Journal of Statistical Physics, Vol. 44, Nos. 3/4, 1986

Reaction–Diffusion Equations for Interacting Particle Systems[1]

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Received January 13, 1986

We study interacting spin (particle) systems on a lattice under the combined influence of spin flip (Glauber) and simple exchange (Kawasaki) dynamics. We prove that when the particle-conserving exchanges (stirrings) occur on a fast time scale of order ε^{-2} the macroscopic density, defined on spatial scale ε^{-1}, evolves according to an autonomous nonlinear diffusion-reaction equation. Microscopic fluctuations about the deterministic macroscopic evolution are found explicitly. They grow, with time, to become infinite when the deterministic solution is unstable.

KEY WORDS: Stirring process; Glauber dynamics; branching processes; hydrodynamic limit; generalized Orenstein–Uhlenbeck processes.

1. INTRODUCTION

Heuristic derivations of time-evolution equations for macroscopic variables, e.g., the particle momentum and energy, and the density of a fluid, usually contain, explicitly or implicitly, the following type of interrelated assumptions[7]: (a) Even systems which are far from thermal

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[6] Partially supported by CNPq Grant No. 201682-83.

[7] The literature on the subject is vast. We cite a few books and reviews: Balescu,[3] Bogoliubov,[7] van Kampen,[30] Forster, [23] Prigogine, [44] and Reichl.[47]

0022-4715/86/0800-0589$05.00/0

equilibrium globally are close to it locally. (b) The macroscopic variables, denoted by $M(r, t)$, which characterize the state of local thermodynamic equilibrium, vary slowly on microscopic temporal and spatial scales. (c) Their change in a macroscopic time interval dt in a macroscopic region dr are (simple) functions of the $M(r, t)$ and their spatial gradients. These assumptions lead to autonomous equations of the form

$$\frac{\partial M(r, t)}{\partial t} = \mathscr{F}(M(r, t), \nabla M, \ldots) \tag{1.1}$$

where $\mathscr{F}$ depends on the type of problems considered (and contains parameters specific to the system considered). Examples include the Navier–Stokes equations for fluids, and diffusion-reaction type equations for chemically reacting mixtures; cf. Smoller.[50]

The agreement between the predictions of equations like (1.1) and observations on real systems leave little doubt of their essential correctness in a great variety of situations. This has led naturally to many attempts at a "microscopic" derivation of these equations. This research has yielded important theoretical and practical results, e.g., the Einstein–Green–Kubo relations for transport coefficients.[19] Despite these successes, however, rigorous mathematical derivations based on realistic microscopic models are still very much beyond our reach.[22]

The situation is much better if one *starts* with the Boltzmann equation, which is itself a deterministic law of form (1.1), for what one may call quasi-macroscopic variables.[11] Once it is accepted, or derived in certain limits,[35] hydrodynamical laws can be obtained from it using Chapman–Enskog expansion methods.[12] These methods fit in (are in fact part of the origin) both the heuristics described above and the rigorous analysis used below.

The first serious mathematical effort to derive the real hydrodynamic equations for simple fluids, using classical dynamics for the motion of their microscopic constituents, was apparently undertaken by Morey.[40] That paper is rather obscure (in every sense of the word) and incomplete, although it contains along the way many important new ideas (including the first rigorous derivation of the convergence of the Mayer fugacity expansion in equilibrium). In particular, it emphasizes the essential element involved in the transition from microscopic to macroscopic evolution equations: the suitable rescaling of space and time. By such rescalings one can take account, in a rigorous mathematical way, of the central fact that there are a very large number of atoms in each drop of macroscopic fluid and that there is a big spread between micro- and macro-time scales. The macroscopic picture is thus a "blurred" one: summing over a large number of elementary events. This brings in the "law of large numbers" which is

crucial for obtaining deterministic autonomous macroscopic equations, like (1.1), not just for averages, but for the almost sure value of quantities which fluctuate on the microscopic scale. Controlling the fluctuations is clearly important if the deterministic equations are to describe what is actually observed in an experiment.

Morey's ideas, combined with the heuristics and Chapman–Enskog methods described earlier, outline a program for derivations of (1.1). The mathematical difficulties encountered in carrying out this program fully are, however, enormous—possibly insurmountable. This has led, in recent years, to the study of special situations and/or special model systems. The special situations include time-dependent fluctuations and "color" diffusion in equilibrium systems.[32,36] The latter concerns the time evolution of the relative concentrations in a fluid mixture whose components differ only by their color, i.e., they are mechanically identical, and the system, ignoring color, is in true thermal equilibrium. Unfortunately, even for these intrinsically linear nonequilibrium phenomena, i.e., the F's in (1.1) are linear in the M's, the problem is too difficult to solve completely except in simplified model systems. These include systems at low density,[36] one-dimensional systems,[32] and those whose microscopic dynamics involve some stochastic elements, e.g., interacting Brownian particles[43,52] and particles on a lattice with hopping dynamics. For some of the latter systems it has in fact been possible to derive nonlinear diffusion equations for the macroscopic particle density,[20,24] the only microscopically conserved quantity in these models. For a review, see De Masi *et al.*,[16], Presutti,[46] and Spohn.[51] [For special mechanical models where (1.1) can be derived, at least partially, see Lanford[35] (low density), Boldrighini *et al.*[6] (one dimension, hard core), and Shuhov and Sukhov[53] (rotators).]

In this paper we generalize the class of stochastic model systems investigated in this context to obtain new interesting macroscopic equations.

We study interacting particle (spin) systems on a lattice under the combined influence of general Glauber (spin flip) and simple exclusion (spin exchange) dynamics.[37]. We prove that when the magnetization-conserving exchanges occur on a microscopic time scale, say of order ε^{-2}, $\varepsilon \ll 1$, the macroscopic density (magnetization) viewed on a spatial scale of order ε^{-1} evolves according to an autonomous nonlinear diffusion-reaction equation,[50],

$$\frac{\partial m(r, t)}{\partial t} = \nabla^2 m + F(m(r, t)) \tag{1.2}$$

where $r \in \mathbb{R}^d$ and $F(m)$ is a polynomial which can be "adjusted" by choosing suitable Glauber rates.

Our analysis is related to, but more complicated than, that used in Ref. 16 for the case when there are only exchanges, which leads to (1.2) with $F(m) = 0$. In all cases the proof involves showing that the microscopic state is close to "local equilibrium." The deviations from local equilibrium produce fluctuations about the deterministic solutions of (1.2). These fluctuations, which we also derive directly from our microscopic dynamics, form a Gaussian field. They have an amplitude of $O(\varepsilon^{d/2})$ around stable solutions of (1.2) but grow exponentially or like a power law around unstable solutions.

The existence of unstable solutions requires, of course, having a nonlinear term in (1.2). The simplest case corresponds to the existence of multiple spatially uniform stationary solutions of (1.2), some of which are stable and other unstable. They can be interpreted (in some cases) as coexisting stationary states of our microscopic lattice system, even before taking the limit $\varepsilon \to 0$. Note that these stationary states are generally *not* equilibrium Gibbs states with *any* finite range (or rapidly decaying) potential. They are nonequilibrium stationary states which can have phase transitions even in one dimension. The study of such stationary nonequilibrium microscopic states is in itself a problem of great interest[31] and was one of the motivations for undertaking the present work. Another motivation is our hope that the analysis of the microscopic fluctuations about the deterministic macroscopic equations will add to our understanding of the instabilities and pattern formation associated with some nonlinear evolution equations.[5] This may be particularly so in cases where stochastic lattice models are invented to simulate physically important and mathematically intractable nonlinear equations, e.g., those describing the motion of an interface between two fluids.[29] (We *hope* that we can find stochastic models leading rigorously to those equations.)

We shall not pursue the above questions too much in the present paper—which is mainly devoted to the derivation of (1.2) and the equations for the fluctuations. We note, however, that Eq. (1.2), or its generalization involving several densities (which we know how to derive), is sometimes used to describe phenomena in biology, population genetics, flame propagation, etc.[2] In some such cases our lattice model with rapid stirrings may in fact model reality, and the fluctuations may then be clearly relevant. The fluctuations play an important role in Ref. 18 where the escape from the unstable equilibrium in a one-dimensional version of the model presented here is studied.

In Haken,[25] and Nicolis and Prigogine,[41] stochastic models of chemical reactions with diffusion are presented. These models are studied in Refs. 1, 33, and 34, where the derivation of Eq. (1.2) is given by means of a mean-field type of limit. In Ref. 33 fluctuations are also studied.

Bramson[8] showed that the solution of Eq. (1.2) when $f(m)=m-m^2$ and when the initial datum is a square step function is equal to the distribution of the position of the maximum displacement of a branching Brownian motion. As we shall see, our model is also strictly related to a branching random walk with hard-core interaction.

We describe our models and give results and an outline of the proofs in Sec. 2. Proofs are presented in Sec. 3 and Sec. 4. A brief report of our result is presented in Ref. 14.

2. MODELS AND RESULTS

2.1. General

Our system is a simple cubic lattice in d dimensions, at each site of which there is a spin $\sigma(x)=\pm 1$, $x\in\mathbb{Z}^d$. We consider two mechanisms by which a configuration of the lattice $\sigma=\{\sigma(x), x\in\mathbb{Z}^d\}$ changes with time: a Glauber dynamics in which a spin flips at a site x, $\sigma\to\sigma^x$, with a rate $c(x;\sigma)$, and a simple exchange dynamics in which unequal spins at neighboring sites x, y, $|x-y|=1$ exchange, $\sigma\to\sigma^{x,y}$, with a rate $\varepsilon^{-2}/2$. This leads to a family of Markov processes on the state space $\mathcal{X}=\{-1,1\}^{\mathbb{Z}^d}$ depending on a parameter ε.

It is sometimes convenient to picture our system as consisting of particles, located at each lattice site, which are of two interconvertible species, $\sigma(x)=\pm 1$, and are stirred rapidly. Equivalently it can be thought of as a lattice gas with occupied and empty sites, $\sigma(x)=\pm 1$, in which particles are born, die, and while alive can jump to nearby empty sites. We shall use all these representations interchangeably and refer to Liggett[37] for a complete description of the interacting particle systems we are considering in this paper.

Given an $\varepsilon>0$ we define a Markov semigroup S_t^ε, $t\geqslant 0$, on $\mathcal{X}$ via its generator L_ε which acts on cylinder functions as a sum of the generators for the Glauber and exchange processes:

$$L_\varepsilon f(\sigma)=L_G f(\sigma)+\varepsilon^{-2}L_E f(\sigma) \tag{2.1a}$$

where

$$L_G f(\sigma)=\sum_{x\in\mathbb{Z}^d} c(x;\sigma)[f(\sigma^x)-f(\sigma)] \tag{2.1b}$$

$$L_E f(\sigma)=\frac{1}{2}\sum_{|x-y|=1}[f(\sigma^{x,y})-f(\sigma)] \tag{2.1c}$$

The semigroup S_t^ε defines on $\mathcal{X}$ a Markov (and Feller) process σ_t^ε which we write as σ_t or $\sigma(\cdot,t)$.

Remark 2.1. The behavior of the exclusion process and the Glauber dynamics is very different. Exclusion conserves magnetization (particle number). It is, in the language of physics, Kawasaki dynamics at infinite temperature. Its stationary measures are ergodic and form a one-parameter family $\{\nu_m, m \in [-1, 1]\}$; ν_m is a translation-invariant product measure $\nu_m(\sigma(0)) = m$.[37] The Glauber dynamics has no such conservation law. Its stationary measures depend on the spin flip rates $c(x; \sigma)$ which we are free to choose pretty much as we want.

We shall assume that $c(x; \sigma)$ is translation invariant and depends only on finitely many $\sigma(y)$'s. This gives generally

$$c(0, \sigma) = \sum_{A \subset \Lambda_0} K_A \sigma_A, \qquad \sigma_A = \prod_{y \in A} \sigma(y) \tag{2.2a}$$

where Λ_0 is some bounded domain, $0 \in \Lambda_0 \subset \mathbb{Z}^d$, and the K_A have to be chosen so that $c(0; \sigma) \geqslant 0$, $\forall \sigma \in \mathcal{X}$. By translation invariance

$$c(x; \sigma) = c(0; \tau_{-x}\sigma) = \sum_A K_A \sigma_{A+x} \tag{2.2b}$$

where τ_{-x} is the shift by $-x$, $(\tau_{-x}\sigma)(y) = \sigma(x + y)$.

Let us now define the magnetization density on the scale ε^{-1} by setting, for $\varepsilon \ll 1$,

$$\int_{\Lambda_r} dr\, m^\varepsilon(r, t; \sigma) = S_t^\varepsilon \left(\varepsilon^d \sum_{x \in \Lambda_r^\varepsilon} \sigma(x) \right) \tag{2.3}$$

where Λ_r and Λ_r^ε are cubes with sides of length δ, $\delta > 0$, and $\delta\varepsilon^{-1}$ respectively, centered on $r \in \mathbb{R}^d$. (We suppress the dependence on δ which is fixed and small and can be taken to be zero *after* $\varepsilon \to 0$). $m^\varepsilon(r, t; \cdot)$ is a random variable whose distribution depends on the initial probability distribution of the system. We shall assume the latter to have good cluster properties and that, as $\varepsilon \to 0$, $\varepsilon^{-d} |\langle \sigma(x) \rangle_\varepsilon - m_0(\varepsilon x)| \to 0$ where $m_0(r)$ is a smooth function of r, $r \in \mathbb{R}^d$, $|m_0(r)| \leqslant 1$.

The change in the magnetization in the region Λ_r^ε in a fixed macroscopic time interval $[0, t]$ will be governed by (a) the number of flips in Λ_r^ε which is proportional to ε^{-d} and (b) the number of exchanges which cause particles to cross the surface of Λ_r^ε. The latter are proportional to $\varepsilon^{-2} |\partial \Lambda_r^\varepsilon|_D$, where $|\partial \Lambda_r^\varepsilon| \sim \varepsilon^{-(d-1)}$ is the surface area and $D \sim \varepsilon$ is the gradient of the magnetization on the microscopic scale. Both effects are thus, for $m^\varepsilon(r, t; \sigma)$, of order unity. The number of exchanges per site behaves, on the other hand, like ε^{-2} in any macroscopic time interval.

Thus, in the limit $\varepsilon \to 0$, when there is a true separation between the microscopic and macroscopic scales, the exchanges whose long-time

asymptotic state is one of uncorrelated spins should cause the spins in the box Λ_r^ε to be distributed independently (with a product measure) at the instantaneous value of the magnetization in Λ_r^ε. The change in this magnetization due to the fluxes across the boundaries of Λ_r^ε and spin flips inside tends to make the spins correlated but their effect, with the scaling chosen, are of $o(\varepsilon)$ compared with the stirring inside Λ_r^ε. Thus, as $\varepsilon \to 0$, $\Lambda_r^\varepsilon \to \infty$, $m^\varepsilon(r, t; \sigma)$ should become a deterministic variable $m(r, t)$ (fluctuations should go to zero), with $m(r, 0) = m_0(r)$, whose time evolution is determined by using the product measure to evaluate the changes it undergoes. This is in fact what happens, as will be seen from the results which follow. These are stated formally in Theorems 1–4 and discussed informally in Remarks 2.2 and 2.3.

For stating the theorems in a precise way we first need some definitions.

Definition 2.1. We will use the following notations:

$$\mu, \upsilon, \ldots, \text{ measures on the state space } \mathscr{X},\ \mu(\cdot) = \text{expectation with respect to } \mu, \tag{2.4a}$$

$$E = D([0, \infty), \mathscr{X}) \text{ is the trajectory space} \tag{2.4b}$$

$$P_\mu^\varepsilon \text{ is the law of the process with initial measure } \mu \tag{2.4c}$$

$$\mathbb{E}_\mu(f) \equiv \mu(S_t^\varepsilon f) \equiv \mu(f(\sigma_t)) \text{ is the expectation with respect to the process with initial measure } \mu \tag{2.4d}$$

Definition 2.2. *Hypothesis on the Initial Measure.*

Let $m_0 : \mathbb{R} \to [-1, 1]$ be a C^3-function with uniformly bounded derivative. Let μ^ε, $\varepsilon \in (0, 1)$ be a family of measures such that

$$\text{(i)} \quad \lim_{\varepsilon \to 0} \sup_{x \in \mathbb{Z}^d} \varepsilon^{-d} |\mu^\varepsilon(\sigma(x)) - m_0(\varepsilon x)| = 0 \tag{2.5}$$

(ii) For every $n \geqslant 2$ there are positive decreasing functions $\phi_n : \mathbb{N} \to \mathbb{R}_+$ such that, for all $A, B \subset \mathbb{Z}^d$, $A \cap B = \phi$,

$$|\mu^\varepsilon(\sigma_A \sigma_B) - \mu^\varepsilon(\sigma_A)\, \mu^\varepsilon(\sigma_B)| \leqslant \phi_n(r_{A,B}) \tag{2.6a}$$

where $r_{A,B} = \text{Min dist}(A, B)$, $n = |A| + |B|$, and

$$\sum_{x \in \mathbb{Z}^d} \phi_n(|x|) < \infty \tag{2.6b}$$

(iii) For every $k \in \mathbb{N}$, $x_i, \ldots, x_k \in \mathbb{Z}^d$, $x_i \neq x_j$, $\forall i \neq j$,

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{Z}^d} \left| \mu^\varepsilon \left(\prod_{i=1}^k \sigma(x_i + x) \right) - m_0(\varepsilon x)^k \right| = 0 \tag{2.7}$$

Theorem 1. Let μ^ε, $\varepsilon \in (0, 1]$ be a family of measures on $\mathscr{X} = \{-1, 1\}^{\mathbb{Z}^d}$ satisfying (i)–(iii) of Definition 2.2. Then for every $t \geqslant 0$, $r \in \mathbb{R}^d$ and for any cylinder function f the following holds:

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon x \to r}} \mu^\varepsilon(S_t^\varepsilon f(\tau_{-x}\sigma)) = \nu_{m(r,t)}(f) \tag{2.8}$$

where ν_m is the translation-invariant product measure on $\mathscr{X}$ with $\nu_m(\sigma(0)) = m$ and $m(r, t)$, $r \in \mathbb{R}^d$, $t \geqslant 0$ is the unique solution of the following reaction-diffusion equation;

$$\left.\begin{aligned} \frac{\partial m}{\partial t} &= \frac{1}{2}\Delta m + F(m) \\ m(r, 0) &= m_0(r) \end{aligned}\right\} \tag{2.9a}$$

$$F(m) = \nu_m(-2\sigma(0)\, c(0, \sigma)) \tag{2.9b}$$

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial r_i^2}, \qquad r = (r_1, \ldots, r_d) \in \mathbb{R}^d \tag{2.9c}$$

Theorem 1 describes the average behavior of the magnetization. The next result is the law of large numbers for the macroscopic magnetization density.

Theorem 2. Let μ^ε, $\varepsilon \in (0, 1]$ be a family of (initial) measures on $\mathscr{X}$ satisfying Definition 2.2. For $\phi \in \mathscr{S}(\mathbb{R}^d)$, define the "magnetization field"

$$X_t^\varepsilon(\phi) = \varepsilon^d \sum_{x \in \mathbb{R}^d} \phi(\varepsilon x)\, \sigma(x, t) \tag{2.10}$$

as a process on $D([0, \infty), \mathscr{S}'(\mathbb{R}))$ and let $\mathscr{P}^\varepsilon$ denote its law as inherited from P_{μ^ε}. Then

$$\mathscr{P}^\varepsilon \xrightarrow[\varepsilon \to 0]{} \mathscr{P} \text{ weakly} \tag{2.11}$$

where $\mathscr{P}$ is the measure having support on a single trajectory given by

$$\int dr\, \phi(r)\, m(r, t) \tag{2.12}$$

$m(r, t)$ being the same as in Theorem 1.

Remark 2.2. Theorem 2 confirms the behavior we expected. It states essentially that the random variable $m^\varepsilon(r, t; \cdot)$ defined in (2.3) converges (weakly) to the deterministic function $m(r, t)$ satisfying (2.9). For the exclusion process it has been proven (see De Masi *et al.*[16] and references

therein) that in the time space scaling $\varepsilon^{-2}t$, $\varepsilon^{-1}r$ the system approaches, as $\varepsilon \to 0$, local equilibrium, i.e., its distribution at time $\varepsilon^{-2}t$ in a region of order ε^{-1} converges to a product (equilibrium) measure with a parameter which varies in space and time according to the linear diffusion equation. Here we are using this scaling in a somewhat different way. In fact, as ε varies, the process is changing. Nevertheless, when ε is small, the distribution at time t of the process with generator L_ε still looks like a product measure but with a time-dependent parameter which feels the effect of the Glauber dynamics and of the fluxes, produced by the stirrings, across the boundaries of Λ_r^ε: the process and the scaling are chosen in such a way that the local magnetization changes in the macroscopic time t by only a finite amount. The exchanges acting in a manner independent of the environment continue to produce a linear diffusion term while the Glauber dynamics produces the polynomial terms in the evolution equation for the magnetization, their exact form depending on $c(x; \sigma)$ via (2.9b).

We next study the microscopic fluctuations about the deterministic macroscopic evolution.

Theorem 3. Let μ^ε satisfy Definition 2.2. For $\phi \in \mathscr{S}(\mathbb{R}^d)$ define the magnetization fluctuation field as

$$Y_t^\varepsilon(\phi) = \varepsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \phi(\varepsilon x)[\sigma(x, t) - \mathbb{E}_{\mu^\varepsilon}(\sigma(x, t))] \tag{2.13}$$

Then $\{Y_t^\varepsilon(\cdot)\}$, considered as a process on $D([0, \infty), \mathscr{S}'(\mathbb{R}^d))$, converges weakly (as $\varepsilon \to 0$) to a generalized Ornstein–Uhlenbeck process $\{Y_t(\cdot)\}$ with distribution P. P is uniquely determined by the condition that the $\{Y_t(\phi)\}$ are centered, that for all $G \in C_0^\infty(\mathbb{R})$

$$G(Y_t(\phi)) - \int_0^t ds\, Y_s(A_s\phi)\, G'(Y_s(\phi)) - \frac{1}{2}\int_0^t ds\, \|B_s\phi\|^2\, G''(Y_s(\phi)) \tag{2.14a}$$

is a P-martingale, and that the law of the $\{Y_0(\phi)\}$ is Gaussian with covariance

$$P(Y_0(\phi)\, Y_0(\psi)) = \int dr\, \phi(r)\, \psi(r)(1 - m_0(r)^2) \tag{2.14b}$$

In (2.14a) G' and G'' denote, respectively, the first and second derivative of G and

$$A_s\phi(r) = \frac{1}{2}\phi'' + \phi(r)\frac{dF}{dm}(m(r, s)) \tag{2.14c}$$

$$\|B_s\phi\|^2 = \int dr(\nabla\phi(r))^2\,(1 - m(r, s)^2) + \int dr\, \phi(r)^2\, 4\nu_{m(r,s)}(c(0, \sigma)) \tag{2.14d}$$

where F is defined in Eq. (2.9b) and $m(r, t)$ is the solution of Eq. (2.9).

We observe that it follows from Eq. (2.14) that the equal time covariance of the limiting process $\{Y_t(\cdot)\}$ is given by

$$P(Y_t(\phi)\, Y_t(\psi)) = P(Y_0(\phi)\, Y_0(\psi)) + \int_0^t ds[\, Y_s(A_s\phi)\, Y_s(\psi) + Y_s(\phi)\, Y_s(A_s\psi)] + \frac{1}{2}\int_0^t ds \langle B_s\psi, B_s\phi\rangle \tag{2.14e}$$

where

$$\langle B_s\psi, B_s\phi\rangle \equiv \int dr[(\nabla\phi)\cdot(\nabla\psi)(1-m(r,s)^2) + \phi(r)\,\psi(r)\,4\nu_{m(r,s)}(c(0,\sigma))] \tag{2.14f}$$

The proof of Theorem 3 uses Theorem 4 below which we state separately because it is an interesting result in its own right (see Remark 2.3).

Theorem 4. For $\phi \in \varphi(\mathbb{R}^d)$ and any f cylinder on $\{-1, 1\}^{\mathbb{Z}^d}$, define

$$Y_t^\varepsilon(\phi; f) = \varepsilon^{d/2} \sum_x \phi(\varepsilon x)[f(\tau_{-x}\sigma_t) - \mathbb{E}_{\mu^\varepsilon}(f(\tau_{-x}\sigma_t))] \tag{2.15}$$

Then for every $0 < \tau' < \tau < \infty$,

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\sup_{\tau'\le t\le\tau} P^\varepsilon\left(\left|\frac{1}{\varepsilon^2 T}\int_t^{t+\varepsilon^2 T} ds\; Y_s^\varepsilon(\phi; f) - Y_s^\varepsilon(a_s^f\phi)\right|^2\right) = 0 \tag{2.16a}$$

where

$$a_s^f\phi(r) = \left[\frac{\partial}{\partial m}\cdot \nu_m(f)\Big|_{m=m(r,s)}\right]\phi(r) \tag{2.16b}$$

Remark 2.3. Roughly speaking, Theorem 3 states that the limiting process describing the magnetization fluctuation field satisfies, in the weak form, the stochastic differential equation

$$d\Phi(r,t) = \left(\frac{1}{2}\nabla^2\Phi + F'(m(r,t))\,\Phi\right)dt + W(r,t) \tag{2.17}$$

where $W(r,t)$ is a "white noise" with the covariance

$$\langle W(r,t)\, W(r',t')\rangle = \delta(t-t')[\nabla_r\nabla_{r'}((1-m(r,t)^2\,\delta(r-r'))^{\cdot} + 4f(m)\,\delta(r-r')] \tag{2.18a}$$

where

$$f(m) = v_m(c(0, \sigma)) \tag{2.18b}$$

The equal time correlations of the fluctuation field

$$c(r, r'; t) \equiv \langle \Phi(r, t)\, \Phi(r', t) \rangle$$

satisfy the following equation. Let

$$c(r, r'; t) = (1 - m^2(r, t))\, \delta(r - r') + \tilde{c}(r, r'; t) \tag{2.19}$$

then $\tilde{c}(r, r'; t)$ satisfies

$$\begin{aligned} \partial_t \tilde{c}_t &= \frac{1}{2}(\partial_r^2 + \partial_{r'}^2)\, \tilde{c}_t + [F'(m(r, t)) + F'(m(r', t))]\, \tilde{c}_t + \delta(r, r') \\ &\times \left[-\left(\frac{\partial m}{\partial r}\right)^2 + 2F'(m)(1 - m^2) + 2mF(m) + 4f(m) \right] \\ \tilde{c}(r, r'; 0) &= 0 \end{aligned} \tag{2.20}$$

Theorem 4 states that the fluctuation fields of any local function become proportional to the fluctuations of the magnetization field. This is a property that one expects to hold in all systems that have conservation laws. It is based on the physically natural idea that *in equilibrium* the fluctuation fields of nonconserved quantities change in time on a much faster scale than the fluctuations of the conserved quantities, hence in a time integral only the component along these fields survives. This property was formulated explicitly in a precise mathematical form by Rost[48] and called the "Boltzmann–Gibbs' principle." It has been proven so far for some models of interacting particle systems in equilibrium; see Brox and Rost,[9] Spohn,[52] and De Masi *et al.*[17] The validity of the principle in nonequilibrium (when a conservation law is present) has been proven for two models—the simple exclusion and the zero-range process (Ferrari *et al.*[20])—and it is questionable whether it should be expected to hold in general. The validity of this principle in our case is even less obvious since the process we are considering has no strictly conserved quantities.

Nevertheless it appears sufficient that in the "microscopic" time $\varepsilon^2 t$ the total magnetization is unchanged, since with large probability no flips occurred in that time interval.

To make the discussions and proofs simpler, we will consider from now on a specific one-dimensional example. We will give all the proofs for this example: the generalization to more dimensions and/or more general models is quite simple. However, when the dimension or the structure of the model has been used in some of the proofs, we will point this out.

We now present the example and conclude this section with an outline of the proofs.

2.1. The Example

Given an $\varepsilon > 0$, we define the one-dimensional Glauber-exclusion process (GE) on the state space $\mathcal{X} = \{-1, 1\}^{\mathbb{Z}}$ via the generator L_ε as in Eq. (2.1) with

$$c(x; \sigma) = 1 - \gamma\sigma(x)[\sigma(x+1) + \sigma(x-1)] + \gamma^2\sigma(x+1)\,\sigma(x-1),$$
$$0 \leqslant \gamma < 1 \tag{2.21a}$$

These rates have the property that if we let

$$\gamma = \tanh \beta J, \qquad \beta > 0 \tag{2.21b}$$

then the (Glauber) dynamics with this generator L_G is reversible for the Gibbs measure μ_β of a one-dimensional Ising model with nearest neighbor interaction J at reciprocal temperature β. The measure μ_β is, as is well known, unique with exponential decay of correlations. In the absence of exchanges, which, as already noted, act as if the system were at infinite temperature, μ_β would be approached in time "exponentially fast."[27]

When the dynamics are combined, $L = L_G + \varepsilon^{-2}L_E$, the stationary state is no longer reversible and its nature is unknown. In particular, we do not know whether it is unique. Its behavior, when $\varepsilon \to 0$, will be related to the deterministic evolution and the fluctuations about it which we now discuss briefly.

Macroscopic Equation

For our example, Eq. (2.9) takes the form

$$\frac{\partial m(q, t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial q^2} m(q, t) + 2(2\gamma - 1)\, m(q, t) - 2\gamma^2 m^3(q, t) \tag{2.22}$$

where q is a one-dimensional variable, $q \in \mathbb{R}$.

Given $m(q, 0) = m_0(q) \in [-1, 1]$, Eq. (2.22) has a unique solution satisfying the integral equation

$$m(q, t) = \int_{-\infty}^{\infty} d\xi\, G(q - \xi, t)\, m_0(\xi) + \int_0^t ds \int_{-\infty}^{\infty} d\xi\, G(q - \xi, t - s)\, F(m(\xi, s) \tag{2.23}$$

where

$$G(q, t) = (2\pi t)^{-1/2} \exp[-q^2/2t]$$
$$F(m) = [2(2\gamma - 1)]\, m - 2\gamma^2 m^3 \tag{2.24}$$

The simplest case to consider is one where the initial magnetization $m_0(q) = m_0$ is spatially uniform. The magnetization at time $t \geqslant 0$ will then be given by

$$m(t) = m_0 + \int_0^t ds\, F(m(s)) \tag{2.25}$$

As $t \to \infty$, $m(t) \to \bar{m}(\gamma)$, one of the roots of the cubic polynomial on the right side of (2.22),

$$\bar{m}[(2\gamma - 1) - \gamma^2 \bar{m}^2] = 0 \tag{2.26}$$

Clearly $\bar{m} = 0$ is a stationary solution. For $\gamma < 0$, corresponding to antiferromagnetic interactions $J < 0$ for the Glauber dynamics in (2.21) as well as for $0 \leqslant \gamma \leqslant 1/2 = \gamma_c$, the $\bar{m} = 0$ solution is unique and stable. When $1 \geqslant \gamma > \gamma_c$, on the other hand, i.e., for ferromagnetic interactions at low temperatures, there are two additional stationary uniform magnetizations,

$$\bar{m} = \pm \gamma^{-1} (\gamma/\gamma_c - 1)^{1/2} = \pm m^* \tag{2.27}$$

m^* can be thought of as the "spontaneous magnetizations"; for $\gamma > \gamma_c$, $\bar{m} = \pm m^*$ are the only stable solutions, $\bar{m} = 0$ becoming unstable at $\gamma = \gamma_c$.

This behavior is most readily understood by writing the polynomial $F(m)$ on the right-hand side of (2.25) as

$$F(m) = V'(m) = -\frac{d}{dm}\left[\frac{\gamma^2}{2} m^4 - \left(\frac{\gamma}{\gamma_c} - 1\right) m^2\right] \tag{2.28}$$

$V(m)$ has the typical form of a mean field free energy with a single minimum for $\beta \leqslant \beta_c$ and a double well form for $\beta > \beta_c$, than $\beta_c J = \gamma_c = 1/2$. The origin of this behavior lies, of course, in the decorrelation introduced by the rapid exchanges. The value of β_c here is, however, different from the usual Curie–Weiss value $\beta J = 1/2$. Mean field theory is, of course, an approximation (a terrible one for $d = 1$) for equilibrium systems with short-range interaction. Its appearance in our system, for $\varepsilon \to 0$, is entirely of dynamical origin—we are very far from equilibrium. This leads us to ask whether the bifurcation in the solution of the macroscopic equation for the magnetization occurs already on the microscopic level before going to the limit $\varepsilon \to 0$. We believe, but cannot prove, that the answer is no for our one-dimensional example (but is so in higher dimensions); cf. also De Masi *et al.*[18]

To study microscopic effects in our present formulation we have to look at the fluctuations about the deterministic solution $m(q, t)$. From

Theorem 3 we have that the "off-diagonal" equal time covariance [see Eq. (2.19)] of $\Phi(\Phi \sim \varepsilon^{-1/2}[\langle \sigma(q/\varepsilon, t)\rangle_\varepsilon - m^e(q/\varepsilon, t; \sigma_t)])$ satisfies, for our example, the equation

$$\frac{\partial}{\partial t}\tilde{c}(q, q'; t) = \frac{1}{2}\left(\frac{\partial^2 \tilde{c}}{\partial q'} + \frac{\partial^2 \tilde{c}}{\partial q}\right) + [F'(m(q, t)) + F'(m(q', t))]\,\tilde{c} + \delta(q-q')\left[-\left(\frac{\partial m}{\partial r}\right)^2 + 2F'(m)(1-m^2) + 2mF(m) + 4f(m)\right] \tag{2.29}$$

where

$$F'(m) = 2\left[\frac{\gamma}{\gamma_c} - 1\right] - 6\gamma^2 m^2 \tag{2.30}$$

$$f(m) = 1 - \gamma(2-\gamma)\, m^2 \tag{2.31}$$

Here $m(q, t)$ is the solution of (2.22), and (2.29) is to be solved with initial condition $\tilde{c}(q, q'; 0) = 0$.

Let us now consider the solution of (2.29) around the stationary state $m_0(q) = m(q, t) = 0$. It is given by

$$\tilde{c}_0(q-q'; t) = 8\gamma \int_0^t ds (4\pi s)^{-1/2} \exp[-(q-q')^2/4s] \cdot \exp[-4(1-\gamma/\gamma_c)\, s] \tag{2.32a}$$

For $\gamma < \gamma_c$,

$$\tilde{c}_0(q-q'; t) \xrightarrow[t\to\infty]{} \frac{\gamma}{(\gamma_c - \gamma)^{1/2}} \exp[-2\,|q-q'|\,(\gamma_c - \gamma)^{1/2}] \tag{2.32b}$$

so the Gaussian field approaches a stationary state with exponentially decaying covariances—on the macroscopic scale. For $\gamma \geqslant \gamma_c$, on the other hand, the covariances given by (2.32) grow without bound, exponentially for $\gamma > \gamma_c$, like $\sqrt{t}$ for $\gamma = \gamma_c$. The macroscopic instability of this solution is thus reflected in the fluctuations. In fact, from (2.29) the criteria for growth of $\tilde{c}$, $F'(m) \geqslant 0$ is precisely the same as for linear instability of stationary solutions of (2.22)—at least for the spatially uniform case. What happens in the nonuniform case, particularly that corresponding to traveling solutions of (2.22)[2,5] which are of interest in pattern formation, will not be discussed here.

2.3. Outline of the Proofs

The proofs of Theorems 1–4 are based on a duality technique that we present in Sec. 3. We refer to Liggett,[37] Sec. 3, for a clear presentation of duality. The key idea of duality is that of looking at the process backwards in time. When applicable this reduces the study of a Markov process on an uncountable space ($\{-1, 1\}^{\mathbb{Z}}$ in our case) to the study of its simpler dual process which generally is a Markov chain on a countable space, e.g., a family of finite subsets of $\mathbb{Z}$ (in our case a slightly more complicated space).

We proceed now to (informally) present a graphic construction of the dual process for the specific example. It is convenient, for this purpose, to rewrite the generator of the process L_ε as follows:

$$L_\varepsilon f(\sigma) = \sum_x \left\{ \sum_{i=1}^{3} \lambda_i \mathbb{1}(\sigma \in A_i(x))[f(\sigma^x) - f(\sigma)] + \frac{\varepsilon^{-2}}{2}[f(\sigma^{x,x+1}) - f(\sigma)] \right\} \tag{2.33a}$$

where

$$\lambda_1 = (1+\gamma)^2, \qquad \lambda_2 = 1-\gamma^2, \qquad \lambda_3 = (1-\gamma)^2, \qquad \lambda_i \geqslant 0 \tag{2.33b}$$

$$\lambda = \sum_{i=1}^{3} \lambda_i > 0 \tag{2.33c}$$

$$A_1(x) = \{\sigma\colon \sigma(x-1) = \sigma(x+1),\ \sigma(x-1) \neq \sigma(x)\} \tag{2.33d}$$

$$A_2(x) = \{\sigma\colon \sigma(x-1) \neq \sigma(x+1)\} \tag{2.33e}$$

$$A_3(x) = \{\sigma\colon \sigma(x-1) = \sigma(x+1) = \sigma(x)\} \tag{2.33f}$$

$$\mathbb{1}\{\sigma \in A_i\} = 1 \quad \text{if} \quad \sigma \in A_i \quad \text{and} \quad 0 \quad \text{if} \quad \sigma \notin A_i \tag{2.33g}$$

From Eq. (2.33) we can realize the G.E. process as follows: At each site of $\mathbb{Z}$ there is a particle with a spin valued $+1$ or -1 according to an initial configuration σ_0. We now associate with each site of $\mathbb{Z}$ three independent Poisson clocks with intensity λ_1, λ_2, and λ_3. A realization of these Poisson processes defines a collection of marks that we call the Glauber marks (of type 1, 2, or 3). A mark of type β at time s means that the clock with intensity λ_β rang at time s. Furthermore, to each pair of nearest-neighbor sites we associate a Poisson clock with intensity $\varepsilon^{-2}/2$. Those define the collection of exchange or stirring marks.

For a fixed configuration ω of marks, we construct the process $\{\sigma_s^\omega, 0 \leqslant s \leqslant t\}$ on $\{-1, 1\}^{\mathbb{Z}} \times [0, t]$ as follows. The particle at site x has a

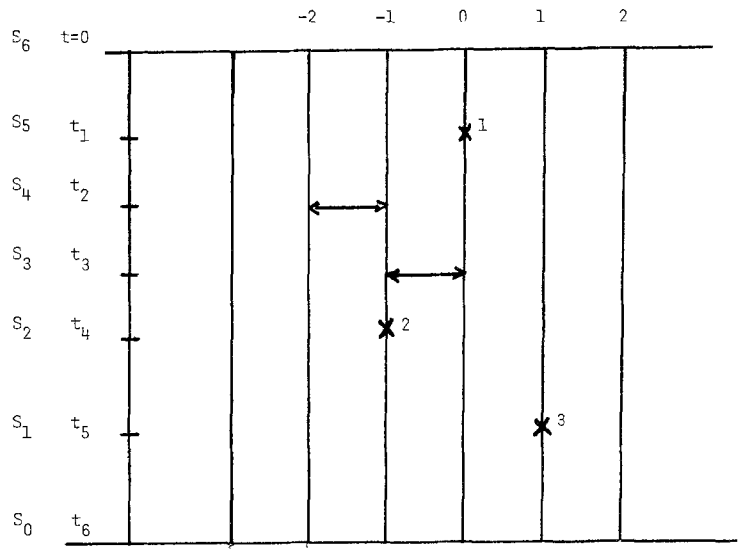

Fig. 1. Graphic realization of the two processes.

spin, say $+1$. It stays at site x with spin 1 up to the first time a mark appears. Two things can happen: (a) The mark is an exchange mark involving (say) x and $x+1$; then the particle jumps to $x+1$ (at the same time the particle sitting at $x+1$ jumps to x) carrying its spin. (b) The mark is a Glauber mark of type β, $\beta = 1, 2$, or 3. Then the particle looks at the neighboring sites: if the configuration belongs to the set $A_\beta(x)$, the particle flips its spin to -1. If not, nothing happens.

This procedure is carried out for all particles independently up to time t. Notice that the exclusion interaction appears automatically in this construction because the particles are using the same exchange marks. This implies that at all times there is exactly one particle at each site.

The Dual Process

Consider a fixed configuration σ_0 at time 0 and a fixed realization of marks in the interval $[0, t]$. Suppose that we now want to know if the spin at site x at time t, $\sigma(x, t)$ is ± 1; we first follow the stirring marks backwards in time up to the first Glauber mark. Assume that this is a type 1 mark at time s at site y. Then $\sigma(x, t) = 1$ iff at time s the configuration σ_s belongs to $\{\sigma(y-1) = 1, \sigma(y+1) = 1\}$ or to $\{\sigma(y-1) \neq \sigma(y+1),\ \sigma(y) = 1\}$. We now have to follow three particles

backward in time and keep in mind that the mark producing the branching was a type 1 mark. Each of these three particles behaves as the particle initially at x. New branches can appear on sites already occupied by an old branch. In this case the two branches "move together." Since with probability one only a finite number of branches can appear in a finite time interval, it is clear that the knowledge of the branching structure (with the type of mark that produced each branching) in $[t, 0]$ and the configuration σ_0 on the (finite) set determined by the branching at time 0 gives us the value of $\sigma(x, t)$. This is essentially the dual branching process for our G.E. dynamics.

In Fig. 1 a realization $\omega \in \Omega$ is represented by marks in $\mathbb{Z} \times \mathbb{R}_+$. The time is flowing down. The cross marks are the Glauber marks (of types 1, 2, and 3), and the double arrows, the exclusion marks. Suppose that at $t = 0$ the initial configuration is $\sigma_0 = \{-2, 0\}$, i.e., the spins of σ_0 are positive at sites -2 and 0 and are negative at all the other sites. Then for this configuration of marks,

$$\sigma_{t_1} = \{-2\}, \quad \sigma_{t_2} = \{-1\}, \quad \sigma_{t_3} = \{0\}, \quad \sigma_{t_4} = \{-1, 0\} = \sigma_{t_5} = \sigma_{t_6}$$

On the other hand, the branching process is realized on the same ω as follows: Suppose that the initial configuration is $Z_{s_0} = \{-1\}$. Then $Z_{s_1} = \{-1\}$, $Z_{s_2} = \{-1, 0, -2\} = Z_{s_3} = Z_{s_4}$, and $Z_{s_5} = \{-2, -1, 0, 1\} = Z_{s_6}$. Thus knowing σ_0 in Z_{s_6} and the branching structure which determines Z_{s_6} is enough to recover the spin value at time t.

To compute the correlation function of, say, n spins one looks at the motion of n (interacting) branching processes. The interaction appears in two ways: (a) by the stirring interaction between the particles, and (b) by the superposition of two branches when new particles are created at a place already occupied by an old branch. We say in the latter case that a Glauber interaction occurred.

We treat the exclusion interaction by suitably coupling a finite number of interacting particles with the same number of independent particles as done in Ref. 16, Sec. 3. In that coupling each interacting particle is "close" to one of the independent ones.

To treat the Glauber interaction, we observe that the probability to create at a fixed time a new particle at a site already occupied is proportional to the probability that a simple random walk is at that site at that fixed time. In our scale this probability is of the order $(\varepsilon^{-2}t)^{-d/2} \sim \varepsilon^d$. We therefore generalize the independent process introduced in Ref. 16 defining an "independent branching process' (see Definition 3.3) and show that to zero order in ε the distribution at time t of the interacting branching process is equal to the distribution of the independent one. At order ε^d a one-body and two-body correlation is present. This is stated in

Proposition 3.4 and is proven in the Appendix. The proof of Proposition 3.4 uses sharp estimates on the simple exclusion process that are a generalization of those obtained in Refs. 15 and 21.

The above analysis implies (see Lemma 3.5) that the distribution at time t of the Glauber exclusion process is to zero order in ε, equal to a product measure. Therefore, to prove Theorems 1 and 2 it is enough to consider $\mu^\varepsilon(\sigma[\varepsilon^{-1}r], t)$, i.e., the expectation of the spin value at time t at the "macroscopic point r." After showing tightness we prove that any limit point of the sequence $\mu^\varepsilon(\sigma([\varepsilon^{-1}r], t))$ satisfies Eq. (2.22) and, therefore, convergence is guaranteed by the uniqueness of the solution of Eq. (2.22); see Smoller.[50]

The proof of Theorems 3 and 4 is also based on Proposition 3.4. In fact, to study fluctuations around the deterministic limit we need to estimate the ε^d-correction. We did not compute explicitly this correction, but it can be shown that it is given by the regular part $\tilde{c}_t$ of the equal time covariance of the limit fluctuation field [see Eq. (2.19)]. This has been proven in Ref. 15 for the exclusion process (see also De Masi *et al.*,[16], Sec. 7).

Finally we mention that the proof of Theorem 3 uses the Holley–Stroock[28] theory of generalized Ornstein–Uhlenbeck processes via martingales. That is, we first show tightness for the process $Y_t^\varepsilon(\phi)$ [see Eq. (2.13)] and then, to guarantee uniqueness, we prove that any limit point satisfies the martingale condition stated in Eq. (2.14). To show this fact, we need Theorem 4. The same techniques are used in Refs. 17, 20, and 52; see also Ref. 16, Sec. 6.

3. DUALITY. THE BRANCHING-EXCLUSION PROCESS

In this section we consider the one-dimensional G.E. process defined via the generator L_ε given in Eq. (2.33). We begin by making precise the duality relation between the G.E. and the branching process described in the last section.

Definition 3.1.

(a) The Branching-Exclusion Process (BEP)

The BEP is a stochastic process with state space $\tilde{\mathcal{X}} = N^{\mathbb{Z}}$. At $t=0$ n particles start their motion from the position

$$x_i \in \mathbb{Z}, \qquad i = 1,..., n,\ x_i < x_j \qquad \text{for} \quad i < j \tag{3.1}$$

Each particle (independently of the others) has a clock[8] (a Poisson process) of rate λ where λ is defined in Eq. (2.33). We call this Poisson process a Glauber clock and label each event time of the Glauber clock by a Glauber mark. A Glauber mark can be one of three kinds: it is of type $\beta \in \{1, 2, 3\}$ with probability λ_β/λ, where λ_β, $\beta \in \{1, 2, 3\}$, is defined in Eq. (2.33).

Each particle moves on the lattice by simple exclusion (stirring) with rate $\varepsilon^{-2}/2$. When a Glauber mark appears the (corresponding) particle creates two new particles in its nearest-neighbor sites. A new (independent) Glauber clock is attached to each of these new particles, which they will keep along their motion. If a particle is created at a site already occupied, the new particle has the same trajectory and the same Glauber clock as the old particle or particles. Therefore, when a Glauber mark appears on this "joint" Glauber clock, the two particles created at the nearest-neighbor sites are descendants of *all* the parent particles. We can have any number of particles on a given site. We call "first class" (respectively "second class") particles those particles which start their motion from an empty site (respectively from a site already occupied) and say that there is a Glauber interaction if there is at least one second-class particle. In what follows, the first- and second-class particles are counted separately even though they move together.

We now define, for any $t \geqslant 0$,

$$\mathcal{N}_t, \text{ total number of particles at time } t \tag{3.2a}$$

$$T_l, \text{ the time when the } l\text{th Glauber mark appears on the Glauber clock of } \textit{any} \text{ of the particles present at time } T_{l-1},\ T_0 = 0 \tag{3.2b}$$

$$\beta_l \in \{1, 2, 3\} \text{ is the type of the } l\text{th Glauber mark} \tag{3.2c}$$

$$\mathcal{M}_t = \tfrac{1}{2}(\mathcal{N}_t - n), \text{ total number of Glauber marks up to time } t \tag{3.2d}$$

Finally

$$\mathcal{T}_t = \{T_l,\ l = 1, \ldots, \mathcal{M}_t\} \tag{3.2e}$$

$$\beta_t = \{\beta_l,\ l = 1, \ldots, \mathcal{M}_t\} \tag{3.2f}$$

(b) The Labeled Branching-Exclusion Process (LBEP)

To establish the duality relationship, we shall label the particles. We choose the following prescription.

Given an initial configuration as in Eq. (3.1), we label with i the particle that starts its motion at the site $x_i \in \mathbb{Z}$, and we call Z_n the vector $Z_n = (x_1, \ldots, x_n)$.

[8] We proceed this way for convenience. It is easy to see that the process constructed with the clocks on the sites, described in the last section, is equivalent to the one constructed with the clock on the particles.

We label $n+1$ and $n+2$ the two particles created at time T_1 [see Eq. (3.2)]: $n+1$ (resp. $n+2$) is the label of the particle that starts its motion at the right (resp. left) of the parent particle. We label $n+3$ and $n+4$ the particles created at time T_2 ans so on. We let

$$x_k(t) = \text{position at time } t \text{ of the } k\text{th particle} \tag{3.3a}$$

and

$$Z(t) = (x_1(t),..., x_{\mathcal{N}_t}(t)) \tag{3.3b}$$

We call the particles with label 1,..., n first generation and, $\forall i \in \{1,..., n\}$, the descendents of the ith particle of the i-family. *Note* that the descendents of second-class particles may belong to more than one family. For any family $i \in \{1,..., n\}$ and any $t \geqslant 0$ we define

$$\mathcal{N}_t(i) = \text{total number of particles of the } i\text{th family at time } t \tag{3.4a}$$

$$\mathcal{M}_t(i) = \text{total number of Glauber marks up to time } t \text{ referring to the particles of the } i\text{th family} \tag{3.4b}$$

$$\mathcal{T}_t(\mathrm{i}) = \{T_l(i),\ l = 1,..., \mathcal{M}_t(i)\} \text{ set of times when a Glauber mark appears referring to the particles of the } i\text{th family} \tag{3.4c}$$

$$\beta_t(i) = \{\beta_l(i),\ l = 1,..., \mathcal{M}_t(i)\} \text{ set of types of the Glauber marks of the } i\text{th family} \tag{3.4d}$$

$$I_t(i) = \text{set of labels of the particles of the } i\text{th family} \tag{3.4e}$$

$$Z(i, t) = \{x_\alpha(t),\ \alpha \in I_t(i)\} \tag{3.4f}$$

Observe that

$$|I_t(i)| = \mathcal{N}_t(i) \tag{3.4g}$$

and

$$\mathcal{N}_t \leqslant \sum_{i=1}^{n} \mathcal{N}_t(i) \tag{3.4h}$$

In Eq. (3.4h) the equality holds if and only if there are no descendents of second-class particles.

We need to define $\forall t \geqslant 0$ a random variable $\mathcal{I}_t = \{\mathcal{I}_t(i),\ i = 1,..., n\}$ in such a way that we can read off from $\mathcal{I}_t$ the branching history up to time t of all the particles of the ith family for any family $i \in \{1,..., n\}$. We avoid giving a precise definition of $\mathcal{I}_t$ and we refer to Harris,[26] Chapter 6, for a possible prescription. In any case, $\mathcal{I}_t$ determines also $I_t(i)\ \forall i \in \{1,..., n\}$. Observe that if $\mathcal{N}_t$, $\mathcal{T}_t$ and $\{Z(s),\ 0 \leqslant s \leqslant t\}$ are given, then $\mathcal{I}_t$ is known. On the other hand, $\mathcal{N}_t$, $\mathcal{T}_t$, and $Z(t)$ do not specify $\mathcal{T}_t$.

Finally, for any $t \geqslant 0$ we define

$$\mathscr{A}_t = (\mathscr{N}_t, \mathscr{T}_t, \beta_t, \mathscr{I}_t) \tag{3.5a}$$

and for each family

$$\mathscr{A}_t(i) = (\mathscr{N}_t(i), \mathscr{T}_t(i), \beta_t(i), \mathscr{I}_t(i)) \tag{3.5b}$$

Therefore the LBEP is a process $(Z(t), \mathscr{A}_t)_{t \geqslant 0}$, where $Z(t) \subset \bigcup_{l \geqslant 1} \mathbb{Z}^l$ is defined in Eq. (3.3) and $\mathscr{A}_t$ is given in Eq. (3.5a).

We will denote by

$P^{\varepsilon}_{Z_n}$ (or simply P_{ε} if no confusion arises) the law of the LBEP (3.6a) and by

$$\begin{aligned} &\mathbb{E}^{\varepsilon}_{Z_n}(\cdot) \text{ (or simply } \mathbb{E}_{\varepsilon}(\cdot)) \text{ the expectation with respect} \\ &\quad \text{to the measure } P^{\varepsilon}_{Z_n} \end{aligned} \tag{3.6b}$$

(c) Spin Variables Associated with the LBEP

Let $Z_n = (x_1, ..., x_n)$, $t \geqslant 0$, and $\sigma_0 \in \{-1, 1\}^{\mathbb{Z}}$ be fixed.

We associate to each particle existing at time s, $0 \leqslant s \leqslant t$, a spin according to the following rule. We begin by giving the spins at time t: all particles of $Z(t)$ which are at x take spin value $\sigma_0(x)$, for any $x \in \mathbb{Z}$. Spins can change only at the times T_l when Glauber marks appear. Assume that we know the values of the spins of all the particles at time "$T_l + 0$" and that particles which are sitting at the same site have the same spin. Then the value of the spins at time "$T_l - 0$" are specified as follows. Let k be the label of the first-class particle for which the Glauber clock rings at time T_l. Then the spin of the kth particle changes at T_l [and with it that of all the second-class particles sitting together the kth one] iff the spins of the particles created at T_l agree with the type β_2 of the Glauber mark. All the other particles keep their spin values [notice, however, that at "$T_l - 0$" there are two particles less than at time "$T_l + 0$," those generated by the T_l mark]. In this way the value of the spins of all the particles are specified in the whole time interval $[0, t]$. Notice that, given $\mathscr{A}_t$, the actual trajectory of the particles of the LBEP enter in the specification of the spins only through the values $Z(t)$ to choose among the spins of σ_0. For the rest, in fact $\mathscr{A}_t$ completely specifies the descendent relationship among particles and which ones are involved at each Glauber mark.

Finally we write

$$\sigma_0(Z(i, t)) = \{\sigma_0(x_\alpha(t)) : \forall \alpha \in I_t(i)\} \in \{-1, 1\}^{\mathscr{N}_t(i)} \tag{3.7}$$

and we let $h_i \equiv h_i(\mathscr{A}_t(i), \sigma_0(Z(i, t)), i \in \{1, ..., n\}$ be the spin value of the particles of the first generation at time 0.

Theorem 3.1. *(Duality).* For any $t \geqslant 0$, $Z_n = (x_1, ..., x_n) \in \mathbb{Z}^n$, and $\sigma \in \mathscr{X}$, there exists a joint probabilistic version of the processes $(\mathscr{A}_t, Z(\cdot, t))$ and $\sigma(\cdot, t)$ with initial configuration Z_n and σ respectively, such that

$$\prod_{i=1}^{n} \sigma(x_i, t) = \prod_{i=1}^{n} h_i(\mathscr{A}_t(i), \sigma(Z(i, t))), \qquad \mathbb{P}^\varepsilon \text{ a.s.} \tag{3.8}$$

where $\mathbb{P}^\varepsilon$ is the law of the joint probabilistic version.

Proof. The proof follows from the joint construction of the G.E. process $\{\sigma_s, s \geqslant 0\}$ and $\{(Z(s), \mathscr{A}_s), s \geqslant 0\}$ on a common probability space $(\Omega, \mathbb{P}^\varepsilon)$. The space $(\Omega, \mathbb{P}^\varepsilon)$ is the direct product of $\prod_{x \in \mathbb{Z}} (\Omega_{x,x+1}, P^\varepsilon_{x,x+1})$ and $\prod_{z \in \mathbb{Z}} (\Omega^\lambda_z, P^\lambda_z)$, where $(\Omega_{x,x+1}, P^\varepsilon_{x,x+1})$ is a Poisson process with rate $\varepsilon^{-2}/2$ for each $x \in \mathbb{Z}$, and $(\Omega^\lambda_z, P^\lambda_z)$ is the direct product of the three Poisson processes with rate λ_β, $\beta \in \{1, 2, 3\}$ defined in Eq. (2.33). We write $\omega \in \Omega$ and $\omega^\lambda_z \in \Omega^\lambda_z$. We only consider $\omega \in \Omega_0 \subset \Omega$, where Ω_0 is the set for which no more than one mark is present at any given time. Since $\mathbb{P}^\varepsilon(\Omega_0) = 1$, with a little abuse of notation we write $\Omega = \Omega_0$.

As already described informally in Sec. 2, given an $\omega \in \Omega$ a trajectory of the G.E. process $\{\sigma_s, 0 \leqslant s \leqslant t\}$ is given by the following prescription. At each site $x \in \mathbb{Z}$ there is a particle with label x. The x-particle follows the bondmarks of ω, carrying along its motion the clock $\omega^\lambda_x \in \omega$. The particles of nearest-neighbor (n.n.) sites exchange their positions at time s if and only if a mark ω occurs on the bond between them at time s. Furthermore, the x-particle flips its spin at time s if and only if at that time the ω^λ_x mark of type β, $\beta \in \{1, 2, 3\}$ appears and the value of its spin and of its nearest neighbor particles at time s is that prescribed by the set $A_\beta(z(s; x))[z(s; x)$ is the position at time s of the x particle$]$.

Let $t \geqslant 0$, $x_1, ..., x_n \in \mathbb{Z}$ be fixed. Any given $\omega \in \Omega$ determines a path of the LEBP starting from $Z_n = (x_1, ..., x_n)$ in the time interval $[0, t]$. Consider that $\tilde{\omega}$, $\tilde{\omega}$ is obtained from ω by restricting ω to $[0, t]$ and reversing the time from t to zero, then use the prescription given in Definition 3.1. (Time reversal is the same as reflection $s \in [0, t] \to \tau = (t - s) \in [0, t]$.) We now observe that in the G.E. process, the value of the spin at site x_i at time t, $\sigma(x_i; t)$, depends on the value of the spins at time zero in the series $x_\alpha(t)$, $\alpha \in I_t(i)$ (see Definition 3.1) and on the "branching history" $\mathscr{A}_t(i)$. This dependence is just that given by the function h_i defined in Definition 3.1c). ∎

Remark 3.2. (i) The duality relation (3.8) is different from the one frequently used. Equation (3.8) implies the following: There exists a function $H(\sigma; Z_n) \equiv \prod_{i=1}^n \sigma(x_i)$ in $\mathscr{X} \times \hat{\mathscr{X}}$, where $\hat{\mathscr{X}}$ is the set of all subsets

of $\mathbb{Z}$, and a function $H(\mathscr{A}, Z_n(t), \sigma)$ in $\Omega \times \mathscr{X}$, where Ω is the *path* space of the LBEP such that

$$\mathbb{E}_{\sigma_0}(H(\sigma_t; Z_n)) = \mathbb{E}_{Z_n}(H(\mathscr{A}_t, Z_n(t), \sigma_0))$$

The difference from the usual definition of duality is that in Eq. (3.2) of Definition 3.1, Chapter 2, of Liggett[37] the function $\hat{H}$ is still a function on $\hat{\mathscr{X}} \times \mathscr{X}$; it is in fact equal to H. Our duality is, therefore, more difficult to deal with. We do not know if it is possible to define some "dual process" in such a way as to obtain the usual duality relationship with the G.E. process.

Independent Branching Process

As noted in Sec. 2c, it is quite natural to compare the BEP with an independent branching process, that is, a process in which the particles move as in a random walk and create particles independently of the others. Proposition 3.4 below states that to zero order in ε the two processes are the same; the correction is of order ε.

Definition 3.3. We define directly the *labeled independent branching process* (LIBP), the definition of the IBP being easily recovered.

For any integer $n \geqslant 1$ and vector $Z_n = (x_1, ..., x_n) \subset \mathbb{Z}$ as in Eq. (3.1), the independent i-particle starts its motion at site $x_1 \in \mathbb{Z}$, $i = 1, ..., n$. Each particle has a Glauber clock of rate λ and the Glauber marks can be of three different species as in Definition 3.1. Each particle moves on the lattice via a random walk with rate $\varepsilon^{-2}/2$. When a Glauber mark appears, the (corresponding) particle creates two new particles in its n.n. sites. These new particles carry along their motion a Glauber clock and so on. The labeling is given with the same prescription as in Definition 3.1.

We call the independent i-family the LIBP starting with the i-particle at time zero and we denote $\forall t \geqslant 0$ by

$$x_k^0(t) = \text{position at time } t \text{ of the independent } k\text{-particle} \tag{3.9a}$$

and

$$Z_n^0(t) = \{x_k^0(t), k = 1, ..., \mathscr{N}_t^0\} \tag{3.9b}$$

We define

$$\mathscr{A}_t^0 = (\mathscr{N}_t^0, \mathscr{T}_t^0, \beta_t^0, \mathscr{I}_t^0) \quad \text{and} \quad \mathscr{A}_t^0(i) \tag{3.9c}$$

in the same way as in Eqs. (3.5). Note, however, that in the LIBP all particles are first-class particles. Finally, using the same rules as in Definition 3.1c, we associate a spin variable h_i^0, $i \in \{1, ..., n\}$ to each particle of the first generation.

In the Appendix we will prove the following proposition.

Proposition 3.4. Let $\{\mu^\varepsilon\}$, $\varepsilon \in (0, 1]$ be a family of (initial) measures on $\mathscr{X}$ defined as in Theorem 2.1. Let $t \geqslant 0$ and $n \geqslant 1$ be fixed. Then there exists a constant $c > 0$ such that the following holds. For any $x_1, \ldots, x_n$ distinct integers and $\varepsilon \in (0, 1]$, there are two families of numbers $\gamma^\varepsilon_{i,j}$, $i \neq j$, $i, j \in \{1, \ldots, n\}$ and γ^ε_i, $i \in \{1, \ldots, n\}$ such that

$$\mathbb{E}_{\mu^\varepsilon}\left(\prod_{i=1}^n \sigma(x_i, t)\right) = \prod_{i=1}^n \mathbb{E}^0_\varepsilon(\sigma(x_i, t)) + \varepsilon \sum_{i=1}^n \gamma^\varepsilon_i \prod_{k \neq i} \mathbb{E}^0_\varepsilon(\sigma(x_k, t)) + \varepsilon \sum_{\substack{i,j=1 \\ i \neq j}} \gamma^\varepsilon_{i,j} \prod_{k \neq i,j} \mathbb{E}^0(\sigma(x_k, t)) + r_\varepsilon \tag{3.10}$$

with

$$\sup_i |\gamma^\varepsilon_i| \leqslant c \qquad \text{and} \qquad \sup_{i,j} |\gamma^\varepsilon_{i,j}| \leqslant c \tag{3.11a}$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} |r_\varepsilon| = 0 \tag{3.11b}$$

In Eq. (3.10) $\mathbb{E}^0_\varepsilon$ denotes the expectation of the r.h.s. of Eq. (3.8) with respect to the law of the IBP.

The proof of Eq. (3.10) is based on a generalization of the coupling techniques and the probability estimates for the exclusion process given in De Masi *et al.*[16] and in Ferrari *et al.*[21].

Proposition 3.4 implies that to zero order in ε the distribution at time t of the G.E. process factorizes, while to first order in ε both a one-body and two-body interaction is present. In the next lemma we state and then prove this property.

Lemma 3.5. Let $\{\mu^\varepsilon\}$ be a family of measures on $\mathscr{X}$ satisfying Definition 2.2. Let $t \geqslant 0$ be fixed. Then the following holds:

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant s \leqslant t} \sup_{x_1, x_2 \in \mathbb{Z}} |\varepsilon^{-1} \mathbb{E}_{\mu^\varepsilon}((\sigma(x_1, t) - p_1)(\sigma(x_2, t) - p_2)| < \infty \tag{3.12a}$$

$$\lim_{\varepsilon \to 0} \sup_{\substack{x_1, \ldots, x_n \in \mathbb{Z} \\ x_i \neq x_j}} \varepsilon^{-1} \left| \mathbb{E}_{\mu^\varepsilon}\left(\prod_{i=1}^n (\sigma(x_i, t) - p_i)\right)\right| = 0, \qquad \forall n \geqslant 3 \tag{3.12b}$$

where

$$p_i = \mathbb{E}_{\mu^\varepsilon}(\sigma(x_i, t)) \tag{3.12c}$$

Proof. From Eq. (3.11a), it follows that

$$\mathbb{E}_{\mu^\varepsilon}\left(\prod_{i=1}^n (\sigma(x_i, t)) - p_i\right) = \prod_{i=1}^{n-1} (p^0_i - p_i) + \varepsilon \sum_{i=1}^n \gamma^\varepsilon_i \prod_{k \neq i}^n (p^0_k - p_k) + \varepsilon \sum_{i,j=1}^n \gamma^\varepsilon_{i,j} \prod_{k \neq i,j} (p^0_k - p_k) + r_\varepsilon \tag{3.13a}$$

where

$$p_i^0 = \mathbb{E}_\varepsilon^0(\sigma(x_i, t)) \tag{3.13b}$$

For $n=2$ we apply Eq. (3.10) twice to $(p_1^0 - p_1)$ and $(p_2^0 - p_2)$; the first two terms in the r.h.s. of Eq. (3.13) give a number r'_ε which by Eq. (3.11a) verifies $|r'_\varepsilon| \leqslant \varepsilon^2$, while the third term in (3.13a) gives $\gamma_{1,2}^\varepsilon$. Equation (3.12a) then follows from Eq. (3.11a). For $n \geqslant 3$ we apply Eq. (3.10) three times and we obtain a number r'_ε which verifies $|r'_\varepsilon| \leqslant \varepsilon^2$ and so Eq. (3.12b) follows. ∎

Remark 3.6. The proof of Theorem 3.1 does not use the particular form of the rates $c(x, \sigma)$ or even the fact that our example is one dimensional: for any finite-range function $c(x, \sigma)$ the construction of the BEP can be done in the same way as in Definition 3.1.

4. PROOF OF THE THEOREMS

We prove Theorems 1–4 for the one-dimensional example, but the proofs can be easily extended to the general case. All the proofs follow easily from Lemma 3.5.

Proof of Theorem 1. From Eq. (3.12) it follows that

$$\lim_{\varepsilon \to 0} \left| \mathbb{E}_{\mu^\varepsilon} \left(\prod_{i=1}^{n} (\sigma(x_i, t)) \right) - \prod_{i=1}^{n} \mathbb{E}_{\mu^\varepsilon}(\sigma(x_i, t)) \right| = 0 \tag{4.1}$$

Therefore to prove Eq. (2.8) it is enough to show that $\forall r \geqslant 0$,

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon x \to r}} \mathbb{E}_{\mu^\varepsilon}(\sigma(x, t)) = m(r, t) \tag{4.2a}$$

where $m(r, t)$, $r \in \mathbb{R}$, $t \geqslant 0$ is the unique solution of the following differential equation:

$$\left.\begin{aligned} \frac{\partial m}{\partial t} &= \frac{1}{2} \frac{\partial^2 m}{\partial r^2} + 2(2\gamma - 1)\, m - 2\gamma^2 m^3 \\ m(r, 0) &= m_0(r) \end{aligned}\right\} \tag{4.2b}$$

To prove (4.2), we first show that $\{\mathbb{E}_{\mu^\varepsilon}(\sigma([\varepsilon^{-1} r], t),\ r \in \mathbb{R},\ t \geqslant 0\}$ ($[q]$ denotes the integer part of q) is an equicontinuous family of functions and therefore it has a limit by subsequences. Afterwards, we will show that every limiting point satisfies Eq. (4.2b) and so uniqueness follows.

Proof of Equicontinuity. We have to show that $\forall \eta > 0$ there exists $\delta > 0$ such that $\forall \xi \in \mathbb{R}$, $\forall t' \geqslant 0$ with $|\xi| \leqslant \delta$, $t' < \delta$ the following holds:

$$\sup_{\varepsilon} \sup_{r \in \mathbb{R}} \sup_{t \geqslant 0} |\mathbb{E}_{\mu^\varepsilon}(\sigma([\varepsilon^{-1} r], t)) - \mathbb{E}_{\mu^\varepsilon}(\sigma[\varepsilon^{-1}(r+\xi)], t+t')| < \eta \quad (4.3)$$

We shall use duality. We consider two particles with labels 1 and 2. They are independent and each one moves according to the BEP as described in Definition 3.1. Particle 1 starts *at time 0* from the position $[\varepsilon^{-1}(r+\xi)]$ while particle 2 starts *at time* t' from $[\varepsilon^{-1} r]$. We denote by $x_1(s)$, $x_2(s)$ the positions at time s of the 1-particle and the 2-particle respectively, letting

$$x_2(s) = [\varepsilon^{-1} r] \qquad \forall s \leqslant t'$$

We shall prove that

$$|\mathbb{E}_{\mu^\varepsilon}(\sigma([\varepsilon^{-1} r], t)) - \mathbb{E}_{\mu^\varepsilon}(\sigma([\varepsilon^{-1}(r+\xi)], t+t'))| \leqslant c_1 t' + c_2(\delta) + (c_3 t^{-1/2} + c_4)(|\xi| + (t')^{1/4}) \quad (4.4a)$$

and also that

$$\text{l.h.s. of Eq. (4.4a)} \leqslant c_1(t+t') + c_2(\delta) + c_5(t^{1/4} + (t')^{1/4} + |\xi|) \quad (4.4b)$$

$$\lim_{\delta \to 0} c_2(\delta) = 0 \quad (4.4c)$$

The first inequality will be applied for "large" t, the second one for "small" t, and together they imply equicontinuity. Hence we are left with the proofs of Eqs. (4.4).

We first reduce ourselves to the equal time case, $t' = 0$. Let T_1 [resp. T_2] be the time when the first Glauber mark appears in the history of particle 1 [resp. 2]; then there exists c_1 so that

$$P(T_1 \leqslant t') \leqslant \tfrac{1}{2} c_1 t' \quad (4.5)$$

For $k > 0$, let

$$A_1(t', \varepsilon, k) = \{|x_1(t') - \varepsilon^{-1}(r+\xi)| \leqslant k\varepsilon^{-1}\sqrt{t'}\} \quad (4.6a)$$

then there exists $\phi(k)$ such that

$$\lim_{k \to \infty} \phi(k) = 0 \quad (4.6b)$$

and

$$P(A_1(t', \varepsilon, k)) \geqslant 1 - \phi(k) \quad (4.6c)$$

By duality we then have

$$\text{l.h.s. of Eq. (4.4a)} \leqslant |\mathbb{E}_\varepsilon(\mathbb{1}(\{T_1 \geqslant t', A_1(t', \varepsilon, k)\})[\mathbb{E}_{\mu^\varepsilon}(\sigma(x_2(t'), t) - \sigma(x_1(t'), t))])| + c_1 t' + 2\phi(k) \tag{4.7}$$

We first consider the case when t is "large." The estimate is based on the fact that if δ is small enough then with large probability $x_1(s)$ and $x_2(s)$ will meet each other before $t+t'$ and before the appearance of a Glauber mark. Such events do not contribute to the r.h.s. of Eq. (4.7). We therefore introduce

$$s_0 = \inf\{s \geqslant t' : x_1(s) = x_2(s)\} \tag{4.8a}$$

$$T = \min(T_1, T_2) \tag{4.8b}$$

$$P(s_0 \geqslant t' + t, A_1(t', \varepsilon, k)) \leqslant c_3[(|\varepsilon^{-1}\xi + \varepsilon^{-1}k\sqrt{t'}|)/\sqrt{\varepsilon^{-2}t}] \tag{4.8c}$$

$$P(s_0 \geqslant T, A_1(t', \varepsilon, k)) \leqslant c_4 2\lambda \int_0^\infty ds\, e^{-2\lambda s}[(|\xi + k\sqrt{t'}|)/\sqrt{s}] \tag{4.8d}$$

Therefore

$$\text{l.h.s. of Eq. (4.4a)} \leqslant (c_1 t' + 2\phi(k)) + c_3[(|\xi| + k\sqrt{t'})/\sqrt{t}] + c_4(|\xi| + k\sqrt{t'}) \tag{4.9}$$

By choosing $k = (t')^{-1/4}$ and setting $c_2(a) = 2\phi(a^{-1/2})$, we obtain Eq. (4.4a).

For t small we shall use the fact that both $x_1(s)$ and $x_2(s)$ do not move "too much" and that the initial datum is smooth. We set

$$A_1(t+t', \varepsilon, k) = \{|x_1(t+t') - \varepsilon^{-1}(r+\xi)| \leqslant \varepsilon^{-1}k\sqrt{t+t'}\} \tag{4.10a}$$

$$A_2(t+t', \varepsilon, k) = \{|x_2(t+t') - \varepsilon^{-1}r| \leqslant \varepsilon^{-1}k\sqrt{t} \tag{4.10b}$$

and as before

$$P(A_1(t+t', \varepsilon, k) \cap A_2(t+t', \varepsilon, k)) \geqslant 1 - 2\phi(k) \tag{4.11a}$$

$$P(T \leqslant t+t') \leqslant c_1(t+t') \tag{4.11b}$$

$$\text{l.h.s. of Eq. (4.4a)} \leqslant c_1(t+t') + 2\phi(k) + \sup|m_0(\varepsilon a) - m_0(\varepsilon b)| \tag{4.12a}$$

where the sup is taken with respect to all a and b such that

$$|a-b| \leqslant \varepsilon^{-1}(|\xi| + k\sqrt{t+t'} + k\sqrt{t})$$

Equation (4.12a) gives Eq. (4.4b) because dm_0/dr is uniformly bounded and we now choose

$$k = \max(t^{-1/4}, (t')^{-1/4}) \tag{4.12b}$$

Remark. The definition of s_0 in $d>1$ is a bit more complicated. We construct a coupling $(\underline{X}(t), \underline{Y}(t))$ on $(\mathbb{Z}^d)^2$ in such a way that each marginal is a continuous time random walk on $\mathbb{Z}^d$. If we define $s_0^i = \inf\{t: x_i(y) = y_i(t)\}$, $i=1,\ldots, d$, the coupling is constructed in such a way that $x_i(t)$ and $y_i(t)$ are independent for $t \leqslant s_0^i$ and $x_i(t) \equiv y_i(t)$ for $t \geqslant s_0^i$. Then we define $s_0 = \max_{i\in\{1,\ldots,d\}} s_0^i$. Thus, after s_0, $\underline{X}(t) = \underline{Y}(t)$ and it is easy to see that, as in one dimension, s_0 satisfies Eq. (4.8).

Proof of Uniqueness. Let $m(r, t)$ be a limiting point of $\mathbb{E}_{\mu^\varepsilon}(\sigma([\varepsilon^{-1}r], t))$. We will show that $m(r, t)$ satisfies the integral equation (2.23). It is well known (see, for, instance Cannon[10]) that there is a unique solution of Eq. (2.23) which is equal to the unique solution of the reaction-diffusion equation (4.2b). From this uniqueness will follow.

To prove that any limiting point of $\mathbb{E}_{\mu^\varepsilon}(\sigma([\varepsilon^{-1}r], t))$ satisfies Eq. (2.23), we use the following identity. For any continuous function f on $\mathcal{X}$

$$S_t^\varepsilon f(\sigma) = T_t^\varepsilon f(\sigma) + \int_0^t ds\, S_{t-s}^\varepsilon L_G(T_s^\varepsilon f(\sigma)) \tag{4.13}$$

where T_t^ε is the semigroup with generator $\varepsilon^{-2}L_E$ and S_t^ε is the one with generator $L_G + \varepsilon^{-2}L_E$. To check Eq. (4.13) it is enough to consider functions f in a core D common to all the generators, L_G, $\varepsilon^{-2}L_E$ and $L_G + \varepsilon^{-2}L_E$. It is not difficult to see that such a core D does exist (see Liggett,[37] Chapter 1, Sec. 3). For f in D, Eq. (4.13) follows from the integration-by part formula.

We use Eq. (4.13) with $f(\sigma) = \sigma([\varepsilon^{-1}r])$ and, observing that $\forall z \in \mathbb{Z}$

$$T_s^\varepsilon \sigma(z) = \sum_x G_s^\varepsilon(z, x)\, \sigma(x) \tag{4.14a}$$

we obtain (see the definition of L_G)

$$\begin{aligned}\mathbb{E}_{\mu^\varepsilon}(\sigma(\varepsilon^{-1}r]; t) = &\sum_{x\in\mathbb{Z}} G_t^\varepsilon([\varepsilon^{-1}r], x)\, m_0(\varepsilon x) + \int_0^t ds \sum_{x\in\mathbb{Z}} G_s^\varepsilon([\varepsilon^{-1}r], x) \\ &\times [-2\mathbb{E}_{\mu^\varepsilon}(\sigma(x; t-s) + 2\gamma(\mathbb{E}_{\mu^\varepsilon}(\sigma(x-1; t-s)) \\ &+ \mathbb{E}_{\mu^\varepsilon}(\sigma(x+1; t-s)) \\ &- 2\gamma^2\mathbb{E}_{\mu^\varepsilon}(\sigma(x-1; t-s)\, \sigma(x+1; t-s)\, \sigma(x; t-s))] \end{aligned} \tag{4.14b}$$

Here $G_s^\varepsilon([\varepsilon^{-1}r], x)$ is the probability that a random walker starting from $[\varepsilon^{-1}r]$ and moving with rate ε^{-2} is at x at time s.

From classical estimates on the sum of independent random variables (see, for instance, Petrov[44]) we have $\forall z \in \mathbb{Z}$

$$\lim_{\varepsilon \to 0} \sum_{y} \left| G_t^{\varepsilon}(z, y) - \frac{\varepsilon}{\sqrt{\pi t}} \exp\left\{ - \frac{(\varepsilon y)^2}{2t} \right\} \right| = 0 \tag{4.15}$$

From Eqs. (4.14) and (4.15) and the Lebesgue dominant convergence theorem, we obtain the first term in the r.h.s. of Eq. (2.23).

From Lemma 3.5 [see also Eq. (4.1)], Eq. (4.14), and the equicontinuity, it follows that along subsequences

$$\lim_{\varepsilon \to 0} |\mathbb{E}_{\mu^{\varepsilon}}(L_{\mathrm{G}} \sigma(x, s)) - 2(2\gamma - 1)\, m(\varepsilon x, s) - 2\gamma^2 m(\varepsilon x, s)^3]| = 0 \tag{4.16}$$

From Eqs. (4.15), (4.16), and the Lebesgue dominant convergence theorem, we have that the expression in the r.h.s. of Eq. (4.14) converges along subsequences (as $\varepsilon \to 0$) to the second term on the r.h.s. of Eq. (2.23). Therefore Theorem 1 follows. ∎

Remark 4.1. For any $\lambda > 0$, let $T_t^{\varepsilon, \lambda}$ be the semigroup with generator $\varepsilon^{-2} L_{\mathrm{E}} + \lambda$ i.e., $T_t^{\varepsilon, \lambda} = \exp(-\lambda t)\, T_t^{\varepsilon}$. Then the same argument as before shows that

$$S_t^{\varepsilon} f(\sigma) = e^{-\lambda t}\, T_t^{\varepsilon} f(\sigma) + \int_0^t ds\, e^{-\lambda s}\, S_{t-s}^{\varepsilon}(L_{\mathrm{G}} + \lambda)(T_s^{\varepsilon} f(\sigma)) \tag{4.17a}$$

Therefore for $f(\sigma) = \sigma([\varepsilon^{-1} r])$ we obtain

$$\begin{aligned} \mathbb{E}_{\mu^{\varepsilon}}(\sigma([\varepsilon^{-1} r], t)) = {} & e^{-\lambda t} \sum_{x \in \mathbb{Z}} G_t^{\varepsilon}([\varepsilon^{-1} r], x)\, m_0(\varepsilon x) \\ & + \int_0^t ds \sum_{x \in \mathbb{Z}} G_s^{\varepsilon}([\varepsilon^{-1} r], x)\, e^{-\lambda s}\, \mathbb{E}_{\mu^{\varepsilon}}(L_{\mathrm{G}} \sigma(x, t-s) \\ & + \lambda \sigma(x, t-s)) \end{aligned} \tag{4.17b}$$

If we choose λ as in Eq. (2.33c), eq. (4.17b) can be interpreted in the following way. A particle with label 1 starts at time 0 from position $[\varepsilon^{-1} r]$ and moves according to the BEP described in Definition 3.1. We denote by $x(s)$ its position at time s. Let T_1 be the time when the first Glauber mark appears: T_1 is distributed according to a Poisson measure of parameter λ. Therefore $e^{-\lambda t} = P(T_1 > t)$ and since in the set $\{T_1 > t\}$ the particle moves up to time t according to a continuous time random walk of intensity ε^{-2}, we easily get the first term in the r.h.s. of Eq. (4.17b). The second term in the r.h.s. of Eq. (4.17b) is obtained by looking at the event $\{T_1 \leqslant t\}$. In

fact, at time T_1 the spin value of the particle changes [does not change] according to both the type of the Glauber mark and the spin values of the particles created in its n.n. sites.

Finally we observe that from Eq. (4.17b) we get that any limiting point $m(r, t)$ of $\mathbb{E}_{\mu^\varepsilon}(\sigma([\varepsilon^{-1}r]; t)$ satisfies the following integral equation:

$$m(r, t) = e^{-\lambda t}\left[\int dq\, G(q-r, t)\, m_0(q) + \int_0^t ds \int dq\, G(q-r, t-s)\, e^{\lambda s} F_\lambda(m(q, s))\right] \tag{4.18a}$$

where $G(q, t)$ is defined in Eq. (2.24) and

$$F_\lambda(m) = [2(2\gamma - 1) + \lambda]\, m - 2\gamma^2 m^3 \tag{4.18b}$$

Obviously the solution of Eq. (4.18) is equal to that of Eq. (2.23).

Theorems 2 and 3 deal respectively with the study of the magnetization field defined by

$$X_t^\varepsilon(\phi) = \varepsilon \sum_x \phi(\varepsilon x)\, \sigma(x, t) \tag{4.19}$$

and of its fluctuation field defined by

$$Y_t^\varepsilon(\phi) = \varepsilon^{-1/2}(X_t^\varepsilon(\phi) - \mathbb{E}_{\mu^\varepsilon}(X_t(\phi)) \tag{4.20}$$

Both $\{X_t^\varepsilon(\cdot)\}$ and $\{Y_t^\varepsilon(\cdot)\}$ are considered as distribution-valued processes, i.e., in the Skorohod space $D([0, \infty), \mathscr{S}'(\mathbb{R}))$. To study these processes we use martingale techniques as presented in Holley and Stroock[28] (see also Metivier[38]) and widely used in the study of hydrodynamical properties of many-particle systems (see De Masi *et al.*, Chapters 4 and 6 and references therein). Even if these arguments are by now standard, we briefly sketch them here for the sake of clarity.

In the proofs of Theorems 2 and 3 we use the following criterion of tightness.

Theorem 4.1. *(Holley-Stroock,[28] Metivier,[38] and Mitoma[39]).* Let P^ε, $\varepsilon \in (0, 1]$, be a family of processes on $D([0, \infty), \mathscr{S}'(\mathbb{R}))$. Let $\{\xi_t(\phi)\}$ be the canonical coordinates on D. Assume that the following three conditions are verified.

(i) For every $\tau \geqslant 0$ and $\phi \in \mathscr{S}(\mathbb{R})$ there exists a constant $c(\tau, \phi) > 0$ such that

$$\sup_\varepsilon \sup_{0 \leqslant t \leqslant \tau} P^\varepsilon(\xi_t(\phi)^2) \leqslant c(\tau, \phi) \tag{4.21}$$

(ii) There are $\gamma_1^\varepsilon(\phi, \tau)$, $\gamma_2(\phi, \tau)$, $\phi \in \mathscr{S}(\mathbb{R})$, $\tau \geqslant 0$ such that

$$M(\phi, \tau) \equiv \xi_\tau(\phi) - \int_0^\tau ds\, \gamma_1^\varepsilon(\phi, s) \tag{4.22a}$$

$$N(\phi, \tau) \equiv M(\phi, \tau)^2 - \int_0^\tau ds\, \gamma_2^\varepsilon(\phi, s) \tag{4.22b}$$

are P^ε-martingales with respect to the canonical filtration $\{\mathscr{M}_\tau\}$, [i.e., $\mathscr{M}_\tau$ is the σ-algebra generated by $\{\xi_s(\psi), 0 \leqslant s \leqslant \tau, \psi \in \mathscr{S}(\mathbb{R})\}$]. Furthermore $\forall \tau \geqslant 0$, $\forall \phi \in \mathscr{S}(\mathbb{R})$ there are constants $c_1(\phi, \tau) > 0$, $c_2(\phi, \tau) > 0$ such that

$$\sup_\varepsilon \sup_{0 \leqslant t \leqslant \tau} P^\varepsilon(\gamma_i^\varepsilon(\phi, t)^2) \leqslant c_i(\tau, \phi), \qquad i = 1, 2 \tag{4.23}$$

(iii) For every $\phi \in \mathscr{S}(\mathbb{R})$, $\tau \geqslant 0$ there exists $\delta(\tau, \phi, \varepsilon)$ so that

$$\lim_{\varepsilon \to 0} \delta(\tau, \phi, \varepsilon) = 0$$

and

$$\lim_{\varepsilon \to 0} P^\varepsilon(\sup_{0 \leqslant t \leqslant \tau} |\xi_t(\phi) - \xi_{t^-}(\phi)| \geqslant \delta(t, \phi, \varepsilon)) = 0 \tag{4.24}$$

Then for every $\tau > 0$, P^ε is tight in $[0, \tau]$ and any limiting point P has support on $C([0, \infty), \mathscr{S}'(\mathbb{R}))$.

Proof of Theorem 2. From Theorem 1 it follows that the process $\{X_t^\varepsilon(\phi)\}$ converges as $\varepsilon \to 0$ to the "deterministic process" $\int dr\, \phi(r)\, m(r, t)$ on the finite distributions. In fact, by Theorem 1 and the Lebesgue dominant convergence theorem, we have

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu^\varepsilon}(X_t^\varepsilon(\phi)) = \int dr\, m(r, t)\, \phi(r) \tag{4.25a}$$

and

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu^\varepsilon}([X_t^\varepsilon(\phi) - \mathbb{E}_{\mu^\varepsilon}(X_t^\varepsilon(\phi))]^2) = 0 \tag{4.25b}$$

From Eqs. (4.25) it follows that $\forall n \geqslant 0$, $\forall \phi_1, \ldots, \phi_n \in \varphi(\mathbb{R})$ $\forall t_1, \ldots, t_n \geqslant 0$ and $\forall \zeta > 0$,

$$\lim_{\varepsilon \to 0} P^\varepsilon\left(\left\{\left| X_{t_i}^\varepsilon(\phi_i) - \int dr\, \phi_i(r)\, m(r, t_i)\right| < \zeta \ \forall i = 1, \ldots, n\right\}\right) = 1 \tag{4.26}$$

Therefore to obtain the proof of Theorem 2 we need to show tightness. This can be done using Theorem 4.1. The proof of (i)–(iii) of Theorem 4.1 is quite similar but much simpler than the one we shall give for the fluctuation field; therefore we omit the details. ▮

Proof of Theorem 3. We use the same argument as the one used in Ref. 20 and in Ref. 15, Sec. 6 to which we refer for the details.

Let $G \in C_0^\infty(\mathbb{R})$. For any $\tau \geqslant 0$ and $\phi \in \mathscr{S}(\mathbb{R})$ we have that

$$G(Y_\tau^\varepsilon(\phi)) - \int_0^\tau ds\, L_\varepsilon G(Y_s^\varepsilon(\phi)) \tag{4.27a}$$

is a P^ε-martingale. In Eq. (4.27) L_ε denotes the generator of the process. From easy computations it follows that

$$L_\varepsilon G(Y_\tau^\varepsilon(\phi)) = (L_\varepsilon Y_\tau^\varepsilon(\phi))\, G'(Y_\tau^\varepsilon(\phi)) + \frac{1}{2}\gamma_2^\varepsilon(\phi, \tau)\, G''(Y_\tau^\varepsilon(\phi)) + r_\varepsilon \tag{4.27b}$$

where

$$G' = \frac{dG}{dq}, \qquad G'' = \frac{d^2G}{dq^2} \tag{4.27c}$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} |r_\varepsilon| = 0 \tag{4.27d}$$

$$\gamma_2^2(\phi, \tau) = \varepsilon \sum_x \phi(\varepsilon x)^2\, 4c(x, \sigma_\tau) + \varepsilon \sum_x \frac{\varepsilon^{-2}}{2} (\phi(\varepsilon x) - \phi(\varepsilon(x+1))^2 \times (\sigma(x, \tau) - \sigma(x+1, \tau))^2 \tag{4.28}$$

Furthermore

$$L_\varepsilon Y_\tau^\varepsilon(\phi) = \gamma_1^\varepsilon(\phi, \tau) + r_1^\varepsilon \tag{4.29}$$

where

$$\gamma_1^\varepsilon(\phi, \tau) = 2(2\gamma - 1)\, Y_\tau^\varepsilon(\phi) - 2\gamma^2 Y_\tau^\varepsilon(\phi; f) + \tfrac{1}{2} Y_\tau^\varepsilon(\phi'') \tag{4.30}$$

$$f(\sigma) = \sigma(-1)\, \sigma(0)\, \sigma(1) - \mathbb{E}_{\mu^\varepsilon}(\sigma(-1)\, \sigma(0)\, \sigma(1)) \tag{4.31a}$$

$$Y_\tau^\varepsilon(\phi; f) = \sqrt{\varepsilon} \sum_x \phi(\varepsilon x)[\tau_{-x} f(\sigma_\tau) - \mathbb{E}_{\mu^\varepsilon}(\tau_{-x} f(\sigma_\tau))] \tag{4.31b}$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} |r_1^\varepsilon| = 0 \tag{4.31c}$$

We first show tightness; then we prove that if P is a limiting point of P^ε then P satisfies the martingale condition stated in Eqs. (2.14). Uniqueness follows from a general theorem proven in Holley and Stroock[28] (see also Stroock and Varadhan[54]).

Proof of Tightness. We use Theorem 4.1. Equation (4.21) follows from Eq. (3.12a). To prove (ii) we observe that from Eq. (4.27) with $G(q)=q$ and $G(q)=q^2$ we obtain Eqs. (4.22a) and (4.22b) respectively, the functions γ_1^ε and γ_2^ε being given in Eqs. (4.30) and (4.28) respectively. Equation (4.23) for γ_2 trivially holds since

$$\gamma_2^\varepsilon(\phi, \tau) \leqslant c \max\{\sup_r (1+r^2)\,|\phi(r)^2|, \sup_r (1+r^2)\,|\phi'(r)^2|\} \tag{4.32}$$

for a suitable $c>0$.

Equation (4.23) for γ_1 follows from Lemma 3.5.

Proof of Uniqueness. We need to prove that if P^ε converges weakly to P, then the term in Eq. (2.14a) is a P-martingale.

Therefore it is enough to show that

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu^\varepsilon}(|\gamma_2^\varepsilon(\phi, \tau) - \|B_\tau \phi\|^2|) = 0 \tag{4.33}$$

and $\forall 0 \leqslant \sigma \leqslant \tau$, for every bounded continuous function ψ_σ measurable with respect to the σ-algebra $\mathscr{M}_\sigma$ ($\mathscr{M}_\sigma$ is defined in Theorem 4.1),

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu^\varepsilon}\left(\psi\left\{\int_\sigma^\tau dt[-Y_t^\varepsilon(A_t\phi)+\gamma_1^\varepsilon(\phi, t)]\, G'(Y_t^\varepsilon(\phi))\right\}\right) = 0 \tag{4.34}$$

Equation (4.33) follows from Lemma 3.5 and Theorem 1, while Eq. (4.34) follows from Theorem 4 that we prove below (see Ref. 16, Sec. 6, pp. 249–250). ∎

Proof of Theorem 4. We will use the same argument (and also the same notations!) as in the proof of Theorem (0.4)′ in Sec. 2 of Ferrari *et al.*[20] We prove Theorem 4 for the correlation functions, i.e., for functions f_n given by

$$f_n(\sigma_t) = \left[\prod_{i=1}^n \sigma(x_i, t) - \mathbb{E}_{\mu^\varepsilon}\left(\prod_{i=1}^n \sigma(x_i, t)\right)\right],$$
$$x_1, \ldots, x_n \in \mathbb{Z},\ x_i \neq x_j,\ \forall i \neq j \tag{4.35}$$

The general case will follow straightforwardly.

First of all we show that

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\sup_{\tau' \leqslant t \leqslant \tau} \mathscr{P}^\varepsilon\left(\left|\frac{1}{\varepsilon^2 T}\int_t^{t+\varepsilon^2 T} ds(Y_s^\varepsilon(\phi; f_n) - Y_s^\varepsilon(a_s^n\phi))\right|^2\right)$$
$$= \lim_{T\to\infty}\lim_{\varepsilon\to 0}\sup_{\tau' \leqslant t \leqslant \tau} \mathscr{P}^\varepsilon\left(\frac{1}{\varepsilon^2 T}\int_t^{t+\varepsilon^2 T} ds \sum_{\substack{\Delta \subset \{1,\ldots,n\} \\ \Delta \neq \varnothing \\ |\Delta|>1}} Y_s^\varepsilon(\hat{\phi}_{\Delta,n}; |\Delta|)\right)^2 \tag{4.36a}$$

where $|\Delta|$ = cardinality of the set Δ, $a_s^n \phi = nm(r, s)^{n-1} \phi(r)$,

$$\hat{\phi}_{\Delta,n}(r) = \phi(r)\, m(r, t)^{n-|\Delta|} \tag{4.36b}$$

$$Y_s^\varepsilon(\psi, k) = \varepsilon \sum_x \psi(\varepsilon x) \left(\prod_{i=1}^{k} (\sigma(x + x_i, s) - p_i^\varepsilon(x, s)) \right) \tag{4.36c}$$

$$p_i^\varepsilon(x, s) = \mathscr{P}^\varepsilon(\sigma(x + x_i, s)) \tag{4.36d}$$

Equation (4.36) follows from the following identity:

$$\prod_{i=1}^{n} \sigma(x + x_i, s) = \sum_{\Delta \subset \{1,\dots,n\}} \prod_{j \notin \Delta} p_j^\varepsilon(x, s) \prod_{i \in \Delta} [\sigma(x + x_i, s) - p_i^\varepsilon(x, s)] \tag{4.37}$$

In fact, the term with $\Delta = \varnothing$ cancels out with the expectation of the quantity in the l.h.s. of Eq. (4.37). Let us consider the terms with $|\Delta| = 1$.

$$\begin{aligned} &\sum_{i=1}^{n} \prod_{j \neq i} p_j^\varepsilon(x, s)(\sigma(x + x_i, s) - p_i^\varepsilon(x, s)) \\ &\quad = \sum_{i=1}^{n} (\sigma(x + x_i, s) - p_i^\varepsilon(x, s)) \prod_{j \neq i} (p_j^\varepsilon(x, s) - m(\varepsilon x, s)) + m(\varepsilon x, s)^{n-1} \\ &\quad \times \sum_{i=1}^{n} (\sigma(x + x_i, s) - p_i^\varepsilon(x, s)) \end{aligned} \tag{4.38}$$

The same argument used in the proof of Theorem 3.1 (see proof of equicontinuity) shows that, given $\tau > 0$, there exists a constant $c(\tau) > 0$ such that for all $x \in \mathbb{Z}$ and s we have

$$\sup_{j \neq i} |p_j^\varepsilon(x, s) - m(\varepsilon x, s)| \leqslant \varepsilon c(\tau) \tag{4.39}$$

Therefore from Eqs. (4.38) and (4.39) we have that the contribution of the terms with $|\Delta| = 1$ to the quantity $Y_s^\varepsilon(\phi; f_n)$ cancels out with $Y_s^\varepsilon(a_s^n \phi)$; in fact

$$\begin{aligned} &\left| \sqrt{\varepsilon} \sum_x \phi(\varepsilon x) \left[\sum_{i=1}^{n} \prod_{j \neq i} p_j^\varepsilon(x, s)(\sigma(x + x_i, s) - p_i^\varepsilon(x, s)) \right. \right. \\ &\quad \left. \left. - nm(\varepsilon x, s)^{n-1} (\sigma(x, s) - p^\varepsilon(x, s)) \right] \right| \\ &\leqslant c(t) \sqrt{\varepsilon} \left(\sup_r |\phi(r)(1 + r^2)| \right) \varepsilon \sum_x \frac{1}{1 + (\varepsilon x)^2} \\ &\quad + \sqrt{\varepsilon} \left| \sum_x \phi(\varepsilon x)\, m(\varepsilon x, s)^{n-1} \left(\sum_{i=1}^{n} (\sigma(x + x_i, s) - p_i^\varepsilon(x, s)) \right. \right. \\ &\quad \left. \left. - n(\sigma(x, s) - p^\varepsilon(x, s)) \right) \right| \end{aligned}$$

$$= \hat{c}\sqrt{\varepsilon} + \sqrt{\varepsilon}\left| \sum_{i=1}^{n} \sum_{x} \phi(\varepsilon(x - x_i))\, m(\varepsilon(x + x_i), s)^{n-1}\, (\sigma(x, s) - p^{\varepsilon}(x, s)) \right.$$

$$\left. - n \sum_{x} \phi(\varepsilon x)\, m(\varepsilon x, s)^{n-1}\, (\sigma(x, s) - p^{\varepsilon}(x, s)) \right|$$

$$\leqslant c'' \sqrt{\varepsilon} + \sqrt{\varepsilon}\, \varepsilon \sum_{x} \psi(\varepsilon x, s) \tag{4.40a}$$

where

$$\psi(r, \tau) = \sup_{r: |r - r'| \leqslant 1} \left| \frac{d}{dr'} \phi(r')\, m(r', \tau)^{n-1} \right| \tag{4.40b}$$

From Eq. (4.36) it follows that Theorem 4 is a consequence of Lemma 4.2 below.

Lemma 4.2. For any $\tau > \tau' > 0$, for any $n \geqslant 2$, $x_1, \ldots, x_n$ distinct integers

$$\lim_{T \to \infty} \lim_{\varepsilon \to 0} \sup_{\tau' \leqslant t \leqslant \tau} \mathscr{P}^{\varepsilon}\left(\left(\frac{1}{\varepsilon^2 T} \int_{t}^{t + \varepsilon^2 T} ds\, Y_s^{\varepsilon}(\phi; n) \right)^2 \right) = 0 \tag{4.41}$$

where $Y_s^{\varepsilon}(\phi; n)$ is defined in Eq. (4.36c)

Proof. We let

$$f_s(x) = \prod_{i=1}^{n} (\sigma(x + x_i, s) - p_i^{\varepsilon}(x, s)) \tag{4.42}$$

The expectation in Eq. (4.41) can be written as

$$\frac{2}{\varepsilon^2 T} \int_{t}^{t + \varepsilon^2 T} ds\, \varepsilon \sum_{x} \phi(\varepsilon x) \frac{1}{\varepsilon^2 T} \int_{0}^{\varepsilon^2 T + t - s} ds' \sum_{y} \phi(\varepsilon y)\, \mathscr{P}^{\varepsilon}(f_s(x)\, f_{s + s'}(y)) \tag{4.43}$$

Therefore, we have to estimate the following quantity:

$$\sum_{y} \phi(\varepsilon y)\, \mathscr{P}^{\varepsilon}(f_s(x)\, f_{s + s'}(y)) \tag{4.44}$$

First of all we notice that in (4.44) the time s' ranges in a microscopic time interval while the time t is "macroscopic." Therefore it may be quite "natural" to use the following argument in estimating (4.44). Use duality until time s', i.e., consider n particles that, starting from $x_i + y$, $i = 1, \ldots, n$, move according to the BEP for a time interval s': since s' is of order $\varepsilon^2 T$, no Glauber marks appeared in that time interval. Then, one is left with

$\mathscr{P}^\varepsilon(f_s(x)\, f_s(\tilde{y}))$, where $\tilde{y} = (y_1(s'),..., y_n(s'))$ gives the positions at time s' of the n particles. Then, using the fact that [see Eq. (3.12b)] $\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathscr{P}^\varepsilon(f_s(x)\, f_s(\tilde{y})) = 0$, Lemma 4.2 follows. The above argument is correct only if $|y-x|$ is big enough. In fact, if $|y-x|$ is small, then the probability that at time s' the particles are in the positions $x_i + x$, $i = 1,..., n$, is not negligible and so we cannot apply Eq. (3.12b). For $|y-x|$ small we use another argument; i.e., we use duality for the time interval s. At this time the process is in local equilibrium in the sense of Theorem 1 and during the microscopic time s' the equilibrium profile remains unchanged. Therefore we fix a number $R > 0$ and we divide the sum (4.44) in two parts as follows:

$$\sum_y \phi(\varepsilon y)\, \mathscr{P}^\varepsilon(f_s(x)\, f_{s+s'}(y)) = C_R^\varepsilon(x, s, s') + D_R^\varepsilon(x, s, s') \tag{4.45a}$$

where

$$C_R^\varepsilon(x, s, s') = \sum_{|y| \leqslant R} \phi(\varepsilon(x+y))\, \mathscr{P}^\varepsilon(f_s(x)\, f_{s+s'}(x+y)) \tag{4.45b}$$

$$D_R^\varepsilon(x, s, s') = \sum_{|y| \geqslant R} \phi(\varepsilon(x+y))\, \mathscr{P}^\varepsilon(f_s(x)\, f_{s+s'}(x+y)) \tag{4.45c}$$

We will estimate C_R^ε and D_R^ε separately and will consider first the limit $\varepsilon \to 0$, then the limit $R \to \infty$, and finally let $T \to \infty$.

Estimate of $C_R^\varepsilon(x, s, s')$. From Theorem 1 we have that

$$|C_R^\varepsilon(x, s, s') - \sum_{|y| \leqslant R} \phi(\varepsilon(y+x))\, \nu_{m(\varepsilon x, \varepsilon^2 s)}(\tilde{f}(0)\, \tilde{f}_{s'}(y)) \leqslant \psi(\varepsilon, R, s') \tag{4.46a}$$

where $\forall \hat{s}$

$$\tilde{f}_{\hat{s}}(z) = \prod_{i=1}^{n} (\sigma(x_i + z, \hat{s}) - m(\varepsilon x, \varepsilon^2 s)) \tag{4.46b}$$

and $\{\psi(\varepsilon, R, \cdot)\}$ is uniformly bounded on $[0, T]$ with

$$\lim_{\varepsilon \to 0} \psi(\varepsilon, R, s') = 0 \tag{4.46c}$$

Let T_1 be the time of the first Glauber mark (see Definition 3.1); then there exists a constant $c > 0$ such that

$$P(T_1 < \varepsilon^2 T) \leqslant c\varepsilon^2 T \tag{4.47}$$

Therefore

$$\Bigg| \sum_{|y| \leqslant R} \phi(\varepsilon(y+x))\, \nu_{m(\varepsilon x, \varepsilon^2 s)}(\tilde{f}(0)\, \tilde{f}_{s'}(y)) - \sum_{|y| \leqslant R} \phi(\varepsilon(y+x)) \cdot \nu_{m(\varepsilon x, \varepsilon^2 s)}(\tilde{f}(0)(S_{s'}^E \tilde{f}(y))) \Bigg| \leqslant \hat{c}\varepsilon^2 T \tag{4.48a}$$

where $\hat{c}$ is a suitable positive constant and

$S_{s'}^E$ is the semigroup of the simple exclusion process with generator L_E (4.48b)

We have that, since $n \geqslant 2$, $\forall \tau \geqslant 0$ there exists a function $\bar{\psi}(\tau)$ such that $\forall m \in [-1, 1]$,

$$\sum_y \left| \nu_m \left(\prod_{i=1}^n (\sigma(x_i) - m)(S_\tau^E \prod_{j=1}^n (\sigma(x_j + y) - m)) \right) \right| \leqslant \bar{\psi}(\tau) \tag{4.49a}$$

and

$$\lim_{\tau \to \infty} \bar{\psi}(\tau) = 0 \tag{4.49b}$$

Therefore

$$\frac{1}{\varepsilon^2 T} \int_0^{\varepsilon^2 T + t - s} ds' \, |C_R^\varepsilon(x, s, s')| \leqslant \hat{c} \varepsilon^2 T + \left[\frac{1}{\varepsilon^2 T} \int_0^{\varepsilon^2 T} \bar{\psi}(s') + \psi(\varepsilon, R, s') \, ds' \right] \sup_r |\phi(r) + 1| \tag{4.50}$$

From Eqs. (4.46c) and (4.49b) it follows that the contribution of C_R^ε is vanishingly small.

Estimate of $D_R^\varepsilon(x, s, s')$. Let T_1 be the time of the first Glauber mark; then by Eq. (4.47) we have

$$|\mathscr{P}^\varepsilon(f_s(x) f_{s+s'}(x+y))| \leqslant |\mathscr{P}^\varepsilon(f_s(x) f_{s+s'}(x+y) \, \mathbb{1}(\{T_1 > s'\}))| + \hat{c} \varepsilon^2 T \tag{4.51}$$

Let $x_i(s')$, $i \in \{1, ..., n\}$ be the position at time s' of the particles starting at time zero from the positions $x_i + y$, $i \in \{1, ..., n\}$. Let

$$B(y) = \mathbb{1}(\{\exists i, j \in \{1, .., n\} : x_i(s') = x_j + x\}) \tag{4.52}$$

Then for $|y| > R$ and R large enough we have

$$P(B(y) \, \mathbb{1}(\{T_1 > \varepsilon^2 T\})) \leqslant \text{const} \, P(\tau_{y/2}^* < T) \tag{4.53a}$$

where τ_x^* is the hitting time at the origin of a random walk which starts at x. We have that

$$\sum_{|y| \geqslant R} P(\tau_{y/2}^* < T) \leqslant \Delta(R^2/T) \tag{4.53b}$$

with

$$\lim_{x \to \infty} \Delta(x) = 0 \tag{4.53c}$$

From Lemma 3.5 it follows that $\forall s \leqslant \tau$,

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \sum_{y} |\phi(x+y)| \; |\mathscr{P}^{\varepsilon}(f_s(x)\, f_{s+s'}(x+y)\, \mathbb{1}\{T_1 > s'\}\, 1 - B(y)| \\
&\quad \leqslant \lim_{\varepsilon \to 0} \varepsilon \sum_{y} |\phi(x+y)| \sup_{\substack{y_i, y_{2n} \\ y_i \neq y_j, i \neq j}} \varepsilon^{-1} |\mathscr{P}^{\varepsilon} \prod_{i=1}^{n} (\sigma(y_i, s) \\
&\qquad - \mathscr{P}^{\varepsilon}(y_i, s)) \prod_{j=n+1}^{2n} (\sigma(y_j, s) \\
&\qquad - \mathscr{P}^{\varepsilon}(y + x_i, s + s')) = 0
\end{aligned} \tag{4.54}$$

From Eqs. (4.51), (4.52), (4.53), and (4.54) we have that for R large enough and $|y| > R$,

$$\lim_{\varepsilon \to 0} |\mathscr{P}^{\varepsilon}(f_s(x)\, f_{s+s'}(x+y))| \leqslant \text{const}\, \Delta\left(\frac{R^2}{T}\right) \tag{4.55}$$

From Eq. (4.53c) it follows that the r.h.s. of Eq. (4.55) vanishes when $R \to \infty$ and this concludes the proof of Lemma 4.2. ∎

APPENDIX

To prove Proposition 3.4 we need to compare the branching-exclusion and the independent branching processes, defined in Definitions 3.1 and 3.3 respectively. For this purpose we couple the two processes, that is, we introduce a suitable joint realization (coupling) Q_{ε} of the LBEP and the LIBP. Initially we shall only use the fact that Q_{ε} is a coupling, hence we postpone the precise definition of Q_{ε} to the time when its particular properties will be needed.

By definition [see Eq. (3.8)], to prove Eq. (3.10) we need to compute

$$S \equiv Q_{\varepsilon}(H - H^0) \tag{A.1a}$$

$$H = \int d\mu^{\varepsilon} \prod_{i=1}^{n} h_i(\mathscr{A}_t(i); \sigma(Z(i, t))) \tag{A.1b}$$

$$H^0 = \int d\mu^{\varepsilon} \prod_{i=1}^{n} h_i(\mathscr{A}_t^0(i); \sigma(Z^0(i, t))) \tag{A.1c}$$

$Q_\varepsilon(\cdot)$ is the expectation with respect to the coupling Q_ε. We prove Eq. (3.10) in the case where the initial measures μ^ε, $\varepsilon \in (0, 1]$ are product measures such that

$$\mu^\varepsilon(\sigma(x)) = m_0(\varepsilon x), \qquad \forall x \in \mathbb{Z} \tag{A.2a}$$

The proof in the case of measure μ^ε, $\varepsilon \in (0, 1]$ satisfying Definition 2.2 is technically more complicated but uses essentially the same arguments (see Remark A.5).

We let

$$m_\pm(\varepsilon x) = \tfrac{1}{2}(1 \pm m_0(\varepsilon x)) \tag{A.2b}$$

Notice that

$$\mu^\varepsilon(\{\sigma(x) = \pm 1\}) = m_\pm(\varepsilon x) \tag{A.2c}$$

To compute S we first consider the case where a Glauber interaction is present. Therefore we first give a precise definition of Glauber interaction and in Lemma A.2 below we estimate its probability. In the sequel we need to distinguish the cases in which there is Glauber interaction between the components of a family and/or between different families. We indicate with $\mathscr{B}$ the characteristic functions of the "bad" events and with $\mathscr{C}$ the "good" ones.

Definition A.1. All the quantities that are not defined here are defined in Sec, 3, Definition 3.1.

When we want to distinguish second class from first-class particles we use greek letters for the former and latin for the latter.

We let

$$\mathscr{C}(t) = \mathbb{1}\{\forall k \in \{1,..., \mathscr{N}_t\}, k \text{ is a first-class particle}\} \tag{A.3}$$

For any $i, j \in \{1,..., n\}$, we let

$$\begin{aligned}\tilde{\mathscr{B}}_{i,j}(t) = \mathbb{1}\{&\text{there is only one second-class particle,}\\ &\alpha \in I_t(i) \text{ and there is } k \in I_t(j) \text{ such}\\ &\text{that } x_\alpha(t) = x_k(t)\}\end{aligned} \tag{A.4a}$$

and we set

$$\mathscr{B}_{i,j}(t) = \tilde{\mathscr{B}}_{i,j}(t) + \tilde{\mathscr{B}}_{j,i}(t) \qquad \text{for} \quad i \neq j \tag{A.4b}$$

and

$$\mathscr{B}_i(t) = \tilde{\mathscr{B}}_{i,i}(t)$$

The remaining cases are given by

$$\mathscr{B}(t) = \mathbb{1}\{\text{there are at least two second-class particles}\} \tag{A.4c}$$

Therefore

$$1 - \mathscr{C}(t) = \sum_{i=1}^{n} (\mathscr{B}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathscr{B}_{i,j}(t)) + \mathscr{B}(t) \tag{A.5}$$

Finally for any set of labels $J \subset \{1,..., n\}$ we let

$$\mathscr{B}_J(t) = \mathbb{1}\left\{\forall k \notin \bigcup_{i \in J} I_t(i),\ k \text{ is a first-class particle}\right\} \tag{A.6}$$

Lemma A.2. There is a positive $d = d(t)$ such that the following holds.

$$\mathbb{E}(\mathscr{C}(t)) \geqslant 1 - d\varepsilon \tag{A.7a}$$

$$\mathbb{E}(\mathscr{B}_{i,j}(t)) \leqslant d\varepsilon, \qquad Ai, j \in \{1,..., n\},\ i \neq j \tag{A.7b}$$

$$\mathbb{E}(\mathscr{B}_i(t)) \leqslant d\varepsilon, \qquad \forall i \in \{1,..., n\} \tag{A.7c}$$

$$\mathbb{E}(\mathscr{B}(t)) \leqslant d\varepsilon^2 \tag{A.7d}$$

Proof. First of all, we observe that Eq. (A.7a) follows from Eq. (A.5) and Eqs. (A.7b)–(A.7d).

Let (see Definition 3.1) $\mathscr{M}_t = 1/2(\mathscr{N}_t - n)$ be the number of Glauber marks up to time t, $\mathscr{N}_t$ being the number of particles present at time t. Let $T_0 = 0$ and T_e, $e = 1,..., M$, be the times when the Glauber marks appear. We let $\hat{P}(\cdot)$ be the expectation w.r.t. the law of $\mathscr{N}_t$, $\{T_e, e = 1,..., \mathscr{M}_t\}$, and $E^{\varepsilon}_M(\cdot)$ be the expectation w.r.t. the law of the LBEP conditioned to $\mathscr{N}_t$, $\{T_e, e = 1,..., \mathscr{M}_t\}$. Here we let $\mathscr{M}_t = M$. $A^{\varepsilon}(t) = \mathbb{1}\{T_e - T_{e-1} \geqslant \varepsilon^2,\ e = 1,..., M\}$.

We have that

$$\hat{P}(A^{\varepsilon}(t)) \geqslant 1 - b\varepsilon^2$$

where b is a suitable positive constant.

To prove Eqs. (A.7) we observe that $\forall i, j \in \{1,..., \mathscr{N}_t\}$, $i \neq j$,

$$\mathbb{E}(\mathscr{B}_i(t)) \leqslant \hat{P}\left(A^{\varepsilon}(t) \sum_{\substack{k_1, k_2 = 1 \\ k_1 \neq k_2}}^{\mathscr{N}_t} \mathbb{E}^{\varepsilon}_M(\mathbb{1}\{x_{k_1}(T_M) = x_{k_2}(T_M)\})\right) + b\varepsilon^2 \tag{A.8a}$$

$$\mathbb{E}(\mathscr{B}_{i,j}(t)) \leqslant \text{r.h.s. of Eq. (A.8a)} \tag{A.8b}$$

$$\mathbb{E}(\mathscr{B}(t)) \leqslant \hat{P}\left(A^{\varepsilon}(t) \sum_{\substack{i,j,k,h=1 \\ i \neq j \neq k}}^{\mathscr{N}_t} \mathbb{E}^{\varepsilon}_M(\mathbb{1}\{x_i(T_M) = x_j(T_M), x_k(T_M) = x_h(T_M)\}) \right) + b\varepsilon^2 \tag{A.8c}$$

From the definitions it follows that $\forall i, j \in \{1,..., \mathscr{N}_t\}$

$$\mathbb{E}^{\varepsilon}_M(\mathbb{1}\{x_i(T_M) = x_j(T_M)\}) \leqslant \mathbb{E}^{\varepsilon}_M(\mathbb{1}\{|x_i(T_{M-1}) - x_j(T_{M-1})| \geqslant 1\} \times P^{\varepsilon}(i, j; M)) + \mathbb{E}^{\varepsilon}_M(\mathbb{1}\{x_i(T_{M-1}) = x_j(T_{M-1})\}) \tag{A.9}$$

where $P^{\varepsilon}(i, j; M)$ is the probability that two particles starting from $x_i(T_{M-1})$ and $x_j(T_{M-1})$ and moving by eclusion with intensity ε^{-2} are at distance one at time $T_M - T_{M-1}$. It is possible to show that there is a constant $\hat{b} > 0$ such that

$$P^{\varepsilon}(i, j; M) \leqslant \varepsilon \hat{b}(T_M - T_{M-1})^{-1/2} + \mathbb{1}(\{T_M - T_{M-1} \leqslant \varepsilon^2\}) \tag{A.10a}$$

Therefore, iterating Eq. (A.9a) up to time 0, from Eqs. (A.8a) and (A.10a) it follows that

$$\mathbb{E}(\mathscr{B}_i(t)) \leqslant \varepsilon \hat{b}\, \hat{P}\left(A^{\varepsilon}(t)\, \mathscr{N}_t \sum_{e=1}^{M} (T_{e+1} - T_e)^{-1/2} \right) + \varepsilon^2 b \tag{A.10b}$$

Using estimates on the branching process we show that

$$\hat{P}\left(\mathscr{N}_t \sum_{e=1}^{M-1} (T_{e+1} - T_e)^{-1/2} \right) \leqslant c(t) < +\infty$$

and from this Eqs. (A.7b) and (A.7c) follow.

To show Eq. (A.7d) we use Eq. (A.8c) and an iterative argument analogous to the previous one. The key estimate are the following. There is a positive constant $\tilde{b}$ such that for any distinct i, j, k, h,

$$P^{\varepsilon}(i, i, k; M) \leqslant \varepsilon^2 \tilde{b}(T_M - T_{M-1})^{-1/2} + \mathbb{1}\{T_M - T_{M-1} \leqslant \varepsilon^2\} \tag{A.11a}$$

$$P^{\varepsilon}(i, j, k, h; M) \leqslant \text{r.h.s. of Eq. (A.11a)} \tag{A.11b}$$

where $P^{\varepsilon}(i, j, k; M)$ (resp. $P^{\varepsilon}(i, j, k, h; M)$) is the probability that three particles (resp. four particles) moving by exclusion with intensity ε^{-2} are at distance one at time $T_M - T_{M-1}$. ∎

Proof of Proposition 4. We separate in S [given in Eq. (A.1)] the contribution due to the Glauber interaction from the one due to the exclusion. Therefore we let

$$S = S_G + S_E \tag{A.12a}$$

$$S_G = Q_\varepsilon((1 - \mathscr{C}(t))(H - H^0)) \tag{A.12b}$$

$$S_E = Q_\varepsilon(\mathscr{C}(t)(H - H^0)) \tag{A.12c}$$

and we will prove that for $q = E, G$

$$S_q = \varepsilon \sum_{i=1}^{n} \Gamma_q^\varepsilon(i) \prod_{j \neq i} \mathrm{E}_\varepsilon^0(\sigma(x_j, t)) + \varepsilon \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \Gamma_q^\varepsilon(i, j) \times \prod_{k \neq i,j} \mathbb{E}_\varepsilon^0(\sigma(x_k, t)) + R_q(\varepsilon) \tag{A.13a}$$

where the sequence of numbers $\Gamma_q^\varepsilon(i)$, $\Gamma_q^\varepsilon(i, j)$, $R_q(\varepsilon)$ verifies

$$\sup_i |\Gamma_q^\varepsilon(i)| \leqslant c, \quad \sup_{i,j} |\Gamma_q^\varepsilon(i, j)| \leqslant c, \quad q = E, G, \qquad \forall \varepsilon > 0 \tag{A.13b}$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} |R_q(\varepsilon)| = 0, \qquad q = E, G \tag{A.13c}$$

From Eqs. (A.13), Eq. (3.10) follows.

In what follows we call a remainder a sequence of numbers $R(\varepsilon)$, $\varepsilon \in (0, 1]$ which verifies Eq. (A.13c) and we call the one-body (resp. two-body) term a sequence of numbers $\Gamma^\varepsilon(i)$, $i \in \{1, \ldots, n\}$, $\varepsilon \in (0, 1]$ (resp. $\Gamma^\varepsilon(i, j)$, $i, j \in \{1, \ldots, n\}$, $\varepsilon \in (0, 1]$) which verifies Eq. (A.13b).

Observation A.3. We emphasize a property of the difference $H - H^0$ that we are going to use often.

Let $J \subset \{1, \ldots, n\}$, let $\mathscr{B}_J(t)$ be the characteristic function defined in Eq. (A.6), and let $N = \mathscr{N}_t$. We denote by

$$N_J = \sum_{j \in J} \mathscr{N}_t(j) \tag{A.14a}$$

$$N - N_J = \sum_{i \notin J} \mathscr{N}_t(i) \tag{A.14b}$$

For $a \in \{-1, 1\}^N$ we let [see Eq. (3.4) for the definition of $I(i)$]

$$a^i = (a_\alpha, \alpha \in I(i)), \qquad i \in \{1, \ldots, n\} \tag{A.14c}$$

$a(J)$ denotes the set $\{a^i, i \in J\}$, where $a^i = (a_\alpha, \alpha \in I(i))$, $a_\alpha = \pm 1$. We define

$$g_J(\varepsilon) = \mathscr{B}_J(t) \sum_{a(J)} \left[\prod_{i \in J} h_i(\mathscr{A}_t(i); a^i) \right] \mu^\varepsilon \left(\prod_{i \in J} \prod_{\alpha \in I(i)} \mathbb{1}(\{\sigma(x_\alpha(t)) = a_\alpha\}) \right) \tag{A.15a}$$

$$g_J^0(\varepsilon) = \sum_{a(J)} \left[\prod_{i \in J} h_i(\mathscr{A}_t^0(i); a^i) \right] \prod_{i \in J} \prod_{\alpha \in I(i)} m_{a_\alpha}(\varepsilon x_\alpha^0(t)) \tag{A.15b}$$

where $\sum_{a(J)}$ means the sum over all the values of the set $a(J) \subset \{-1, 1\}^{N_J}$. Then the following holds.

$$\mathscr{B}_J(t)(H - H^0) = [g_J(\varepsilon) - g_J^0(\varepsilon)] \sum_{a(J^c)} h_{J^c}^0 M_\varepsilon^0(J^c) + g_J(\varepsilon) \sum_{a(J^c)} h_{J^c}^0 [M_\varepsilon(J^c) - M_\varepsilon^0(J^c)] \tag{A.16a}$$

where $J^c = \{1, ..., n\} \setminus J$, $\sum_{a(J^c)}$ means the sum over all the values of the set $a(J^c) \subset \{-1, 1\}^{N - N_J}$ and

$$h_{J^c}^0 = \prod_{i \notin J} h_i(\mathscr{A}_t^0(i); a^i) \tag{A.16b}$$

$$M_\varepsilon(J^c) = \prod_{i \notin J} \prod_{\alpha \in I(i)} m_{a_\alpha}(\varepsilon x_\alpha(t)) \tag{A.16c}$$

$$M_\varepsilon^0(J^c) = \prod_{i \notin J} \prod_{\alpha \in I(i)} m_{a_\alpha}(\varepsilon x_\alpha^0(t)) \tag{A.16d}$$

The proof of Eq. (A.16) uses the particular form of the coupling Q_ε. In fact, since μ^ε is a product measure, it is easy to see that from the definition of the function H it follows that

$$\mathscr{B}_J(t)\, H = g_J(\varepsilon) \sum_{a(J^c)} h_{J^c} M_\varepsilon(J^c)$$

where

$$h_{J^c} = \prod_{i \notin J} h_i(\mathscr{A}_t(i); a^i)$$

The coupling Q_ε will be defined (see Definition A.3 below) in such a way that $\mathscr{B}_J(t)\, h_{J^c} = \mathscr{B}_J(t)\, h_{J^c}^0$. Therefore Eq. (A.16) follows by adding and subtracting $g_J(\varepsilon) \sum_{a(J^c)} h_{J^c}^0 M_\varepsilon^0(J^c)$. Finally we observe that

$$Q_\varepsilon \left([g_J(\varepsilon) - g_J^0(\varepsilon)] \left[\sum_{a(J^c)} h_{J^c}^0 M_\varepsilon^0(J^c) \right] \right) = Q_\varepsilon(g_J(\varepsilon) - g_J^0(\varepsilon)) \prod_{i \notin J} \mathbb{E}^0(\sigma(x_i, t)) \tag{A.17}$$

Computation of S_G. We use the partition given in Eq. (A.5); therefore we have

$$S_G = \sum_{i=1}^n Q_\varepsilon(\mathscr{B}_i(t)(H - H^0)) + \sum_{\substack{i,j=1 \\ i \neq j}}^n Q_\varepsilon(\mathscr{B}_{i,j}(t)(H - H^0)) + R_G^1(\varepsilon) \tag{A.18a}$$

where

$$R^1_G(\varepsilon) = Q_\varepsilon(\mathscr{B}(t)(H - H^0)) \tag{A.18b}$$

From Lemma A.2, Eq. (A.7d), it follows that

$$|R^1_G(\varepsilon)| \leqslant 2d\,\varepsilon^2 \tag{A.19}$$

Therefore $R^1_G(\varepsilon)$ is a remainder. To compute the first and second term in the r.h.s. of Eq. (A.18), we use Eq. (A.16) with $J = \{i\}$ and $J = \{i, j\}$ respectively. Therefore we have [see also Eq. (A.17)]

$$\sum_{i=1}^{n} Q_\varepsilon(\mathscr{B}_i(t)(H - H^0)) = \varepsilon \sum_i \Gamma^\varepsilon_G(i) \prod_{j \neq i} \mathbb{E}^0_\varepsilon(\sigma(x_j, t)) + R^2_G(\varepsilon) \tag{A.20a}$$

$$\sum_{i=1}^{n} Q_\varepsilon(\mathscr{B}_{i,j}(t)(H - H^0)) = \varepsilon \sum_{i,j} \Gamma^\varepsilon_G(i, j) \prod_{k \neq i,j} \mathbb{E}^0_\varepsilon(\sigma(x_k, t)) + R^3_G(\varepsilon) \tag{A.20b}$$

where

$$\Gamma^\varepsilon_G(i) = \varepsilon^{-1} Q_\varepsilon(g_i(\varepsilon) - g^0_i(\varepsilon)) \tag{A.20c}$$

$$\Gamma^\varepsilon_G(i, j) = \varepsilon^{-1} Q_\varepsilon(g_{\{i,j\}}(\varepsilon) - g^0_{\{i,j\}}(\varepsilon)) \tag{A.20d}$$

$$R^2_G(\varepsilon) = \sum_{i=1}^{n} Q_\varepsilon\left(g_i(\varepsilon) \sum_{a(\{i\}^c)} h^0_{\{i\}^c}[M_\varepsilon(\{i\}^c) - M^0_\varepsilon(\{i\}^c)] \right) \tag{A.20e}$$

$$R^3_G(\varepsilon) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} Q_\varepsilon\left(g_{\{i,j\}}(\varepsilon) \sum_{a(\{i,j\}^c)} h^0_{\{i,j\}^c}[M_\varepsilon(\{i,j\}^c) - M^0_\varepsilon(\{i,j\}^c)] \right) \tag{A.20f}$$

From Lemma A.2, Eqs. (A.7b) and (A.7c), it follows that

$$|\Gamma^\varepsilon_G(i)| \leqslant 2d, \qquad |\Gamma^\varepsilon_G(i, j)| \leqslant 2d$$

Therefore we are left with the proof that $R^1_G(\varepsilon)$ and $R^2_G(\varepsilon)$ are remainders. This proof uses an estimate on the difference between the position at time t of particles moving by exclusion with the corresponding ones moving by random walk. This estimate follows from the definition of Q_ε: therefore we give the proof that $R^1_G(\varepsilon)$ and $R^2_G(\varepsilon)$ are remainders after Definition A.3 and Lemma A.4.

Definition A.3.

(a) *Construction of the coupling* Q_ε. The coupling Q is defined by matching the trajectories of the labeled IBP to the trajectories of the labeled BEP. At each trajectory $\{(Z^0(s), \mathscr{A}^0_s), s \in [0, t]\}$ we associate a tra-

jectory $(Z(s), \mathscr{A}_s)$, $s \in [0, t]$ in the following way. We let $Z(0) = Z^0(0)$. By the definition of the IBP at the times $T_l^0 \in \mathscr{T}_t^0$, $l = 1,..., \frac{1}{2}(\mathscr{N}_t^0 - n)$, two independent particles are created by, let us say, the kth independent particle. Then two interacting particles are created at time T_l^0 if and only if the kth interacting particle is a "first class" particle, otherwise nothing happens at time T_l^0 for the interacting process.

In each time interval (T_l^0, T_{l+1}^0) we let $Q_\varepsilon = Q$, where Q is the coupling defined in Ref. 16, pp. 189–191. This coupling can be realized as follows. We can define a one-to-one correspondence between the displacements of the independent and interacting particles during the time intervals (T_l^0, T_{l+1}^0), $l = 1,..., \frac{1}{2}(\mathscr{N}_t^0 - n)$ as follows. The interacting kth particle has the same displacements as the kth independent one with two exceptions. If the kth interacting particle is a second-class particle, then it moves together with the "corresponding" first-class particle and therefore it is not coupled to any of the independent particles. If the kth independent particle has a displacement that would take the kth interacting one in top of, let us say, the jth interacting particle, then, if $k > j$ (resp. $k < j$), nothing happens (resp. the kth and jth particles interchange positions).

At time t we label the particles in such a way that the $\mathscr{N}_t$ independent and interacting particles have the same labels given according to the order with which they are created, i.e., with the same rules as in Definitions 3.1 and 3.3. The "exceeding" $\mathscr{N}_t^0 - \mathscr{N}_t$ independent particles have a label different from the others.

Finally, we observe that the distribution of $\mathscr{N}_t^0$ can be explicitly computed in the following way (see, for instance, Chapter 2, p. 111 of Snyder). For $m = 2N + n$, $N = 0, 1, 2,...$, we define

$$P_m(t) = P(\mathscr{N}_t^0 = m) \tag{A.21a}$$

then $P_m(t)$ satisfies

$$P_m(t) = \lambda(m-2) \int_0^t ds\, P_{m-2}(s)\, e^{-m\lambda(t-s)} \tag{A.21b}$$

Now it is easy to check that the solution of Eq. (A.21b) is given by

$$P(\mathscr{N}_t = 2N + n) = e^{-\lambda t m} c_N (1 - e^{-2\lambda t})^N, \qquad N = 0, 1, 2,... \tag{A.21c}$$

$$c_N = c_{N-1} \frac{n + (2N-2)}{2N}, \qquad N = 1, 2,..., c_0 = 1 \tag{A.21d}$$

From Eq. (A.21), it follows that for any integer $q \geqslant 0$

$$\langle (\mathrm{N}_t^0)^q \rangle < \infty \tag{A.21e}$$

where $\langle \cdot \rangle$ denotes expectation with respect to the law given in Eq. (A.21c). Equation (A.21e) will be used often in what follows.

(b) Now we define a way to measure the distance in $\mathbb{Z}$ between the positions at time $t \geqslant 0$ of the interacting and independent particles that are coupled to each other.

Let $\mathscr{A}_t^0$ be given. We let $N = \frac{1}{2}(\mathscr{N}_t^0 - n)$ and $T_{N+1}^0 = t$. For any $l = 1, \ldots, N$ and for any given configuration $(Z(T_l^0), Z^0(T_l^0))$ we define a family of random variables $D_{k_1,k_2}^l(T_{l+1}^0)$ for any k_1 and k_2 such that $k_1 > k_2$ and both the k_1th and k_2th interacting particles are first-class particles. The $D_{k_1,k_2}^l(T_{l+1}^0)$ are r.v. measurable with respect to the σ-algebra generated by $\{(Z(s), Z^0(s)), T_l^0 < s \leqslant T_{l+1}^0\}$.

The definition of D_{k_1,k_2}^l is the same as that given in Definition 3.6 of Ref. 16 and we report it with the same notations. The variable D_{k_1,k_2}^l counts as positive or negative the times s in the interval $[T_l, T_{l+1}]$ when

(A.10) (a) $|x_{k_1}(s) - x_{k_2}(s)| = 1$

(b) either $x_{k_1}^0(s)$ or $x_{k_2}^0(s)$ has a displacement at time s

(c) either $x_{k_1}(s) + [x_{k_1}^0(s+) - x_{k_1}^0(s-)] = x_{k_2}(s)$ or $x_{k_2}(s) + [x_{k_2}^0(s+) - x_{k_2}^0(s-)] = x_{k_1}(s)$

We then define for each such times s

$$D_{k_1,k_2}^l(s) = D_{k_1,k_2}^l(s+)$$

$$D_{k_1,k_2}^l(s+) - D_{k_1,k_2}^l(s-) = [x_{k_1}(s+) - x_{k_1}(s-)] + [x_{k_1}^0(s+) - x_{k_1}^0(s-)]$$

Finally, we observe that by definition of the coupling, for any "first class" particle k_1

$$x_{k_1}(t) - x_{k_1}^0(t) = x_{k_1}(T_N^0) - x_{k_1}^0(T_N^0) + \sum_{k_2 < k_1} D_{k_1,k_2}^N(t) \tag{A.22a}$$

and iteratively

$$x_{k_1}(T_l^0) - x_{k_1}^0(T_l^0) = x_{k_1}(T_{l-1}^0) - x_{k_1}^0(T_{l-1}^0) + \sum_{k_2 < k_1}^{l} D_{k_1,k_2}^l(T_l^0), \qquad \forall l \geqslant l_{k_1} \tag{A.22b}$$

where l_{k_1} is such that the k_1th particle starts its motion at time $T_{l_{k_1}}^0$, and $\sum_{k_2 < k_1}^l$ means the sum over the labels of all the "first-class" particles present at time T_l^0. We observe that Eq. (A.22b) can be iterated up to time 0. In fact, let k be the label of the first-class parent particle of the k_1 particle; then

$$x_{k_1}(T_{l_{k_1}}^0) - x_{k_1}^0(T_{l_{k_1}}^0) = x_k(T_{l_{k_1}}^0) - x_k(T_{l_{k_1}}^0) \tag{A.22c}$$

Lemma A.4.

(i) For any β, $\frac{1}{2}<\beta<1$, there are constants $d_2>0$, $d_2'>0$ such that the following holds. Define

$$\mathscr{E}_t(\beta)=\mathbb{1}(\{\forall k \text{ such that } k \text{ is a first class particle, } |x_k(t)-x_k^0(t)|\leqslant \mathscr{N}_t^2\varepsilon^{-\beta}\}) \tag{A.23a}$$

Then

$$Q_\varepsilon(\mathscr{E}_t(\beta))\geqslant 1-d_2\exp\{-d_2'\varepsilon^{-(\beta-1/2)}\} \tag{A.23b}$$

(ii) For any bounded function f

$$\lim_{\varepsilon\to 0}|\mathbb{E}_\varepsilon(f(Z_n(t)))-\mathbb{E}_\varepsilon^0(f(Z_n^0(t)))|=0 \tag{A.24}$$

Proof.

(i) From the definitions it follows that the inequality (A.23b) can be proved in the same way as for the exclusion process and, therefore, it is a straightforward consequence of Eq. (3.11) of Ref. 16. We briefly sketch the main arguments used in the proofs and we refer to the above paper for the details. From Eqs. (A.22) it follows that we have to estimate the random variable $D_{k,j}^l(T_l)$. Let k and j be given.

We take conditional expectation with respect to $\mathscr{A}_t^0$ and with respect to the trajectory of the coupled process for all the times $s\notin(T_l^0, T_{l+1}^0)$. Then define

$$n_{k,j}\equiv n_{k,j}(l)\equiv \#\{s\in[T_l^0, T_{l+1}^0]: |x_k(s)-x_j(s)|=1\} \tag{A.25a}$$

where $\#\{\cdot\}$ is the cardinality of the set $\{\cdot\}$. One observes that typically

$$n_{k,j}\sim(\varepsilon^{-2}(T_{l+1}^0-T_l^0))^{1/2}\sim\varepsilon^{-1}\sqrt{t} \tag{A.25b}$$

On the other hand, the law of the random variable $D_{k,j}^l$ conditioned to $n_{k,j}$ is the law of a continuous-time simple random walk at time $n_{k,j}$; therefore $D_{k,j}$ behaves like $(n_{k,j})^{1/2}$. Then from Eq. (A.25b)

$$D_{k,j}\sim\varepsilon^{-1/2} \tag{A.25c}$$

The dependence on $\mathscr{N}_t$ in Eq. (A.23a) is due to the sum over l and over j in Eq. (A.22), the exponential bound on the probability that $D_{k,j}^l>\varepsilon^{-1/2}$ conditioned to $\mathscr{A}_t^0$ and the history outside the time interval (T_l^0, T_{l+1}^0) being independent of $\mathscr{A}_t^0$.

(ii) Using Lemma A.2, the proof of Eq. (A.24) can be done in the same way as for the exclusion process (see Ref. 16, Proposition 3.3, p. 193). For $d>1$ we use an argument analogous to the remark after the proof of equicontinuity in Theorem 1. ∎

Proof that R^2_G and R^3_G [see Eq. (A.20)] are remainders. We give the proof for $R^2_G(\varepsilon)$, the one for $R^3_G(\varepsilon)$ being the same.

$$|R^2_G(\varepsilon)| \leqslant \sum_{i=1}^{n} Q_\varepsilon \left(\mathscr{B}_i(t) \sum_{a(\{i\}^c)} \left| \prod_{k \notin I(i)} m_{a_k}(\varepsilon x_k(t)) - \prod_{k \notin I(i)} m_{a_k}(\varepsilon x^0_k(t)) \right| \right) \tag{A.26}$$

Equation (A.26) follows from the fact that

$$|g_i(\varepsilon)| \leqslant \mathscr{B}_i(t)$$
$$|h^0_{\{i\}}| \leqslant 1$$

Since $0 \leqslant m_\pm(r) \leqslant 1$, $\forall r \in \mathbb{R}$, it is not difficult to show that

$$\left| \prod_{k \notin I(i)} m_{a_k}(\varepsilon x_k(t)) - \prod_{k \notin I(i)} m_{a_k}(\varepsilon x^0_k(t)) \right| \leqslant \max_{k \notin I(i)} |m_{a_k}(\varepsilon x_k(t)) - m_{a_k}(\varepsilon x^0_k(t))| \left[\prod_{k_1 \notin I(i)} m_{a_{k_1}}(\varepsilon x_{k_1}(t)) + \prod_{k_1 \notin I(i)} m_{a_{k_1}}(\varepsilon x^0_{k_1}(t)) \right] \tag{A.27}$$

Since $\forall r \in \mathbb{R}$, $m_+(r) + m_-(r) = 1$, we have that

$$\sum_{a(\{i\}^c)} \left[\prod_{k_1 \notin I(i)} m_{a_{k_1}}(\varepsilon x_{k_1}(t)) + \prod_{k_1 \notin I(i)} m_{a_{k_1}}(\varepsilon x^0_{k_1}(t)) \right] = 2 \tag{A.28a}$$

Let $\beta > 1/2$ and let $\mathscr{E}_t(\beta)$ be the characteristic function defined in Eq. (A.23a). We have that

$$\max_k |m_{a_k}(\varepsilon x_k(t)) - m_{a_k}(\varepsilon x^0_k(t))| \leqslant \bar{m}(\varepsilon^{1-\beta} + \max_k |m_{a_k}(\varepsilon x_k(t)) - m_{a_k}(\varepsilon x^0_k(t))| (1 - \mathscr{E}_t(\beta))) \tag{A.28b}$$

where

$$\bar{m} = \sup_r \left\{ \max \left(\frac{d}{dr} m_+(r), \frac{d}{dr} m_-(r) \right) \right\}$$

From Lemma A.4, Eq. (A.23b) and Eqs. (A.27), (A.28) and Lemma A.2, Eq. (A.7c),

$$\text{e.h.s. of Eq. (A.26)} \leqslant b\varepsilon^{1-\beta} \sum_{i=1}^{n} \mathbb{E}(\mathscr{B}_i(t)) \leqslant \hat{b}\varepsilon^{2-\beta} \tag{A.29}$$

where b, $\hat{b}$ are suitable positive constants. ∎

Computation of S_E. From Eqs. (A.15), (A.16), and the definition of Q_ε it follows that

$$S_E = Q_\varepsilon\left(\mathscr{C}(t) \sum_{a \in \{-1,1\}^N} h^0 \left[\prod_{k=1}^{N} m_{a_k}(\varepsilon x_k(t)) - \prod_{k=1}^{N} m_{a_k}(\varepsilon x_k^0(t))\right]\right) \tag{A.30}$$

where $N = \mathscr{N}_t^0$ and $h^0 = \prod_{i=1}^{N} h_i(\mathscr{A}_t^0(i); a^i)$. We fix $\beta \in (1/2, 2/3]$ and we expand the function $\sum_a h^0 \prod_{k=1}^{N} m_{a_k}(\varepsilon x_k^0(t))$ up to the third order around the point $(\varepsilon x_1(t), \ldots, \varepsilon x_{N^0}(t))$. We obtain

$$S_E = S_E^1 + S_E^2 + R_E^1(\varepsilon) + R_E^2(\varepsilon) \tag{A.31}$$

where

$$S_E^1 \equiv Q_\varepsilon\left(\mathscr{C}(t) \sum_a h^0 \sum_{k=1}^{N} \prod_{k_1 \neq k} m_{a_{k_1}}(\varepsilon x_{k_1}(t)) \times m_{a_k}^{(1)}(\varepsilon x_k(t))\, \varepsilon(x_k(t) - x_k^0(t))\right) \tag{A.32a}$$

$$S_E^2 \equiv Q_\varepsilon\left(\mathscr{C}(t) \sum_a h^0 \sum_{k=1}^{N} \prod_{k_1 \neq k} m_{a_{k_1}}(\varepsilon x_{k_1}(t)) \times m_{a_k}^{(2)}(\varepsilon x_k(t))\, \varepsilon^2(x_k(t) - x_k^0(t))^2\right) \tag{A.32b}$$

$$R_E^1(\varepsilon) \equiv Q_\varepsilon\left(\mathscr{C}(t) \sum_a h^0 \sum_{i \neq j=1}^{N} \prod_{k \neq i,j} m_{a_k}(\varepsilon x_k(t))\, m_{a_i}^{(1)}(\varepsilon x(t)) \times m_{a_j}^{(1)}(\varepsilon x_j(t)) \frac{\varepsilon^2}{2} (x_i(t) - x_i^0(t))(x_j(t) - x_j^0(t))\right) \tag{A.32c}$$

$$R_E^2(\varepsilon) \equiv Q_\varepsilon\left(\mathscr{C}(t) \sum_a h^0 \sum_{\{n_i\}} \times \prod_{i=1}^{N} m_{a_i}^{(n_i)}(\varepsilon \bar{x}_i) \frac{1}{6} (\varepsilon(x_i(t) - x_i^0(t)))^{n_i}\right) \tag{A.32d}$$

$m_{\pm}^{(k)}$ is the kth derivative of $m_{\pm}$; $(\bar{x}_1,..., \bar{x}_N)$ is a point in between $(x_1(t),..., x_N(t))$ and $(x_1^0(t),..., x_N^0(t))$; $\sum_{\{n_i\}}$ means the sum over all integers n_i, $i=1,..., N$ such that $\sum_i n_i = 3$.

We estimate the quantities in the right-hand side of Eq. (A.31) separately: we show that $R_{\mathrm{E}}^1(\varepsilon)$ and $R_{\mathrm{G}}^2(\varepsilon)$ are remainders while S_{E}^1 and S_{E}^2 are given by

$$S_{\mathrm{E}}^q = \varepsilon \sum_{i=1}^{n} \hat{\Gamma}_q^{\varepsilon}(i) \prod_{i \notin j} \mathbb{E}^0(\sigma(x_i, t)) + \varepsilon \sum_{i,j=1}^{n} \hat{\Gamma}_q^{\varepsilon}(i, j) \times \prod_{k \neq i,j} \mathbb{E}_{\varepsilon}^0(\sigma(x_k, t)) + R_q^{\mathrm{E}}(\varepsilon), \qquad q = 1, 2 \tag{A.33}$$

where $\hat{\Gamma}_q^{\varepsilon}(i)$ (resp. $\hat{\Gamma}_q^{\varepsilon}(i, j)$) is a one-body (resp. two body) term and $R_q^{\mathrm{E}}(\varepsilon)$ is a remainder, $q=1, 2$. From this Eq. (A.13) follows.

Proof that $R_2^{\mathrm{E}}(\epsilon)$ is a Remainder. First we observe that for any $\{n_i\}$ such that $\sum_i n_i = 3$,

$$\sum_{a \in \{-1, 1\}^N} \prod_{i=1}^{N} m_{a_i}^{(n_i)}(\varepsilon \bar{x}_i) \leqslant \bar{m} 2^3 \tag{A.34a}$$

where

$$\bar{m} = \max_{k \in \{1,2,3\}} (\sup_r m_{\pm}^{(k)}(r)) \tag{A.34b}$$

Therefore we have

$$|R_2^{\mathrm{E}}(\varepsilon)| \leqslant \bar{m} \left\{ Q_{\varepsilon} \left(\mathscr{C}(t) \sum_{\{n_i\}} (\varepsilon |x_i(t) - x_i^0(t)|)^{n_i} \mathscr{E}_t(\beta) \right) + \mathbb{E}_{\varepsilon} \left(\mathscr{C}(t) \sum_{\{n_i\}} (\varepsilon |x_i(t) - x_i^0(t)|)^{n_i} (1 - \mathscr{E}_t(\beta)) \right) \right\} \tag{A.35a}$$

Using Eqs. (A.7a) and (A.23b), we have

$$\varepsilon^{-1} |R_2^{\mathrm{E}}(\varepsilon)| \leqslant b \varepsilon^{2-3\beta} \langle (\mathscr{N}_t^0)^3 \rangle \tag{A.35b}$$

where b is a positive constant and $\langle (\mathscr{N}_t^0)^3 \rangle < \infty$. [see Eqs. (A.21)]. Therefore, since $\beta < 2/3$, $R_2^{\mathrm{E}}(\varepsilon)$ is a remainder.

Proof of Eq. (A.33) for S_{E}^1. We take conditional expectation with respect to $\mathscr{A}_t^0$ and we write $\sum_{k=1}^{N} = \sum_{i=1}^{n} \sum_{\alpha \in I^0(i)}$; therefore we have

$$S_{\mathrm{E}}^1 = \varepsilon \sum_{i=1}^{n} \hat{P} \left(\sum_{\alpha \in I^0(i)} Q_{\varepsilon}^0(\mathscr{C}(t) F_t^1[x_\alpha(t) - x_\alpha^0(t)]) \right) \tag{A.36}$$

where

$$Q_\varepsilon^0(\cdot) = Q_\varepsilon(\ |\mathscr{A}_t^0) \tag{A.37a}$$

$$\hat{P} \text{ is the law of } \mathscr{A}_t^0 \tag{A.37b}$$

$$F_t^1 = \sum_a h^0 \prod_{k \neq \alpha} m_{a_k}(\varepsilon x_k(t))\, m_{a_\alpha}^{(1)}(\varepsilon x_\alpha(t)) \tag{A.37c}$$

To compute $Q_\varepsilon^0(\mathscr{C}(t)\, F_t^1(x_\alpha(t) - x_\alpha^0(t)))$ we use iteratively Eq. (A.22). Therefore, we have to compute

$$Q_\varepsilon^0\left(\mathscr{C}(t)\, F_t^1 \sum_{k<\alpha} D_{\alpha,k}^l(T_{l+1}^0)\right) \quad \text{for} \quad l \geqslant l_\alpha \tag{A.38a}$$

and

$$Q_\varepsilon^0(\mathscr{C}(t)\, F_t^1[x_\alpha(T_{l_\alpha}^0) - x_\alpha^0(T_{l_\alpha}^0)]) \tag{A.38b}$$

To compute (A.38b) we have to consider $x_{\hat{\alpha}}(T_{l_\alpha}^0) - x_{\hat{\alpha}}^0(T_{l_\alpha}^0)$, where $\hat{\alpha}$ is the parent particle of the α-particle. Since, by definition, $\hat{\alpha} \in I^0(i)$, the iteration on (A.38b) gives still quantities like (A.38a). Therefore we have to evaluate only (A.38a). On the other hand, since there is $\mathscr{C}(t)$, (A.38a) can be computed in the same way as for the exclusion process. Therefore, from Lemma 5.2 of Ref. 15 it follows that

$$Q_\varepsilon^0\left(\mathscr{C}(t)\, F_t^1 \sum_{k<\alpha} D_{\alpha,k}^l(T_{l+1})\right)$$

$$= \sum_{k<\alpha} Q_\varepsilon^0(\mathscr{C}(t)\, F_t^1\, \mathbb{1}(\{n_l(\alpha, k) \geqslant 1\})\, \chi_l(\alpha, k)) \tag{A.39a}$$

where

$$n_l(\alpha, k) = \#(\{s \in (T_l^0, T_{l+1}^0) : |x_\alpha(s) - x_k(s)| = 1\}) \tag{A.39b}$$

and

$$\chi_l(\alpha, k) = \begin{cases} 1 & \text{if} \quad \operatorname{sgn}[x_\alpha(T_l^0) - x_k(T_l^0)] = \operatorname{sgn}[x_\alpha(T_{l+1}^0) - x_k(T_{l+1}^0)] \\ -1 & \text{otherwise} \end{cases} \tag{A.39c}$$

$\operatorname{sgn} r = \text{sign of } r.$

Equation (A.39) concludes the estimate of S_{E}^1. In fact, from Eq. (A.39), it follows that at the end of the iteration we are left with

$$S_{\mathrm{E}}^1 = \varepsilon \sum_{i=1}^n \hat{P}\left(\sum_{\substack{\alpha_1, \alpha_2 \in I^0(i) \\ \alpha_2 < \alpha_1}} \sum_{l=1}^N Q_\varepsilon^0(\mathscr{C}(t)\, F_t^1\, \hat{\chi}_l(\alpha_1, \alpha_2))\right)$$

$$+ \varepsilon \sum_{\substack{i,j=1 \\ i \neq j}}^n \hat{P}\left(\sum_{\alpha_1 \in I^0(i)} \sum_{\substack{\alpha_2 \in I^0(j) \\ \alpha_2 < \alpha_1}} \sum_{l=1}^N Q_\varepsilon^0(\mathscr{C}(t)\, F_t^1\, \hat{\chi}_l(\alpha_1, \alpha_2))\right) \tag{A.40}$$

where $N = \frac{1}{2}(\mathcal{N}_t^0 - n)$ and $\hat{\chi}_l(\alpha_1, \alpha_2)$ is a bounded function of the trajectories of the α_1 and α_2 interacting particles. Note that in writing Eq. (A.40) the fact has been used that if $\alpha_2 < \alpha_1$ then the α_2 particle starts its motion before the α_1 particle, i.e., $l_{\alpha_2} \leqslant l_{\alpha_1}$ [see Eq. (A.22)]. Furthermore, the function $\hat{\chi}_l(\alpha_1, \alpha_2)$ can be explicitly computed: $\hat{\chi}_l(\alpha_1, \alpha_2) = \mathbb{1}\{n_l(\alpha_1, \alpha_2) \geqslant 1\}\, \chi_l(\alpha_1, \alpha_2)$, for $l \geqslant l_{\alpha_1}$; for $l_{\hat{\alpha}_1} \leqslant l \leqslant l_{\alpha_1}$, we have $\hat{\chi}_l(\alpha_1, \alpha_2) = 0$ if $\alpha_2 > \hat{\alpha}_1$ and $\hat{\chi}_l(\alpha_1, \alpha_2) = \mathbb{1}\{n_l(\hat{\alpha}_1, \alpha_2) \geqslant 1\}\, \chi_l(\hat{\alpha}_1, \alpha_2)$ if $\alpha_2 < \alpha_1$ ($\hat{\alpha}_1$ is the parent particle of the α_1 particle), and so on.

Since F_t and $\hat{\chi}_l$ are bounded functions, from Lemma A.2, Eq. (A.7a) and Lemma A.4, Eq. (A.24), it follows that

$$\lim_{\varepsilon \to 0} |Q_\varepsilon^0(\mathscr{C}(t)\, F_t^1 \hat{\chi}_l(\alpha_1, \alpha_2)) - Q_\varepsilon^0(F_t^0 \hat{\chi}_l^0(\alpha_1, \alpha_2))| = 0 \tag{A.41}$$

where F_t^0 and $\hat{\chi}_l^0$ are the same functions as F_t^1 and $\hat{\chi}_l$ but evaluated in the configuration of the independent process. Finally, since $\langle (\mathcal{N}_t^0)^2 \rangle < \infty$ [see Eq. (A.21)], by the Lebesgue dominant convergence theorem, from Eqs. (A.40) and (A.41), Eq. (A.33) for S_{E}^1 follows. ∎

Proof of Eq. (A.33) for S_{E}^2. We use the same argument as in the estimate of S_{E}^1.

$$S_{\mathrm{E}}^2 = \varepsilon \sum_{i=1}^{n} \hat{P}\left(\sum_{\alpha \in I^0(i)} Q_\varepsilon^0(\mathscr{C}(t)\, F_t^2 \varepsilon (x_\alpha(t) - x_\alpha^0(t))^2) \right) \tag{A.42}$$

where Q_ε^0 and $\hat{P}$ are defined in Eqs. (A.37a) and (A.37b) respectively and

$$F_t^2 = \sum_a h^0 \prod_{k \neq \alpha} m_{a_k}(\varepsilon x_k(t))\, m_{a_\alpha}^{(2)}(\varepsilon x_\alpha(t)) \tag{A.43}$$

We use Eq. (A.22); therefore we have to compute

$$Q_\varepsilon^0\left(\mathscr{C}(t)\, F_t^2 \varepsilon \sum_{k<\alpha} D_{\alpha,k}^l (T_{l+1}^0)^2 \right) \tag{A.44a}$$

$$\varepsilon Q_\varepsilon^0\left(\mathscr{C}(t)\, F_t^2 \sum_{k_1 < \alpha} \sum_{\substack{k_2 < \alpha \\ k_2 \neq k_1}} D_{\alpha,k_1}^l (T_{l+1}^0)\, D_{\alpha,k_2}^l (T_{l+1}^0) \right) \tag{A.44b}$$

$$\varepsilon Q_\varepsilon^0\left(\mathscr{C}(t)\, F_t^2 (x_\alpha(T_l^0) - x_\alpha(T_l^0)) \sum_{k<\alpha} D_{\alpha,k}^l (T_{l+1}^0) \right) \tag{A.44c}$$

We show that only (A.44a) gives a nonvanishing contribution to S_{E}^2. From Lemma 5.2 of Ref. 15 it follows that

$$Q_\varepsilon^0(\mathscr{C}(t)\, F_t^2 \varepsilon D_{\alpha,k}^l (T_{l+1}^0)^2) = Q_\varepsilon^0(\mathscr{C}(t)\, F_t^2 \varepsilon n_l(\alpha, k)) \tag{A.45a}$$

$$\varepsilon\,|Q_\varepsilon^0(\mathscr{C}(t)\,F_t^2\,D_{\alpha,k_1}^l(T_{l+1}^0)\,D_{\alpha,k_2}^l(T_{l+1}^0))\leqslant 2\varepsilon Q_\varepsilon^0(\mathscr{C}(t)\,|F_t^2|) \tag{A.45b}$$

$$\begin{aligned}&\varepsilon Q_\varepsilon^0(\mathscr{C}(t)\,F_t^2(x_\alpha(T_l^0)-x_\alpha^0(T_l^0))\,D_{\alpha,k}^l(T_{l+1}^0))\\&\quad=Q_\varepsilon^0(\mathscr{C}(t)\,F_t^2(x_\alpha(T_l^0)-x_\alpha^0(T_l^0))\,\mathbb{1}\{n_l(\alpha,k)\geqslant 1\}\,\chi_l(a,k))\end{aligned} \tag{A.45c}$$

where $n_l(\alpha, k)$ and $\chi_l(\alpha, k)$ are defined in Eqs. (A.39b) and (A.39c) respectively.

From Eq. (A.45b) it follows that there is $b>0$ such that

$$|(\text{A.44b})|\leqslant\varepsilon Q_\varepsilon^0(\mathscr{C}(t)\,|F_t^2|)\leqslant\varepsilon b \tag{A.46}$$

From Eq. (A.45c) and Lemma A.4, it follows that

$$\begin{aligned}|(\text{A.44c})|&\leqslant\varepsilon Q_\varepsilon^0(\mathscr{C}(t)\,|F_t^2|\;\;|x_\alpha(T_l^0)-x_\alpha^0(T_l^0)|\;\mathscr{E}_t(\beta))\\&\quad+\hat{d}\exp\{-d'\varepsilon^{-(\beta-1/2)}\}\\&\leqslant\hat{b}\varepsilon^{1-\beta}\end{aligned} \tag{A.47}$$

where $\hat{d}$, $\hat{d}'$, and $\hat{b}$ are suitable positive constants. To evaluate (A.44a), we use Lemma 5.4 of Ref. 15 where it has been proven that for any bounded function f_t

$$\lim_{\varepsilon\to 0} Q_\varepsilon^0(f_t\varepsilon n_l(\alpha,k)-f_t^0\varepsilon n_l^0(\alpha,k))=0 \tag{A.48a}$$

and

$$\lim_{\varepsilon\to 0} Q_\varepsilon^0(f_t^0\varepsilon n_l^0(\alpha,k))<\infty \tag{A.48b}$$

where f_t^0 is the same function as f_t but evaluated in the configuration of the independent process. We use Eqs. (A.48) for $f_t=\mathscr{C}(t)\,F_t^2$. From Eqs. (A.45a) and (A.48) it follows that the same argument used in the estimate of S_{E}^1 shows that Eq. (A.33) for S_{E}^2 holds.

Proof That $R_{\mathrm{E}}^1(\epsilon)$ [see Eq. (A.32c)] is a remainder. The proof follows by using Eq. (A.22) and the same iterative argument used in the estimate of S_{E}^2. In fact, the iteration gives quantities like the ones in (A.44b) and (A.44c) and also the following:

$$\varepsilon Q_\varepsilon^0\left(\mathscr{C}(t)\,F_t\sum_{\eta_1<\alpha_1}\sum_{\eta_2<\alpha_2}D_{\alpha_1,\eta_1}^l(T_{l+1}^0)\,D_{\alpha_2,\eta_2}(T_{l+1}^0)\right) \tag{A.49}$$

In Lemma 5.2 of Ref. 15 it has been proven that

$$\begin{aligned}Q_\varepsilon^0(\mathscr{C}(t)\,F_t D_{\alpha_1,\eta_1}^l(T_{l+1})\,D_{\alpha_2,\eta_2}^l(T_{l+1}))&=Q_\varepsilon^0(\mathscr{C}(t)\,F_t\mathbb{1}\{n_l(\alpha_1,\eta_1)\geqslant 1\\&\quad\chi_l(\alpha_1,\eta_1)\,\mathbb{1}\{n_l(\alpha_2,\eta_2)\geqslant 1\}\,\chi_l(\alpha_2,\eta_2))\end{aligned} \tag{A.50}$$

Therefore from Eq. (A.50) it follows that $R_{\mathrm{E}}^1(\varepsilon)$ is a remainder. ∎

Remark A.5. In the same way as for the exclusion process (see Theorem 5.1 of Ref. 15) it can be proven that $\forall t \geqslant 0$ fixed, the distribution at time t of the G.E. process with starting measure μ^ε satisfying Definition 3.2 (also at first order in ε) is equal to the distribution at time t with starting measure a product measure satisfying Eq. (A.2). Therefore the proof of Proposition 3.4 in the general case follows from the one given above.

ACKNOWLEDGMENT

We thank Errico Presutti, Herman Rost, and Herbert Spohn for some very useful discussions.

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