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Flux Fluctuations in the One Dimensional Nearest Neighbors Symmetric Simple Exclusion Process

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Let $J(t)$ be the the integrated flux of particles in the symmetric simple exclusion process starting with the product invariant measure ν_ρ with density ρ. We compute its rescaled asymptotic variance:

$$\lim_{t\to\infty} t^{-1/2}\mathbb{V}J(t)=\sqrt{2/\pi}\,(1-\rho)\,\rho$$

Furthermore we show that $t^{-1/4}J(t)$ converges weakly to a centered normal random variable with this variance. From these results we compute the asymptotic variance of a tagged particle in the nearest neighbor case and show the corresponding central limit theorem.

KEY WORDS: Symmetric simple exclusion process; flux fluctuations.

RESULTS

The nearest neighbors symmetric simple exclusion process describes the evolution of particles sitting at the sites of $\mathbb{Z}$ evolving as follows. At most one particle is allowed at each site. If there is a particle at a given site, at rate one the particle chooses one of its nearest neighbor sites with probability $1/2$ and attempts to jump to this site. The jump is effectively realized if the destination site is empty; if not, the jump is suppressed. A formal definition using Poisson processes is given below. The generator of the process is given by

$$Lf(\eta)=\tfrac{1}{2}\sum_{x\in\mathbb{Z}}\,[f(\eta^{x,x+1})-f(\eta)] \tag{1}$$

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where $\eta^{x,x+1}(x)=\eta(x+1)$, $\eta^{x,x+1}(x+1)=\eta(x)$ and $\eta^{x,x+1}(y)=\eta(y)$ for $y\neq x$, $x+1$. For each $\rho\in[0,1]$ the product measure ν_ρ with density ρ is invariant for the process.

For an initial configuration η let the integrated flux of particles $J(t)=J^\eta(t)$ be the number of particles to the left of the origin at time zero and to the right of it at time t minus the number of particles to the right of the origin at time 0 and to the left of it at time t.

Fix $\rho\in(0,1)$ and let the initial configuration have law ν_ρ. Let

$$\mathbb{V}J(t)=\mathbb{E}^{\nu_\rho}J(t)^2=:\int d\nu_\rho(\eta)(J_t^\eta)^2$$

(Notice that $\mathbb{E}^{\nu_\rho}J(t)=0$.)

We prove the following asymptotics for the variance:

$$\lim_{t\to\infty} t^{-1/2}\mathbb{V}J(t)=\sqrt{2/\pi}\,(1-\rho)\,\rho:=\sigma^2 \tag{2}$$

We then prove the following central limit theorem for the integrated flux:

$$t^{-1/4}J(t)\ \text{ converges weakly to } \mathcal{N}(0,\sigma^2) \tag{3}$$

where $\mathcal{N}(0,\sigma^2)$ is a centered normal random variable with variance σ^2.

Finally, let $X(t)$ be the position of a tagged particle interacting by exclusion. We show that if the initial configuration is chosen with the product measure ν_ρ, then

$$\lim_{t\to\infty} t^{-1/2}\mathbb{E}^{\nu_\rho}(X(t)-\rho^{-1}J(t))^2=0 \tag{4}$$

An immediate consequence of (2), (3) and (4) is that, defining $\mathbb{V}X(t)=\mathbb{E}^{\nu_\rho}(X(t))^2$, the asymptotic variance of the tagged particle is

$$\lim_{t\to\infty} t^{-1/2}\mathbb{V}X(t)=\sqrt{2/\pi}\,\frac{1-\rho}{\rho}:=\bar{\sigma}^2 \tag{5}$$

and the tagged particle satisfies a central limit theorem:

$$t^{-1/4}X(t)\text{ converges weakly to } \mathcal{N}(0,\bar{\sigma}^2) \tag{6}$$

The limits (5) and (6) were proven by Arratia.[1]

To prove the above results we use the stirring motion representation of the symmetric exclusion process introduced by Harris[2] and used by Arratia to prove (5) and (6).

THE STIRRING PROCESS

The stirring process $z(i, t) \in \mathbb{Z}$, $i \in \mathbb{Z}$, is defined as follows. At time $t = 0$ put a (labeled) particle at each site and define $z(i, 0) = i$ for all $i \in \mathbb{Z}$. With each bond $(x, x+1)$, $x \in \mathbb{Z}$ associate a Poisson process (clock) with parameter $1/2$. When the clock rings at the bond $(x, x+1)$ the particles at those sites interchange their positions. $z(i, t)$ is the position at time t of the particle sitting at i at time 0. Given an initial configuration $\eta \in \mathscr{X}$, it is possible to define the simple exclusion process η_t in terms of the stirring process by setting

$$\eta_t(x) = \mathbf{1}\{x \in \{z(i, t) : \eta(i) = 1\}\} \tag{7}$$

FIRST PROOF OF (2)

In terms of the stirring process, we define the following random variables.

$$K^+(t) = \sum_{i \leqslant 0} \mathbf{1}\{z(i, t) > 0\}; \qquad K^-(t) = \sum_{i > 0} \mathbf{1}\{z(i, t) \leqslant 0\} \tag{8}$$

where $\mathbf{1}\{\cdot\}$ is the characteristic function of the set $\{\cdot\}$. The variable $K^+(t)$ is the number of stirring particles starting at the left of the point $1/2$ and sitting at time t at the right of $1/2$. The variable $K^-(t)$ is the number of stirring particles starting at the right of the point $1/2$ and sitting at time t at the left of $1/2$. Since at all times all sites are occupied by one stirring particle, each crossing of the point $1/2$ from left to right involves a simultaneous crossing in the opposite direction and viceversa. So $K^+(t) - K^-(t)$ is constant in t and since $K^+(0) = K^-(0) = 0$, $K^+(t) = K^-(t) := K(t)$, for all $t \geqslant 0$. In the stirring process the representation of $J(t)$ is given by

$$J(t) = \sum_{i \leqslant 0} \mathbf{1}\{z(i, t) > 0\}\, \eta(i) - \sum_{i > 0} \mathbf{1}\{z(i, t) \leqslant 0\}\, \eta(i) \tag{9}$$

Let $i_1 < i_2 < \cdots < i_{K(t)} \leqslant 0$ be the random sites for which $z(i_k, t) > 0$ and $0 < j_1 < j_2 < \cdots < j_{K(t)}$ be the random sites for which $z(j_k, t) \leqslant 0$. Define $B^+(k) = \eta(i_k)$ and $B^-(k) = \eta(j_k)$ and $A(k) = B^+(k) - B^-(k)$. Thus

$$J(t) = \sum_{k=1}^{K(t)} A(k) \tag{10}$$

Assume η is distributed according to the product measure ν_ρ. Then the variables $B^+(k)$, $B^-(k)$ and $K(t)$ are independent. Hence $A(k)$ are iid independent of $K(t)$ with law

$$\mathbb{P}(A(k)=1)=\mathbb{P}(A(k)=-1)=\rho(1-\rho); \qquad \mathbb{P}(A(k)=0)=1-2\rho(1-\rho) \tag{11}$$

Thus $\mathbb{E}A(k)=0$, $\mathbb{E}A(k)^2=2\rho(1-\rho)$ and by independence, using (10) we have

$$\mathbb{E}^{\nu_\rho}J(t)^2=\mathbb{E}A(k)^2\,\mathbb{E}K(t) \tag{12}$$

To compute $\mathbb{E}K(t)$ write

$$\mathbb{E}K(t)=\sum_{i\leqslant 0}\mathbb{P}(z(i,t)>0)=\sum_{i\geqslant 0}\mathbb{P}(z(0,t)>i)=\mathbb{E}(z(0,t))^+$$

But $z(0,t)$ is a symmetric random walk, thus, since $t^{-1}\mathbb{E}z(0,t)^2$ is uniformly integrable,

$$\lim_{t\to\infty}t^{-1/2}\mathbb{E}K(t)=\frac{1}{\sqrt{2\pi}} \tag{13}$$

Thus, using (12) we obtain (2).

SECOND PROOF OF (2)

From the definition we have

$$J(t)-\int_0^t \tfrac{1}{2}(\eta_s(0)-\eta_s(1)):=M(t) \tag{14}$$

where $M(t)$ is a martingale with variance

$$\mathbb{E}^{\nu_\rho}M(t)^2=t\rho(1-\rho) \tag{15}$$

As in De Masi *et al.*,[3,4] from the time invariance of ν_ρ and the fact that $J(t)$ is an anti-symmetric random variable, it follows that

$$\mathbb{E}^{\nu_\rho}J(t)^2=t\rho(1-\rho)-\tfrac{1}{2}\int_0^t ds(t-s)\int\nu_\rho(d\eta)(\eta(0)-\eta(1))\,\mathbb{E}(\eta^\eta_s(0)-\eta^\eta_s(1)) \tag{16}$$

where η_s^η is the exclusion process with initial configuration η. From the reversibility and the translation invariance of ν_ρ,

$$\int \nu_\rho(d\eta)(\eta(0)-\eta(1))\ \mathbb{E}(\eta_s^\eta(0)-\eta_s^\eta(1))$$
$$= 2\int \nu_\rho(d\eta)(\eta(0)\ \mathbb{E}\eta_s^\eta(0)-\eta(0)\ \mathbb{E}\eta_s^\eta(1)) \tag{17}$$

Calling L the generator of the process we have that

$$L\eta(0)=\tfrac{1}{2}[\eta(1)-\eta(0)]+\tfrac{1}{2}[\eta(-1)-\eta(0)] \tag{18}$$

Therefore, using once more translation invariance

$$2\int \nu_\rho(d\eta)(\eta(0)\ \mathbb{E}\eta_s^\eta(0)-\eta(0)\ \mathbb{E}\eta_s^\eta(1)) = -2\frac{d}{ds}\left(\int \nu_\rho(d\eta)\ \eta(0)[\mathbb{E}\eta_s^\eta(0)-\rho]\right) \tag{19}$$

We use (17) and (19) in the second term on the right hand side of (16) then, integrating by parts, we get

$$\int_0^t (t-s)\frac{d}{ds}\left(\int \nu_\rho(d\eta)\ \eta(0)\ \mathbb{E}\eta_s^\eta(0)-\rho^2\right)$$
$$= -t\rho(1-\rho)+\int_0^t\int \nu_\rho(d\eta)\ (\eta(0)-\rho)\ \mathbb{E}\eta_s^\eta(0) \tag{20}$$

From (16) and (20) we finally get

$$\mathbb{E}^{\nu_\rho}J(t)^2=\int_0^t\int \nu_\rho(d\eta)\ (\eta(0)-\rho)\ \mathbb{E}\eta_s^\eta(0) \tag{21}$$
$$=\rho(1-\rho)\ R_t(0) \tag{22}$$

where $R_t(0)$ is the expected amount of time spent at the origin up to time t for a continuous time symmetric random walk starting at zero. Finally,

$$\lim_{t\to\infty} t^{-1/2}R_t(0)=\sqrt{2/\pi} \tag{23}$$

PROOF OF (3)

To show (3) from (10) it is enough to show that

$$C(t):=t^{-1/4}\left(\sum_{k=1}^{K(t)} A(k)-\sum_{k=1}^{t^{1/2}/\sqrt{2\pi}} A(k)\right)\to 0 \qquad \text{as} \quad t\to\infty \tag{24}$$

in measure. Using Chebishev inequality we have, for any $c > 0$,

$$\mathbb{P}(C(t) > c) \leqslant \frac{\mathbb{E}A(k)^2}{c^2}\,\mathbb{E}\left|\frac{K(t)}{t^{1/2}} - \frac{1}{\sqrt{2\pi}}\right| \to 0 \qquad \text{as} \quad t \to \infty \tag{25}$$

The limit goes to zero because $K(t)$ is the sum of negatively correlated 0–1 random variables and so $\mathbb{V}K(t) \leqslant \mathbb{E}K(t) \sim \sqrt{t}$ Arratia[1] and by Schwarz inequality.

PROOF OF (4)

We use a lattice version of a result of Dürr *et al.*[5] for an infinite ideal gas of point particles on $\mathbb{R}$. Suppose that the initial configuration η is distributed according to the invariant measure ν_ρ. Fix $t \geqslant 0$. For $k \geqslant 0$ let $Y_k(t)$ be the position of the kth particle of η_t to the right of $1/2$, with $Y_0(t) \leqslant 0$. For $k < 0$ let $Y_k(t)$ be the position of the $-(k+1)$th particle of η_t to the left of $1/2$. (When time goes on the particles change these labels.) It is easy to see that at time t the tagged particle (which at time $t = 0$ is labeled 0) is the $J(t)$th particle, that is:

$$X(t) = Y_{J(t)}(t) \tag{26}$$

By the ergodicity (under translations) and stationarity of ν_ρ we have that

$$\lim_{n \to \infty} n^{-1} Y_n(t) = \rho^{-1}, \qquad \mathbb{P}^{\nu_\rho}\text{-almost surely} \tag{27}$$

One can then prove (as in Lemma 2.8 of Dürr *et al.*[5]) that

$$\lim_{t \to \infty} t^{1/2} \mathbb{E}^{\nu_\rho}(Y_{J(t)}(t) - \rho^{-1} J(t))^2 = 0 \tag{28}$$

REMARK

This work was written when the authors visited Rutgers University in 1985 and was kept unpublished for more than 15 years. We decided to publish it now for three reasons. The first proof of (2) is an application of Arratia's method, but it is not written anywhere; in fact, it is easier first to compute the variance of the flux and then, as a corollary, the variance of the tagged particle than vice versa. The second proof of (2) is the unique application we know of the method of De Masi *et al.*[3, 4] that works for a subdiffussive process. Finally, the flux in the simple exclusion process is isomorphic to a $1+1$ dimensional interface. The role of the entropic repulsion when this interface interacts by exclusion with a wall has been studied by Dunlop *et al.*,[6] who compare the asymptotic variance of the flux for the process starting with the deterministic configuration ...101010... with the stationary process studied here.

Presumably our result can be obtained using the fact that the asymptotic behavior of the current can be deduced from the hydrodynamic behavior of the symmetric simple exclusion and the asymptotics of the density fluctuation field at equilibrium. This technique has been introduced by Rost and Vares[7] and applied to the zero range process by Landim *et al.*[8, 9] and Landim and Volchan.[10] We are not aware of any application of this argument to our case and after consulting Landim and Olla it seems that their results do not cover, at least automatically, ours. We thank an anonymous referee, Errico Presutti, Claudio Landim and Stefano Olla for pointing out this possibility.

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