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Separation versus diffusion in a two species system

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Abstract. We consider a finite number of particles that move in $\mathbb{Z}$ as independent random walks. The particles are of two species that we call a and b. The rightmost a-particle becomes a b-particle at constant rate, while the leftmost b-particle becomes a-particle at the same rate, independently. We prove that in the hydrodynamic limit the evolution is described by a nonlinear system of two PDE's with free boundaries.

1 Introduction

We consider a two-species particle system in $\mathbb{Z}$, the species, also called colors, are indicated by a and b. We suppose that at time 0 the species are partially separated with a rightmost a-particle at a site denoted by X_0 and a leftmost b-particle at a site $Y_0 < X_0$. The evolution is such that if we are "color blind" we just see independent symmetric random walks which jump at rate one on the nearest neighbor sites. As particles keep their color during their random walk motion this means that the a and b species diffuse in $\mathbb{Z}$. In our model however, particles may also change color with the following mechanism. Independently at rate $\lambda > 0$ the rightmost a-particle becomes a b-particle and the leftmost b-particle becomes an a-particle. If the evolution consisted only of this color exchanges, then eventually a and b would separate, but this is contrasted in our model by the random walk motion of the particles which drives toward homogenization.

The motivation behind this paper is to understand how much the species separate as time evolves when both random walks and color exchange are acting, in particular to determine the evolution of the difference $X_t - Y_t$, with X_t and Y_t the positions at time t of the rightmost a-particle and leftmost b-particle, respectively. In this paper, we begin this program by looking at the hydrodynamic scale: we take $\lambda = \varepsilon\kappa$, $\kappa > 0$, and scale space and time diffusively ($x \to r = \varepsilon x$, $x \in \mathbb{Z}$, $t \to \tau = \varepsilon^2 t$). We assume that the initial distribution is such that the densities of the two species approach in the limit $\varepsilon \to 0$ a macroscopic profile and that the total mass is macroscopically finite. These two assumptions imply that the total number of particles is of order ε^{-1}, see Section 2 below for a precise definition of the initial condition.

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Under the above hypothesis, we prove convergence as $\varepsilon \to 0$ to a nonlinear system of two PDE's with free boundaries provided the solution exists.

Despite its simplicity, the rule at which species mutate creates a very nonlocal interaction: to find the rightmost b-particle it is necessary to know the whole configuration of b-particles. Here the interaction is "topological rather than metric", as the influence on a particle i of a particle j does not depend on their distance but rather depends on whether j is to the right or left of i. Stochastic evolutions with similar nonlocal interactions have been considered to model problems from different fields such as queuing theory, Atar, Biswas and Kaspi (2014), statistical mechanics of open systems (currents and Fourier law), Carinci et al. (2014a), De Masi et al. (2011) and pinned interface motions, Lacoin (2014).

2 Model and results

We thus consider a system of colored particles on $\mathbb{Z}$. Both the initial distribution and evolution depend on a scaling parameter $\varepsilon > 0$. We are interested in the hydrodynamic limit when $\varepsilon \to 0$ and space and time are rescaled diffusively.

The initial condition

The initial macroscopic profile is described by a pair (u, v) of nonnegative functions on $\mathbb{R}$ which are interpreted as the macroscopic particle densities of the a and b species, respectively. We suppose that $(u, v) \in \mathcal{U}$:

$$\begin{aligned} \mathcal{U} = \big\{(u, v) \in C_0\big(\mathbb{R}, \mathbb{R}^2_+\big) : \text{support } u = (L, R), \text{support } v = (D, E);\\ L < D < R < E, u, v > 0 \text{ in their support}\big\}. \end{aligned} \tag{2.1}$$

The total "macroscopic mass" of the two species is denoted by

$$M_{\text{tot}} = \int (u + v).$$

The macroscopic profiles (u, v) are approximated by particle configurations using a scaling parameter $\varepsilon > 0$. For each $\varepsilon > 0$ the initial configuration has $M := [\varepsilon^{-1} M_{\text{tot}}]$ particles. Their positions $\underline{x} = (x_1, \dots, x_M)$ are random, they are independently identically distributed with parameters

$$P^{\varepsilon}[x_i = x] = Z_\varepsilon^{-1}\big[u(\varepsilon x) + v(\varepsilon x)\big], \qquad Z_\varepsilon = \sum_x \big[u(\varepsilon x) + v(\varepsilon x)\big]. \tag{2.2}$$

Conditioned on $\underline{x}$ we add independently a color $\sigma_i \in \{a, b\}$ to each particle i, by setting

$$P^{\varepsilon}[\sigma_i = a|\underline{x}] = \frac{u(\varepsilon x_i)}{u(\varepsilon x_i) + v(\varepsilon x_i)}. \tag{2.3}$$

It is convenient for technical purposes to label the particles but the physically relevant quantities are the occupation numbers

$$\xi_{\underline{x},\underline{\sigma}}(y) = \sum_{i=1}^{M} \mathbf{1}_{x_i=y,\sigma_i=a}, \qquad \eta_{\underline{x},\underline{\sigma}}(y) = \sum_{i=1}^{M} \mathbf{1}_{x_i=y,\sigma_i=b}, \qquad y \in \mathbb{Z}, \tag{2.4}$$

where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_M)$. We then say that $(\underline{x}, \underline{\sigma})$ and $(\underline{x}', \underline{\sigma}')$ are equivalent if

$$\xi_{\underline{x},\underline{\sigma}} = \xi_{\underline{x}',\underline{\sigma}'}, \qquad \eta_{\underline{x},\underline{\sigma}} = \eta_{\underline{x}',\underline{\sigma}'}, \tag{2.5}$$

which means that one can be obtained from the other by exchanging colors of particles at the same site.

It easily follows from the above definitions that under P^ε, $(\varepsilon\xi_{\underline{x},\underline{\sigma}}, \varepsilon\eta_{\underline{x},\underline{\sigma}})$ converges weakly in probability to (u, v) as $\varepsilon \to 0$. Our main results will be to extend the result to positive times and identify the limit.

The positions time evolution

If we disregard the color of the particles, we just see a system of independent random walks denoted by $\underline{x}(t) = (x_1(t), \ldots, x_M(t))$, $t \geq 0$. The $x_i(t)$ are symmetric independent random walks on $\mathbb{Z}$ which jump at rate 1 on nearest neighbor sites. We denote by $\mathscr{P}^\varepsilon$ the law of this process.

We shall next define how the colors change in time. To this end, we first define the label of the rightmost a- and leftmost b-particles denoted, respectively, by $i_a(\underline{x}, \underline{\sigma})$ and $i_b(\underline{x}, \underline{\sigma})$.

Definition 2.1. We denote the total number of a- and b-particles, respectively, by

$$h_a(\underline{\sigma}) = \sum_i \mathbf{1}_{\sigma_i=a}, \qquad h_b(\underline{\sigma}) = M - h_a(\underline{\sigma}). \tag{2.6}$$

If $h_a(\underline{\sigma}) > 0$, we define $i_a(\underline{x}, \underline{\sigma}) = i$ if $\sigma_i = a$ and for any $j \neq i$ with $\sigma_j = a$, either $x_j < x_i$ or, if $x_j = x_i$, then $j < i$. Analogously if $h_b(\underline{\sigma}) > 0$, $i_b(\underline{x}, \underline{\sigma}) = i$ if $\sigma_i = b$ and if $\sigma_j = b$, either $x_j > x_i$, or if $x_j = x_i$, then $j < i$. We also define the operators $H^{\text{right}}(\underline{x}, \underline{\sigma}) =: (\underline{x}, \underline{\sigma}')$, $H^{\text{left}}(\underline{x}, \underline{\sigma}) =: (\underline{x}, \underline{\sigma}'')$ where $\underline{\sigma}' = \sigma$ if $h_a(\underline{\sigma}) = 0$, $\underline{\sigma}'' = \sigma$ if $h_b(\underline{\sigma}) = 0$. Otherwise $\underline{\sigma}'$ and $\underline{\sigma}''$ are obtained from $\underline{\sigma}$ by changing $\sigma_{i_a(\underline{x},\underline{\sigma})}$ into b and, respectively, $\sigma_{i_b(\underline{x},\underline{\sigma})}$ into a.

The evolution of colors is determined by the clock rings of the following Poisson processes.

Definition 2.2. Given $\varepsilon > 0$ and $j > 0$ we define the probability space $(\Omega, \mathbb{P}^\varepsilon)$. Ω is the set of $\omega = (\underline{s}, \underline{\ell})$ where $\underline{s} = (s_1, s_2, \ldots)$, $s_k \leq s_{k+1}$ is an ordered sequence of times, and $\underline{\ell} = (\ell_1, \ell_2, \ldots)$, $\ell_k \in \{\text{right, left}\}$ is a sequence of marks.

$\mathbb{P}^\varepsilon$ is the product probability law of a Poisson process of intensity $2\varepsilon\kappa$ for the time sequences $\underline{s}$ and of a Bernoulli process with parameter $1/2$ for the mark sequences $\underline{\ell}$. In the sequel, we will consider strictly increasing sequences of time $\underline{s}$ since these have $\mathbb{P}^\varepsilon$-probability one. We denote by $\mathcal{P}^\varepsilon = \mathscr{P}^\varepsilon \times \mathbb{P}^\varepsilon$ the joint law of the random walk $\underline{x}$ and of ω.

The color time evolution

Given $\varepsilon > 0$, $\underline{x}(t)$, $t \geq 0$, and $\omega = (\underline{s}, \underline{\ell})$ we define the "càdlàg" trajectory $\underline{\sigma}(t)$ by saying that colors are unchanged except at the times s_k: at these times the configuration is updated by applying H^{right} or H^{left} according to $\ell_k = \text{right}$ or $\ell_k = \text{left}$, respectively. We denote by $(\underline{x}(t), \underline{\sigma}(t))$ positions and colors of particles at time t.

The main results in this paper are Theorems 2.1 and 2.2 below.

Theorem 2.1. *Under the above assumptions on the initial data there are nonnegative continuous functions $(\bar{u}(\cdot,t), \bar{v}(\cdot,t))$ equal to $(u,v) \in \mathcal{U}$ at $t = 0$ and such that for any $t > 0$*

$$(\varepsilon\xi_{\underline{x}(\varepsilon^{-2}t),\underline{\sigma}(\varepsilon^{-2}t)}, \varepsilon\eta_{\underline{x}(\varepsilon^{-2}t),\underline{\sigma}(\varepsilon^{-2}t)}) \to \big(\bar{u}(\cdot,t), \bar{v}(\cdot,t)\big),$$

as $\varepsilon \to 0$ weakly in probability.

Since for all $s \geq 0$, $\xi_{\underline{x}(s),\underline{\sigma}(s)}(x) + \eta_{\underline{x}(s),\underline{\sigma}(s)}(x) = \sum_i \mathbf{1}_{x_i(s)=x}$ and the $x_i(\cdot)$ are independent random walks, we know, see, for instance, De Masi and Presutti (1991), that

$$\varepsilon[\xi_{\underline{x}(\varepsilon^{-2}t),\underline{\sigma}(\varepsilon^{-2}t)} + \eta_{\underline{x}(\varepsilon^{-2}t),\underline{\sigma}(\varepsilon^{-2}t)}] \to w(\cdot,t),$$

as $\varepsilon \to 0$ weakly in probability with w the solution of the linear heat equation $w_t = \frac{1}{2}w_{rr}$ and initial condition $u + v$. Thus, it is enough for Theorem 2.1 to prove convergence of $\xi_{\underline{x}(\varepsilon^{-2}t),\underline{\sigma}(\varepsilon^{-2}t)}$ alone.

The proof is reported in Section 4, it follows the same strategy used in De Masi, Ferrari and Presutti (2015) and then in Carinci et al. (2014a). Namely we first introduce auxiliary processes for which the hydrodynamic limit can be computed and then prove by stochastic inequalities that the true process is sandwiched between the auxiliary ones and that the inequalities become equalities in the limit. The first part is easy (as the auxiliary processes are essentially independent random walks) and we just sketch it in Section 4. The proof of the stochastic inequalities is instead quite involved and given in full details in the next section, being one of the most important parts of the paper.

Theorem 2.1 only states the existence of the hydrodynamic limit for all macroscopic times $t \geq 0$. It does not give its properties nor specifies the hydrodynamic equations. On the other hand, one may guess that the latter are given by the following system of two equations

$$
\begin{aligned}
&u_t = \frac{1}{2}u_{rr} + \kappa\delta_{V_t}, \qquad r < U_t;\\
&u(r,0) = u(r), \qquad u(U_t,t) = 0, \qquad -\frac{1}{2}u_r\big(U_t^-,t\big) = \kappa,\\
&v_t = \frac{1}{2}v_{rr} + \kappa\delta_{U_t}, \qquad r > V_t;\\
&v(r,0) = v(r), \qquad v(V_t,t) = 0, \qquad -\frac{1}{2}v_r\big(V_t^+,t\big) = -\kappa,
\end{aligned}
\tag{2.7}
$$

where $U_0 = R$ and $V_0 = D$, see (2.1), and δ_x is the Dirac-delta.

Equations (2.7) is a system of two free boundary equations as the domains $(-\infty, U_t)$ where $u(r,t)$ is defined and (V_t, ∞) where $v(r,t)$ is defined are also unknowns to be determined.

By the Dirichlet condition $u(r,t)$ can be extended continuously past U_t by setting $u(r,t) \equiv 0$ for all $r \geq U_t$ so that U_t is the rightmost-end point of the interval where $u > 0$, it thus corresponds to the macroscopic position of the rightmost particle. Analogous interpretation is given to V_t. In the particle system a-particles are created at rate $\varepsilon\kappa$ at the position of the leftmost b-particle, correspondingly the equation for u has a source term $\kappa\delta_{V_t}$, with an analogous interpretation for $\kappa\delta_{U_t}$. Finally, the boundary condition $-\frac{1}{2}u_r(U_t^-,t) = \kappa$ just says that the outgoing mass flux of u is equal to κ which is the macro-analogue of the rate at which a-particles disappear (changing into b-particles), analogous interpretation holds for the term $-\frac{1}{2}v_r(V_t^+,t) = -\kappa$.

The two equations are coupled by the Dirac-delta terms which involve the free boundary terms U_t and V_t which make the problem highly nonlinear.

We did not find in the literature the above system of free boundary problems. We notice however that (2.7) is similar to the free boundary PDE studied and for which local and sometimes global existence and uniqueness are proved, see for instance Fasano (2008). It is then conceivable that the same techniques might be applied to our equation, but we did not pursue this issue here, so we assume existence of a solution and we prove that this solution coincides with the limit of our particles evolution.

We thus suppose that for some positive time interval $[0, T]$ there is a regular solution of (2.7). By regular, we mean that the functions U_t, V_t of (2.7) are $C^1[0, T]$; that $u(r,t)$, $v(r,t)$ have the differentiability properties required by (2.7), and finally that $(u(\cdot,t), v(\cdot,t)) \in \mathcal{U}$ for all $t \in [0, T]$.

Theorem 2.2. *Assume there is $T > 0$ so that a regular solution of (2.7) exists in the above sense in $[0, T]$. Then this solution coincides with the hydrodynamic limit $(\bar{u}(\cdot,t), \bar{v}(\cdot,t))$ of Theorem 2.1 restricted to $t \in [0, T]$.*

We prove Theorem 2.2 in Section 5. The proof has some similarities with the proofs in Carinci et al. (2014a) but it requires new ideas and it is the other most important point of the paper together with the proof of the microscopic inequalities.

3 Microscopic inequalities

As already mentioned stochastic inequalities play a fundamental role in our proof. Let ξ' and ξ be nonnegative, integer valued functions on $\mathbb{Z}$ with compact support.

Definition 3.1. We say that $\xi' \preccurlyeq \xi$ if for all $x \in \mathbb{Z}$

$$F(x;\xi') \le F(x;\xi), \qquad F(x;\xi) = \sum_{y \ge x} \xi(y). \tag{3.1}$$

We also say that $(\underline{x}', \underline{\sigma}') \preccurlyeq (\underline{x}, \underline{\sigma})$ if $\xi_{\underline{x}',\underline{\sigma}'} \preccurlyeq \xi_{\underline{x},\underline{\sigma}}$ (observe that the inequality remains valid if we replace a configuration by an equivalent one, see (2.5)).

Recalling Definitions 2.1 and 2.2, we first introduce the following sets.

Definition 3.2. We call $\mathcal{X}_t$, $t > 0$ the set of all $(\underline{\sigma}, \omega)$ such that

$$\begin{aligned} N_a(t) &:= h_a(\underline{\sigma}) + \sum_k \mathbf{1}_{s_k \le t}(\mathbf{1}_{\ell_k=\text{left}} - \mathbf{1}_{\ell_k=\text{right}}) > 0, \\ N_b(t) &:= M - h_a(\underline{\sigma}) + \sum_k \mathbf{1}_{s_k \le t}(\mathbf{1}_{\ell_k=\text{right}} - \mathbf{1}_{\ell_k=\text{left}}) > 0. \end{aligned} \tag{3.2}$$

In $\mathcal{X}_t$ there are always both a- and b-particles in the time interval $[0,t]$. In the next section, see Lemma 3.2, we prove that $\mathcal{P}^\varepsilon(\mathcal{X}_t) \to 1$ as $\varepsilon \to 0$.

The setup

Throughout this section, we fix $\varepsilon > 0$, $\delta > 0$, a time interval $[0, \varepsilon^{-2}\delta]$, a random walk trajectory $\underline{x}(t) = (x_1(t), \dots, x_M(t))$, $t \in [0, \varepsilon^{-2}\delta]$ and an element $(\underline{\sigma}, \omega) \in \mathcal{X}_{\varepsilon^{-2}\delta}$.

The auxiliary evolutions

They are denoted by $(\underline{x}(t), \underline{\sigma}^{(\delta,\pm)}(t))$ and are defined by anticipating or postponing the color changes at the initial, respectively final, time. Namely, given $(\underline{\sigma}, \omega) \in \mathcal{X}_{\varepsilon^{-2}\delta}$ we let $\underline{\sigma}^{(\delta,+)}(t)$ be the left continuous with right limits function obtained by setting $\underline{\sigma}^{(\delta,+)}(t) = \underline{\sigma}^{(\delta,+)}(0^+)$ for $t \in (0, \varepsilon^{-2}\delta]$ and

$$(\underline{x}(0^+), \underline{\sigma}^{(\delta,+)}(0^+)) = \prod_{i=1}^m H^{\ell_i}(\underline{x}(0), \underline{\sigma}). \tag{3.3}$$

Similarly $\underline{\sigma}^{(\delta,-)}(t)$ is the right continuous with left limits function obtained by setting $\underline{\sigma}^{(\delta,-)}(t) = \underline{\sigma}$ for all $t \in [0, \varepsilon^{-2}\delta)$ while

$$\big(\underline{x}(\varepsilon^{-2}\delta), \underline{\sigma}^{(\delta,-)}(\varepsilon^{-2}\delta)\big) = \prod_{i=1}^{m} H^{\ell_i}\big(\underline{x}(\varepsilon^{-2}\delta), \underline{\sigma}\big). \tag{3.4}$$

Theorem 3.1. *In the above setup, given* $(\underline{\sigma}, \omega) \in \mathcal{X}_{\varepsilon^{-2}\delta}$ *and* $\underline{\sigma}'$ *such that* $h_a(\sigma') = h_a(\sigma)$ *and* $(\underline{x}(0), \underline{\sigma}') \preccurlyeq (\underline{x}(0), \underline{\sigma})$, *we have*

$$\big(\underline{x}(\varepsilon^{-2}\delta), \sigma'^{(\delta,-)}(\varepsilon^{-2}\delta)\big) \preccurlyeq \big(\underline{x}(\varepsilon^{-2}\delta), \sigma(\varepsilon^{-2}\delta)\big), \tag{3.5}$$

$$\big(\underline{x}(\varepsilon^{-2}\delta), \sigma'(\varepsilon^{-2}\delta)\big) \preccurlyeq \big(\underline{x}(\varepsilon^{-2}\delta), \sigma^{(\delta,+)}(\varepsilon^{-2}\delta)\big). \tag{3.6}$$

In (3.5) $\sigma'^{(\delta,-)}(\varepsilon^{-2}\delta)$ *is the auxiliary evolution associated to* $(\underline{\sigma}', \omega)$, *and in* (3.6) $\sigma'(\varepsilon^{-2}\delta)$ *is the true evolution associated to* $(\underline{\sigma}', \omega)$. *The evolutions on the right-hand side of* (3.5) *and* (3.6) *are, respectively, the true and the auxiliary* $(\delta, +)$*-evolutions associated to* $(\underline{\sigma}, \omega)$.

We shall prove Theorem 3.1 in the remaining part of this section by constructing joint processes (that we call couplings by an abuse of notation) which exploit the fact that the above inequalities remain valid if we exchange colors of particles at the same site.

The coupling is determined by specifying the colors of each $x_i(t)$ in the two processes, the one associated to $(\underline{\sigma}', \omega)$ and the one associated to $(\underline{\sigma}, \omega)$: they have same positions and same ω. Thus, the configurations in the coupled process are systems of particles with two colors: $(\underline{x}, \Sigma)$, $\Sigma = (\underline{\sigma}, \underline{\sigma}')$. We call $(x_i, \sigma_i, \sigma_i')$, $i = 1, \dots, M$, the specification of particle i. With the aim of establishing stochastic inequalities, we split the particles of $(\underline{x}, \Sigma)$ into "married pairs", "singletons", and "discrepancies" using the following notions:

- i is a a-singleton or a b-singleton if it has specification (x_i, a, a), respectively, (x_i, b, b);
- i is married with j if i has specification (x_i, a, b) and j has specification (x_j, b, a) with $x_i > x_j$; (i, j) are then said to be a "married pair";
- i is a (b, a)-discrepancy or a (a, b)-discrepancy if it has specification (x_i, b, a) or (x_i, a, b), respectively, and it is not in a married pair.

We shall say that a quadruple (P, S, I, J) is a "splitting" of $(\underline{x}, \Sigma)$ if P is a set of married pairs, S a set of singletons, I a set of (b, a) discrepancies, J a set of (a, b) discrepancies and each particle is either in one (and only one) of the pairs in P or if it is not in any of the pairs then it is in one (and only one) of the other three sets. Of course there are in general many ways to split $(\underline{x}, \Sigma)$ into a quadruple (P, S, I, J), we want splittings with as less discrepancies as possible, as it follows from the following lemma which will be extensively used in the sequel (its proof is an immediate consequence of the definitions and omitted).

Lemma 3.2. *Let (P, S, I, J) be a splitting of $(\underline{x}, \Sigma)$, $\Sigma = (\underline{\sigma}, \underline{\sigma}')$ with $I = J = \varnothing$. Then $(\underline{x}, \underline{\sigma}') \preccurlyeq (\underline{x}, \underline{\sigma})$. Vice versa if $(\underline{x}, \underline{\sigma}') \preccurlyeq (\underline{x}, \underline{\sigma})$ there exists a splitting of $(\underline{x}, \Sigma)$, $\Sigma = (\underline{\sigma}, \underline{\sigma}')$, such that $I = J = \varnothing$.*

The coupling will be defined by specifying the evolution $(\underline{x}(t), \Sigma(t))$ and its splitting $(P(t), S(t), I(t), J(t))$.

The map R

Let (s_k, s_{k+1}) be an interval between events of the Poisson process and let (P, S, I, J) the quadruple at time s_k. Let t^* be the first time after s_k when $x_i(t^*) = x_j(t^*)$ for some $(i, j) \in P$. We then set $(P(t), S(t), I(t), J(t)) = (P, S, I, J)$ for $t < \min\{t^*, s_{k+1}\}$ and if $t^* < s_{k+1}$ we set $P(t^*) = P \setminus (i, j)$ and put $i, j \in S(t^*)$ with i a a-singleton and j a b-singleton (we have used here the fact that we may exchange colors of particles at a same site). By iteration, the evolution is extended till time s_{k+1} with a new configuration $\underline{x}'$ and with a new splitting (P', S', I', J').

The set of possible $\underline{x}'$, (P', S', I', J') obtained in this way is characterized by the following requests: $I' = I$, $J' = J$, $P' \subseteq P$ with $S' \setminus S$ made by all labels i and j of the pairs which have disappeared. $\underline{x}'$ has the only constraint that $x'_i > x'_j$ if $(i, j) \in P'$. We denote by $\mathcal{R}$ the collection of all maps R such that $R(\underline{x}, P, S, I, J)$ has the above properties. The important points for the sequel are: (i) the discrepancies are unchanged under any $R \in \mathcal{R}$ and (ii) the identity map is in $\mathcal{R}$.

The C-maps

They describe the changes of colors which involve, according to cases, the particles $i_a(\underline{x}, \sigma)$, $i_b(\underline{x}, \sigma)$, $i_a(\underline{x}, \sigma')$ and $i_b(\underline{x}, \sigma')$. Due to such changes the splitting quadruple $(P; S; I; J)$ associated to $(\underline{x}, \Sigma)$ will be modified into a new quadruple $(P'; S'; I'; J')$, in the way described below:

C_1^{right}: shorthand $i = i_a(\underline{x}, \sigma)$:
- (a) if there is j such that $(i, j) \in P$ then $P' = P \setminus (i, j)$, $S' = S \cup i$, $I' = I \cup j$, $J' = J$;
- (b) if $i \in S$ then $S' = S \setminus i$, $I' = I \cup i$, $J' = J$, $P' = P$;
- (c) if $i \in J$ then $S' = S \cup i$, $J' = J \setminus i$, $I' = I$ and $P' = P$.

C_1^{left}: shorthand $i = i_b(\underline{x}, \sigma)$:
- (a) if there is j such that $(j, i) \in P$, then $P' = P \setminus (j, i)$, $S' = S \cup i$, $J' = J \cup j$, $I' = I$;
- (b) if $i \in S$ then $S' = S \setminus i$, $J' = J \cup i$, $I' = I$, $P' = P$;
- (c) if $i \in I$ then $S' = S \cup i$, $I' = I \setminus i$, $J' = J$ and $P' = P$.

C_2^{right}: shorthand $i = i_a(\underline{x}, \sigma')$ and k the largest label in I if $I \neq \varnothing$:
- (a) if there is j such that $(j, i) \in P$ and $I \neq \varnothing$, then $P' = P \setminus (j, i) \cup (j, k)$, $S' = S \cup i$, $I' = I \setminus k$, $J' = J$; if instead $I = \varnothing$, then $P' = P \setminus (j, i)$, $S' = S \cup i$, $I' = I = \varnothing$, $J' = J \cup j$;

(b) if $i \in S$ and $I \neq \varnothing$ then $S' = S \setminus i$, $I' = I \setminus k$, $J' = J$, $P' = P \cup (i, k)$; if instead $I = \varnothing$, then $S' = S \setminus i$, $P' = P$, $I' = I$ and $J' = J \cup i$;
(c) if $i \in I$ then $I' = I \setminus i$, $P' = P$, $S' = S \cup i$ and $J' = J$.

C_2^{left}: shorthand $i = i_b(\underline{x}, \sigma')$ and k the largest label in J if $J \neq \varnothing$:
(a) if there is j such that $(i, j) \in P$ and $J \neq \varnothing$, then $P' = P \setminus (i, j) \cup (k, j)$, $S' = S \cup i$, $J' = J \setminus k$, $I' = I$; if instead $J = \varnothing$, then $P' = P \setminus (i, j)$, $S' = S \cup i$, $J' = J = \varnothing$, $I' = I \cup j$;
(b) if $i \in S$ and $J \neq \varnothing$ then $S' = S \setminus i$, $J' = J \setminus k$, $I' = I$, $P' = P \cup (k, i)$; if instead $J = \varnothing$, then $S' = S \setminus i$, $P' = P$, $J' = J$ and $I' = I \cup i$;
(c) if $i \in J$ then $J' = J \setminus i$, $P' = P$, $S' = S \cup i$ and $I' = I$.

Remark 3.1. The subscript 1, 2, reminds that the C operator acts on the first component σ, respectively, the second one, σ'. The above properties of the C_2 operators follow from the definitions of i_a and i_b allowing for the formation of married pairs which are instead not used for the C_1 operators. Recall that our goal is to prove that at the end I and J are empty, in this respect the C_1 operators are dangerous, as they may increase by 1 the cardinality of I (with C_1^{right}) or J (with C_1^{left}) while the C_2 are recovery operators as they decrease by 1 the cardinality of I (with C_2^{right}) or J (with C_2^{left}) when I and J are nonempty. This is behind the proof of the next theorem which, as we shall see after its proof, yields as a corollary the proof of Theorem 3.1.

Theorem 3.3. *Let (P, S, I, J), $I = J = \varnothing$, be a quadruple associated to $(\underline{x}, \Sigma)$. Then for any nonnegative integer m, any sequences $(R_1, \ldots, R_m)$, $(R'_1, \ldots, R'_m)$ of elements of $\mathcal{R}$,*

$$\big(\underline{x}^*, P^*, S^*, I^*, J^*\big) := \big(C_2 R'\big)_m (C_1 R)_m (\underline{x}, P, S, I, J) \tag{3.7}$$

has $I^* = J^* = \varnothing$ where we have used the notation for $q \le m$: $(C_1 R)_q = C_1^{\ell_q} R_q \cdots C_1^{\ell_1} R_1$ and $(C_2 R')_q = C_2^{\ell_q} R'_q \cdots C_2^{\ell_1} R'_1$.

Proof. Observe that the elements of $\mathcal{R}$ change only the sets P and S, thus to prove the theorem we only need to consider the C-maps. For $q \le m$, we call I_q and J_q the discrepancies of $(C_1 R)_q(\underline{x}, P, S, I, J)$ and we define

$$N_{\le q}^{\text{right}} = \sum_{i=1}^{q} \mathbf{1}_{\ell_i = \text{right}}, \qquad N_{\le q}^{\text{left}} = \sum_{i=1}^{q} \mathbf{1}_{\ell_i = \text{left}}.$$

For $q > m$, we call I_q, J_q the discrepancies of $(C_2 R')_{q-m} (C_1 R)_m (\underline{x}, P, S, I, J)$ and we set

$$N_{>q}^{\text{right}} = \sum_{i=q+1}^{2m} \mathbf{1}_{\ell_{i-m} = \text{right}}, \qquad N_{>q}^{\text{left}} = \sum_{i=q+1}^{2m} \mathbf{1}_{\ell_{i-m} = \text{left}}.$$

We prove below that

$$N^{\text{right}}_{\le q} - |I_q| = N^{\text{left}}_{\le q} - |J_q| \ge 0, \qquad q \le m, \tag{3.8}$$

$$N^{\text{right}}_{> q} - |I_q| = N^{\text{left}}_{> q} - |J_q| \ge 0, \qquad q > m \tag{3.9}$$

and observe that if we put $q = 2m$ in (3.9) we get $I_{2m} = J_{2m} = \varnothing$, so that the theorem follows from (3.8)–(3.9).

Proof of (3.8). (3.8) trivially holds for $q = 0$ so that proceeding by induction we suppose that (3.8) holds with $q - 1 < m$. Take for instance, $\ell_q =$ left. Then $N^{\text{left}}_{\le q} = N^{\text{left}}_{\le q-1} + 1$ while $N^{\text{right}}_{\le q} = N^{\text{right}}_{\le q-1}$. Recalling the definition of C_1^{left}, in case (a) or (b) $|J_q| = |J_{q-1}| + 1$, and $|I_q| = |I_{q-1}|$; while in case (c) $|J_q| = |J_{q-1}|$ and $|I_q| = |I_{q-1}| - 1$, thus in all cases (3.8) holds with q. The case when $\ell_q =$ right is analogous and omitted.

Proof of (3.9). As before we proceed by induction observing first that (3.9) holds for $q = m$. In fact by definition $N^{\ell}_{>m} = N^{\ell}_{\le m}$ for $\ell =$ right and left. We then assume (3.9) holds for $q - 1 \in (m, 2m)$. Suppose for instance that $\ell_q =$ left. Then $N^{\text{left}}_{>q} = N^{\text{left}}_{>q-1} - 1$ while $N^{\text{right}}_{>q} = N^{\text{right}}_{>q-1}$. Recalling the definition of C_2^{left}, in case (a) or (b) if $J_{q-1} \ne \varnothing$ then $|J_q| = |J_{q-1}| - 1$, and $|I_q| = |I_{q-1}|$; if instead $J_{q-1} = \varnothing$ then $|J_q| = |J_{q-1}|$ and $|I_q| = |I_{q-1}| + 1$. In case (c) $|J_q| = |J_{q-1}| - 1$ and $|I_q| = |I_{q-1}|$, thus in all cases (3.9) holds with q. The case when $\ell_q =$ right is analogous and omitted. □

Proof of Theorem 3.1. Given ω, σ and σ' as in the statement of Theorem 3.1, we use Lemma 3.2 to construct a splitting (P, S, I, J) such that $I = J = \varnothing$. Let m be such that $s_m \le \varepsilon^{-2}\delta$ and $s_{m+1} > \varepsilon^{-2}\delta$.

Proof of (3.5). For $q = 1, \dots, m$ let R_q be the maps corresponding to the times intervals (s_q, s_{q+1}) and let R'_1 be the map corresponding to the time interval $(s_m, \varepsilon^{-2}\delta)$. Furthermore let $R'_q =$ identity for all $q = 2, \dots, m$. Then (3.7) is a splitting of $(\underline{x}(t), \sigma(\varepsilon^{-2}\delta), \sigma'^{(\delta,-)}(\varepsilon^{-2}\delta))$. From Theorem 3.3, we then have that $I^* = J^* = \varnothing$ and thus by Lemma 3.2 we get (3.5).

Proof of (3.6). We let $R_q =$ identity for all $q = 1, \dots, m$ and instead, for $q = 1, \dots, m$, R'_q are the maps corresponding to the times intervals (s_q, s_{q+1}). Finally, R'_{m+1} is the map corresponding to the time interval $(s_m, \varepsilon^{-2}\delta)$. Then

$$(\underline{x}^*, P^*, S^*, I^*, J^*) := R'_{m+1}(C_2 R')_m R_{m+1}(C_1 R)_m(\underline{x}, P, S, I, J)$$

is a splitting of $(\underline{x}(t), \sigma^{(\delta,+)}(\varepsilon^{-2}\delta), \sigma'(\varepsilon^{-2}\delta))$. Since R'_{m+1} does not change the sets of discrepancies, from Theorem 3.3 we get that $I^* = J^* = \varnothing$ which, by Lemma 3.2 concludes the proof of (3.6). □

4 Proof of Theorem 2.1

For any $\varepsilon > 0$, we choose an initial configuration $(\underline{x}, \underline{\sigma})$ with law P^ε (as described in Section 2) and study its evolution $(\underline{x}(t), \underline{\sigma}(t))$ for a fixed time interval $[0, T]$. We do not have a good knowledge of $(\underline{x}(t), \underline{\sigma}(t))$ (just that the process is well defined). The information needed to prove Theorem 2.1 will be gained by studying two auxiliary processes $(\underline{x}(t), \underline{\sigma}^{(\delta,\pm)}(t))$ (which start at time 0 from $(\underline{x}, \underline{\sigma})$ as the true process) and by using the inequalities of the previous section to compare the true and the auxiliary processes.

Thus the first step is to extend the definition of the auxiliary processes to the whole time interval $[0, T]$. This is done in Definition 4.1 below by iterating the definition given in the last section to the intervals $[(k-1)\varepsilon^{-2}\delta, k\varepsilon^{-2}\delta]$, $k \le K$, K the smallest integer such that $K\varepsilon^{-2}\delta \ge T$. To this purpose we consider the set $\mathcal{X}_{K\varepsilon^{-2}\delta}$ defined in the previous section (see Definition 3.2) and we prove below that with large probability we can restrict our analysis to trajectories in $\mathcal{X}_{K\varepsilon^{-2}\delta}$.

Lemma 4.1. *There is a positive constant c independent of ε (but it may depend on δ and T) such that*

$$\mathcal{P}^\varepsilon[\mathcal{X}_{K\varepsilon^{-2}\delta}] \ge 1 - e^{-c\varepsilon^{-1}}, \tag{4.1}$$

where $\mathcal{P}^\varepsilon$ is defined in Definition 2.2.

Proof. Call $Z = \int(u+v)$ and $p_a := \frac{1}{Z}\int u \in (0,1)$. By (2.3) and (2.2),

$$P^\varepsilon[\sigma_i = a] = \frac{1}{Z_\varepsilon}\sum_x u(\varepsilon x), \qquad Z_\varepsilon = \sum_x \big[u(\varepsilon x) + v(\varepsilon x)\big]$$

and since the σ_i are independent variables, given $\zeta > 0$ such that $0 < p_a - \zeta < p_a + \zeta < 1$ we have for $\varepsilon > 0$ small enough

$$P^\varepsilon\big[\big|h_a(\sigma) - \varepsilon^{-1}p_a\big| < \zeta\big] \ge 1 - e^{-c\varepsilon^{-1}},$$

with c a suitable positive constant. Recalling (3.2), the number $N_a(t)$ of a-particles at time t is a nearest neighbor symmetric random walk with jump intensity $2\varepsilon\kappa$, until the time when $N_a(t)$ reaches either 0 or M. Thus,

$$\mathcal{P}^\varepsilon\big[N_a(t) \in (0, M) \text{ for all } t \le K\varepsilon^{-2}\delta\big] \ge 1 - e^{-c\varepsilon^{-1}}$$

with c a new suitable constant. □

Definition 4.1. Chose an initial configuration $(\underline{x}, \underline{\sigma})$ as above, fix a $(\underline{\sigma}, \omega) \in \mathcal{X}_{K\varepsilon^{-2}\delta}$ and a trajectory $\underline{x}(t)$, $t \le K\varepsilon^{-2}\delta$. We call m_k, $k = 0, \dots, K$ the positive integers such that $k\varepsilon^{-2}\delta \le s_{m_k+1} < s_{m_k+2} < \cdots < s_{m_{k+1}}$. We also call $t_k = k\varepsilon^{-2}\delta$.

We then define $\underline{\sigma}^{(\delta,+)}(t)$ as the left continuous with right limits function obtained by setting $\underline{\sigma}^{(\delta,+)}(t) = \underline{\sigma}^{(\delta,+)}(t_k+)$ for $t \in (t_k, t_{k+1}]$ and

$$\big(\underline{x}(t_k+), \underline{\sigma}^{(\delta,+)}(t_k+)\big) = \prod_{i=m_k+1}^{m_{k+1}} H^{\ell_i}\big(\underline{x}(t_k), \underline{\sigma}^{(\delta,+)}(t_k)\big), \qquad t_k = k\varepsilon^{-2}\delta.$$

Similarly $\underline{\sigma}^{(\delta,-)}(t)$ is the right continuous with left limits function obtained by setting $\underline{\sigma}^{(\delta,-)}(t) = \underline{\sigma}^{(\delta,-)}(t_k)$ for all $t \in [t_k, t_{k+1})$, while at $t_{k+1} = (k+1)\varepsilon^{-2}\delta$

$$\big(\underline{x}(t_{k+1}), \underline{\sigma}^{(\delta,-)}(t_{k+1})\big) = \prod_{i=m_k+1}^{m_{k+1}} H^{\ell_i}\big(\underline{x}(t_{k+1}), \underline{\sigma}^{(\delta,-)}(t_k)\big).$$

An immediate corollary of Theorem 3.1 is the following.

Corollary 4.2. *In $\mathcal{X}_{K\varepsilon^{-2}\delta}$ setting $t_k = k\varepsilon^{-2}\delta$ we have for all $k \le K$*

$$\big(\underline{x}(t_k), \sigma^{(\delta,-)}(t_k)\big) \preccurlyeq \big(\underline{x}(t_k), \sigma(t_k)\big) \preccurlyeq \big(\underline{x}(t_k), \sigma^{(\delta,+)}(t_k)\big), \tag{4.2}$$

where all the above evolutions start from the same initial datum $(\underline{x}, \underline{\sigma})$.

Proof. The number $N_a(k\varepsilon^{-2}\delta)$ of a-particles at time $k\varepsilon^{-2}\delta$ is the same in all the three evolutions. This is evidently true for $k = 0$ because they all start from the same configuration and the claim follows because

$$N_a\big((k+1)\varepsilon^{-2}\delta\big) - N_a\big(k\varepsilon^{-2}\delta\big) = \sum_{i=m_k+1}^{m_{k+1}} (\mathbf{1}_{\ell_i=\text{right}} - \mathbf{1}_{\ell_i=\text{left}}).$$

The corollary then follows from Theorem 3.1. □

Next step is to prove that $(\underline{x}(k\varepsilon^{-2}\delta), \sigma^{(\delta,\pm)}(k\varepsilon^{-2}\delta))$ have a limit as $\varepsilon \to 0$. The limit will be described by the following macroscopic evolutions.

Definition 4.2. For $u, v \in L_1(\mathbb{R}, \mathbb{R}_+)$ and $\delta > 0$ let $R_\delta(u)$ and $D_\delta(v)$ be such that

$$\int_{R_\delta(u)}^{\infty} u(r)\, dr = \kappa\delta, \qquad \int_{-\infty}^{D_\delta(v)} v(r)\, dr = \kappa\delta. \tag{4.3}$$

We define $K^{(\delta)}(u, v) = (u', v')$ with

$$\begin{aligned} u'(r) &= \mathbf{1}_{(-\infty, R_\delta(u)]}(r) u(r) + \mathbf{1}_{(-\infty, D_\delta(v)]}(r) v(r), \\ v'(r) &= \mathbf{1}_{[D_\delta(v), +\infty)}(r) v(r) + \mathbf{1}_{[R_\delta(u), +\infty)}(r) u(r). \end{aligned} \tag{4.4}$$

Denote by $G_t \star u$ the convolution of the Gaussian kernel with a function u:

$$G_t(r, r') = \frac{e^{-(r-r')^2/2t}}{\sqrt{2\pi t}}, \qquad G_t \star u = \int G_t(r, r') u(r')\, dr'. \tag{4.5}$$

With an abuse of notation, we write $G_t \star (u, v) \equiv (G_t \star u, G_t \star v)$. We define the "barriers" $S^{(\delta,\pm)}_{n\delta}(u, v)$, $n \in \mathbb{N}$, by setting $S^{(\delta,\pm)}_0(u, v) = (u, v)$, and $\forall n \ge 1$

$$\begin{aligned} S^{(\delta,-)}_{n\delta}(u, v) &= K^{(\delta)} G_\delta \star S^{(\delta,-)}_{(n-1)\delta}(u, v), \\ S^{(\delta,+)}_{n\delta}(u, v) &= G_\delta \star K^{(\delta)} S^{(\delta,+)}_{(n-1)\delta}(u, v). \end{aligned} \tag{4.6}$$

We denote by $(u^{(\delta,\pm)}_{n\delta}, v^{(\delta,-)}_{n\delta}) = S^{(\delta,\pm)}_{n\delta}(u, v)$.

Theorem 4.3. *For any $k \le K$ and any δ small enough*

$$\varepsilon \xi_{(\underline{x}(k\varepsilon^{-2}\delta), \underline{\sigma}^{(\delta,\pm)}(k\varepsilon^{-2}\delta))} \to S^{(\delta,\pm)}_{k\delta}(u, v)$$

as $\varepsilon \to 0$ weakly in probability.

The auxiliary processes are essentially independent random walk evolutions with an additional colors change at finitely many times, $k\varepsilon^{-2}\delta$, $k \le K$. The convergence of the random walk evolutions can be established in a very strong form which allows to control the positions of the rightmost a- and leftmost b-particles. The argument is rather lengthy but essentially analogous to that in Carinci et al. (2014a) and for brevity we omit it.

Theorem 4.4. *There exist continuous functions $\bar{u}(r, t)$, $\bar{v}(r, t)$, $r \in \mathbb{R}$, $t \in [0, T)$, also denoted by $(\bar{u}(r, t), \bar{v}(r, t)) = S_t(u, v)$ such that $S_0(u, v) = (u, v)$ and for any $t \in [0, T)$:*

$$\lim_{n\to\infty} S^{(\delta,\pm)}_{2^{-n}t}(u, v) = S_t(u, v), \tag{4.7}$$

uniformly in the compacts and in L^1.

We refer to Section 8 of Carinci et al. (2014a) where an analogous statement has been proved. Fix t, by (4.2) and Theorem 4.3 with $\delta = 2^{-n}t$, for any $r \in \mathbb{R}$, in probability

$$\limsup_{\varepsilon\to 0} \varepsilon \sum_{y \ge \varepsilon^{-1} r} \xi_{(\underline{x}(\varepsilon^{-2}2^{-n}t), \underline{\sigma}(\varepsilon^{-2}2^{-n}t))}(y) \le \int_r^{+\infty} u_t^{(2^{-n}t,+)}, \tag{4.8}$$

$$\int_r^{+\infty} u_t^{(2^{-n}t,-)} \le \liminf_{\varepsilon\to 0} \varepsilon \sum_{y \ge \varepsilon^{-1} r} \xi_{(\underline{x}(\varepsilon^{-2}2^{-n}t), \underline{\sigma}(\varepsilon^{-2}2^{-n}t))}(y). \tag{4.9}$$

Theorem 2.1 then follows because by (4.7), the integrals in (4.8) and (4.9) converge as $n \to \infty$ to the same limit $\int_r^{+\infty} \bar{u}(r', t)\, dr'$. Details are omitted.

5 Macroscopic inequalities

In this section, we assume that for some $S > 0$ there exists a solution $(\mu(\cdot, t), U_t)$, $(\nu(\cdot, t), V_t)$, $t \in [0, S]$ of the free boundary problem (2.7). We assume that this solution is regular in the sense specified before Theorem 2.2.

The main result of this section is Theorem 5.1 below that states that, modulo an error exponentially small in δ, $(\mu(\cdot, t), \nu(\cdot, t))$ is in between the barriers $S_{n\delta}^{(\delta,\pm)}(\mu_0, \nu_0) \equiv (u_{n\delta}^{(\delta,\pm)}, v_{n\delta}^{(\delta,\pm)})$, $\mu_0 = \mu(\cdot, 0)$, $\nu_0 = \nu(\cdot, 0)$. The inequalities are the macroscopic analogue of the microscopic ones.

Theorem 5.1. *There is δ_0 so that the following holds. There are constants c and c' so that for all $\delta < \delta_0$, for all $k \le \delta^{-1} S$ and for all $r \in \mathbb{R}$ we have*

$$F\big(r; u_{k\delta}^{(\delta,-)}\big) - kc'e^{-c\delta^{-1}} \le F\big(r; \mu(\cdot, k\delta)\big) \le F\big(r; u_{k\delta}^{(\delta,+)}\big) + kc'e^{-c\delta^{-1}}, \tag{5.1}$$

where $F(r; g) = \int_r^{+\infty} g$.

We first prove Theorem 2.2 as a corollary of Theorem 5.1.

Proof of Theorem 2.2. Fix a $t \le S$ and consider $k =$ integer part of $\delta^{-1} t$, then take the limit $\delta \to 0$ in (5.1) using Theorem 4.4 we then get that $(\mu(\cdot, t), \nu(\cdot, t))$ coincide with $(\bar{u}(\cdot, t), \bar{v}(\cdot, t))$ of Theorem 2.1. □

We prove in Section 5.2 the lower bound and in Section 5.3 the upper bound in (5.1) for $k = 1$, finally, in Section 5.4 we prove that we can reduce the generic step to this case. We first need to state properties of the regular solutions that will be used in the sequel.

5.1 Properties of a regular solution

The regular solution $(\mu(\cdot, t), U_t)$, $(\nu(\cdot, t), V_t)$, $t \in [0, S]$ is related to the law $P_{r',s}$ of a Brownian motion $\{B_t, t \ge s\}$ that starts from $r' \in \mathbb{R}$ at time $s \in [0, S]$ in the following way, see, for instance, Karatzas and Shreve (1991). First, define the stopping times

$$\tau_s^U = \inf\{t \ge s : B_t \ge U_t\}, \qquad \tau_s^V = \inf\{t \ge s : B_t \le V_t\}. \tag{5.2}$$

Then for any $t \in [0, S]$ and any interval $I \subset \mathbb{R}$

$$\begin{aligned} &\int_I \mu(r, t)\, dr \\ &\quad = \int \mu_0(r') P_{r',0}\big(B_t \in I; \tau_0^U > t\big) + \kappa \int_0^t P_{V_s,s}\big(B_t \in I; \tau_s^U > t\big), \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\int_I \nu(r, t)\, dr \\ &\quad = \int \nu_0(r') P_{r',0}\big(B_t \in I; \tau_0^V > t\big) + \kappa \int_0^t P_{U_s,s}\big(B_t \in I; \tau_s^V > t\big). \end{aligned} \tag{5.4}$$

We call $P_{r,s}(\tau_s^U \in dt)$ and $P_{r,s}(\tau_s^V \in dt)$ the law of the stopping times (5.2).

Lemma 5.2. *For all $t \in [0, S]$ we have*

$$\int \mu_0(r) P_{r,0}\big(\tau_0^U \le t\big)\, dr + \kappa \int_0^t P_{V_s,s}\big(\tau_s^U \le t\big)\, ds = \kappa t, \tag{5.5}$$

$$\int \nu_0(r) P_{r,0}\big(\tau_0^V \le t\big)\, dr + \kappa \int_0^t P_{U_s,s}\big(\tau_s^V \le t\big)\, ds = \kappa t. \tag{5.6}$$

Moreover, there are C and C' depending on the constant $c > U_t - V_t$ such that for all δ small enough the following holds. For all $r^* \in \mathbb{R}$ and $t \le \delta$

$$\begin{aligned} \Big| &\kappa \int_0^t P_{V_s,s}\big(B_t \ge r^*;\, \tau_s^U > t\big)\, ds \\ &- \int_0^t \int \nu_0(r) P_{r,0}\big(\tau_0^V \in ds\big) P_{V_s,s}\big(B_t \ge r^*;\, \tau_s^U > t\big)\, dr \Big| \le C' e^{-C\delta^{-1}}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \Big| &\kappa \int_0^t P_{U_s,s}\big(B_t \le r^*;\, \tau_s^V > t\big)\, ds \\ &- \int_0^t \int \mu_0(r) P_{r,0}\big(\tau_0^V \in ds\big) P_{U_s,s}\big(B_t \le r^*;\, \tau_s^V > t\big)\, dr\, ds \Big| \le C' e^{-C\delta^{-1}}. \end{aligned} \tag{5.8}$$

Proof. From (5.3), we have

$$\begin{aligned} &\int \mu(r,t)\, dr \\ &\quad = \int \mu_0(r) P_{r,0}\big(\tau_0^U > t\big)\, dr + \kappa \int_0^t P_{V_s,s}\big(\tau_s^U > t\big)\, ds \\ &\quad = \int \mu_0(r)\, dr + \kappa t - \int \mu_0(r) P_{r,0}\big(\tau_0^U \le t\big)\, dr - \kappa \int_0^t P_{V_s,s}\big(\tau_s^U \le t\big)\, ds. \end{aligned}$$

Since the total mass is conserved this yields (5.5). The proof of (5.6) is analogous. Differentiating equations (5.5) and (5.6) and noticing that $P_{r,0}(\tau_0^U \in dt)$ is absolutely continuous with respect to the Lebesgue measure, we get

$$\kappa = \int \mu_0(r) P_{r,0}\big(\tau_0^U \in dt\big)\, dr + \kappa \int_0^t P_{V_s,s}\big(\tau_s^U \in dt\big)\, ds, \tag{5.9}$$

$$\kappa = \int \nu_0(r) P_{r,0}\big(\tau_0^V \in dt\big)\, dr + \kappa \int_0^t P_{U_s,s}\big(\tau_s^V \in dt\big)\, ds. \tag{5.10}$$

We now use (5.10) to rewrite κ on the right-hand side of (5.3) as follows

$$\begin{aligned} &\kappa \int_0^t P_{V_s,s}\big(B_t \ge r^*;\, \tau_s^U > t\big)\, ds \\ &\qquad = \int_0^t \int \nu_0(r) P_{r,0}\big(\tau_0^V \in ds\big) P_{V_s,s}\big(B_t \ge r^*;\, \tau_s^U > t\big)\, dr \\ &\qquad\quad + \int_0^t \int_0^s \kappa P_{U_{s'},s'}\big(\tau_{s'}^V \in ds\big) P_{V_s,s}\big(B_t \ge r^*;\, \tau_s^U > t\big)\, ds'. \end{aligned} \tag{5.11}$$

There are $C, C' > 0$ so that for all $0 \le s' < s < \delta$

$$P_{U_{s'},s'}\big(\tau_{s'}^V < s\big) \le C' e^{-C\delta^{-1}}, \qquad P_{V_{s'},s'}\big(\tau_{s'}^U < s\big) \le C' e^{-C\delta^{-1}}. \tag{5.12}$$

To prove (5.7), we observe that the last term in (5.11) is bounded by (5.12). The proof of (5.8) is analogous by using (5.4) and (5.9). □

5.2 Lower bound in the first time interval

Here we prove the first inequality in (5.1) for $k = 1$ observing that in the proof we only use that the evolution $S_\delta^{(\delta,-)}(\mu_0, \nu_0)$ has same initial datum as the regular solution. More precisely, we prove that for all $r^* \in \mathbb{R}$

$$F\big(r^*; \mu(\cdot, \delta)\big) = \int_{r^*}^\infty \mu(r, \delta)\, dr \ge \int_{r^*}^\infty u_\delta^{(\delta,-)}(r)\, dr - 3C' e^{-C\delta^{-1}}, \tag{5.13}$$

with C' and C as in Lemma 5.2.

By definition $u_\delta^{(\delta,-)} = 1_{(-\infty,R)} G_\delta \star \mu_0 + \mathbf{1}_{(-\infty,D]} G_\delta \star \nu_0$ with R, D so that

$$\int_R^\infty G_\delta \star \mu_0 = \kappa\delta, \qquad \int_{-\infty}^D G_\delta \star \nu_0 = \kappa\delta. \tag{5.14}$$

By using the law of the Brownian motion, we write

$$\begin{aligned}\int_{r^*}^\infty u_\delta^{(\delta,-)} &= \int \mu_0(r) P_{r,0}\big(B_\delta \in (r^*, R)\big)\, dr + \int \nu_0(r) P_{r,0}\big(B_\delta \in [r^*, D)\big)\, dr \\ &= \int \mu_0(r) P_{r,0}\big(B_\delta \ge r^*\big) - \kappa\delta + \int \nu_0(r) P_{r,0}\big(B_\delta \in [r^*, D)\big)\, dr.\end{aligned} \tag{5.15}$$

Using (5.3) and (5.5), we get

$$\int_{r^*}^\infty \mu(r, \delta)\, dr \ge \int \mu_0(r) P_{r,0}\big(B_\delta \ge r^*\big) + \kappa \int_0^\delta P_{V_s,s}\big(B_{\delta-s} \ge r^*\big) - \kappa\delta. \tag{5.16}$$

Thus, if $r^* > D$ from (5.15) and (5.16) we get (5.13). We then assume that $r^* \le D$ and observe that by (5.7) and (5.12)

$$\begin{aligned}\kappa \int_0^\delta P_{V_s,s}\big(B_{\delta-s} \ge r^*\big) &\ge \kappa \int_0^\delta P_{V_s,s}\big(B_{\delta-s} \ge r^*; \tau_s^U > \delta\big) \\ &\ge \int \nu_0(r) P_{r,0}\big(B_\delta \ge r^*; \tau_0^V \le \delta\big)\, dr - 2C' e^{-C\delta^{-1}}.\end{aligned} \tag{5.17}$$

By (5.6) and (5.12), $\int \nu_0(r) P_{r,0}(\tau_0^V \le \delta)\, dr \ge \kappa\delta - C' e^{-C\delta^{-1}}$. Thus,

$$\begin{aligned}&\int \nu_0(r) P_{r,0}\big(B_\delta \ge r^*; \tau_0^V \le \delta\big) \\ &\quad \ge \kappa\delta - \int \nu_0(r) P_{r,0}\big(B_\delta \le r^*; \tau_0^V \le \delta\big) - C' e^{-C\delta^{-1}} \\ &\quad \ge \kappa\delta - \int \nu_0(r) P_{r,0}\big(B_\delta \le r^*\big)\, dr - C' e^{-C\delta^{-1}}.\end{aligned} \tag{5.18}$$

Then from (5.17), (5.18) and the definition of D we get

$$\begin{aligned}\kappa\int_0^\delta P_{V_s,s}\big(B_{\delta-s}\geq r^*\big)&\geq\kappa\delta-\int\nu_0(r)P_{r,0}\big(B_\delta\leq r^*\big)\,dr-3C'e^{-C\delta^{-1}}\\&=\int\nu_0(r)P_{r,0}\big(B_\delta\in[r^*,D]\big)\,dr-3C'e^{-C\delta^{-1}},\end{aligned}\tag{5.19}$$

concluding the proof of (5.13).

5.3 Upper bound in the first time interval

Here we give the proof of the upper bound in (5.1) for $k=1$. Call R_0 and D_0 the points such that

$$\int_{R_0}^\infty\mu_0(r)\,dr=\kappa\delta,\qquad\int_{-\infty}^{D_0}\nu_0(r)\,dr=\kappa\delta\tag{5.20}$$

and call $u_2=\mu_0-u_1$, $v_2=\nu_0-v_1$, where

$$u_1(r)=\mu_0(r)\mathbf{1}_{(R_0,+\infty)}(r),\qquad v_1(r)=\nu_0(r)\mathbf{1}_{(-\infty,D_0]}(r).\tag{5.21}$$

Thus v_1 and u_1 have mass $\kappa\delta$ and by definition

$$u_\delta^{(\delta,+)}=G_\delta\star[u_2+v_1],\qquad v_\delta^{(\delta,+)}=G_\delta\star[v_2+u_1].$$

From (5.3), we get that the inequality $F(r^*;\mu(\cdot,\delta))\leq F(r^*;u_\delta^{(\delta,+)})+m$ can be written as

$$\begin{aligned}\int\mu_0(r)P_{r,0}\big(B_\delta\geq r^*;\,\tau_0^U>\delta\big)\,dr+\kappa\int_0^\delta P_{V_s,s}\big(B_{\delta-s}\geq r^*;\,\tau_s^U>\delta\big)\,ds\\\leq\int\big[u_2(r)+v_1(r)\big]P_{r,0}\big(B_\delta\geq r^*\big)\,dr+m.\end{aligned}\tag{5.22}$$

We prove below (5.22) for $m=4C'e^{-C\delta^{-1}}$ with C' and C as in Lemma 5.2. Since $\mu_0=u_1+u_2$ we have

$$\begin{aligned}&\int\mu_0(r)P_{r,0}\big(B_\delta\geq r^*;\,\tau_0^U>\delta\big)\,dr\\&\quad=\int u_2(r)P_{r,0}\big(B_\delta\geq r^*\big)\,dr+\int u_1(r)P_{r,0}\big(B_\delta\geq r^*;\,\tau_0^U>\delta\big)\,dr\\&\qquad-\int u_2(r)P_{r,0}\big(B_\delta\geq r^*;\,\tau_0^U\leq\delta\big)\,dr.\end{aligned}\tag{5.23}$$

From (5.7), (5.12) and using that $\nu_0=v_1+v_2$ we have

$$\begin{aligned}&\kappa\int_0^\delta P_{V_s,s}\big(B_{\delta-s}\geq r^*;\,\tau_s^U>\delta\big)\\&\quad\leq\int\nu_0(r)P_{r,0}\big(B_\delta\geq r^*;\,\tau^D\leq\delta\big)\,dr+2C'e^{-C\delta^{-1}}\end{aligned}\tag{5.24}$$

$$= \int v_2(r) P_{r,0}\big(B_\delta \geq r^*; \tau^V \leq \delta\big)\, dr - \int v_1(r) P_{r,0}\big(B_\delta \geq r^*; \tau^V > \delta\big)\, dr$$
$$+ \int v_1(r) P_{r,0}\big(B_\delta \geq r^*\big)\, dr + 2C' e^{-C\delta^{-1}}.$$

From (5.23) and (5.24), we get

$$\begin{aligned} F\big(r^*; \mu(\cdot, \delta)\big) \leq & \int [u_2(r) + v_1(r)] P_{r,0}\big(B_\delta \geq r^*\big)\, dr + 2C' e^{-C\delta^{-1}} \\ & + \int u_1(r) P_{r,0}\big(B_\delta \geq r^*; \tau_0^U > \delta\big)\, dr \\ & - \int u_2(r) P_{r,0}\big(B_\delta \geq r^*; \tau_0^U \leq \delta\big)\, dr \\ & + \int v_2(r) P_{r,0}\big(B_\delta \geq r^*; \tau^V \leq \delta\big)\, dr \\ & - \int v_1(r) P_{r,0}\big(B_\delta \geq r^*; \tau^V > \delta\big)\, dr. \end{aligned} \tag{5.25}$$

In Lemma 5.3 below we prove that the last two terms on the right-hand side of (5.25) are bounded by $C' e^{-C\delta^{-1}}$ thus concluding the proof of (5.22).

Lemma 5.3. *Let u_i and v_i, $i = 1, 2$ be as in* (5.21), *then for all $r^* \in \mathbb{R}$*

$$\begin{aligned} & \int u_1(r) P_{r,0}\big(B_\delta \geq r^*; \tau_0^U > \delta\big) \\ & \quad \leq \int u_2(r) P_{r,0}\big(B_\delta \geq r^*; \tau_0^U \leq \delta\big) + C' e^{-C\delta^{-1}}, \end{aligned} \tag{5.26}$$

$$\begin{aligned} & \int v_1(r) P_{r,0}\big(B_\delta \geq r^*; \tau_0^V > \delta\big) \\ & \quad \geq \int v_2(r) P_{r,0}\big(B_\delta \geq r^*; \tau_0^V \leq \delta\big) - C' e^{-C\delta^{-1}}. \end{aligned} \tag{5.27}$$

Proof. We only prove (5.26) since the proof (5.27) is completely analogous. From (5.5), we get

$$\int [u_1(r) + u_2(r)] P_{r,0}\big(\tau_0^U \leq \delta\big)\, dr + \kappa \int_0^\delta P_{V_s,s}\big(\tau_s^U \leq \delta\big)\, ds = \kappa\delta = \int u_1(r)\, dr,$$

thus

$$\begin{aligned} & \int u_1(r)\big[1 - P_{r,0}\big(\tau_0^U \leq \delta\big)\big]\, dr \\ & \quad = \int u_2(r) P_{r,0}\big(\tau_0^U \leq \delta\big)\, dr + \kappa \int_0^\delta P_{V_s,s}\big(\tau_s^U \leq \delta\big). \end{aligned} \tag{5.28}$$

We call

$$\alpha(r) = P_{r,0}\big(\tau_0^U \le \delta\big), \qquad \beta(s) = P_{V_s,s}\big(\tau_s^U \le \delta\big) \tag{5.29}$$

and from (5.28) we get

$$Z := \int u_2(r)\alpha(r) + \kappa \int_0^\delta \beta(s) = \int u_1(r)\big[1-\alpha(r)\big]\,dr. \tag{5.30}$$

We call $\lambda_r(ds)$ the law of τ_0^U conditioned to the event $\tau_0^U \le \delta$ when the Brownian motion starts from r at time 0 and write

$$\begin{aligned} &\int u_2(r) P_{r,0}\big(B_\delta \ge r^*;\, \tau_0^U \le \delta\big)\,dr \\ &\qquad = \int u_2(r)\alpha(r) \int_0^\delta \lambda_r(ds) P_{U_s,s}\big(B_\delta \ge r^*\big). \end{aligned} \tag{5.31}$$

We denote by $\nu_s(ds')$ the law of τ_s^U conditioned to the event $\tau_s^U \le \delta$ when the Brownian motion starts from V_s at time s and write

$$\begin{aligned} &\kappa \int_0^\delta P_{V_s,s}\big(B_{\delta-s} \ge r^*;\, \tau_s^U \le \delta\big) \\ &\qquad = \kappa \int_0^\delta \beta(s) \int_s^\delta \nu_s(ds') P_{V'_s,s'}\big(B_{\delta-s'} \ge r^*\big). \end{aligned} \tag{5.32}$$

From (5.30), (5.31) and (5.32) it follows that there exists a nonnegative measure $g(dt)$ on $[0,\delta]$, so that $\int_0^\delta g(dt) = Z$ and

$$\begin{aligned} &\int u_2(r) P_{r,0}\big(B_\delta \ge r^*;\, \tau_0^U \le \delta\big)\,dr + \kappa \int_0^\delta P_{V_s,s}\big(B_{\delta-s} \ge r^*;\, \tau_s^U \le \delta\big) \\ &\qquad = \int_0^\delta g(dt) P_{U_t;t}\big(B_{\delta-t} \ge r^*\big). \end{aligned} \tag{5.33}$$

Thus since by (5.30) the measures $u_1(r)[1-\alpha(r)]\,dr$ and $g(dt)$ have same mass Z then, by the isomorphism of Lebesgue measures, Rohlin (1952), there is a map $\Gamma : \mathbb{R} \to [0,\delta]$ so that

$$\int_0^\delta g(dt) P_{U_t;t}\big[B_\delta \ge r^*\big] = \int u_1(r)\big[1-\alpha(r)\big] P_{U_{\Gamma(r)};\Gamma(r)}\big(B_\delta \ge r^*\big)\,dr. \tag{5.34}$$

We use the following inequality proved in Carinci et al. (2014b) (see the proof of (5.36) in this paper). If $\gamma = (\gamma(t), t \ge 0)$ is a C^1-curve then for all $\delta > 0$

$$P_{r;0}\big[B_\delta \ge r \,|\, \tau_0^\gamma > \delta\big] \le P_{\gamma_t;t}[B_\delta \ge r] \qquad \forall r \le \gamma(0),\, t \in [0,\delta], \tag{5.35}$$

where τ_0^γ is the hitting time of the curve γ.

By (5.35) and (5.12), from (5.33) and (5.34) we get

$$\begin{aligned}
&\int u_1(r) P_{r,0}\big(B_\delta \ge r^*; \tau_0^U > \delta\big)\\
&\quad = \int u_1(r)[1-\alpha(r)] P_{r,0}\big(B_\delta \ge r^* | \tau_0^U > \delta\big)\\
&\quad \le \int u_1(r)[1-\alpha(r)] P_{U_{\Gamma(r)};\Gamma(r)}\big(B_\delta \ge r^*\big)\\
&\quad = \int u_2(r) P_{r,0}\big(B_\delta \ge r^*; \tau_0^U \le \delta\big)\,dr + \kappa \int_0^\delta P_{V_s,s}\big(B_{\delta-s} \ge r^*; \tau_s^U \le \delta\big)\\
&\quad \le \int u_2(r) P_{r,0}\big(B_\delta \ge r^*; \tau_0^U \le \delta\big)\,dr + C' e^{C\delta^{-1}}.
\end{aligned}$$

This concludes the proof of (5.26). □

5.4 Properties of the barriers

The function $w(\cdot,t) = \mu(\cdot,t) + \nu(\cdot,t)$ is the solution of the heat equation:

$$w(r,t) = (G_t \star w_0)(r), \qquad r \in \mathbb{R}, t \ge 0, \qquad w_0 = \mu(\cdot,0) + \nu(\cdot,0). \tag{5.36}$$

Observe that not only the total mass $\int w_0 = M_{\mathrm{tot}}$ is conserved but also $\int \mu(r,t) = \int \mu(r,0) =: M_0$ for all t. Given $\phi \in L_1(\mathbb{R}, \mathbb{R}_+)$ we call

$$\mathcal{B}(\phi, M_0) := \left\{ (u,v) \in \mathcal{U} : u(r) + v(r) = \phi(r), \forall r \in \mathbb{R}, \text{ and } \int_{\mathbb{R}} u = M_0 \right\}. \tag{5.37}$$

Below we will use the above definition with $\phi = w(\cdot, n\delta)$, because from the definitions it follows that

$$u_{n\delta}^{(\delta,\pm)}(r) + v_{n\delta}^{(\delta,\pm)}(r) = w(r, n\delta) \qquad \forall r \in \mathbb{R}, \forall n \le \delta^{-1} T \tag{5.38}$$

and also that for all $n \le \delta^{-1} T$

$$\int_{\mathbb{R}} u_{n\delta}^{(\delta,\pm)} = \int_{\mathbb{R}} \mu(r,0) = M_0, \qquad \int_{\mathbb{R}} v_{n\delta}^{(\delta,\pm)} = \int_{\mathbb{R}} \nu(r,0) = M_{\mathrm{tot}} - M_0. \tag{5.39}$$

Definition 5.1. Given two pairs (u',v'), $(u,v) \in \mathcal{B}(\phi, M_0)$ and a number $m \ge 0$, we define

$$\big(u',v'\big) \prec (u,v) \text{ modulo } m \quad \text{iff} \quad \forall r \in \mathbb{R} : F\big(r;u'\big) \le F(r;u) + m. \tag{5.40}$$

If $m = 0$ we say that $(u',v') \preccurlyeq (u,v)$.

At the end of this subsection we will prove that (5.1) for all $k \ge 1$ follows from the one step estimates of Sections 5.2 and 5.3. We first prove that the evolutions $S_\delta^{(\delta,\pm)}$ preserve the order in the case $m = 0$.

Lemma 5.4. *Let* $(u', v'), (u, v) \in \mathcal{B}(\phi, M_0)$

$$\textit{if } \big(u', v'\big) \preccurlyeq (u, v) \qquad \textit{then } S_\delta^{(\delta,\pm)}\big(u', v'\big) \preccurlyeq S_\delta^{(\delta,\pm)}(u, v). \tag{5.41}$$

Moreover $S_\delta^{(\delta,\pm)}(u', v')$ *and* $S_\delta^{(\delta,\pm)}(u, v)$ *belong to* $\mathcal{B}(G_\delta \star \phi, M_0)$.

Proof. We first prove that K_δ is nondecreasing with respect to $\preccurlyeq$. Calling $(\bar{u}', \bar{v}') = K_\delta(u', v')$ and $(\bar{u}, \bar{v}) = K_\delta(u, v)$ we have

$$\begin{aligned} \bar{u}' &= u'\mathbf{1}_{(-\infty,\mathcal{R}')} + v'\mathbf{1}_{(-\infty,\mathcal{D}')}, \qquad & \bar{v}' &= u'\mathbf{1}_{[\mathcal{R}',+\infty)} + v'\mathbf{1}_{(\mathcal{D}',+\infty)}, \\ \bar{u} &= u\mathbf{1}_{(-\infty,\mathcal{R})} + v\mathbf{1}_{(-\infty,\mathcal{D})}, \qquad & \bar{v} &= u\mathbf{1}_{[\mathcal{R},+\infty)} + v\mathbf{1}_{(\mathcal{D},+\infty)}, \end{aligned} \tag{5.42}$$

where $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{R}$ and $\mathcal{R}'$ are the points such that

$$\int_{\mathcal{R}'}^{\infty} u' = \kappa\delta = \int_{-\infty}^{\mathcal{D}'} v', \qquad \int_{\mathcal{R}}^{\infty} u = \kappa\delta = \int_{-\infty}^{\mathcal{D}} v. \tag{5.43}$$

Since $(u', v') \preccurlyeq (u, v)$ we have that $\mathcal{D} \le \mathcal{D}' \le \mathcal{R}' \le \mathcal{R}$. Furthermore $K_\delta(u', v')$ and $K_\delta(u, v)$ are both in the set $B(\phi, M)$. Using this fact we get

$$\begin{aligned} &\int_{\mathcal{D}}^{\mathcal{D}'} [u' + v'] + \int_{\mathcal{D}'}^{\mathcal{R}'} u' \\ &\quad = \int_{\mathcal{D}}^{\infty} [u' + v'] - \int_{\mathbb{R}} v' = \int_{\mathcal{D}}^{\infty} [u + v] - \int_{\mathbb{R}} v = \int_{\mathcal{D}}^{R} u. \end{aligned} \tag{5.44}$$

For $r \le \mathcal{D}$, from (5.44) we get

$$F\big(r; \bar{u}'\big) = \int_r^{\mathcal{D}'} \big[u' + v'\big]\,dr + \int_{\mathcal{D}'}^{\mathcal{R}'} u' = \int_r^{\mathcal{D}} \phi + \int_{\mathcal{D}}^{R} u = F(r; \bar{u}).$$

Analogous computations show that $F(r; \bar{u}') \le F(r; \bar{u})$ for $r \le \mathcal{D}'$. For $r > \mathcal{D}'$

$$F\big(r; \bar{u}'\big) = \int_r^{+\infty} u' - \kappa\delta \le \int_{r^*}^{\mathcal{R}} u = F(r; \bar{u}).$$

Thus $F(r; \bar{u}') \le F(r; \bar{u})$ for all $r \in \mathbb{R}$ and this concludes the proof of the monotonicity of K_δ. Recalling the definitions, to conclude the proof of the lemma it is enough to show that also the convolution with G_δ is nondecreasing with respect to $\preccurlyeq$. This fact is a simple adaptation of the proof of Lemma 2.6 of Carinci et al. (2014a) and thus we omit its proof. □

The following proposition, proved in Appendix, will allow us to reduce the inequalities modulo $m > 0$ to the ones with $m = 0$.

Proposition 5.5. *There is* $m_0 > 0$ *so that for all* $m \in (0, m_0)$ *the following holds. Let* (u', v'), $(u, v) \in \mathcal{B}(\phi, M_0)$ *be such that* $(u', v') \prec (u, v)$ *modulo* $m < M_0$, $m > 0$.

1. *There is* $(f^\star, g^\star) \in \mathcal{B}(\phi, M)$ *such that* $(u, v) \preccurlyeq (f^\star, g^\star)$, $(u', v') \preccurlyeq (f^\star, g^\star)$ *and*

$$S_\delta^{(\delta,+)}(f^\star, g^\star) \prec S_\delta^{(\delta,+)}(u, v) \text{ modulo } 2m. \tag{5.45}$$

2. *There is* $(f_\star, g_\star) \in \mathcal{B}(\phi, M)$ *so that* $(f_\star, g_\star) \preccurlyeq (u', v')$, $(f_\star, g_\star) \preccurlyeq (u, v)$ *and*

$$S_\delta^{(\delta,-)}(u', v') \prec S_\delta^{(\delta,-)}(f_\star, g_\star) \text{ modulo } 2m. \tag{5.46}$$

As a consequence of the above proposition, we now prove (5.1). We will use the following notation:

$$\big(\mu(\cdot, (k+1)\delta), \nu(\cdot, (k+1)\delta)\big) = T_\delta\big(\mu(\cdot, k\delta), \nu(\cdot, k\delta)\big). \tag{5.47}$$

Proof of Theorem 5.1. As a consequence of the estimates in Sections 5.2 and 5.3, we have that for all k, letting $(\hat u, \hat v) := T_{k\delta}(\mu_0, \nu_0)$

$$S_\delta^{(\delta,-)}(\hat u, \hat v) \prec T_\delta(\hat u, \hat v) \prec S_\delta^{(\delta,+)}(\hat u, \hat v) \text{ modulo } m := \bar c e^{-C\delta^{-1}} \tag{5.48}$$

with $\bar c = 4C'$ and C and C' as in Lemma 5.2.

Observing that (5.36), (5.38) and (5.39) imply that for all k, $S_{k\delta}^{(\delta,\pm)}(\mu_0, \nu_0)$ and $T_{k\delta}(\mu_0, \nu_0)$ belong to $\mathcal{B}(w_{k\delta}, M_0)$, by (5.48) with $k = 0$ we can use 1 of Proposition 5.5 with $\phi = w(\cdot, \delta)$, $(u', v') = T_\delta(\mu_0, \nu_0)$ and $(u, v) = S_\delta^{(\delta,+)}(\mu_0, \nu_0)$. Thus from Lemma 5.4 and (5.45) we get

$$\begin{aligned} S_\delta^{(\delta,+)}\big(u', v'\big) \preccurlyeq S_\delta^{(\delta,+)}\big(f^*, g^*\big) \prec S_\delta^{(\delta,+)}(u, v) \\ = S_{2\delta}^{(\delta,+)}(\mu_0, \nu_0) \text{ modulo } 2m. \end{aligned} \tag{5.49}$$

We apply (5.48) with $(\hat u, \hat v) = (\mu(\cdot, \delta), \nu(\cdot, \delta))$

$$T_{2\delta}(\mu_0, \nu_0) = \big(\mu(\cdot, 2\delta), \nu(\cdot, 2\delta)\big) \prec S_\delta^{(\delta,+)}\big(u', v'\big) \text{ modulo } m \tag{5.50}$$

that together with (5.49) proves the upper bound in (5.1) for $k = 2$ and $c' = 3\bar c$. By using 2 of Proposition 5.5 and Lemma 5.4 we similarly get the lower bound in (5.1) for $k = 2$ and $c' = 3\bar c$. Theorem 5.1 follows from the iteration of the above procedure. □

Appendix

Proof of Proposition 5.5. Let H and Z be the points so that

$$\int_{-\infty}^{H} u(r)\, dr = m, \qquad \int_{Z}^{+\infty} v(r)\, dr = m.$$

Since $(u, v) \in \mathcal{U}$ for m_0 small enough we have that $H < Z$. We define

$$f^\star = u + v\mathbf{1}_{[Z,+\infty)} - u\mathbf{1}_{(-\infty,H]}, \qquad g^\star = v + u\mathbf{1}_{(-\infty,H]} - v\mathbf{1}_{[Z,+\infty)}. \tag{A.1}$$

Obviously $(f^\star, g^\star) \in \mathcal{B}(\phi, M_0)$ and $(u, v) \preccurlyeq (f^\star, g^\star)$.

If $r \le H$ then $F(r; f^\star) = \int_{\mathbb{R}} u(r)\,dr = M \ge F(r; u')$. For $r \in [H, Z]$ by using that $(u', v') \prec (u, v)$ modulo m we get

$$F\big(r; f^\star\big) = \int_r^{+\infty} u(r)\,dr + m \ge F\big(r; u'\big) - m + m \qquad \forall r \in [H, Z].$$

Finally since $g^\star = 0$ for all $r > Z$ and $(f^\star, g^\star) \in \mathcal{B}(\phi, M_0)$ we have that $f^\star(r) = \phi(r) = u'(r) + v'(r)$ for all $r > Z$ and therefore $F(r; f^\star) \ge F(r; u')$ for all $r \ge Z$. Thus, $(u', v') \preccurlyeq (f^\star, g^\star)$.

To prove (5.45), recalling that $S_\delta^{(\delta,+)} = G_\delta K_\delta$, we first compare $(\bar{f}^\star, \bar{g}^\star) := K_\delta(f^\star, g^\star)$ with $(\bar{u}, \bar{v}) := K_\delta(u, v)$. Let $D^\star$, $R^\star$ and D, R be the points such that

$$\begin{aligned} \int_{R^\star}^{+\infty} f^\star(r)\,dr = \kappa\delta = \int_{-\infty}^{D^\star} g^\star(r)\,dr, \\ \int_R^{+\infty} u(r)\,dr = \kappa\delta = \int_{-\infty}^{D} v(r)\,dr. \end{aligned} \tag{A.2}$$

By definition of K_δ,

$$\begin{aligned} \bar{u} &= u\mathbf{1}_{(-\infty,R]} + v\mathbf{1}_{(-\infty,D]}, & \bar{v} &= u\mathbf{1}_{[R,+\infty)} + v\mathbf{1}_{[D,+\infty)}, \\ \bar{f}^\star &= f^\star\mathbf{1}_{(-\infty,R^\star]} + g^\star\mathbf{1}_{(-\infty,D^\star]}, & \bar{g}^\star &= f^\star\mathbf{1}_{[R^\star,+\infty)} + g^\star\mathbf{1}_{[D^\star,+\infty)}. \end{aligned}$$

Since $(u, v) \preccurlyeq (f^\star, g^\star)$ we have $D^\star \le D \le R \le R^\star$ and since $(u, v) \in \mathcal{U}$ then for m_0 small enough we have that $H < D$ and $Z > R$ that implies $\int_R^{R^\star} u(r)\,dr \le m$ analogously $\int_{D^\star}^{D} v(r)\,dr \le m$. Let $r \le D^\star$, then since $(f^\star, g^\star) \in \mathcal{B}(\phi, M_0)$,

$$F\big(r; \bar{f}^\star\big) = \int_r^{D^\star} \phi\big(r'\big)\,dr' + \int_{D^\star}^{R^\star} f^\star\big(r'\big)\,dr'.$$

Since $\bar{u}(r') = \phi(r')$ for $r' \le D$, using the definition (A.1) we have that

$$\begin{aligned} \int_{D^\star}^{R^\star} f^\star &= \int_{D^\star}^{D} u + \int_D^{R^\star} u + \int_{Z\wedge R^\star}^{R^*} v \\ &\le \int_{D^\star}^{D} \phi - \int_{D^\star}^{D} v + \int_D^R u + \int_R^{R^\star} u + \int_{Z\wedge R^\star}^{R^\star} v \le \int_{D^\star}^{R} \bar{u} + 2m. \end{aligned}$$

Thus, $F(r; \bar{f}^\star) \le F^+(r; \bar{u}) + 2m$ for all $r \le D^\star$. For $r > D^\star$

$$F\big(r; \bar{f}^\star\big) = \int_r^{R^\star} f^\star = \int_r^{R^\star} u + \int_{Z\wedge R^\star}^{R^\star} v \le \int_r^{\mathcal{R}} u + 2m = F(r; \bar{u}) + 2m.$$

Thus,

$$\big(\bar{f}^\star, \bar{g}^\star\big) = K_\delta\big(f^\star, g^\star\big) \prec K_\delta(u, v) = (\bar{u}, \bar{v}) \text{ modulo } 2m. \tag{A.3}$$

We are left with the proof of the analogous inequality for the convolution with G_δ. We call $C_\pm$ the point such that

$$\int_{C_+}^{+\infty} \bar{f}^\star(r)\,dr = 2m, \qquad \int_{-\infty}^{C_-} \bar{g}^\star(r)\,dr = 2m$$

and we let $f = \bar{f}^\star \mathbf{1}_{(-\infty,C_+)} + \bar{g}^\star \mathbf{1}_{(-\infty,C_-)}$ and $g = \bar{f}^\star \mathbf{1}_{[C_+,+\infty)} + \bar{g}^\star \mathbf{1}_{[C_-,+\infty)}$. Then, by definition $(f,g) \preccurlyeq (\bar{f}^\star, \bar{g}^\star)$ and it is not difficult to check that $(f,g) \preccurlyeq (\bar{u},\bar{v})$. Since G_δ is nondecreasing with respect to $\preccurlyeq$ (see the proof of Lemma 5.4) we have that $(G_\delta \star f, G_\delta \star g) \preccurlyeq (G_\delta \star \bar{u}, G_\delta \star \bar{v})$. On the other hand,

$$F\big(r, G_\delta \star \bar{f}^\star\big) = F(r, G_\delta \star f) + F\big(r, G_\delta \star \big(\bar{f}^\star - f\big)\big) \le F(r, G_\delta \star \bar{u}) + 2m.$$

Thus $G_\delta K_\delta(f^\star, g^\star) \prec G_\delta K_\delta(u,v)$ modulo $2m$ which proves (5.45) and thus concludes the proof of 1.

We define

$$\begin{aligned} f_\star &= u' + v'\mathbf{1}_{(-\infty,Z')} - u'\mathbf{1}_{[H',+\infty)}, \\ g_\star &= v' + u'\mathbf{1}_{[H',+\infty)} - v'\mathbf{1}_{(-\infty,Z')}, \end{aligned} \tag{A.4}$$

where H' is such that $\int_{H'}^{+\infty} u'(r)\,dr = m$ and Z' is such that $\int_{-\infty}^{Z'} v'(r)\,dr = m$. By definition $(f_\star, g_\star) \in \mathcal{B}(\phi, M_0)$ and $(f_\star, g_\star) \preccurlyeq (u',v')$. We next observe that for $r \le Z'$

$$F(r; f_\star) = \int_{-\infty}^{+\infty} f_\star - \int_{-\infty}^{r} [u'+v'] = M_0 - \int_{-\infty}^{r} [u+v] \le M_0 - \int_{-\infty}^{r} u = F(r;u).$$

For $r \in [Z', H']$ we have

$$F(r; f_\star) = \int_r^{+\infty} u'(r)\,dr - m \le F(r;u).$$

And finally for $r \ge H'$, $F(r; f_\star) = 0 \le F(r;u)$, which concludes the proof that $(f_\star, g_\star) \preccurlyeq (u,v)$. To prove (5.46), we first write

$$\begin{aligned} G_\delta \star f_\star &= G_\delta \star u' + G_\delta \star \big(v'\mathbf{1}_{(-\infty,Z')}\big) - G_\delta \star \big(u'\mathbf{1}_{[H',+\infty)}\big), \\ G_\delta \star g_\star &= G_\delta \star v' + G_\delta \star \big(u'\mathbf{1}_{[H',+\infty)}\big) - G_\delta \star \big(v'\mathbf{1}_{(-\infty,Z']}\big). \end{aligned} \tag{A.5}$$

Let D', R', $D_\star$ and $R_\star$ be the points such that

$$\int_{R_\star}^{+\infty} G_\delta \star f_\star = \kappa\delta = \int_{-\infty}^{D_\star} G_\delta \star g_\star, \qquad \int_{R'}^{+\infty} G_\delta \star u' = \kappa\delta = \int_{-\infty}^{D'} G_\delta \star v'.$$

From the fact that the convolution with G_δ preserves the inequality, we have $D' \le D_\star \le R_\star \le R'$. Furthermore using (A.5), we get

$$\int_{R_\star}^{R'} G_\delta \star u'(r)\,dr \le m, \qquad \int_{D'}^{D_\star} G_\delta \star v'(r)\,dr \le m. \tag{A.6}$$

Recalling the definition of K_δ we call $(\bar{f}_\star, \bar{g}_\star) := K_\delta(G_\delta \star f_\star, G_\delta \star g_\star)$ and $(\bar{u}', \bar{v}') := K_\delta(G_\delta \star u', G_\delta \star v')$. For $r \le D'$, using that $(f_\star, g_\star)$ and (u',v') are both in the set $\mathcal{B}(\phi, M_0)$ and (A.6) we have for $r \le D'$

$$\begin{aligned}\int_r^{+\infty} \bar{u}' &= \int_r^{D'} G_\delta \star \phi + \int_{D'}^{D_\star} G_\delta \star u' + \int_{D_\star}^{R'} G_\delta \star u' \\ &\le \int_r^{D_\star} G_\delta \star \phi + \int_{D_\star}^{R_\star} G_\delta \star u' + m \\ &\le \int_r^{D_\star} G_\delta \star \phi + \int_{D_\star}^{R_\star} G_\delta \star f_\star + 2m = F(r; \bar{f}_\star) + 2m.\end{aligned}$$

Analogously, for $r > D'$

$$\begin{aligned}\int_r^{+\infty} \bar{u}' &\le \int_r^{D_\star} G_\delta \star u' + \int_{D_\star}^{R_\star} G_\delta \star u' + m \\ &\le \int_r^{D_\star} G_\delta \star \phi + \int_{D_\star}^{R_\star} G_\delta \star f_\star + 2m = F(r; \bar{f}_\star) + 2m.\end{aligned}$$

This proves (5.46) and concludes the proof of the proposition. □

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