepsilon_{j;s}\|_\infty$ itself and this allows us to decouple the hierarchy of equations (3.1).

Let $j \geqslant 2$; then

$$
\begin{aligned}
&|(V_{j,t-s} C^{\varepsilon}_{j;j+1} f^{\varepsilon}_{j+1;s})(x_1, v_1,..., x_j, v_j)| \\
&\leqslant \sum_{k=1}^{j} \int dy_1 \cdots dy_j \prod_{h=1}^{j} H_{t-s}(x_h - v_h(t-s), y_h) \\
&\quad \times (N-j) \int dy_{j+1}\, \chi^{\varepsilon}(k, j+1) [f^{\varepsilon}_{j+1;s}(y_1, v_1,..., y_k, -v_k,..., y_{j+1}, -v_{j+1}) \\
&\quad + f^{\varepsilon}_{j+1;s}(y_1, v_1,..., y_k, v_k,..., y_{j+1}, v_{j+1})] \\
&\leqslant \sum_{k=1}^{j-1} \int dy_1 \cdots dy_k\, dy_{k+2} \cdots dy_j \prod_{h \neq k+1} H_{t-s}(x_h - v_h(t-s), y_h) \frac{c_\sigma}{(t-s)^{1/2}} \\
&\quad \times (N-j) \int dy_{j+1} \int dy_{k+1} [f^{\varepsilon}_{j+1;s}(y_1, v_1,..., y_k, -v_k,..., y_{j+1}, -v_{j+1}) \\
&\quad + f^{\varepsilon}_{j+1;s}(y_1, v_1,..., y_k, v_k,..., y_{j+1}, v_{j+1})]\, \chi^{\varepsilon}(k, j+1) \\
&\quad + \int dy_2 \cdots dy_j \prod_{h=2}^{j} H_{t-s}(x_h - v_h(t-s), y_h) \frac{c_\sigma}{(t-s)^{1/2}} \\
&\quad + (N-j) \int dy_{j+1} \int dy_1 [f^{\varepsilon}_{j+1;s}(y_1, v_1,..., y_{j+1}, -v_{j+1}) \\
&\quad + f^{\varepsilon}_{j+1;s}(y_1, v_1,..., y_{j+1}, v_{j+1})]\, \chi^{\varepsilon}(j, j+1) \\
&\leqslant \frac{c_\sigma}{(t-s)^{1/2}}\, j \, \|f^{\varepsilon}_{j;s}\|_\infty
\end{aligned}
\tag{3.7}
$$

The last inequality follows from the symmetry and the compatibility condition

$$
\sum_{v_{j+1}} \int dy_{j+1}\, f^{\varepsilon}_{j+1;s}(y_1, v_1,..., y_{j+1}, -v_{j+1}) = f_{j;s}(y_1, v_1,..., y_j, v_j)
\tag{3.8}
$$

By (3.4), (3.6), and (3.7) we get

$$
\|f^{\varepsilon}_{j;t}\|_\infty \leqslant \|f_{j,0}\|_\infty + c_\sigma \int_0^t ds \frac{j}{(t-s)^{1/2}} \|f^{\varepsilon}_{j;s}\|_\infty
\tag{3.9}
$$

yielding

$$
\|f^{\varepsilon}_{j;t}\|_\infty \leqslant \|f_{j,0}\|_\infty \exp(c_\sigma t j^2), \qquad j \geqslant 2
\tag{3.10}
$$

The same estimate is easily obtained also for $j = 1$ using (3.8).

Notice that in the above proof we have used that σ is positive [the constant c_σ in (3.8) diverges as $1/\sqrt{\sigma}$ when $\sigma \to 0$]. We have also exploited the one-dimensional nature of the model; in more dimensions we would have a factor $(t-s)^{-d/2}$ ($d=$ the dimension) out of the heat kernel, hence a divergence for $d>1$.

We shall now prove that keeping σ fixed, we can construct a sequence $f_j^{\varepsilon_{n(k)}}$ which converges uniformly on the compacts to a limit h_j^σ which satisfies the equation

$$(D_t + \sigma \Delta) h_j^\sigma = C_{j;j+1} h_{j+1}^\sigma \tag{3.11}$$

To this purpose, we show that $\{f_{j;t}^\varepsilon\}$ is an equicontinuous family in $\mathbf{x}$ and t over a time interval $[\delta, T]$ with δ and T fixed and positive [note that from (3.10) we already know that $\{f_{j;t}^\varepsilon\}$ is equibounded].

By all the previous estimates [see Eqs. (3.3)] we have, for $i=1,..., j$,

$$\|\partial_x f_{j;t}^\varepsilon\|_\infty \leqslant \frac{c_\sigma}{\sqrt{\delta}} \|f_{j;0}^\varepsilon\|_\infty + \int_0^t ds \frac{c_\sigma}{(t-s)^{1/2}} j[\|f_{j;s}^\varepsilon\|_\infty + \|f_{j+1;s}^\varepsilon\|_\infty] \tag{3.12}$$

and this together with (3.10) implies that $\partial_{x_i} f_{j;t}^\varepsilon$ is equibounded in $[\delta, T]$. Furthermore, for any t, t' in $[\delta, T]$ with $t'<t$ and for any bounded function f, one has

$$\begin{aligned}
&|(V_{j,t} - V_{j,t'}) f(x_1, v_1,..., x_j, v_j)| \\
&\quad \leqslant cj \|f\|_\infty \int_{t'}^t ds \max\left(\|\Delta_x H_s\|_\infty, \sum_{i=1}^j \|\nabla_{x_i} H_s\|_\infty\right) \\
&\quad \leqslant c_\sigma j \|f\|_\infty \frac{t-t'}{\delta}
\end{aligned} \tag{3.13}$$

Hence, by (3.10),

$$\|f_{j;t}^\varepsilon - f_{j;t'}^\varepsilon\|_\infty \leqslant c_\sigma \frac{t-t'}{\delta} \tag{3.14}$$

with c_σ only depending on j, T, and the initial datum. This proves the equicontinuity of $f_{j;t}^\varepsilon$ for t in $[\delta, T]$.

By a diagonal procedure, for any sequence $\varepsilon_n \to 0$, a subsequence $\varepsilon_{n(k)}$ can be extracted such that the corresponding $f_{j;t}^{\varepsilon_{n(k)}}$ converges to some limit $h_{j;t}^\sigma$ for $t \in (0, T)$. Since $f_{j;0}^{\varepsilon_{n(k)}} = f_{j,0}$ for any k, then the convergence holds for all t in $[0, T]$. We want now to show that $h_{j;t}^\sigma$ satisfies Eq. (3.11) for $t \in [0, T]$. We have

$$h_{j;t}^\sigma = V_{j,t} f_{j,0} + \lim_{k\to\infty} \int_0^t ds\, V_{j,t-s}(G_j^{*\varepsilon} f_{j;s}^{\varepsilon_{n(k)}} + C_{j;j+1}^{\varepsilon_{n(k)}} f_{j+1;s}^{\varepsilon_{n(k)}}) \tag{3.15}$$

By (3.5), (3.7), and (3.10) the integrand is bounded by an integrable function of time, so that by the dominated convergence theorem, we can interchange the limit with the integral. We note that the first term in the integral goes to zero with ε, by (3.5). Moreover,

$$\begin{aligned} &\| C^{\varepsilon_{n(k)}}_{j;j+1} f^{\varepsilon_{n(k)}}_{j+1;s} - C_{j;j+1} h^{\sigma}_{j+1;s} \|_{\infty} \\ &\quad \leqslant \| (C^{\varepsilon_{n(k)}}_{j;j+1} - C_{j;j+1}) f^{\varepsilon_{n(k)}}_{j+1;s}) \|_{\infty} + \| C_{j;j+1} (h^{\sigma}_{j+1;s} - f^{\varepsilon_{n(k)}}_{j+1;s}) \|_{\infty} \end{aligned} \tag{3.16}$$

Since the estimate (3.12) together with (3.10) implies that the first term in the rhs of (3.16) goes to zero as $\varepsilon_{n(k)}$ goes to zero, we can conclude that $h^{\sigma}_{j;t}$ satisfies Eq. (3.11).

The initial value problem associated with the equation

$$(D_t f_t)(x, v) = f_t(x, -v)^2 - f_t(x, v)^2 + \sigma \Delta f_t(x, v) \tag{3.17}$$

has a unique solution f^{σ}_t (see ref. 8) which satisfies the bound

$$\| f^{\sigma}_t \|_{\infty} \leqslant \| f^{\sigma}_0 \|_{\infty} \qquad \text{for} \quad \sigma \geqslant 0 \tag{3.18}$$

As a consequence of algebraic manipulations, the functions

$$f^{\sigma}_{j;t}(x_1, v_1, ..., x_j, v_j) = \prod_{i=1}^{j} f^{\sigma}_t(x_i, v_i) \tag{3.19}$$

solve (3.11), which also has a unique solution. In fact, applying the Lanford's argument (cf. Section 4) to this context, one easily finds that there exists a unique solution to (3.11) for a time interval depending only on the supremum norm of the initial datum. This solution factorizes if the initial datum does. By the estimate (3.18) we know that the factorizing solutions do not increase the supremum norm, so that we can iterate the argument to prove the uniqueness of solutions with factorizing initial data. Thus, we conclude that

$$\lim_{k \to \infty} f^{\varepsilon_{n(k)}}_{j;t}(x_1, v_1, ..., x_j, v_j) = \prod_{i=1}^{j} f^{\sigma}_t(x_i, v_i) \tag{3.20}$$

To conclude the proof of the theorem, it is enough to show that

$$\lim_{\sigma \to 0} f^{\sigma}_t(x, v) = f_t(x, v) \tag{3.21}$$

This is standard, so we only outline the proof. To underline the dependence on σ, we denote by V^{σ}_t ($\sigma \geqslant 0$) the semigroup $V_{1,t}$ defined in (3.2). First we assume $f_0 \in C^1([0, 1]^2)$. From the estimate

$$
\begin{aligned}
&|(V_t^\sigma - V_t^0) f_0(x, v)| \\
&\quad \leqslant \left| \int_0^\sigma d\eta\, \partial_\eta V_t^\eta f_0(x, v) \right| \\
&\quad \leqslant \left| \int_0^\sigma d\eta \int dy\, \partial_\eta H_{\eta t}(y, x)\, f_0(x, v) \right| \\
&\quad \leqslant \int_0^\sigma d\eta\, t \sup_y \|\partial_y H_{\eta t}(y, \cdot)\|_1\, \|\partial_x f_0\|_\infty \\
&\quad \leqslant \int_0^\sigma d\eta \left(\frac{t}{\eta}\right)^{1/2} c\, \|\partial_x f_0\|_\infty
\end{aligned} \tag{3.22}
$$

we obtain, for $t \leqslant T$,

$$
\begin{aligned}
\|f_t - f_t^\sigma\|_\infty \leqslant{}& c T^{1/2} \sigma^{1/2} \|\partial_x f_0\|_\infty + c T^{3/2} \sigma^{1/2} \sup_{t \leqslant T} (\|f_t\|_\infty, \|\partial_x f_t\|_\infty) \\
&+ c \int_0^t ds\, (\|f_s^\sigma\|_\infty + \|f_s\|_\infty)\, \|f_s^\sigma - f_s\|_\infty
\end{aligned} \tag{3.23}
$$

Condition (3.18), together with the obvious integral inequality

$$
\begin{aligned}
\|\partial_x f_t\| \leqslant{}& V_t^0 \|\partial_x f_0\|_\infty \\
&+ 2 \int_0^t ds\, \|V_{t-s}^0 \{f_s(x, -v)\, \partial_x f_s(x, -v) + f_s(x, v)\, \partial_x f_s(x, v)\}\|_\infty
\end{aligned} \tag{3.24}
$$

provides an *a priori* bound on $\|\partial_x f_t\|_\infty$ so that, by the Gronwall lemma applied to (3.23), we conclude that

$$
\|f_t - f_t^\sigma\|_\infty \leqslant c \sqrt{\sigma} \tag{3.25}
$$

with c depending only on T and the supremum norm of f_0 and $\partial_x f_0$. The general case, i.e., $f_0 \in C^0([0, 1]^2)$, can easily be recovered by a density argument, since f_t^σ is continuous in L_∞, uniformly in $\sigma\delta$ with respect to the initial datum.

4. CONCLUDING REMARKS

To derive the Carleman equation, we added a Brownian motion with diffusion coefficient σ to each particle, so that the correlation functions satisfy the following equations [see (2.12) and (2.11)]

$$
(D_t - G_j^\varepsilon) f_j^\varepsilon = C_{j;j+1}^\varepsilon f_{j+1}^\varepsilon \tag{4.1a}
$$

$$
G_j^\varepsilon = G_j^{*\varepsilon} + \sigma \Delta \tag{4.1b}
$$

In Theorem 2.1 we have shown that there is a sequence $\sigma(\varepsilon)$ which makes the term $G_j^\varepsilon f_j^\varepsilon$ in (4.1) vanishingly small in the limit $\varepsilon \to 0$. The method used for proving Theorem 2.1 does not specify how fast $\sigma(\varepsilon)$ should vanish. This is even less satisfactory because, on one hand, $\sigma(\varepsilon)$ depends on the initial datum $f(\cdot, 0)$ and, moreover, there is a natural choice for $\sigma(\varepsilon)$, namely $\sigma(\varepsilon) = \varepsilon$, as we shall discuss later, and, on the other hand, there are reasons to conjecture that $\sigma(\varepsilon) = \varepsilon^a$ should work as well if $a < 2$, as we are going to see now.

This last consideration is based on a proof of Theorem 2.1 for short times which only requires $\lim_{\varepsilon \to 0}(\varepsilon/\sqrt{\sigma}) = 0$. We use the Lanford approach: to this purpose we define, for any j and t, $W_{j,t}$ as the semigroup generated by the operator

$$L_{\sigma,j} \equiv -\sum_{i=1}^{j} v_j \frac{\partial}{\partial x_i} + \sigma \sum_{i=1}^{j} \frac{\partial^2}{\partial x_i^2} + G_j^{*\varepsilon} \tag{4.2}$$

so that f_j^ε satisfies the following integral equation [compare it with (3.1)]

$$f_{j;t}^\varepsilon = W_{j,t} f_0 + \int_0^t ds\, W_{j,t-s} C_{j;j+1}^\varepsilon f_{j+1;s}^\varepsilon \tag{4.3}$$

We can iterate (4.3); since $\|W_{j,t}\|_\infty \leqslant 1$, we obtain a convergent series at "short times." In fact, from (4.3) we have

$$\|f_{j;t}^\varepsilon\|_\infty \leqslant \sum_{n=0}^{\infty} c^n \frac{t^n}{n!} j \cdots (j+n-1)\, \|f_0\|_\infty^{j+n} \tag{4.4}$$

which converges for $t \leqslant T$ where T is some suitably chosen positive number.

From (4.4) it follows that in order to prove Theorem 2.1 for $t \leqslant T$, it is enough to show that the series for $f_{j;t}^\varepsilon$ converges term by term to $\prod_1^j f_t$ with f_t solving (2.16). Observe that up to now we have not used at all the diffusion: (4.4) is in fact true also for $\sigma = 0$. On the other hand, if for any bounded, continuous function F

$$\lim_{\varepsilon \to 0} \|W_{j,t} F - V_{j,t} F\|_\infty = 0 \tag{4.5}$$

where $V_{j,t}$ is defined in (3.2), then we are reduced to the analysis of the series expansion for $f_{j;t}^\varepsilon$ with $V_{j,t}$ in place of $W_{j,t}$ and by the same arguments as those discussed at the end of Section 3 we can conclude that

$$\lim f_{j;t}^\varepsilon = \prod_1^j f_t, \qquad \forall t \leqslant T \tag{4.6}$$

where f_t solves Carleman.

So we are left with the proof of (4.5), where the presence of σ is crucial, as it should be. The difference between $W_{j,t}$ and $V_{j,t}$ comes from the presence of the interaction among the j particles described by $G_j^{*\varepsilon}$. By definition this interaction is present whenever two Brownian particles (with diffusion constant σ) are at a distance less than ε. Proceeding as in (3.6), realizing that $c_\sigma = c/\sqrt{\sigma}$, we obtain

$$\begin{aligned} \|W_{j,t}F - V_{j,t}F\|_\infty &\leqslant c \int_0^t ds \, \|V_{j,t-s} G_j^{*\varepsilon} W_{j,s} F\|_\infty \\ &\leqslant c \|F\|_\infty j^2 (\varepsilon/\sqrt{\sigma}) \sqrt{t} \end{aligned} \tag{4.7}$$

Therefore, in order to have (4.5), it sufficies that $\lim_{\varepsilon \to 0} (\varepsilon/\sqrt{\sigma}) = 0$.

As mentioned earlier, there is a natural choice for $\sigma(\varepsilon)$, namely $\sigma(\varepsilon) = \varepsilon$, as we are going to explain. Consider a system of N Brownian particles in the interval $[0, \varepsilon^{-1}]$ (with periodic boundary conditions). The generator of the process is

$$-\sum_{i=1}^N v_i \frac{\partial}{\partial x_i} + \sum_{i=1}^N \Delta_i + \alpha G_N \tag{4.8}$$

where, $\alpha > 0$, G_N stands for G_N^{*1}, i.e., $\varepsilon = 1$, and $\chi^\varepsilon(i, j)$ is replaced by $\chi(i, j)$, where

$$\begin{aligned} \chi(i, j) &= 1 \qquad \text{if} \quad d(x_i, x_j) \leqslant 1 \quad \text{and} \quad v_i = v_j \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{4.9}$$

From a physical point of view, this is the microscopic representation of our model. If we want to investigate its macroscopic behavior, we set $N \cong \varepsilon^{-1}$ to have finite densities and we rescale space and time in the following way:

$$r = \varepsilon x, \qquad \tau = \varepsilon t \tag{4.10}$$

so that the (macroscopic) variable r varies in $[0, 1]$, as in the model we have been considering so far.

Defining

$$\nu_\tau^\varepsilon(r_1, v_1, \ldots, r_N, v_N) = \varepsilon^{-N} \mu_t^\varepsilon(x_1, v_1, \ldots, x_N, v_N) \tag{4.11}$$

where μ^ε is the probability distribution of the system in microscopic variables, i.e., the distribution evolved according to the dynamics generated by (4.8), we obtain

$$D_\tau \nu_\tau^\varepsilon = \varepsilon \Delta \nu_\tau^\varepsilon + \frac{\alpha}{\varepsilon} G_N \nu_\tau^\varepsilon \tag{4.12}$$

We now require that the number of collisions per unit macroscopic time is finite, while the particle density is constant ($N \cong \varepsilon^{-1}$). Hence, we set $\alpha = \varepsilon$ and in the limit $\varepsilon \to 0$ we reproduce exactly the model considered in the introduction but with $\sigma = \varepsilon$. This shows that a standard Brownian motion at the microscopic level automatically disappears in the hydrodynamic limit, its diffusion coefficient in macroscopic units vanishing proportionally to ε.

The Carleman equation is therefore derived by rescaling the interaction ($\alpha = \varepsilon$). The same is true for the Boltzmann equation, which cannot be derived (at least formally) by means of a pure space-time scaling: one also needs to reduce the collision rate. The hypothesis $\alpha = \varepsilon$ can be regarded as a rarefaction hypothesis, which gives a finite mean free path. Consequently, the continuum limit $\varepsilon \to 0$ describes the kinetic regime given by the Carleman equation. It is only on a longer time scale that the hydrodynamic regime appears; we need to wait for a time so long that the number of collisions per particle becomes infinite. At such times the collision term in the Carleman equation forces the local equilibrium current to vanish, so that the first nontrivial hydrodynamic equation is diffusive. While all this is proven,[5,6] by taking the hydrodynamic limit in the Carleman equation, it is still an open and interesting question whether such behavior holds in the particle model.

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