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# INVARIANCE PRINCIPLE FOR REVERSIBLE MARKOV PROCESSES WITH APPLICATION TO DIFFUSION IN THE PERCOLATION REGIME

A. De Masi[1], P.A. Ferrari[2], S. Goldstein[3] and W.D. Wick[4]

ABSTRACT. We present an invariance principle for antisymmetric functionals of a reversible Markov process which immediately implies convergence to Brownian motion for a wide class of random motion in random environments. We here apply it to establish convergence to Brownian motion for a walker moving in the infinite cluster of the two dimensional bond percolation model.

## 1. INTRODUCTION.

Recently there has been considerable interest, stimulated by physical applications, in the study of the asymptotic motion of a tagged particle in a random (static or dynamic) environment. In symmetric cases one expects the motion to be asymptotically Brownian, with a finite diffusion matrix, which one would like to compute as explicitly as possible. In this note we discuss an invariance principle applicable when the process of the environment as seen from the tagged particle is reversible in time. The main theorems - which are useful in other contexts as well - are parsimonious in hypotheses; in particular, there are no mixing properties or spectral conditions assumed on the variables which might be difficult to check in practice. As a result we are able to improve upon several results in the literature, and give short proofs of many other results. Details will appear in [ 2 ].

Before discussing the abstract theorems we mention in passing a typical example to which our methods apply (to be discussed more fully below). On the integer lattice $\mathbb{Z}^d$ let there be given rates (or conductivities) $a_e(x) > 0$ for the bond $(x,x+e)$ , $x \in \mathbb{Z}^d$ , $e \in \mathbb{Z}^d$ , $|e| = 1$ . Let $X_t$ be the position of a

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particle started at $t = 0$ at $x = 0$ and performing a random walk with these rates. Assume in addition that the $a_e(x)$ are chosen from a translation-invariant, ergodic distribution (random bond conductivities). Then, with only moment conditions on the rates, we prove that $\varepsilon X_{t\varepsilon^{-2}}$ tends, in distribution as $\varepsilon \to 0$, to a Brownian motion with a finite, non-degenerate diffusion matrix. (Compare [1], [7], [8], in which $0 < b_1 < a_e(x) < b_2 < \infty$ is assumed). We can even treat the case when $P[a_e(x) > 0] = p < 1$, assuming that $p > p_c$ (percolation threshold) and conditioning on the percolation cluster containing the origin. In other words, we prove an invariance principle for a random walk on an infinite percolation cluster.

2. <u>ABSTRACT THEOREMS</u>. Invariance principles for anti-symmetric functionals of reversible Markov processes.

Let $\xi_t$, $t \in \mathbb{Z}$ on $\mathbb{R}$ (discrete or continuous time) be a time-reversible Markov chain with state space $X$. We are interested in proving an invariance principle for a functional $X_{[0,t]}$ of paths of the process (depending on $\xi_\tau$, $0 \le \tau \le t$), which has a certain symmetry property. Let $R_{t/2}$ be reflection in the midpoint of the interval, defined by $(R\xi)_\tau = \xi_{t-\tau}$. We call $X_{[0,t]}$ <u>anti-symmetric</u> if

$$X_{[0,t]} \circ R_{t/2} = -X_{[0,t]} , \tag{2.1}$$

with a similar definition for a general closed interval $I = [a,b]$.

Anti-symmetric variables arise naturally in many contexts. For instance, let $X_{[0,t]}$ be the increment made, in the interval $[0,t]$, by one component of a multi-component process. Alternatively, let $\xi_t$ be a jump process on $X$, let $x$ and $y$ be points in $X$ and let $X_{[0,t]}$ be the "current" from $x$ to $y$, i.e. the number of jumps from $x$ to $y$ minus the number from $y$ to $x$. In an infinite particle system $X_{[0,t]}$ might be the current across a bond.

We first present the discrete time version of our theorem whose only assumptions concern symmetry and square integrability and in which the antisymmetry appears in simplest form.

THEOREM 2.1. Let $\xi_i$, $i = 0,1,2,\ldots$ be an ergodic reversible Markov process with state space $X$ and initial distribution $\mu$, $Y : X \times X \to \mathbb{R}$ be antisymmetric : $Y(\xi,\eta) = -Y(\eta,\xi)$, let $Y_i \equiv Y(\xi_{i-1},\xi_i)$ be square integrable.

Define

$$X_n = \sum_{i=1}^{n} Y_i \tag{2.2}$$

$$X_t^\varepsilon = \varepsilon X_{[\varepsilon^{-2}t]}$$

Let $\mathbb{E}_D$ be the expectation with respect to the Brownian motion $W_D$ starting at the origin with variance $Dt$ and let $\mathbb{E}_\xi$ be the expectation with respect to the Markov process $\xi_i$ starting at $\xi$. Then for all bounded continuous function $F$ on $D(\mathbb{R},\mathbb{R})$

$$\mathbb{E}_\xi(F(X_\cdot^\varepsilon)) \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}_D(F) \tag{2.3}$$

in $\mu$-measure, where $D$ is given by

$$D = \mathbb{E}(X_1^2) - 2\langle\varphi(1-T)^{-1}\varphi\rangle \tag{2.4}$$

with $\varphi(\xi) = \mathbb{E}_\xi(X_1)$ and where $T$ is the (time one) transition operator.

In particular, as $\varepsilon \to 0$ the process $X_\cdot^\varepsilon$ converges (in distribution) weakly to $W_D(\cdot)$. □

(2.5) We call the convergence given in (2.3) convergence in distribution in $\mu$ probability.

The continuous time version of our invariance principle applies to a family $X_I$, indexed by closed intervals $I = [a,b]$ and taking values in $\mathbb{R}^d$, which in addition to being antisymmetric (i.e. each $X_I$ is antisymmetric) satisfies certain additional natural properties (except for A.1, automatically satisfied in practice) which we list below.

(A.1) $X_I$ is square-integrable for each $I$

(A.2) The family $X_I$ is covariant

$$X_{[a,b]} \circ \theta_\tau = X_{[a+\tau,b+\tau]}$$

where $\theta_\tau$ denotes the time-shift operator, defined by $(\theta_\tau\xi)_t = \xi_{t+\tau}$

(A.3) The family $X_I$ is additive :

$$X_I + X_{I'} = X_{I\cup I'}$$

a.s. if $\mathring{I} \cap \mathring{I}' = \emptyset$. ($\mathring{I}$ is the interior of $I$) .

We also require that $X_t \equiv X_{[0,t]}$ has a version with paths in $D(\mathbb{R},\mathbb{R}^d)$ .

THEOREM 2.2. Let $\xi_t$, $t \in \mathbb{R}$, be an ergodic reversible Markov process with initial measure $\mu$. Let $X_I$ be an anti-symmetric family of $\mathbb{R}^d$-valued random variables satisfying (A.1)-(A.3) . Assume in addition that the mean forward drift velocity exists and is square-integrable, more precisely, that

$$\lim_{\delta\to 0^+} \delta^{-1}\,\mathbb{E}_\mu[X_{[0,\delta]}/F_o] = \varphi \tag{2.6}$$

exists weakly in $L^1(\mu)$ and $\varphi \in L^2(\mu)$ ($\varphi$ is vector-valued). Then the following hold :

(i) The velocity auto-correlation function is integrable, that is, for each i,j

$$\int_o^\infty |\langle\varphi_i T_t \varphi_j\rangle|\,dt < \infty$$

where $T_t$ is the Markov semigroup determined by the process $\xi_t$, and $\langle\cdot\rangle$ is the expectation w.r.t. $\mu$, where $\varphi_i = e_i.\varphi$, with $e_i$, $i = 1,\dots,d$ an orthonormal basis on $\mathbb{R}^d$.

(ii) Let $X_t^\varepsilon \equiv \varepsilon X_{[0,\varepsilon^{-2}t]}$. Then as $\varepsilon \to 0$

$$X_\cdot^\varepsilon \to W_{\mathbb{D}}(\cdot) \tag{2.7}$$

in distribution in $\mu$-probability, where $W_{\mathbb{D}}(\cdot)$ is a d-dimensional Brownian motion starting at zero with covariance matrix $\mathbb{D}t$ and where

(iii) $\mathbb{D}$ is given by

$$D_{ij} = C_{ij} - 2\int_o^\infty \langle\varphi_i T_t \varphi_j\rangle\,dt \tag{2.8a}$$

$$D_{ij} = C_{ij} - 2\langle\varphi_i L^{-1}\varphi_j\rangle \tag{2.8b}$$

where $\mathbb{C} = (C_{ij})$ is determined by

$$e^T.\mathbb{C}.e = \lim_{\delta\to 0^+} \delta^{-1}\,\mathbb{E}_\mu(e.X_{[0,\delta]})^2 \tag{2.9}$$

and $-L$ is the generator of the semigroup $T_t : L^2(\mu) \to L^2(\mu)$.

$\mathbb{C}$ is positive definite and finite.

(iv) Let $X_t = X_{[0,t]}$. Then

$$\lim_{t\to 0} \frac{\mathbb{E}(X_{t,i}^2)}{t} = D_{ii} \qquad \square$$

REMARKS. (1) Ergodicity is required only for part (ii) of the theorem. (2) $\mathbb{D}$ degenerate is not excluded. We call the limiting Brownian motion non-degenerate if all the eigenvalues of $\mathbb{D}$ are positive. (3) By averaging with respect to $\mu$ it is immediate from (2.7) (see eq. (2.3)) that

$X^{\varepsilon}_{\cdot} \underset{\varepsilon\to 0}{\to} W_{\mathbb{D}}(\cdot)$ in distribution (under the law $\mathbb{P}_{\mu}$ of the process $\xi_t$ starting from the stationary distribution $\mu$ ).

The proof of the theorem (2.1) builds on earlier work [6], [3]. The gist of the proof is to write $\varepsilon X_{t\varepsilon^{-2}}$ as the sum of a square-integrable martingale and a small remainder, and then to apply the martingale Central limit theorem [4]. The first step in the proof is, roughly speaking, to establish (2.8), simultaneously proving the integrability of the velocity-autocorrelation function, which is (equivalent to) the main assumption in earlier work. (Familiarity with Spohn's Green-Kubo formulas for interacting particle systems [12] lead us to this formula).

More precisely we first derive the equation obtained from (2.8) by $\lim_{t\to\infty} \mathbb{E}_{\mu}(X_{t,i}X_{t,j})$ , which we later identify with $D_{ij}$ . Assume, for notational convenience, that $X_t \in \mathbb{R}$ . The key observation is that from the easily obtained decompostion $X_t = S_t + M_t$ where $S_t = \int_0^t \varphi(\xi_s)ds$ and $M_t$ is a square integrable martingale with stationary increments, it follows that $\mathbb{E}_{\mu}(M_t^2) = \mathbb{E}_{\mu}(X_t^2) + \mathbb{E}_{\mu}(S_t^2)$, since the cross term $2\,\mathbb{E}_{\mu}(X_t S_t)$ vanishes by symmetry. Here we use the fact that the measure $\mathbb{P}_{\mu}$ is symmetric under time reversal $R_{t/2}$ (i.e. $\mu$ is reversible) while $S_t$ is manifestly symmetric, and by assumption $X_t$ is antisymmetric.

We turn next to the problem of obtaining bounds on the diffusion matrix (in particular a positive lower bound). Note that from eq. (2.8.b) and writing $D$ for $D_{ii}$ , $\varphi$ for $\varphi_i$ and $C$ for $C_{ii}$ , an upper bound is immediate : since $L$ is non-negative, $D \leq C$ . For a lower bound one can often exploit the following situation. Suppose $L = L_1 + L_2$ , with $-L_i$ , $i = 1,2$ generators of reversible Markov processes. Then operator monotonicity of the inverse function suggests the lower bound :

$$D \geq C - 2(\varphi L_i^{-1} \varphi) \tag{2.10}$$

which is often useful in practice and frequently has an interesting probabilistic meaning. We mention several : (i) For reversible infinite particle systems e.g. interacting Brownian particles [11], [10], [13] (2.10) can be used to prove that the diffusion constant for a particle moving in a random, moving environment is greater than that for the same particle in a "frozen" environment. (ii) When $-L$ generates a multi-dimensional process and $-L_1$ generates a process in fewer dimensions, (2.10) implies that $D$ is an increasing function of dimension.

3. RANDOM WALK IN THE PERCOLATION CLUSTER.

In this section we use the theorem 2.2 to establish the diffusive behavior of a random walker on an inhomogeneous lattice, i.e. the invariance principle with non degenerate $\mathbb{D}$ . The model is the following. In the two dimensional lattice $\mathbb{Z}^d$ with each bond $(x,x+e)$ , $x,e \in \mathbb{Z}^2$, $|e| = 1$ associate random rates $a_e(x) \geq 0$ (sometimes we write $a_i(x)$ instead of $a_{e_i}(x)$) . A configuration of rates is called an environment and the space of environments we denote by $X = \{a : \mathbb{Z}^2 \times \{e_1,e_2\} \to [0,\infty)\}$ .

Let $\mu$ a measure on $X$ satisfying

C.1. $a_i(x)$ are identically distributed and mutually independent.

Given a configuration $a \in X$ , put a particle at the origin and let it move to its nearest neighbours with rates given by $a$ . That is, define the jump process $X(a,t)$ , $t \in \mathbb{R}$ as the Markov process with state space $\mathbb{Z}^2$ and transition probabilities determined by

$$P(X(a,t+h) = x + e \,/\, X(a,t) = x) = a_e(x)h + 0(h) \tag{3.1}$$

$$P(X(a,t+h) = x \,/\, X(a,t) = x) = 1 - \sum_{|e|=1} a_e(x)h + 0(h)$$

We are interested in establishing the convergence to Brownian Motion of the rescaled process $\varepsilon X(a,\varepsilon^{-2}t)$ as $\varepsilon \to 0$ .

Given $a \in X$ , let

$$W(x,a) = \text{connected cluster of } x \text{ in } W(a) =: \{\text{bonds } (x,x+e) \text{ with } a_e(x) > 0\} \tag{3.2}$$

It is clear that for configurations $a \in X$ for which $|W(0,a)| < \infty$ ($|A|$ is the cardinality of the set $A$ ), the position $X(a,t)$ of the particle will be bounded for all $t$ . So $|W(0,a)| = \infty$ is the minimal condition for convergence to Brownian Motion.

It is well known [ 5 ] that if

C.2. $\mu\{a_1(0) > 0\} = p > \frac{1}{2}$ (3.3)

then $\mu\{|W(0,a)| = \infty\} = \theta_p > 0$ . Define

$$\hat{\mu} =: \mu\{\cdot/|W(0,a)| = \infty\} \quad . \tag{3.4}$$

Consider also the following conditions on $\mu$ , which will be discussed later :

C.3. $\mu\{a_1(0) > b\} = 0$ for some $b > 0$ .

C.4. $\mu(a_1(0)^{-1}/a_1(0) > 0) > 0$ .

The main result of this section is the following

THEOREM 3.1. Let $\mu$ a measure on the space of environments $X$ satisfying conditions C.1-4 above, and let $X(a,t)$ be the jump Markov process with state space $\mathbb{Z}^2$ , $X(a,0) \equiv 0$ and transition probabilities given by eq. (3.1) above.

Then

(A) as $\varepsilon \to 0$, $\varepsilon X(a,\varepsilon^{-2}.) \to W_{\hat{\mathbf{D}}}(\cdot)$ in distribution in $\hat{\mu}$-probability (see Def. 2.5), where $\hat{\mu} = \mu(\cdot/|W(0,.)| = \infty)$ and $W_{\hat{\mathbf{D}}}(t)$ is two dimensional Brownian motion with covariance matrix $\hat{\mathbf{D}}$ , where

(B) $\hat{\mathbf{D}} = \hat{D}I$ , where $I$ is the identity matrix, and

$$\theta_p \hat{D} = D =: 2\mu(a_1(0)) - 2\int_0^\infty dt\, \mu[(a_1(0) - a_1(-e_1)) \cdot \mathbf{E}_a(a_1(X(a,t)) - a_1(X(a,t) - e_1))] \tag{3.5}$$

(C) $\hat{D} > 0$ □

REMARKS.

1) Convergence to (non degenerate) Brownian Motion of the process $X(a,t)$ was established by [ 1 ], [ 7 ], [ 8 ] for any ergodic translation invariant $\mu$ such that

$$0 < b_1 < a_i(x) < b_2 < \infty \quad \text{for all } x \in \mathbb{Z}^2 , \quad i = 1,2 .$$

In [ 2 ] the result is extended up to $b_1 = 0$ and $b_2 = \infty$ with only the conditions that $\mu(a_i(0)) < \infty$ , and $\mu(a_i(0)^{-1}) < \infty$ . This is done by introducing a related discrete time process and invoking Theorem 2.1, but we prefer not to introduce this approach here and so we require that $b_2 < \infty$ and apply Thm. 2.2 directly. We also need $b_2 < \infty$ as a technical requirement for proving that $\hat{D} > 0$ (see Lemma 3.1 below).

2) The condition that $\mu$ is a product measure is needed in two parts : first to prove that the induced process $a(t)$ (the environment as seen from $X(a,t)$) is ergodic with respect to $\hat{\mu}$ . Second to establish the positivity of $\hat{D}$ . Here we use strong results of percolation theory, namely we relate $\hat{D}$ to the electric current flowing through the environment $a$ when a potential gradient is applied.

The condition $\mu(a_1(0)^{-1} / a_1(0) > 0) < \infty$ is needed only to establish the positivity of $\hat{D}$, since it appears as a divisor in the Kesten's lower bound for the lim inf of the current.

3) For the case $a_i(x) > 0$, the result of convergence to Brownian Motion is independent of the dimension if $0 < a_i(x)$ and $\mu[a_i(x)] < \infty$. In [ 2 ] the inequalities $\mu(a_1(0))^{-1} \leq D \leq \mu(a_1(0))$ (known in special case) are derived generally for rotation (by $\pi/2$ ) invariant $\mu$, while if the measure is not rotation invariant similar equalities are proven for each of the coefficients. It is also proven there that the diffusion coefficient(s) $D$ $(D_i)$ is a non decreasing function of the dimension.

Proof. (A) We place the problem within the framework of Theorem 2.2. The role of the Markov process $\xi_t$ is played by the process $a(t) \in X$ defined by

$$\begin{aligned} a(0) &\equiv a \\ a(t) &= \tau_{X(a,t)} a \end{aligned} \tag{3.6}$$

where $\tau_x$ is the translation by $x$ on $X$. $a(t)$ is clearly reversible for both $\mu$ and $\hat{\mu}$.

Moreover, $X(a,t)$ is an adapted (to $a(t)$) antisymmetric process. Since the rates are bounded, the function $\varphi_i = a_i(0) - a_i(-e_i)$ is well defined and $X_t$ is square integrable. So, the only hypothesis of Theorem 2.2 left to check is the ergodicity of $\hat{\mu}$ with respect to the process $a(t)$. This follows from the fact that in two dimensions there exists only one infinite cluster $\mu$ a.s., and the ergodicity of the random environment under translations (see [2]) □

(B) The fact that $\mathbb{D}$ is a multiple of the identity is an easy consequence of the rotation invariance of $\mu$.

The equation $\theta_p \hat{D} = D$ can be obtained as follows : in view of (2.8), by decomposing $\mu$ into its ergodic components (for the process $a(t)$) we obtain from the definition, eq. (3.5), of $D$, its ergodic decomposition $D = \int \nu(d\beta) D_\beta$. Since the only ergodic component of $\mu$ with positive diffusion constant is $\hat{\mu}$, the equation follows (see [ 2 ] for details). □

The idea of the proof of (C) is the following. We consider periodic configurations with period $2N$ and call $D_N$ the diffusion coefficient (i.e. $\lim \frac{E(X_t^2)}{t}$ ) for the corresponding random walk. Since with positive probability all the rates are strictly positive one expects that $D_N > 0$, which can, in fact, be easily shown. The problem left is to see if (a) as $N$ increases $D_N$

remains bounded away from 0 and b) $D_N$ and D are suitably related.

The relationship between $D_N$ and D is easily established by looking at the "explicit" formula for D in eq. (3.5). In that formula the integrand of the second term is positive and involves E (X(t)) . For each t fixed the expectation $E_N(\varphi(X(t))$ related to the periodic configurations converges to $E(\varphi(X(t))$. So we can use Fatou's lemma to obtain the inequality.

$$D \geq \overline{\lim}\, D_N \quad .$$

The positivity of $\overline{\lim}\, D_N$ is not so clear : here deep percolation problems enter. We solve this problem without really solving it ! In fact we prove that

$$\frac{1}{2} D_N = \sigma_N \tag{3.7}$$

where $\sigma_N$ is the current flowing in a box of size N when a unit potential difference is established, which is better thought of as the effective conductivity. Then we observe that Kesten (and also Grimmet and Kesten) solved our problem : they proved that $\underline{\lim}\, \sigma_N > 0$ . [5], [9].

Thus to prove (C) we define the periodic random walk, its diffusion coefficient $D_N$ ; we establish that $D \geq \overline{\lim}\, D_N$ and then define the conductivity $\sigma_N$ and determine its relationship with $D_N$ .

DEFINITION. <u>The periodic random walk</u>.

Let $B_N \subset \mathbb{Z}^2$ be the box

$$B_N = \{x = (x_1,x_2) \in \mathbb{Z}^2 \; : \; -N \leq x_1 \leq N \quad , \quad -N \leq x_2 \leq N-1\}$$

Given a configuration $a \in X$ of rates, let $\tilde{a}$ be the periodic rate configuration, with period 2N , (on bonds linking nearest neighbours points) in the strip

$$\Sigma_N = \{(x_1,x_2) \in \mathbb{Z}^2 \; : \; -N \leq x_2 \leq N-1\}$$

which agrees with a on the box $B_N$ , except that the vertical bonds on $x_1 = \pm N$ are given infinite rates (this is required to obtain a simple relationship between $D_N$ and $\sigma_N$ ).

Since bonds leading out of $\Sigma_N$ on $x_2 = N-1$ and on $x_2 = -N$ do not exist for $\tilde{a}$ , $\Sigma_N$ has "reflecting boundary conditions" on top and bottom. In order to construct a random walk on $\Sigma_N$ with rate $\tilde{a}$ , it is necessary to identify the points of $x_1 = -N$ , and similarly for the points of $x_1 = N, 3N, -3N, \ldots$ but as long as we consider only functions which agree on equivalent points, this may safely be ignored, provided we bear in mind that from all points on e.g. $x_1 = N$ , the rates $\tilde{a}$ to the points $(N+1, x_2)$ $[(N-1, x_2)]$ are now $\frac{1}{2N} a_1(-N_1 x_2)$ $[\frac{1}{2N} a_1(N-1, x_2)]$ . Denote by $\tilde{X}_a^N(x,t)$ (or simply $\tilde{X}^N(t)$ if no confusion is likely) the Markov process on $\Sigma_N$ defined by the rates $\tilde{a}_1$ starting from $x$ , and let $X^N(t)$ be the $x_1$-component of $\tilde{X}^N(t)$ . Denote by $Y_a^N(x,t)$ (or simply $Y^N(t)$) the Markov process $\tilde{X}^N(t)/\sim$ , where $(x_1, x_2) \sim (\bar{x}_1, \bar{x}_2) \Leftrightarrow \bar{x}_1 - x_1 = 0 \mod 2N$ and $\bar{x}_2 = x_2$ . $Y^N(t)$ may be regarded as a Markov process with state space $\tilde{B}_N$ , the box $B_N$ with the left side identified with the right (i.e., $(-N, x_2)$ identified with $(N, x_2)$) . Let $m_N$ be the uniform distribution on $\tilde{B}_N$ . Then with respect to $m_N$ , $Y^N(t)$ is clearly a reversible Markov process, in terms of which $\tilde{X}^N(t)$ and hence $X^N(t)$ , can be realized. Under this realization

$$X^N(t) = \text{number of jumps of } Y^N(s) \text{ in the positive horizontal direction minus number of jumps of } Y^N(s) \text{ in the negative horizontal direction during the time interval } [0,t] . \tag{3.8}$$

(In the above definition the jump from $N-1$ to $-N \equiv N$ [respectively to $N-1$ from $N \equiv -N$] , is considered as a unit positive resp. negative jump). Note that this is well defined if $N \geq 2$ which we will assume. We now need some notation. The rates are most naturally regarded as bond functions $(x,y) \equiv b \to \tilde{a}(b)$ the rate for going from $x$ to $y$ . (Though the original $a(b)$ was symmetric, this may no longer be the case for $\tilde{a}$ , because of the redefinition of the rates for leaving $x_1 = \ldots, -N, N, 3N, \ldots$ necessitated by granting infinite "conductivities (i.e. rates) to the vertical bonds on these lines : e.g. $\tilde{a}((N, x_2), (N+1, x_2')) = \tilde{a}((N+1, x_2'), (N, x_2))$ may fail). For any bond $b = (x,y)$ , let $\nabla_b f = f(y) - f(x)$. Let $B_x = \{(x,y)\}$ be the set of bonds emanating from $x$ .

Then the generator of the process $\tilde{X}_a^N(x,t)$ is given by

$$L_a f(x) = \sum_{b \in B_x} \tilde{a}(b) \nabla_b f \tag{3.9}$$

while the generator for the process $Y_a^N(x,t)$ may be identified with the restriction of $L_a$ to periodic functions. Here, and from now, "periodic" means periodic with period $2N$ (in the $x_1$-direction).

Let $\varphi_a^N = L_a x_1$ , and note that

$$\varphi_a^N(x) = (\nabla_1^- a_1)(x) \equiv a_1(x) - a_1(x - e_1) \tag{3.10}$$

for $x \in B_N \; |x_1| \neq N$ .

Denote by $P_a^N$ [resp $P_{a,x}^N$ ] the law of the process $Y^N(t)$ with starting measure $m_N$ [resp. $\delta_x$] .

Let

$$\begin{aligned} D_N(a) &= 2m_N(a_1(x)) - 2\int_0^\infty ds\; m_N(\varphi_a^N\, \mathbf{E}_{a,x}^N \varphi(Y^N(s))) \\ &= 2m_N(a_1(x)) + 2m_N(\varphi_a^N L_a^{-1} \varphi_a^N) \end{aligned} \tag{3.11}$$

REMARKS. It follows easily from Theorem 2.2 and the Remark (1) which follows it, with $Y^N(t)$ playing the role of the Markov process $\xi(t)$ , $m_N$ the role of $\mu$ and $X^N(t)$ the role of the antisymmetric functional $X_{[0,t]}$ that the process $X^N(t)$ converges to a mixture of Brownian motion with (average) diffusion constant $D_N(a)$ , but we don't need this fact.

LEMMA 3.1. Let $\mu$ be a translation invariant probability measure on $X$ such that $\mu\{a_i(x) > b\} = 0$ for some $b < \infty$ for all $i = 1,\dots,d$ .Then the following hold

(i) For any $t > 0$ fixed

$$\lim_{N\to\infty} \mu[m_N(\varphi_a^N\, \mathbf{E}_{a,x}^N \varphi_a^N(Y^N(t)))] = \mu[\varphi_1(a)\, \mathbf{E}_a \varphi_1(a(t))]$$

where $\varphi_a^N$ is defined in eq. (3.10) and $\varphi_i(a) = a_i(0) - a_i(-e_i)$

(ii) $D \geq \overline{\lim}_{N\to\infty} \mu(D_N(a))$

where the "diffusion coefficient" $D$ for the infinite system is defined in eq. (3.5) and $D_N$ , the one for the finite system in eq. (3.11). □

<u>Proof</u>. Part (ii) follows from (i). In fact, if (i) holds we have

$$\begin{aligned} \int_0^\infty dt\; \mu[\varphi_1(a)\, \mathbf{E}_a \varphi_1(a(t))] &= \int_0^\infty dt\; \lim_{N\to\infty} \mu[m_N(\varphi^N\, \mathbf{E}_{a,x}^N \varphi^N(Y^N(t)))] \\ &\leq \underline{\lim}_{N\to\infty} \int_0^\infty dt\; \mu[m_N(\varphi^N\, \mathbf{E}_{a,x}^N \varphi^N)] \\ &= \underline{\lim}_{N\to\infty} \mu[\int_0^\infty dt\; m_N(\varphi^N\, \mathbf{E}_{a,x}^N \varphi^N)] \end{aligned}$$

where the inequality follows by Fatou's lemma from the fact that the integrand on the right hand side is positive, since $m_N$ is reversible. Furthermore by translation invariance of $\mu$ ,

$$\mu[m_N(a_1(x))] = 2\mu[a_1(0)]$$

Now, by the formulas for $D$ and $D_N$ :

$$D = 2\mu[a_1(0)] - 2\int_0^\infty dt\ \mu[\varphi_1(a)\ \mathbf{E}_a\varphi_1(a(t))] \geq$$

$$\geq 2\mu[a_1(0)] - \varlimsup_{N\to\infty} 2\mu[\int_0^\infty m_N(\varphi_a^N\ \mathbf{E}_{a,x}^N\varphi_a^N)] = \lim_{N\to\infty}\mu(D_N(a))$$

Part (i) follows easily from the fact that until the boundary of $B_N$ is reached the periodic random walk and the original random walk agree (provided they start from the same place in the same immediate environment). □

DEFINITION. <u>The effective conductivity</u>.

Let $a \in X$ be fixed. The effective conductivity $\sigma_N(a)$ is the current established across $B_N$ by a unit potential difference. Let $V$ be the corresponding potential. It satisfies

$$L_a V(x) = 0 \qquad -N < x_1 < N \tag{3.12a}$$

$$V(-N,x_2) = 1 \qquad V(N,x_2) = 0 \quad , \quad -N \leq x_2 \leq N-1 \tag{3.12b}$$

Equation (3.12a) merely expresses current balance at $x$ (see eq. 3.13 below). The solution of (3.12) is not unique, but all solutions give rise to the same current

$$\sigma_N(a) \equiv \sum_{x_2=-N}^{N-1} a_1(x_1,x_2)(\nabla_1^+ V)(x_1,x_2) \tag{3.13}$$

where

$$(\nabla_1^+ V)(x) = V(x+e_1) - V(x) \quad .$$

This expression is independent of $x_1$ , provided (3.12a) is satisfied, and thus we have

$$\sigma_N(a) = 2N\ m_N(a_1\nabla_1^+ V) \tag{3.14}$$

Let $\tilde{V}$ be the natural extension of $V$ to all of $\Sigma_N$ , not the periodic extension but one producing periodic current :

$$\tilde{V}(x) = V(x) \quad \text{for} \quad x \in B_N \quad . \quad \tilde{V}(x_1+2N,x_2) = \tilde{V}(x_1,x_2) - 1 \quad ;$$

Because the left and right boundaries of $B_N$ have been made "superconducting",

$\tilde{V}$ satisfies

$$L_a\tilde{V}(x) = 0 \qquad \text{all } x \in \Sigma_N ,$$

since all this is now required at e.g. $x_1 = -N$ is that the current into this line, from the left, equals the current out, into the right, which is already guaranteed by the fact that the expression (3.13) for $\sigma_N(a)$ is independent of $x_1$. The relationship between $D_N(a)$ and $\sigma_N(a)$ now follows easily:

Proposition 3.1. For any $a \in X$

$$D_N(a) = 2\sigma_N(a)$$

where $D_N$ is defined in eq. (3.11) and $\sigma_N$ in eq. (3.13).

Proof. Consider the function $f = x_1+2N\tilde{V}-N$ on $\Sigma_N$ (where the constant $-N$ is added so that, for convenience $f = 0$ on $x_1 = \pm N$ ). Since $\varphi_a^N = L_a x_1$, it follows that $f$ is a periodic function satisfying

$$L_a f = \varphi_a^N .$$

Thus

$$\begin{aligned} D_N(a) &= 2m_N(a_1) + 2m_N(\varphi_a^N L_a^{-1}\varphi_a^N) \\ &= 2m_N(a_1) + 2m_N(\varphi_a^N f) \\ &= 2m_N(a_1) + 2m_N((\nabla_1^- a_1) f) . \end{aligned}$$

"Integrating" by parts, and using the translation invariance of $m_N$, (i.e., periodicity) we obtain that

$$\begin{aligned} D_N(a) &= 2m_N(a_1) - 2m_N(a_1\nabla_1^+ f) \\ &= 2m_N(a_1) - 2m_N(a_1(1 + 2N\nabla_1^+ V) \\ &= 2\cdot 2Nm_N(a_1\nabla_1^+ V) = 2\sigma_N(a) . \end{aligned}$$ □

Proof of Thm. 3.1 (C) . It follows from Lemma 3.1 (ii), Proposition 3.1 and Fatou's lemma that

$$D \geq \overline{\lim_{N\to\infty}}\, \mu(D_N(a) = \overline{\lim_{N\to\infty}}\, \mu(\sigma_N(a)) \geq \mu(\underline{\lim_{N}}\, \sigma_N(a))$$

Moreover it has been proven in [ 5 ] and [ 9 ] that there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 \leq \underline{\lim}\ \sigma_N(a) \leq \overline{\lim}\ \sigma_N(a) \leq c_2 \tag{3.15}$$

$\mu$ a.s. Thus $D \geq c_1 > 0$ . □

REMARKS.

1) Independence is required only for eq. (3.15).

2) For boxes other than squares, and for higher dimensions, it remains true that $D_N(a) = 2\sigma_N(a) = m(a_1 \overset{+}{\nabla}_1 V)$, where $V(x_-) = 2N$ , $V(x_+) = 0$ and the box extends from $x_1 = x_-$ to $x_1 = x_+$ and $x_+ - x_- = 2N$ . Thus $\sigma_N(a)$ is the effective local conductivity : the current per unit cross section per unit average electric field.

CLARIFICATION. Contrary to what was announced in the 1984 Conference on The Mathematics of Phase Transitions (Bowdoin College, June 1984), the problem of the existence of the limit of the expected current, i.e. $\lim_{N\to\infty} \langle \sigma_N(a) \rangle$ remains open in the percolation case.

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Permanent addresses

For A.D.M. : Università dell'Aquila, Dipartimento di Matematica, 67100 L'Aquila, Italy.

For P.A.F. : Instituto de Matemática e Estatística, Universidade de São Paulo, Cx Postal 20570, São Paulo, Brasil.

For S.G. : Dept of Math., Rutgers University, New Brunswick, N.J. 08903, USA.

For D.W.W. : Math. Dept., Univ. of Washington, GN-50 Seattle, Wa.98115, USA