Hydrodynamic Limit for Interacting Neurons

A. De Masi · A. Galves · E. Löcherbach · E. Presutti

Abstract This paper studies the hydrodynamic limit of a stochastic process describing the time evolution of a system with N neurons with mean-field interactions produced both by chemical and by electrical synapses. This system can be informally described as follows. Each neuron spikes randomly following a point process with rate depending on its membrane potential. At its spiking time, the membrane potential of the spiking neuron is reset to the value 0 and, simultaneously, the membrane potentials of the other neurons are increased by an amount of potential $\frac{1}{N}$. This mimics the effect of chemical synapses. Additionally, the effect of electrical synapses is represented by a deterministic drift of all the membrane potentials towards the average value of the system. We show that, as the system size N diverges, the distribution of membrane potentials becomes deterministic and is described by a limit density which obeys a non linear PDE which is a conservation law of hyperbolic type.

Keywords Hydrodynamic limit · Piecewise deterministic Markov process · Biological neural nets

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1 Introduction

This paper studies the hydrodynamic limit of a continuous time stochastic process describing a system of interacting neurons. The system we consider is made of N neurons whose state is specified by $U^N(t) = (U_1^N(t), \ldots, U_N^N(t))$, $t \geq 0$, $U^N(t) \in \mathbb{R}_+^N$, for some fixed integer $N \geq 1$. Each $U_i^N(t)$ models the membrane potential of neuron i at time t, for $i = 1, \ldots, N$. Neurons interact either by *chemical* or by *electrical* synapses. Our model does not consider external stimuli.

Chemical synapses can be described as follows. Each neuron spikes randomly following a point process with rate depending on the membrane potential of the neuron. At its spiking time, the membrane potential of the spiking neuron is reset to a reversal potential. At the same time, simultaneously, the other neurons, which do not spike, receive an additional amount of potential $\frac{1}{N}$ which is added to their membrane potential.

Electrical synapses occur through *gap-junctions* which allow neurons in the brain to communicate directly. This induces an attraction between the values of the membrane potentials and, as a consequence, a drift of the system towards its average membrane potential.

Our model is a continuous time version of a new class of biological neuronal systems introduced recently by [1]. The model considered in [1] is a non Markovian system consisting of an infinite number of interacting chains where each component has memory of variable length and where each neuron is represented through its spike train. In the present paper we add gap junctions to this system and we adopt an equivalent description via the membrane potential of each neuron, leading to a Markovian process.

The number of neurons in the brain is huge and often neurons have similar properties (see [2], Chap. 1.5.1). Therefore we assume that we are in an idealized situation where all neurons have identical properties, leading to a mean field description. The mean field assumption appears in the following aspects. For the chemical synapses it is translated into the fact that when a neuron spikes the membrane potential of any other neuron increases by $1/N$. For the electrical synapses, the mean field type assumption implies that the drift felt by each neuron potential is described by a linear attraction towards the average membrane potential of the system.

We regard the state of the neurons $U^N(t) = (U_1^N(t), \ldots, U_N^N(t))$ as a distribution of $1/N$ valued Dirac *masses* placed at the *positions* $U_1^N(t), \ldots, U_N^N(t)$. The main result of the present paper, presented in Theorem 2, is that in the limit as $N \to \infty$ this membrane potential distribution becomes deterministic and it is described by a density $\rho_t(r)$. More precisely, in the limit, for any interval $I \subset \mathbb{R}_+$, $\int_I \rho_t(r)dr$ is the limit fraction of neurons whose membrane potentials are in I at time t. The limit density $\rho_t(r)$ is proved to obey a non linear PDE which is a conservation law of hyperbolic type.

The usual approach to prove hydrodynamic limits in mean field systems is to show that propagation of chaos holds. In our case this amounts to prove that the membrane potentials $U_i^N(t)$ and $U_j^N(t)$ of any pair i and j of neurons get uncorrelated as $N \to \infty$. However, at each time that another neuron fires, it instantaneously affects both U_i^N and U_j^N by changing them with an additional amount $1/N$. Thus U_i^N and U_j^N are correlated, and propagation of chaos comes only by proving first that the firing activity of the other neurons—by propagation of chaos—is essentially deterministic. We are thus caught in a circular argument and it is not clear *a priori* that propagation of chaos holds. It is for this reason that in this paper we introduce an auxiliary process $Y^{(\delta)}$ which is a good approximation of the true process in the $N \to \infty$ limit, and for which it is easy to prove the hydrodynamic limit. Once the convergence for $Y^{(\delta)}$ is proved, we can then conclude by letting $\delta \to 0$.

Our model is an example of the class of processes introduced by [3] under the name of *piecewise deterministic Markov processes*. Processes in this class combine a deterministic continuous motion (in our case, due to the electrical synapses) with discontinuous, random jump events (in our case, the spike events). This is not the first time that piecewise deterministic Markov processes are used in the modelization of neuronal systems, see for instance [4] and [5] in which processes of this type appear, however in a different context.

The mean field approach intending to replace individual behavior in large homogeneous systems of interacting neurons by the mean behavior of the neuronal population has a long tradition in the frame of neural networks, see e.g. Chap. 6 of [2] or [6] and the references therein. Most of the models used in the literature are either based on rate models where randomness comes in through random synaptic weights (see e.g. [7] or [8]); or they are based on populations of integrate and fire neurons which are diffusion models in either finite or infinite dimension, see for instance [9] or [10]. The model we consider is reminiscent of integrate-and-fire models but firing does not occur when reaching a fixed threshold, and the membrane potential is not described by a diffusion process. In particular, the equation which we obtain is different from usual population density equations obtained for integrate-and-fire neurons as considered e.g. in Chap. 6.2.1 of [2].

Our paper is organized as follows. In Sect. 2 we introduce the process and state the main results, Theorem 1, Theorem 2 and Theorem 3. Theorem 1 guarantees the existence of the process and gives upper bounds on the values of the potentials U^N which are uniform in N. Theorems 2 and 3 give existence and properties of the hydrodynamic limit.

Proofs are organized as follows: we first study the system under very restrictive assumptions on the firing rate f, in such a case the proof of Theorem 1 becomes trivial and is given in Sect. 3. Even with such an assumption on f the proof of Theorems 2 and 3 remains rather complex. In Sect. 4, tightness of the sequence of processes indexed by N is proved. Section 5 introduces the sequence of auxiliary processes, and Sect. 6 states the hydrodynamic limit theorem for this sequence; the proof is postponed to Appendix 3. Section 7 concludes the proof of Theorems 2 and 3. In the Appendix we extend the result to general firing rates f. The main point is the proof of Theorem 1 which is given in Appendix 1, together with some upper bounds for the maximal membrane potential of the process in the case of unbounded firing rate functions. In Appendix 2 we prove that the auxiliary process is close to the original one, if both are suitably coupled. Finally, in Appendix 3, the hydrodynamic limit for the auxiliary process is rigorously proved.

2 Model Definition and Main Results

We consider a Markov process

$$U^N(t) = (U_1^N(t), \ldots, U_N^N(t)), \ t \geq 0,$$

taking values in $\mathbb{R}_+^N$, for some fixed integer $N \geq 1$, whose generator is given for any smooth test function $\varphi : \mathbb{R}_+^N \to \mathbb{R}$ by

$$L\varphi(x) = \sum_{i=1}^N f(x_i) \left[\varphi(x + \Delta_i(x)) - \varphi(x)\right] - \lambda \sum_i \left(\frac{\partial \varphi}{\partial x_i}(x) \left[x_i - \bar{x}\right]\right), \qquad (2.1)$$

where

$$(\Delta_i(x))_j = \left\{ \begin{array}{ll} \frac{1}{N} & j \neq i \\ -x_i & j = i \end{array} \right\}, \quad \bar{x} = \frac{1}{N} \sum_{i=1}^N x_i \qquad (2.2)$$

and where $\lambda \geq 0$ a positive parameter. Assume that

Assumption 1 $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is strictly positive for $x > 0$ and non-decreasing. Moreover, $f(0) = 0$ and f is not flat, i.e. for any fixed $u \in]0, 1[$,

$$\lim_{x \to 0} \inf \frac{f(ux)}{f(x)} > 0.$$

The function $f(x) = x^p$, $p > 0$, satisfies the above assumption. We can also consider functions $f(x) = e^{\nu x} - 1$, for some $\nu > 0$.

In (2.1), the first term describes random jumps at rate $f(x_i)$ due to spiking of neurons having potential x_i. The function f is therefore called firing rate or spiking rate of the system. The second term, due to electrical synapses (gap junctions), describes a deterministic time evolution tending to attract the neurons to the common average potential.

Our first theorem proves the existence of the process and gives some a priori estimates on the maximal membrane potential. In order to state these results, we introduce the following notation. Let $N_i(t), t \geq 0$, be the simple point process on $\mathbb{R}_+$ which counts the jump events of neuron i up to time t and let

$$N(t) = \sum_{i=1}^{N} N_i(t) \tag{2.3}$$

be the total number of jumps seen before time t. For any $x \in \mathbb{R}^N$, we define $\|x\| = \max_{i=1,\ldots,N} x_i$. In this way,

$$\left\| U^N(t) \right\| = \max_{i=1,\ldots,N} U_i^N(t)$$

is the maximal membrane potential at time t.

Theorem 1 *Let f be a firing rate function satisfying Assumption 1.*

1. *For any $N \geq 1$ and any $x \in \mathbb{R}_+^N$ there exists a unique strong Markov process $U^N(t)$ taking values in $\mathbb{R}_+^N$ starting from x whose generator is given by (2.1).*
2. *Denote by $P_x^{(N,\lambda)}$ the probability law under which the process $U^N(t)$ starts from the initial configuration $U^N(0) = x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$. Then for any $A > 0$ and $T > 0$ there exists B such that*

$$\sup_{x:\|x\| \leq A} P_x^{(N,\lambda)} \left[\sup_{t \leq T} \|U^N(t)\| < B \right] \geq 1 - ce^{-CN}, \tag{2.4}$$

where c and C are suitable constants.

The proof of Theorem 1 is given in the Appendix 1.

We now give the main result of this paper. It shows that the process converges in the hydrodynamic limit, as $N \to \infty$, to a specified evolution which will be defined below. Since the space where $U^N(t)$ takes values changes with N it is convenient to identify configurations $U^N(t)$ with the associated empirical measure in the following way. Let $\mathcal{M}$ be the space of all probability measures on $\mathbb{R}_+$. To any $x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$ we associate the element of $\mathcal{M}$ given by

$$\mu_x = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}. \tag{2.5}$$

μ_x has the nice physical-biological interpretation of being the distribution of membrane potentials of the neurons.

We suppose that for all N, $U_i^N(0) = x_i^N$, $i = 1, \ldots, N$, such that the following assumption is satisfied.

Assumption 2 $x_1^N, \ldots, x_N^N$ are i.i.d. random variables, distributed according to $\psi_0(x)dx$ on $\mathbb{R}_+$. Here, ψ_0 is a smooth probability density on $\mathbb{R}_+$ with compact support $[0, R_0]$ such that the following properties are verified.

1. $\psi_0 > 0$ on $[0, R_0[$.
2. $\psi_0 \equiv 0$ on $[R_0, \infty[$.
3. $\psi_0(x) \geq c(x - R_0)^2$, $c > 0$, in a left neighborhood of R_0.

The above assumption can be weakened, see Remark 5 below. Condition 3. could be relaxed to other rates of decay to 0 near R_0. We will eventually extend the definition of ψ_0 to the whole line by putting $\psi_0(x) = \psi_0(0)$ for all $x < 0$.

Identifying $U^N(t)$ with the associated probability measure $\mu_{U^N(t)}$, we may identify the process with the element $\mathbb{R}_+ \ni t \to \mu_{U^N(t)}$ of the Skorokhod space $D(\mathbb{R}_+, \mathcal{S}')$, where $\mathcal{S}$ is the Schwartz space of all smooth functions $\phi : \mathbb{R} \to \mathbb{R}$. We write $\mu_{U^N_{[0,T]}}$ for the restriction of this process to $[0, T]$ which is an element of $D([0, T], \mathcal{S}')$. Our next theorem states that $\mu_{U^N_{[0,T]}}$ converges to a deterministic limit density $(\rho_t(x)dx)_{t \in [0,T]}$. We can easily guess the equation satisfied by $\rho_t(x)$. In fact if $\rho_t(x)$ is the limit density then the limit total firing rate per unit time p_t and the limit average membrane potential $\bar{\rho}_t$ are

$$p_t = \int_0^\infty f(x)\rho_t(x)dx, \quad \bar{\rho}_t = \int_0^\infty x\rho_t(x)dx. \tag{2.6}$$

Thus

$$V(x, \rho_t) := -\lambda(x - \bar{\rho}_t) + p_t \tag{2.7}$$

is the velocity field, namely the limit drift that neurons have at time t and at energy x, the first term being the attraction to the average membrane potential of the system, due to the gap junction effect, the second one the drift produced by the other neurons spiking. Besides such a mass transport we have also a loss of mass term $f(x)\rho_t(x)$ due to spiking so that we should expect that for smooth $\rho_t(x)$

$$\frac{\partial}{\partial t}\rho_t + \frac{\partial}{\partial x}(V\rho_t) = -f\rho_t, \quad x > 0, t > 0. \tag{2.8}$$

However (2.8) does not determine the solution, it must be complemented by boundary conditions:

$$\rho_0(x) = u_0(x), \quad \rho_t(0) = u_1(t). \tag{2.9}$$

u_0 is specified by the problem: $u_0 = \psi_0$, u_1 instead must be derived together with (2.8). It turns out from our analysis that

$$u_1(t) = \frac{p_t}{V(0, \rho_t)} = \frac{p_t}{p_t + \lambda\bar{\rho}_t}. \tag{2.10}$$

(2.10) follows from conservation of mass as it will be discussed after the definition of weak solutions of (2.8)–(2.9). Indeed if $u_0(0) \neq u_1(0)$, i.e. $\psi(0) \neq \frac{p_0}{V(0, \psi_0)}$, then $\rho_t(x)$ cannot be continuous, hence the necessity of a weak formulation of (2.8)–(2.9).

Definition 1 A real valued function $\rho_t(x)$ defined on $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$ is a weak solution of (2.8)–(2.9) if for all smooth functions $\phi(x)$, $\mathbb{R}_+ \ni t \to \int \phi(x)\rho_t(x)dx$ is continuous, differentiable in $t > 0$ and

$$\frac{d}{dt} \int_0^\infty \phi(x)\rho_t(x)dx - \int_0^\infty \phi'(x)V(x, \rho_t)\rho_t(x)dx - \phi(0)V(0, \rho_t)u_1(t)$$

$$= -\int_0^\infty \phi(x)f(x)\rho_t(x)dx, \tag{2.11}$$

$$\int_0^\infty \phi(x)\rho_0(x)dx = \int_0^\infty \phi(x)u_0(x)dx,$$

where $V(x, \rho_t)$ is given by (2.7) with p_t and $\bar{\rho}_t$ as in (2.6).

Let us now give a heuristic derivation of (2.10). Observe that if ρ_t is the limit density of our neuron system then, by definition, at all times $t \geq 0$

$$\int_0^\infty \rho_t(x)dx = 1. \tag{2.12}$$

Recalling that $V(x, \rho_t)$ is the limit velocity field, we have that the rate at which mass enters into $(0, \infty)$ is $V(0, \rho_t)u_1(t)$ while the rate at which mass leaves $(0, \infty)$ is p_t (due to spiking). Mass conservation then indicates that $V(0, \rho_t)u_1(t) = p_t$ for almost all t, hence (2.10).

As we shall see in the next theorem the limit density solves (2.11) and it can be quite explicitly computed by using the method of characteristics. The characteristics are curves along which the solution is transported, they are defined by the equation

$$\frac{dx(t)}{dt} = V(x(t), \rho_t). \tag{2.13}$$

The solution of (2.13) in the time interval $[s, t]$, $0 \leq s \leq t$, with value x at time s is denoted by $\varphi_{s,t}(x)$, $x \in \mathbb{R}_+$, and it has the following expression:

$$\varphi_{s,t}(x) = e^{-\lambda(t-s)}x + \int_s^t e^{-\lambda(t-u)}[\lambda\bar{\rho}_u + p_u]du. \tag{2.14}$$

Now our main result reads as follows.

Theorem 2 *Grant Assumptions* 1 *and* 2. *For any fixed $T > 0$,*

$$\mathcal{L}(\mu_{U_{[0,T]}^N}) \xrightarrow{w} \mathcal{P}_{[0,T]} \tag{2.15}$$

(weak convergence in $D([0, T], \mathcal{S}')$) as $N \to \infty$, where $\mathcal{P}_{[0,T]}$ is the law on $D([0, T], \mathcal{S}')$ supported by the distribution valued trajectory ω_t given by

$$\omega_t(\phi) = \int_0^\infty \phi(x)\rho_t(x)dx, \quad t \in [0, T],$$

for all $\phi \in \mathcal{S}$.

Here, $\rho_t(x)$ is the unique weak solution of (2.8)–(2.9) with $u_0 = \psi_0$ and u_1 as in (2.10). Moreover, $\rho_t(x)$ is a continuous function of (x, t) in $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(\varphi_{0,t}(0), t), t \in \mathbb{R}_+\}$ where it is differentiable in x and t and the derivatives satisfy (2.8). Moreover for any $t \geq 0$, $\rho_t(x)$ has compact support in x and

$$\rho_t(0) = \frac{p_t}{p_t + \lambda\bar{\rho}_t}, \quad \int \rho_t(x)dx = 1. \tag{2.16}$$

Its explicit expression for $x \geq \varphi_{0,t}(0)$ is:

$$\rho_t(x) = \psi_0\left(\varphi_{0,t}^{-1}(x)\right) \exp\left\{-\int_0^t [f - \lambda](\varphi_{s,t}^{-1}(x))ds\right\}, \tag{2.17}$$

and for any $x = \varphi_{s,t}(0)$ for some $0 < s \leq t$,

$$\rho_t(x) = \frac{p_s}{p_s + \lambda\bar{\rho}_s} \exp\left\{-\int_s^t [f(\varphi_{s,u}(0)) - \lambda]du\right\}. \tag{2.18}$$

Theorem 3 *Grant Assumptions 1, 2 and suppose that*

$$\psi_0(0) = \frac{p_0}{p_0 + \lambda\bar{\psi}_0}, \text{ where } p_0 = \int_0^\infty f(x)\psi_0(x)dx \text{ and } \bar{\psi}_0 = \int_0^\infty x\psi_0(x)dx. \tag{2.19}$$

Then $\rho_t(x)$ is continuous in $\mathbb{R}_+ \times \mathbb{R}_+$.

We give some comments on the above result. We first compare our result with classical "population density equations" obtained in integrate-and-fire models as for instance described in [2]. In our second remark, we discuss condition (2.19).

Remark 1 In case $\lambda = 0$, (2.8) reads as follows.

$$\begin{cases} \partial_t\rho_t(x) = -p_t\partial_x\rho_t(x) - f(x)\rho_t(x), & x > 0, x \neq \varphi_{0,t}(0), \\ \rho_t(0) = 1 & \text{for all } t \geq 0. \end{cases}$$

This equation is different from usual population density equations which are obtained for integrate-and-fire neurons as considered e.g. in Chap. 6.2.1 of [2], see in particular their formula (6.14). As in integrate-and-fire models, also in our model spiking neurons are reset to a reversal potential (which equals 0); but spiking does not create Dirac-masses at the reset value. This is due to the Poissonian mechanism giving rise to spiking in our model. The loss of mass at time t due to spiking of neurons having potential height x is therefore described by the term $-f(x)\rho_t(x)$.

At the same time, spiking induces a deterministic drift $p_t dt$ for those neurons that are not spiking. In particular, a neuron having initially potential 0 at time t will have potential $\approx p_t h$ after a time $t + h$, for $h << 1$ small. Hence, during $[t, t + h]$, there is creation of an interval $[0, p_t h]$ at the beginning of the support in which no non-spiking neurons are present. At the same time, there are approximately $p_t h$ neurons that spike during $[t, t + h]$ which invade this initial interval. This implies that the initial density of neurons at the border $x = 0$ is of height 1. This initial condition is different from the usual initial condition obtained in integrate-and-fire models.

Remark 2 The condition (2.19) ensures that the limit density $\rho_t(x)$ does not have a discontinuity at the point $x = \varphi_{0,t}(0)$. This point $\varphi_{0,t}(0)$ is the point where two densities are pieced together: on the one hand the density of neurons that did not yet spike up to time t, which is given by formula (2.17), and on the other hand the density of neurons that have already spiked, given by (2.18). Without condition (2.19), the convergence result still holds true, but $\rho_t(x)$ will have a (single) jump at $x = \varphi_{0,t}(0)$; in particular, it is not a strong solution of the nonlinear PDE. However, even without condition (2.19), for any $t > 0$, (2.16) holds true.

To separate the difficulties we shall first prove Theorem 1 and Theorem 2 under a very restrictive assumption on f:

Assumption 3 f is a positive C^1-function satisfying Assumption 1. f is non-decreasing, Lipschitz continuous, bounded and constant for all $x \geq x^{**}$ for some $x^{**} > 0$. We shall denote by $f^* = \|f\|_\infty$ the sup norm of f.

The proof of Theorem 1 under Assumption 3 is easy, it is given in the next section. In the successive sections we shall prove Theorems 2 and 3 under Assumption 3. In Sect. 4, tightness of the sequence of processes indexed by N is proved. In Sect. 5 we introduce a sequence of auxiliary processes which are discrete time models and for which it is easier to prove the hydrodynamical limit which is done in Sect. 6. Section 7 will then conclude the proof of Theorems 2 and 3 under Assumption 3.

In the Appendix we shall prove Theorem 1 in its original formulation (i.e. dropping Assumption 3) and then Theorem 2. However this last step is trivial because the estimate (2.4) implies that with probability going to 1 as $N \rightarrow \infty$ all the membrane potentials are uniformly bounded in the time interval $[0, T]$ that we are considering. It is then possible to replace the true f with one satisfying Assumption 3 and which differs only for potential values larger than those reached by the true process, so that we can use what was already proved under Assumption 3. The precise argument is given at the end of the Appendix.

3 Energy Bounds Under Assumption 3

Exploiting Assumption 3 we shall prove a statement stronger than in Theorem 1.

Proposition 1 *Let f satisfy Assumption 3 and call $f^* = \|f\|_\infty$.*

1. *For any $N \geq 1$ and any $x \in \mathbb{R}_+^N$ there exists a unique strong Markov process $U^N(t)$ starting from x taking values in $\mathbb{R}_+^N$ whose generator is given by (2.1).*
2. *Calling $N(t)$ the total number of fires in the time interval $[0, t]$ we have*

$$N(t) \leq N^*(t) \quad stochastically \tag{3.20}$$

where $N^*(t)$ is a Poisson process with intensity Nf^*.

3. $\sup_{t \leq T} \|U^N(t)\| \leq \|U^N(0)\| + \dfrac{N(t)}{N}$ *and for any $T > 0$ there exist positive constants c and C such that for any N and any $U^N(0)$:*

$$P_{U^N(0)}^{(N,\lambda)}\left[\sup_{t \leq T} \|U^N(t)\| \leq \|U^N(0)\| + 2f^*T \right] \geq 1 - ce^{-CTN}. \tag{3.21}$$

Proof The existence of the process for each fixed N is now trivial as the firing rates are bounded. The variable $N(t)$ is stochastically upper bounded by $N^*(t) := \sum_{i=1}^N n_i(t)$, where $(n_i(t))$ are i.i.d. Poisson processes of intensity f^*. $N^*(t)$ is therefore a Poisson process with intensity Nf^*. We have

$$\sup_{t \leq T} \|U^N(t)\| \leq \|U^N(0)\| + \frac{N(t)}{N},$$

because each firing event increases the rightmost neuron by $\frac{1}{N}$, while, in between firing events, the rightmost neuron is attracted to the average membrane potential of the process and thus decreases. (3.21) then follows from item 2. because $\{N(T) \geq B\}$ is an increasing event and thus the bound is reduced to large deviations for a Poisson variable, details are omitted. □

4 Tightness

With this section we begin the proof of Theorems 2 and 3 (under Assumption 3). We start by proving tightness of the sequence of laws of $\mu_{U^N_{[0,T]}}$.

Proposition 2 *Grant Assumption 3. Suppose that $U^N(0) = x^N$ is such that Assumption 2 is verified. Then the sequence of laws of $\mu_{U^N_{[0,T]}}$ is tight in $D(\mathbb{R}_+, \mathcal{S}')$.*

Proof For any test function $\phi \in \mathcal{S}$ and all $t \in [0, T]$, we write,

$$\left\langle U^N(t), \phi \right\rangle = \frac{1}{N} \sum_i \phi\left(U_i^N(t) \right) = \int \phi(x) \mu_{U^N(t)}(dx). \tag{4.22}$$

By [11] it is sufficient to prove the tightness of $\langle U^N(t), \phi \rangle, t \in [0, T] \in D([0, T], \mathbb{R})$ for any fixed $\phi \in \mathcal{S}$. In order to do so, we shall use a well known tightness criterion, see for instance Theorem 2.6.2 of [12], which requires that the L^2 norms of the "compensators" of $\langle U^N(t), \phi \rangle$ are finite. The compensators are

$$\gamma_1^N(t) = L\left\langle U^N(t), \phi \right\rangle, \quad \gamma_2^N(t) = L\left\langle U^N(t), \phi \right\rangle^2 - 2\left\langle U^N(t), \phi \right\rangle L\left\langle U^N(t), \phi \right\rangle, \tag{4.23}$$

where L is the generator given by (2.1). The criterion requires that there exists a constant c so that

$$\sup_{t \leq T} E[\gamma_1^N(t)]^2 \leq c, \quad \sup_{t \leq T} E[\gamma_2^N(t)]^2 \leq c. \tag{4.24}$$

The proof of the criterion is based on the fact that

$$M_t^N = \langle U^N(t), \phi \rangle - \int_0^t \gamma_1^N(s)ds \text{ and } (M_t^N)^2 - \int_0^t \gamma_2^N(s)ds$$

are martingales. To prove (4.24) we start by calculating $\gamma_1^N(t) = \frac{1}{N} \sum_i L\phi(U_i^N(t))$. We have

$$\gamma_1^N(t) = \frac{1}{N} \sum_i \left[\sum_{j \neq i} f\left(U_j^N(t) \right) \left[\phi\left(U_i^N(t) + \frac{1}{N} \right) - \phi\left(U_i^N(t) \right) \right] \right.$$
$$+ f\left(U_i^N(t) \right) \left[\phi\left(0 \right) - \phi(U_i^N(t)) \right] \bigg]$$
$$+ \frac{\lambda}{N} \sum_i \phi'\left(U_i^N(t) \right) \left[\bar{U}_N(t) - U_i^N(t) \right],$$

where $\bar{U}_N(t) = \langle U^N(t), id \rangle$ is the average of the $U_i^N(t)$. Expanding the discrete derivative, we get

$$\gamma_1^N(t) = \left\langle U^N(t), f \right\rangle \left\langle U^N(t), \phi' \right\rangle - \left\langle U^N(t), f\phi \right\rangle + \phi(0) \left\langle U^N(t), f \right\rangle$$
$$+ \lambda \left[\left\langle U^N(t), \phi' \right\rangle \left\langle U^N(t), id \right\rangle - \left\langle U^N(t), \psi \right\rangle \right] + O\left(\frac{1}{N} \right),$$

where $\psi(x) = x\phi'(x)$ and

$$O\left(\frac{1}{N} \right) = \frac{1}{N} \sum_i \left[\sum_{j \neq i} f\left(U_j^N(t) \right) [\phi\left(U_i^N(t) + \frac{1}{N} \right) - \phi\left(U_i^N(t) \right) - \frac{1}{N}\phi'\left(U_i^N(t) \right)] \right].$$

Since ϕ, ϕ' and ϕ'' are bounded as well as f (thanks to Assumption 3) there is a constant c so that

$$\left|\gamma_1^N(t)\right| \leq c\Big(1 + \langle U^N(t), id\rangle + |\langle U^N(t), \psi\rangle|\Big) \leq c'\Big(1 + \frac{1}{N}\sum_i U_i^N(t)^2\Big).$$

By Proposition 1, $\sup_{t \leq T} E[\gamma_1^N(t)^2] \leq c$ for a constant c not depending on N.

The proof of (4.24) for $\gamma_2^N(t)$ is simpler. We write $L = L_{\text{fire}} + L_\lambda$, where $L_{\text{fire}}\phi$ and $L_\lambda\phi$ are given by the first, respectively second, term on the right hand side of (2.1). Since L_λ acts as a derivative we have

$$L_\lambda\langle U^N(t), \phi\rangle^2 - 2\langle U^N(t), \phi\rangle L_\lambda\langle U^N(t), \phi\rangle = 0$$

as can be easily checked. We have

$$\frac{1}{N^2}\sum_{i,j} L_{\text{fire}}(\phi(U_i^N(t))\phi(U_j^N(t)))$$

$$= \frac{1}{N^2}\sum_{i\neq j}\left[\sum_{k\neq i,j} f\left(U_k^N(t)\right)\left[\phi\left(U_i^N(t) + \frac{1}{N}\right)\phi\left(U_j^N(t) + \frac{1}{N}\right)\right.\right.$$

$$- \phi\left(U_i^N(t)\right)\phi\left(U_j^N(t)\right)\Big]$$

$$+ f\left(U_i^N(t)\right)\left[\phi(0)\phi\left(U_j^N(t) + \frac{1}{N}\right) - \phi\left(U_i^N(t)\right)\phi\left(U_j^N(t)\right)\right]$$

$$+ f\left(U_j^N(t)\right)\left[\phi(0)\phi\left(U_i^N(t) + \frac{1}{N}\right) - \phi\left(U_i^N(t)\right)\phi\left(U_j^N(t)\right)\right]\Big]$$

$$+ \frac{2}{N^2}\sum_i\left[\sum_{k\neq i} f\left(U_k^N(t)\right)\left[\phi^2\left(U_i^N(t) + \frac{1}{N}\right) - \phi^2\left(U_i^N(t)\right)\right]\right.$$

$$+ f\left(U_i^N(t)\right)\left[\phi^2(0) - \phi^2\left(U_i^N(t)\right)\right]\Big].$$

The same arguments used earlier show that the L^2-norm of this term is bounded uniformly in $t \in [0, T]$ and in N. The L^2-norm of $-2\langle U^N(t), \phi\rangle L_{\text{fire}}\langle U^N(t), \phi\rangle$ is also bounded uniformly because $|\langle U^N(t), \phi\rangle| \leq c$ and we have already proved the bound for $L_{\text{fire}}\langle U^N(t), \phi\rangle$. We have thus proved (4.24) and finished the proof. Observe that taking into account the signs we could prove that $\gamma_2^N(t) \to 0$ as $N \to \infty$. $\square$

5 Coupling the True with an Auxiliary Process

The natural step after having proved tightness is to prove propagation of chaos. This is however not so simple in our model because the firing of a neuron (i.e. when its membrane potential jumps) affects *simultaneously the state* of the other neurons and not just their jumping rates, as usual in mean field models. For this reason we follow a different strategy here. In order to overcome this difficulty, we introduce an auxiliary process which is from one side a good approximation of the true one in the $N \to \infty$ limit, and which, from the other side, is easy to handle in the same limit. The auxiliary process is defined in the present section where we prove that it is close to the true process uniformly in N. In Sect. 6 we study the hydrodynamic

limit for the approximating process. Section 7 will then conclude the proof of Theorems 2 and 3.

5.1 The Auxiliary Process

We work under Assumption 3 throughout the whole section. We fix a time mesh $\delta > 0$ and approximate the process $U^N(t)$ for fixed N by a process which is constant on time intervals $[n\delta, (n+1)\delta[$, $n \geq 0$. Since N is fixed we shall drop the superscript N from $U^N(t)$ unless ambiguities may arise.

The auxiliary process is denoted by $Y^{(\delta)}(n\delta)$ and is defined at discrete times $n\delta$, $n \in \mathbb{N}$, such that $(Y^{(\delta)}(n\delta))_{n \in \mathbb{N}}$ is a Markov chain. Its transition probability describes a process where neurons fire with constant firing rate $f(y_i)$ in the time interval $[n\delta, (n+1)\delta[$. Moreover, all firing events after the first one are suppressed. Finally, the new configuration of neurons at time $(n+1)\delta$ is obtained by first letting the neurons evolve (for a time δ) under the action of the gap-junction interaction and then taking into account the effect of the firings at the end of the time interval. The precise definition is given now.

We put $Y^{(\delta)}(0) = U(0)$ and then proceed by induction on n. Conditionally on $Y^{(\delta)}(n\delta) = y = (y_1, \ldots, y_N)$, we choose N independent exponential random variables $\tau_1, \ldots, \tau_N$, which are independent of anything else, having intensities $f(y_i)$, $i = 1, \ldots, N$, respectively. We put

$$\Phi_i(n) = 1_{\{\tau_i \leq \delta\}}, \ 1 \leq i \leq N, \ q = \frac{1}{N} \sum_{i=1}^{N} \Phi_i(n); \tag{5.25}$$

hence neurons i such that $\Phi_i(n) = 1$ spike during $[n\delta, (n+1)\delta[$, all other neurons do not spike during that interval. Notice that we keep constant the firing intensity of the neurons. We write

$$\varphi_{\bar{y},t}(y_i) = e^{-\lambda t} y_i + (1 - e^{-\lambda t})\bar{y}, \ 0 \leq t \leq \delta, \ \ \bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i \tag{5.26}$$

for the deterministic flow attracting position y_i to $\bar{y}$ and set

$$Y_i^{(\delta)}((n+1)\delta) = \varphi_{\bar{y},\delta}(y_i) + q, \ \ \text{for all } i \text{ such that } \Phi_i(n) = 0. \tag{5.27}$$

Thus neurons which do not fire follow the deterministic flow. Moreover, we suppose that they feel the additional potential q, generated by spiking of other neurons, only at the end of the interval $[n\delta, (n+1)\delta[$.

Let us now describe the evolution of the Nq neurons that fire. Let $i_1, \ldots, i_{Nq}$ be the labels of neurons such that $\Phi_{i_j}(n) = 1$, $j = 1, \ldots, Nq$, ordered in such a way that $\tau_{i_j} > \tau_{i_{j+1}}$. We then assign the position

$$Y_{i_{Nq}}^{(\delta)}((n+1)\delta) = \varphi_{\bar{y},\delta}(0) + \left(q - \frac{1}{N}\right) = (1 - e^{-\lambda\delta})\bar{y} + \left(q - \frac{1}{N}\right) \tag{5.28}$$

to the first neuron which has fired. This is the position of a neuron starting from potential 0 at time $n\delta$, evolving according to the flow and receiving an additional potential $q - \frac{1}{N}$ at time $(n+1)\delta$, due to the influence of the other spiking neurons (whose number is $Nq - 1$).

The remaining $Nq - 1$ neurons that spike are distributed uniformly in the following manner. We put

$$d_n = \frac{\varphi_{\bar{y},\delta}(0) + \left(q - \frac{1}{N}\right)}{Nq - 1}, \ \text{if } Nq - 1 > 0, \ d_n = \varphi_{\bar{y},\delta}(0), \ \text{if } Nq - 1 = 0, \tag{5.29}$$

and

$$Y_{i_j}^{(\delta)}((n+1)\delta) = (j-1)d_n, \ j = 1, \ldots, Nq - 1. \tag{5.30}$$

Remark 3 The definition of the auxiliary process $Y^{(\delta)}$ has to be such that $Y^{(\delta)}$ is $\delta-$close to the original process. Therefore, we have some freedom in choosing the distribution of the spiking neurons in the auxiliary process and the above definitions (5.29) and (5.30) could be changed. However, the above choice is convenient for our purpose; we will see later that this precise choice enables us to produce strong convergence of the associated empirical measures to the limit equation, see also Remark 4 below.

The analogue of Proposition 1 holds for the auxiliary process as well and it is straightforward to see that

Proposition 3 *The variables $\sum_{i=1}^{N} \Phi_i(n)$ are stochastically bounded by Poisson variables of intensity $Nf^*\delta$, $f^* := \|f\|_\infty$.*

As a consequence, proceeding as in the proof of Proposition 1, for any T there is C so that for any initial datum x with $Y^{(\delta)}(0) = x$,

$$P_x\left[\sup_{n:n\delta \le T} \|Y^{(\delta)}(n\delta)\| \le \|x\| + 2f^*T\right] \le e^{-CNT}. \tag{5.31}$$

5.2 Coupling the Auxiliary and the True Process

In order to show that $Y^{(\delta)}$ is close to the original process, we couple the two Markov chains $(U(n\delta))_{n\ge 0}$ and $(Y^{(\delta)}(n\delta))_{n\ge 0}$ in such a way that neurons in both processes spike together as often as possible and such that the pair $(U(n\delta), Y^{(\delta)}(n\delta))$, $n \in \mathbb{N}$, is a Markov chain taking values in $\mathbb{R}_+^{N \times N}$.

We start with $Y^{(\delta)}(0) = U(0)$. For any $n = 0, 1, \ldots$, given $(U(n\delta), Y^{(\delta)}(n\delta))$, the values of $(U((n+1)\delta), Y^{(\delta)}((n+1)\delta))$ will be chosen according to the simulation algorithm given below. The algorithm uses the following variables.

- $(x, y) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$ and $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$. The strings x and y represent the state of the neurons in the two processes and $\bar{x}$ gives the average potential of x.
- Independent random times $\tau_i^1 \in (0, +\infty)$, $\tau_i^2 \in (0, +\infty)$ and $\tau_i \in (0, +\infty)$, for all $i = 1, \ldots, N$. These variables will determine the times of possible updates.
- $m = (m_1, \ldots, m_N) \in \{0, 1\}^N$. The variable m_i indicates the occurrence of a spike for neuron i in the auxiliary process.
- $K \in \{0, \ldots, N\}$. The variable K counts the number of spikes in the auxiliary process.
- $j = (j_1, \ldots, j_N) \in \{0, 1, \ldots, N\}^N$. The variable j_i is the label of the neuron associated with the $i-$th occurrence of a spike in the auxiliary process.
- $L \in [0, \delta]$. The variable L indicates the remaining time after every update of the variables. The simulation algorithm stops when $L = 0$.

The deterministic flow attracting position x_i to the average potential $\bar{x}$, given in (5.26), will appear in the algorithm. For convenience of the reader we recall its definition here

$$\varphi_{\bar{x},t}(x_i) = e^{-\lambda t}x_i + (1 - e^{-\lambda t})\bar{x}, \ 0 \le t \le \delta, \ \ \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

Before proceeding further, let us explain the coupling. Given $U(n\delta) = x = (x_1, \ldots, x_N)$ and $Y^{(\delta)}(n\delta) = y = (y_1, \ldots, y_N)$, we start by associating to each neuron i two independent

stopping times τ_i^1 and τ_i^2. Here, τ_i^1 has intensity $f(\varphi_{\bar{x},t}(x_i)) \wedge f(y_i)$ and τ_i^2 is of intensity $|f(\varphi_{\bar{x},t}(x_i)) - f(y_i)|$. Stopping times associated to different neurons are independent. If τ_i^1 rings first, then U_i and $Y_i^{(\delta)}$ spike together, and the coupling is successful. However, if τ_i^2 rings first, then either U_i spikes and $Y_i^{(\delta)}$ does not (this happens if $U_i > Y_i^{(\delta)}$, details are given in lines $17 - 22$ of the algorithm) or vice versa. Once neuron i has spiked in the auxiliary process we set $m_i = 1$ and do not consider any spikes for neuron i in the auxiliary process any more. Therefore, the next time to be considered for neuron i is simply the next spiking time in the original process which is of intensity $f(\varphi_{\bar{x},t}(x_i))$. This time is called τ_i in our algorithm. All stopping times are only taken into account if they appear during the time interval $[0, \delta]$ that we consider.

Our algorithm is given below. In the remainder of this section we shall prove that this is indeed a good coupling of the two processes.

Algorithm 1 Coupling algorithm

1: *Input:* $(U(n\delta), Y^{(\delta)}(n\delta)) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$
2: *Output:* $(U((n+1)\delta), Y^{(\delta)}((n+1)\delta)) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$
3: *Initial values:* $(x, y) \leftarrow (U(n\delta), Y^{(\delta)}(n\delta))$, $K \leftarrow 0, L \leftarrow \delta, m_i \leftarrow 0$, for all $i = 1, \ldots, N$, $j_i \leftarrow 0$,
   for all $i = 1, \ldots, N$
4: **while** $L > 0$ **do**
5:   For $i = 1, \ldots, N$, choose independent random times

  - $\tau_i^1 \in (0, +\infty)$ with intensities $f(\varphi_{\bar{x},t}(x_i)) \wedge f(y_i)$
  - $\tau_i^2 \in (0, +\infty)$ with intensities $|f(\varphi_{\bar{x},t}(x_i)) - f(y_i)|$
  - $\tau_i \in (0, +\infty)$ with intensities $f(\varphi_{\bar{x},t}(x_i))$
  - $R = \inf\limits_{1 \leq i \leq N; m_i = 0} (\tau_i^1 \wedge \tau_i^2) \wedge \inf\limits_{1 \leq i \leq N; m_i = 1} \tau_i.$

7:   **if** $R \geq L$ **then**
8:     *Stop situation:*
9:     $x_i \leftarrow \varphi_{\bar{x},L}(x_i)$ for all $i = 1, \ldots, N$
10:    $L \leftarrow 0$
11:    $y_i \leftarrow \varphi_{\bar{y},\delta}(y_i) + \frac{K}{N}$ for all $i = 1, \ldots, N$ such that $m_i = 0$
12:    $y_{j_1} \leftarrow \varphi_{\bar{y},\delta}(0) + \frac{K-1}{N}$, if $K \geq 1$
13:    $y_{j_k} \leftarrow (K - k) \left[\frac{\varphi_{\bar{y},\delta}(0)+(K-1)/N}{K-1} 1_{\{K>1\}} \right]$ for all $k = 2, \ldots, K$
14:   **else if** $R = \tau_i^1 < L$ **then**
15:     $m_i \leftarrow 1$, $K \leftarrow K + 1$, $j_K \leftarrow i$, $L \leftarrow (L - R)$
16:     $x_i \leftarrow 0$ and $x_j \leftarrow \varphi_{\bar{x},R}(x_j) + \frac{1}{N}$, for all $j \neq i$
17:   **else if** $R = \tau_i^2 < L$ **then**
18:     **if** $f(y_i) > f(\varphi_{\bar{x},R}(x_i))$ **then**
19:       $m_i \leftarrow 1$, $K \leftarrow K + 1$, $j_K \leftarrow i$, $L \leftarrow (L - R)$, $x_j \leftarrow \varphi_{\bar{x},R}(x_j)$ for all j
20:     **else if** $f(y_i) \leq f(\varphi_{\bar{x},R}(x_i))$ **then**
21:       $L \leftarrow (L - R)$, $x_i \leftarrow 0$ and $x_j \leftarrow \varphi_{\bar{x},R}(x_j) + \frac{1}{N}$, for all $j \neq i$
22:     **end if**
23:   **else if** $R = \tau_i < L$ **then**
24:     $L \leftarrow (L - R)$
25:     $x_i \leftarrow 0$ and $x_j \leftarrow \varphi_{\bar{x},R}(x_j) + \frac{1}{N}$, for all $j \neq i$
26:   **end if**
27: **end while**
28: $(U((n+1)\delta), Y^{(\delta)}((n+1)\delta)) \leftarrow (x, y)$.
29: **return** $(U((n+1)\delta), Y^{(\delta)}((n+1)\delta))$.

5.3 Closeness Between the Auxiliary and the True Process

The main result in this section, Theorem 4 below, states that the auxiliary and the true processes are close to each other. This means that for most neurons, the potentials in the two processes are close to each other (proportionally to δ), while the remaining ones constitute a small fraction of the totality (also proportional to δ).

Definition 2 A label $i \in \{1, \ldots, N\}$ is called "good at time $k\delta$" if for all $n = 1, \ldots, k$ the following is true.

Either $\Phi_i(n - 1) = 0$ and U_i has not fired during the whole time interval $[(n - 1)\delta, n\delta]$.
Or $\Phi_i(n - 1) = 1$ and U_i has fired exactly once in the time interval $[(n - 1)\delta, n\delta]$.
We call $\mathcal{G}_n$ the set of good labels at time $n\delta$ and $M_n = N - |\mathcal{G}_n|$ the cardinality of its complement. If $i \in \mathcal{G}_k$ we call $D_i(k) := |U_i(k\delta) - Y_i^{(\delta)}(k\delta)|$. Finally, we set

$$\theta_n = \max\{D_i(k), \ i \in \mathcal{G}_n, k \leq n\}.$$

Then the following holds.

Theorem 4 *Under Assumption 3, for any fixed $T > 0$, there exist $\delta_0 > 0$ and a constant C depending on f^* and on T such that for all $\delta \leq \delta_0$, with probability $\geq 1 - e^{-CN\delta^2}$,*

$$\theta_n \leq C\delta \quad and \quad \frac{M_n}{N} \leq C\delta \quad for \ any \ fixed \ n \ such \ that \ n\delta \leq T.$$

Strategy of proof It is clear that $M_{n-1} \leq M_n \leq \cdots$, because there is no recovery from not being a good label. We shall first prove that till when $\theta_n \leq c\delta$ the increments $M_n - M_{n-1} \leq c'\delta^2 N$. In fact a label i becomes bad at $n\delta$ if in the time interval $((n-1)\delta, n\delta)$ there are either two or more fires of $U_i(\cdot)$ (which cost $O(\delta^2)$) or else the clock τ_i^2 (recall the algorithm given in Sect. 5.2) rings, which also costs $O(\delta^2)$. Since $n\delta \leq T$ the sum of the increments is then bounded by $c\delta$ as desired. Thus there may be of order $c\delta N$ neurons which fire quite differently in the two processes but this produces a change for the potential of the good labels of the order of $\frac{1}{N}(c\delta N)T$ which is also what is claimed in the theorem. The above heuristic argument can be made rigorous; the precise proof is given in the Appendix 2.

5.4 Corollaries

We conclude the section with a corollary of the above results which will be used in the analysis of the hydrodynamic limit $N \to \infty$. Recall that by considering the associated empirical measures (2.5), we interpret $U(t)$ and $Y^{(\delta)}(t)$ as elements of $\mathcal{S}'$.

Definition 3 We introduce the space $\mathcal{F}$ of smooth functions $\phi(m)$, $m \in \mathcal{S}'$, which have the form

$$\phi(m) = h(m[a_1], \ldots, m[a_k]), \ k \text{ a positive integer}, \tag{5.32}$$

where $h(r_1, \ldots, r_k)$ is a smooth function on $\mathbb{R}^k$, uniformly Lipschitz continuous with Lipschitz constant c_h, i.e.

$$|h(r_1, \ldots, r_k) - h(r_1', \ldots, r_k')| \leq c_h \left(\sum_{i=1}^{k} |r_i - r_i'| \right). \tag{5.33}$$

The functions a_i, $i = 1, \ldots, k$, in (5.32) are C^∞−functions on $\mathbb{R}$, each one with compact support contained in $\{|x| \leq c\}$, $c > 0$.

Let c' be an upper bound for the derivatives $|a_i'(r)|$, $i = 1, \ldots, k$. We also introduce

$$\mathcal{T} = \left\{ t \in [0, T] : t = n2^{-k}T, k, n \in \mathbb{N} \right\}. \tag{5.34}$$

Recall that $P_x^{(N,\lambda)}$ denotes the law under which $U(\cdot)$ starts from $U(0) = x$. Denote by $S_x^{(\delta,N,\lambda)}$ the law under which its approximation $Y^{(\delta)}(\cdot)$ starts from x at time 0, and write $Q_x^{(\delta,N,\lambda)}$ for the probability law governing the coupled process defined above. By abuse of notation, we shall also denote the associated expectations by $P_x^{(N,\lambda)}$, $S_x^{(\delta,N,\lambda)}$ and $Q_x^{(\delta,N,\lambda)}$.

Proposition 4 *Let $t \in \mathcal{T}$, $\delta \in \{2^{-l}T, l \in \mathbb{N}\}$ such that $t = \delta n$ for some positive integer n. Let ϕ as in (5.32) with constants c_h, c and c'. Then, with C as in Theorem 4 (C is independent of δ)*

$$|P_x^{(N,\lambda)}[\phi(\mu_{U(t)})] - S_x^{(\delta,N,\lambda)}[\phi(\mu_{Y^{(\delta)}(t)})]| \leq kc_h c' \frac{C}{\delta^2} e^{-C\delta^2 N} c + \delta(2kc_h c'C). \tag{5.35}$$

Proof The left hand side of (5.35) is not changed if we replace $U(t)$ and $Y^{(\delta)}(t)$ by $U^*(t)$ and $Y^{(\delta),*}(t)$ which are defined by setting

$$U_i^*(t) = \min\{U_i(t), c\}, \quad Y_i^{(\delta),*}(t) = \min\{Y_i^{(\delta)}(t), c\},$$

c as in Definition 3. Let ϕ be as in (5.32), then by (5.33)

$$|\phi(\mu_{U(t)}) - \phi(\mu_{Y^{(\delta)}(t)})| = |\phi(\mu_{U^*(t)}) - \phi(\mu_{Y^{(\delta),*}(t)})| \leq kc_h c' \frac{1}{N} \sum_{i=1}^N \left| U_i^*(t) - Y_i^{(\delta),*}(t) \right|.$$

Hence

$$|P_x^{(N,\lambda)}[\phi(\mu_{U(t)})] - S_x^{(\delta,N,\lambda)}[\phi(\mu_{Y^{(\delta)}(t)})]| \leq kc_h c' Q_x^{(\delta,N,\lambda)}\left[\frac{1}{N} \sum_{i=1}^N |U_i^*(t) - Y_i^{(\delta),*}(t)| \right].$$

Let $e^{-C\delta^2 N}$ be the bound on the bad events in the estimates of the coupled process, obtained in Theorem 4. Then

$$Q_x^{(\delta,N,\lambda)}\left[\frac{1}{N} \sum_{i=1}^N |U_i^*(t) - Y_i^{(\delta),*}(t)| \right] \leq \frac{C}{\delta^2} e^{-C\delta^2 N} c + 2C\delta \tag{5.36}$$

where we used that $|U_i^*(t) - Y_i^{(\delta),*}(t)| \leq c$. □

6 Hydrodynamic Limit for the Auxiliary Process

The main result of this section is given in Theorem 5 below. It states that the auxiliary process converges in the hydrodynamic limit to the evolution defined in Sect. 6.1. When necessary we shall make explicit the dependence on N writing $Y^{(\delta)} = Y^{(\delta,N)}$. We then suppose that for all $\delta \in \{2^{-l}T, l \in \mathbb{N}\}$, $Y^{(\delta,N)}(0) = x^N$, where x^N satisfies Assumption 2 of Sect. 2. We will then show that the law of $\mu_{Y^{(\delta,N)}}$ converges weakly to a process supported by a single trajectory.

6.1 The Limit Trajectory of the Auxiliary Process

In this subsection we describe the limit law of $\mu_{Y^{(\delta,N)}}$ denoted by $\rho_{\delta n}^{(\delta)}(r)$. We start with some heuristic considerations which will motivate the expression which defines $\rho_{\delta n}^{(\delta)}(r)$ and which foresee the way we shall prove convergence to $\rho_{\delta n}^{(\delta)}(r)$.

   Heuristics Consider an interval $I = [a, b] \subset \mathbb{R}_+$ of length ℓ and center r. We choose $\ell = N^{-\alpha}$, $\alpha > 0$ and properly small. The density of the initial configuration x^N in I is the average $\mu_{x^N}(I)$; our Assumption 2 ensures that $\mu_{x^N}(I) = \frac{|x^N \cap I|}{N} \approx \psi_0(r)|I|$. At time δ the neurons initially in I and which do not fire will be in the interval $J = [a', b']$ having center denoted by r'. Here, recalling (5.25) and (5.26) for notation,

$$a' = \varphi_{\bar{x}^N, \delta}(a) + q^N, \quad b' = \varphi_{\bar{x}^N, \delta}(b) + q^N,$$

where $q^N = q$ is the proportion of neurons that have fired, see (5.25). By the definition of $\varphi_{\bar{x}^N, \delta}$, $|J| = b' - a' = e^{-\lambda\delta}|I|$. The only neurons in J at time δ are those initially in I which do not fire, hence their number is approximately $|x^N \cap I| e^{-f(r)\delta}$. Thus

$$\rho_\delta^{(\delta)}(r')|J| \approx \mu_{Y_\delta^{(\delta,N)}}(J) \approx e^{-f(r)\delta}\psi_0(r)|I|, \quad \rho_\delta^{(\delta)}(r') \approx e^{\lambda\delta}e^{-f(r)\delta}\psi_0(r),$$

which gives $\rho_\delta^{(\delta)}(r')$ in terms of $\rho_0(r) = \psi_0(r)$, once we consider $r = r(r')$ which is given by

$$r = \varphi_{\bar{x}^N, \delta}^{-1}\left(r' - q^N\right) \approx \varphi_{\bar{\psi}_0, \delta}^{-1}\left(r' - p_0^{(\delta)}\delta\right) = \varphi_{\bar{\psi}_0, \delta}^{-1}(r') - e^{\lambda\delta}p_0^{(\delta)}\delta,$$

where $\bar{\psi}_0 = \int x\psi_0(x)dx$, $p_0^{(\delta)} = \int \psi_0(x)\frac{1-e^{-\delta f(x)}}{\delta}dx$ are obtained by letting $N \to \infty$. The inverse of $\varphi_{\bar{x}, \delta}(\cdot)$, see (5.26), is

$$\varphi_{\bar{x}, \delta}^{-1}(x) = e^{\lambda\delta}\left(x - (1 - e^{-\lambda\delta})\bar{x}\right). \tag{6.37}$$

The above gives a formula for $\rho_\delta^{(\delta)}(r')$ for all

$$r' \geq r_0' = \varphi_{\bar{x}^N, \delta}(0) + q^N = (1 - e^{-\lambda\delta})\bar{x}^N + q^N \approx (1 - e^{-\lambda\delta})\bar{\psi}_0 + p_0^{(\delta)}\delta;$$

r_0' is the same as in (5.28). The definition of $Y_\delta^{(\delta,N)}$ is such that all the neurons which have fired are put uniformly in $[0, r_0']$, thus

$$\rho_\delta^{(\delta)}(r') \approx \frac{p_0^{(\delta)}\delta}{p_0^{(\delta)}\delta + (1 - e^{-\lambda\delta})\bar{\rho}_0}, \quad r' \leq r_0'.$$

   Definition of the limit trajectory The definition of $\rho_{n\delta}^{(\delta)}(r)$ will extend and formalize the above definitions to all $n\delta \leq T$. We put $\rho_0^{(\delta)}(x) = \psi_0(x)$, where ψ_0 is a smooth probability density on $\mathbb{R}_+$ satisfying Assumption 2. We then proceed inductively in n such that $n\delta \leq T$. Suppose that $\rho_{n\delta}^{(\delta)}$ has already been defined. Then we put

$$p_{n\delta}^{(\delta)} := \int_0^\infty \rho_{n\delta}^{(\delta)}(x)\frac{1 - e^{-\delta f(x)}}{\delta}dx, \tag{6.38}$$

$$\bar{\rho}_{n\delta}^{(\delta)} := \int_0^\infty x\rho_{n\delta}^{(\delta)}(x)dx, \tag{6.39}$$

and we define for all $x \geq r_n = (1 - e^{-\lambda \delta}) \bar{\rho}_{n\delta}^{(\delta)} + p_{n\delta}^{(\delta)} \delta$,

$$\rho_{(n+1)\delta}^{(\delta)}(x) = e^{\lambda \delta} \rho_{n\delta}^{(\delta)} \left(\varphi_{\bar{\rho}_{n\delta}^{(\delta)},\delta}^{-1}(x) - e^{\lambda \delta} p_{n\delta}^{(\delta)} \delta \right) e^{-f\left(\varphi_{\bar{\rho}_{n\delta}^{(\delta)},\delta}^{-1}(x) - e^{\lambda \delta} p_{n\delta}^{(\delta)} \delta \right) \delta}. \qquad (6.40)$$

Finally we put

$$\rho_{(n+1)\delta}^{(\delta)}(x) \equiv \frac{p_{n\delta}^{(\delta)} \delta}{p_{n\delta}^{(\delta)} \delta + (1 - e^{-\lambda \delta}) \bar{\rho}_{n\delta}^{(\delta)}} \quad \text{for all } x \in]-\infty, r_n[. \qquad (6.41)$$

In this way, $\rho_{(n+1)\delta}^{(\delta)}$ are probability densities on $\mathbb{R}_+$ for all n, i.e.

$$1 = \int_0^\infty \rho_{(n+1)\delta}^{(\delta)}(x) dx. \qquad (6.42)$$

Remark 4 The fact that we have extended the definition of $\rho_{(n+1)\delta}(x)$ to $\mathbb{R}_-$ will be useful in the sequel. Notice that as $\delta \to 0$, (6.41) reads as $\rho_{(n+1)\delta}(0) \sim \frac{p_{n\delta}^{(\delta)}}{p_{n\delta}^{(\delta)} + \lambda \bar{\rho}_{n\delta}^{(\delta)}}$ which corresponds to (2.16).

Notice that if $\rho_{n\delta}^{(\delta)}$ has support $] - \infty, R_n]$, then the support of $\rho_{(n+1)\delta}^{(\delta)}$ is included in $] - \infty, R_{n+1}]$, where

$$R_{n+1} = e^{-\lambda \delta} R_n + p_{n\delta}^{(\delta)} \delta + (1 - e^{-\lambda \delta}) \bar{\rho}_{n\delta}^{(\delta)}.$$

This leads to the following definition.

Definition 4 (*Edge*) We call R_0 the edge of the profile ρ_0 and

$$R_n = e^{-\lambda \delta} R_{n-1} + p_{(n-1)\delta}^{(\delta)} \delta + (1 - e^{-\lambda \delta}) \bar{\rho}_{(n-1)\delta}^{(\delta)} \qquad (6.43)$$

the edge of $\rho_{n\delta}^{(\delta)}$.

Noticing that

$$p_{n\delta}^{(\delta)} \leq \int \rho_{n\delta}^{(\delta)}(x) \frac{1 - e^{-\delta f^*}}{\delta} dx = \frac{1 - e^{-\delta f^*}}{\delta} \leq f^* \quad \text{and} \quad \bar{\rho}_{(n-1)\delta}^{(\delta)} \leq R_{n-1}, \quad (6.44)$$

it then follows that

$$R_n \leq R_{n-1} + f^* \delta \leq R_0 + f^* n \delta \leq R_0 + f^* T, \qquad (6.45)$$

since $n\delta \leq T$. Hence the supports of $\rho_{n\delta}^{(\delta)}$ are uniformly bounded. By iterating (6.40) and by using the explicit form of the inverse flow $\varphi_{\bar{x},\delta}^{-1}(x)$, we get the explicit representation

$$\rho_{(n+1)\delta}^{(\delta)}(x) = e^{\lambda(n+1)\delta} \psi_0 \left(e^{\lambda(n+1)\delta} x - (1 - e^{-\lambda \delta}) \sum_{k=0}^{n} e^{\lambda(k+1)\delta} \bar{\rho}_{k\delta}^{(\delta)} - \sum_{k=0}^{n} e^{\lambda(k+1)\delta} p_{k\delta}^{(\delta)} \delta \right)$$

$$\exp \left\{ -\sum_{k=0}^{n} \delta f \left(e^{\lambda(h+1-k)\delta} x - (1 - e^{-\lambda \delta}) \sum_{h=k}^{n} e^{\lambda(h+1-k)\delta} \bar{\rho}_{k\delta}^{(\delta)} - \sum_{h=k}^{n} e^{\lambda(h+1-k)\delta} p_{h\delta}^{(\delta)} \delta \right) \right\}, \qquad (6.46)$$

for all

$$x \geq x_{n+1}^* = e^{-\lambda(n+1)\delta} \left((1 - e^{-\lambda\delta}) \sum_{k=0}^{n} e^{\lambda(k+1)\delta} \bar{\rho}_{k\delta}^{(\delta)} + \sum_{k=0}^{n} e^{\lambda(k+1)\delta} p_{k\delta}^{(\delta)} \delta \right), \quad (6.47)$$

where ψ_0 is the initial density. Notice that for all $x > x_{n+1}^*$, $\rho_{(n+1)\delta}^{(\delta)}(x)$ is continuous in x. On $[0, x_{n+1}^*[$, however, discontinuities are introduced. The following proposition shows that they are of order δ.

Proposition 5 *There exists a constant C depending only on f^*, $\|f\|_{Lip}$, T and R_0, such that*

$$\left| p_{n\delta}^{(\delta)} - p_{(n-1)\delta}^{(\delta)} \right| + \left| \bar{\rho}_{n\delta}^{(\delta)} - \bar{\rho}_{(n-1)\delta}^{(\delta)} \right| \leq C\delta,$$

for all n such that $n\delta \leq T$.

Proof The proof is straightforward, using the Lipschitz property of f and the fact that the supports of $\rho_{n\delta}^{(\delta)}$ are uniformly bounded. □

The following is a direct consequence of the definition of $\rho_{n\delta}^{(\delta)}$ and of Proposition 5.

Corollary 1 *There exists a constant c such that for any $\delta = 2^{-k}T$ with k large enough, for any n, l with $n\delta \leq T, l\delta \leq T$,*

$$\left| \rho_{n\delta}^{(\delta)}(x) - \rho_{n\delta}^{(\delta)}(y) \right| \leq c(|x - y| \vee \delta) \text{ for all } x, y \in [0, x_n^*[, \quad (6.48)$$

and

$$\left| \rho_{n\delta}^{(\delta)}(x) - \rho_{n\delta}^{(\delta)}(y) \right| \leq c|x - y| \text{ for all } x, y \in [x_n^*, \infty[.$$

Moreover, for all $n, l \geq 0$,

$$\left| \rho_{n\delta}^{(\delta)}(x) - \rho_{l\delta}^{(\delta)}(x) \right| \leq c|n - l|\delta \text{ for any fixed } x \in [x_n^* \vee x_l^*, \infty[\cup]0, x_n^* \wedge x_l^*[. \ (6.49)$$

Finally, if ψ_0 satisfies the additional assumption (2.19), then also

$$\left| \rho_{n\delta}^{(\delta)}(x_n^*+) - \rho_{n\delta}^{(\delta)}(x_n^*-) \right| \leq c\delta$$

and

$$\left| \rho_{n\delta}^{(\delta)}(x) - \rho_{l\delta}^{(\delta)}(x) \right| \leq c|n - l|\delta$$

for all $x > 0$.

Finally, the following result will also be used in the sequel.

Proposition 6 *There exists $\delta_0 > 0$ and a constant C depending on f^*, δ, $\|f\|_{Lip}$, T and R_0, such that*

$$\rho_{n\delta}^{(\delta)}(0) \geq C\psi_0(0),$$

for all $n > 0$ such that $n\delta \leq T$, for all $\delta \leq \delta_0$.

Proof In what follows, C will be a constant that might change from line to line. Thanks to our assumptions imposed on the function f, there exists a constant C such that

$$f(ux) \geq Cf(x) \text{ for all } x \in [0, R_0 + f^*T], u \in [e^{-\delta_0\lambda}, 1]. \qquad (6.50)$$

Then, using (6.38) and (6.40) and the fact that f is non decreasing, for all $\delta \leq \delta_0$,

$$
\begin{aligned}
p_{(n+1)\delta}^{(\delta)} &\geq C \int_0^\infty f(x)\rho_{(n+1)\delta}^{(\delta)}(x)dx \\
&\geq C \int_0^\infty f\left(e^{-\lambda\delta}x + (1 - e^{-\lambda\delta})\bar{\rho}_{n\delta}^{(\delta)} + p_{n\delta}^{(\delta)}\delta\right)\rho_{n\delta}^{(\delta)}(x)e^{-f(x)\delta}dx \\
&\geq Ce^{-f^*\delta} \int_0^\infty f(e^{-\lambda\delta}x)\rho_{n\delta}^{(\delta)}(x) \geq Ce^{-f^*\delta} \int_0^\infty f(x)\rho_{n\delta}^{(\delta)}(x) \geq Cp_{n\delta}^{(\delta)}.
\end{aligned}
$$

In particular,

$$p_{n\delta}^{(\delta)} \geq Cp_0,$$

where C depends on δ and where $p_0 = \int_0^\infty f(x)\psi_0(x)dx$. On the other hand, Proposition 5 implies that

$$p_{n\delta}^{(\delta)} \leq p_0 + Cn\delta, \quad \bar{\rho}_{n\delta}^{(\delta)} \leq \bar{\psi}_0 + Cn\delta,$$

for all n with $n\delta \leq T$, which implies that

$$\rho_{(n+1)\delta}^{(\delta)}(0) \geq \frac{p_{n\delta}^{(\delta)}}{p_{n\delta}^{(\delta)} + \lambda\bar{\rho}_{n\delta}^{(\delta)}} \geq C\frac{p_0}{p_0 + \lambda\bar{\psi}_0 + CT} = C \geq C\psi_0(0).$$

$\square$

6.2 Discretization of the Membrane Potentials

Let $(Y^{(\delta)}(n\delta))_{n \leq T/\delta}$ be the auxiliary process defined in Sect. 5.1, starting from $x = x^N$ according to Assumption 2 such that $\|x\| \leq R_0$. Recalling the definition of $\Phi_i(n)$ in (5.25) we put

$$q(n\delta) = \frac{\sum_{i=1}^N \Phi_i(n)}{N}, \quad V(n\delta) = \frac{q(n\delta)}{\delta}, \quad \bar{Y}^{(\delta)}(n\delta) = \frac{\sum_{i=1}^N Y_i^{(\delta)}(n\delta)}{N} \qquad (6.51)$$

and then define the sequence of random edges $R_0' = R_0$,

$$R_n' := e^{-\lambda\delta}R_{n-1}' + V((n-1)\delta)\delta + (1 - e^{-\lambda\delta})\bar{Y}^{(\delta)}((n-1)\delta). \qquad (6.52)$$

We will compare $Y^{(\delta)}(n\delta)$ and the limit $\rho_{n\delta}^{(\delta)}$ within small intervals, starting to explore the respective supports $[0, R_n']$ and $[0, R_n]$ from the right border of the support (edge). Doing so, we are sure to compare configurations of neurons in both process that correspond and that have evolved in the same fashion in the two processes, with high probability.

In order to do so, we introduce a mesh of $\mathbb{R}_+$ which depends on N and on time, where times have the form $t = n\delta$, $t \leq T$. The meshes at different times will be related as in the heuristic considerations in the beginning of this section.

Definition 5 (*Membrane potential mesh*) Let $0 < \alpha \ll \frac{1}{6}$. Given N, let $r \in [\frac{1}{2}, 1]$ be such that R_0 is an integer multiple of $rN^{-\alpha}$. We then partition $(-\infty, R_0]$ into intervals

$$\mathcal{I}_0 = \{I_{i,0}, i \geq 1\}, \quad I_{i,0} =\,]R_0 - i\ell, R_0 - (i-1)\ell], \quad \ell = rN^{-\alpha}, \qquad (6.53)$$

and define $\mathcal{I}'_0 = \{I'_{i,0}, i \geq 1\}$ by setting $I'_{i,0} = I_{i,0}$ so that at time 0, $\mathcal{I}'_0 = \mathcal{I}_0$. At times $n\delta$ we define $\mathcal{I}_{n\delta} = \{I_{i,n}, i \geq i\}$ and $\mathcal{I}'_{n\delta} = \{I'_{i,n}, i \geq 1\}$ as the sequences of intervals

$$I_{i,n} :=]R_n - e^{-\lambda\delta n}i\ell, \, R_n - e^{-\lambda\delta n}(i-1)\ell], \quad I'_{i,n} :=]R'_n - e^{-\lambda\delta n}i\ell, \, R'_n - e^{-\lambda\delta n}(i-1)\ell].$$
(6.54)

The strategy is to compare the "mass" of $\mu_{Y^{(\delta)}(n\delta)}$ in $I'_{i,n}$ and the mass in the corresponding interval $I_{i,n}$ (with same i) for the limit $\rho^{(\delta)}_{n\delta}$. We shall prove that for most intervals the corresponding masses are close to each other in a sense to be made precise below. In order to do this properly, we need to specify the mass distributions in $\{x < 0\}$ and to define "bad" intervals where the masses may differ. We start with the former. We have already extended the density $\rho^{(\delta)}_{n\delta}(x)$ to $x < 0$, see (6.41). For the neurons we proceed analogously and extend $\mu_{Y^{(\delta)}(n\delta)}$ to the negative axis by adding an infinite mass

$$\left(\mu_{Y^{(\delta)}(n\delta)}\right)_{|\,]-\infty,0[} := \frac{1}{N}\sum_{i=1}^{\infty}\delta_{-id_n}, \quad n\delta \leq T$$
(6.55)

where in agreement with (5.29)

$$d_n = \frac{(1 - e^{-\lambda\delta})\bar{Y}^{(\delta)}(n\delta) + \left(\delta V(n\delta) - \frac{1}{N}\right)}{N\delta V(n\delta) - 1}.$$
(6.56)

Notice that the choice (6.55) corresponds exactly to the initial configuration given in (5.30). We introduce the following quantities for all i, n.

$$N'_{i,n} = N\mu_{Y^{(\delta)}(n\delta)}(I'_{i,n}), \, N_{i,n} = N\int_{I_{i,n}}\rho^{(\delta)}_{n\delta}(x)\,dx, \, \mathrm{w}_i = \int_{I_{i,0}}\psi_0(x)\,dx, \quad (6.57)$$

where we extend the definition of ψ_0 to $\mathbb{R}_-$ by putting $\psi_0(x) = \psi_0(0)$ for all $x < 0$. Notice that since $\psi_0 \geq c(x - R_0)^2, c > 0$, in a left neighborhood of R_0,

$$\mathrm{w}_i \geq c\ell^3,$$
(6.58)

while, "away" from R_0, $\mathrm{w}_i \geq c\ell$, for some $c > 0$. Finally we define the "bad" intervals as follows.

Definition 6 (*Bad intervals*) $I_{i,n}$ is *bad*, if there is $n_0 \leq n$ such that (at least) one of the following four properties holds.

1. $I_{i,n_0} \cap \{x < 0\} \neq \emptyset$ and $I_{i,n_0} \cap \{x > 0\} \neq \emptyset$.
2. $I'_{i,n_0} \cap \{x < 0\} \neq \emptyset$ and $I'_{i,n_0} \cap \{x > 0\} \neq \emptyset$.
3. $I_{i,n_0} \subset \{x < 0\}$ and $I'_{i,n_0} \subset \{x > 0\}$.
4. $I'_{i,n_0} \subset \{x < 0\}$ and $I_{i,n_0} \subset \{x > 0\}$.

$I'_{i,n}$ is bad if $I_{i,n}$ is bad. An interval is *good* if it is not bad.

6.3 Hydrodynamic Limit

In order to compare $Y^{(\delta)}(n\delta), n \leq \delta^{-1}T$, and $\rho^{(\delta)} := (\rho^{(\delta)}_{n\delta}, n \leq \delta^{-1}T)$, we introduce the following distance.

Definition 7 (*Distances*) We define for any $n \leq \delta^{-1}T$,

$$B_n := \text{number of bad intervals in}\,\mathcal{I}_n$$
(6.59)

and set

$$
d_n(Y^{(\delta)}, \rho^{(\delta)}) := B_n + \max_{I_{i,n} \text{ good}, I_{i,n} \subset \mathbb{R}_+} \frac{|N'_{i,n} - N_{i,n}|}{\mathrm{w}_i N \ell}
$$

$$
+ \frac{\delta}{\ell} |V((n-1)\delta) - p^{(\delta)}_{(n-1)\delta}| + \frac{\left| \bar{Y}^{(\delta)}(n\delta) - \bar{\rho}^{(\delta)}_{n\delta} \right|}{\ell}. \tag{6.60}
$$

$d_n(Y^{(\delta)}, \rho^{(\delta)})$ depends on the times $\tau_j(k)$, $j = 1, \ldots, N$, $k \le n-1$, where the $\tau_j(k)$ are the times which enter in the definition of $\Phi_j(k)$, see (5.25). Let

$$
\mathcal{F}_n = \sigma\{\tau_j(k), j = 1, \ldots, N, k \le n-1\}
$$

be the σ-algebra generated by these variables. Observe that $Y^{(\delta)}_{\delta n}$, $\bar{Y}^{(\delta)}(n\delta)$ and $V((n-1)\delta)$ are $\mathcal{F}_n$-measurable. We prove in Theorem 5 below that with large probability (going to 1 as $N \to \infty$) the distances $d_n(Y^{(\delta)}, \rho^{(\delta)})$ are bounded for all n such that $n\delta \le T$. Loosely speaking this is due to the fact that the auxiliary process is defined in terms of independent exponential random variables and that the initial configuration is made of i.i.d. random variables. The bounds on $d_n(Y^{(\delta)}, \rho^{(\delta)})$ are given by coefficients κ_n which do not depend on N but have a very bad dependence on δ for small δ. δ however is a fixed parameter in this section and by the way d_n is defined, the bounds imply that $Y^{(\delta)}$ and $\rho^{(\delta)}$ become very close in most of the space as $N \to \infty$ (and keeping δ fixed).

Theorem 5 *Grant Assumptions* 2 *and* 3. *There exist* $\kappa_n > 0$, $\gamma \in]0, 1[$, *a sequence* $c_1(n) \in \mathbb{R}_+$ *which is increasing in n, and a constant* $c_2 > 0$ *such that*

$$
S^{(\delta, N, \lambda)}_x \left[d_n(Y^{(\delta)}, \rho^{(\delta)}) \le \kappa_n \right] \ge 1 - c_1(n) e^{-c_2 N^\gamma}, \tag{6.61}
$$

for all n such that $n\delta \le T$.

The proof of Theorem 5 is given in the Appendix 3.

Remark 5 We prove Theorem 5 under the strong Assumption 3 which can be weakened. Indeed, it is sufficient to impose (6.61) for $n = 0$. Recalling that by the definition of $\mathcal{I}_0$ all its intervals are good at time $n = 0$, Assumption 3 clearly implies (6.61) for $n = 0$.

Remark 6 Theorem 5 gives strong convergence of $\mu_{Y^{(\delta)}(t)}$ to $\rho^{(\delta)}_t(x)dx$. Indeed, (6.61) implies the convergence of the "densities" $\frac{N'_{i,n}}{N\ell}$ (notice that $\mathrm{w}_i \le \ell \to 0$ as $N \to \infty$).

As a corollary of Theorem 5 we obtain the desired convergence

Corollary 2 (Hydrodynamic limit for the approximating process) *Under the conditions of Theorem* 5, *let* $t \in \mathcal{T}$, $\delta \in \{2^{-lT}, l \in \mathbb{N}\}$ *such that* $t = \delta n$ *for some positive integer n. Then almost surely, as* $N \to \infty$,

$$
\mu_{Y^{(\delta)}(t)} \xrightarrow{w} \rho^{(\delta)}_t(x)dx.
$$

7 Hydrodynamic Limit for the True Process

We can now conclude the proof of Theorem 2. The convergence in the hydrodynamic limit will be proved as a consequence of Proposition 2 and of Corollary 2, which proves the convergence for the approximating process, of (6.48), (6.49).

Given $T > 0$ let $\mathcal{T} := \{n\delta \leq T : n \in \mathbb{N}, \ \delta = 2^{-k}T, k \in \mathbb{N}\}$. Since $\mathcal{T}$ is countable and $p_t^{(\delta)}, \bar{\rho}_t^{(\delta)}$ are bounded, there exist bounded functions $p_t^{(0)}$ and $\bar{\rho}_t^{(0)}$ on $\mathcal{T}$ and a subsequence $(k_n)_n$ so that for all $t \in \mathcal{T}$

$$p_t^{(0)} = \lim_{n\to\infty} p_t^{(2^{-k_n}T)}, \quad \bar{\rho}_t^{(0)} = \lim_{n\to\infty} \bar{\rho}_t^{(2^{-k_n}T)}.$$

By Proposition 5 $p_t^{(0)}$ and $\bar{\rho}_t^{(0)}$ are continuous in $\mathcal{T}$ and thus extend uniquely to continuous functions on $[0, T]$ which are denoted by the same symbol; moreover, using again Proposition 5 and denoting below by δ elements of the form $2^{-k_n}T$:

$$\lim_{\delta\to 0} \sup_{n: n\delta \leq T} \sup_{t\in[n\delta,(n+1)\delta)} \left(|p_{n\delta}^{(\delta)} - p_t^{(0)}| + |\bar{\rho}_{n\delta}^{(\delta)} - \bar{\rho}_t^{(0)}| \right) = 0. \tag{7.62}$$

Define for any $t \in [0, T]$

$$x_t^{*,0} = e^{-\lambda t}\left(\lambda \int_0^t e^{\lambda s}\bar{\rho}_s^{(0)}ds + \int_0^t e^{\lambda s}p_s^{(0)}ds \right)$$

and, to underline the dependence on δ, rewrite the x_n^* defined in (6.47) as $x_{n\delta}^{*,\delta}$. Then, using (7.62),

$$\lim_{\delta\to 0} \sup_{n: n\delta \leq T} \sup_{t\in[n\delta,(n+1)\delta)} |x_{n\delta}^{*,\delta} - x_t^{*,0}| = 0. \tag{7.63}$$

Denoting below by ϵ elements in $\{2^{-k}, k \in \mathbb{N}\}$, by (7.63) for any such ϵ there is δ_ϵ so that for any $\delta < \delta_\epsilon$ the following holds. For all $t = n\delta \leq T$ if $|x - x_t^{*,0}| \geq \epsilon$ then $x - x_t^{*,\delta}$ has the same sign as $x - x_t^{*,0}$. We can then use (6.48) and (6.49) and a Ascoli-Arzelà type of argument to deduce that $\rho_t^{(\delta)}(x)$ converges in sup norm by subsequences to a continuous function $\rho_t(x)$, $t \in \mathcal{T}$, $|x - x_t^{*,0}| \geq \epsilon$ with compact support. By continuity $\rho_t(x)$ extends to all $t \in [0, T]$, $|x - x_t^{*,0}| \geq \epsilon$. By a diagonalization procedure we extend the above to all x, t with $t \in [0, T]$ and $x \neq x_t^{*,0}$. Then by (6.42), (6.38) and (6.39) for any $t \in \mathcal{T}$

$$1 = \int_0^\infty \rho_t(x)dx, \quad p_t^0 = \int_0^\infty \rho_t(x)f(x)dx$$

and

$$\bar{\rho}_t^{(0)} = \int_0^\infty x\rho_t(x)dx,$$

which, by continuity extend to all $t \in [0, T]$. Thus p_t^0 and $\bar{\rho}_t^{(0)}$ coincide with p_t and $\bar{\rho}_t$ given in (2.6) and we shall hereafter drop the superscript 0. Finally by taking the limit $\delta \to 0$ in (6.40) and (6.41) we prove that $\rho_t(x)$ satisfies (2.17)–(2.18).

We shall next prove that $\rho_t(x)$ is a weak solution of (2.8)–(2.9) with $u_0 = \psi_0$ and u_1 as in (2.10).

Lemma 1 *If $\rho_t(x)$ is given by* (2.17)–(2.18) *then for any test function ϕ*

$$\int_0^\infty \phi(x)\rho_t(x)dx = \int_0^\infty \psi_0(x)e^{-\int_0^t f(\varphi_{0,s}(x)ds)}\phi(\varphi_{0,t}(x))dx$$

$$+ \int_0^t p_s e^{-\int_s^t f(\varphi_{s,s'}(0)ds')}\phi(\varphi_{s,t}(0))ds. \tag{7.64}$$

Proof Calling $x_t^* = \varphi_{0,t}(0)$ we write

$$\int_0^\infty \phi(x)\rho_t(x)dx = \int_0^{x_t^*} \phi(x)\rho_t(x)dx + \int_{x_t^*}^\infty \phi(x)\rho_t(x)dx. \tag{7.65}$$

In the second integral on the right hand side we make the change of variables $x \to y$ where $\varphi_{0,t}(y) = x$. Recalling (2.14), we have

$$\frac{dx}{dy} = e^{-\lambda t}.$$

Using that $\rho_t(x)$ is given by (2.17) we can then check that the second integral on the right hand side of (7.65) is equal to the first integral on the right hand side of (7.64).

In the first integral on the right hand side of (7.65) we make the change of variables $x \to s$ where $\varphi_{s,t}(0) = x$. Using once more (2.14), we have

$$\frac{d\varphi_{s,t}(0)}{ds} = -V(0, \rho_s)e^{-\lambda(t-s)}.$$

Using this and recalling that that $\rho_t(x)$ is given by (2.18) we then complete the proof of the lemma.   □

It follows from (7.64) that for any test function ϕ, $\int \phi\rho_t dx$ is differentiable in t and that its derivative satisfies (2.11). Moreover by choosing $\phi = f$ and $\phi = x$ in (7.64) we then deduce that p_t and $\bar\rho_t$ are differentiable and from this that $\rho_t(x)$ is differentiable in t and x in the open set $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(t, x) : x = \varphi_{0,t}(0)\}$. Hence by (2.11), $\rho_t(x)$ satisfies (2.8) in such a set and the boundary conditions (2.9) with $u_0 = \psi_0$ and u_1 as in (2.10).

We shall next prove uniqueness for (2.11). As a consequence the limit $\rho_t(x)$ we have found using compactness does not depend on the converging subsequences, we therefore have full convergence. It is convenient to rewrite (2.11) as follows. For all $\phi \in C^1(\mathbb{R}_+, \mathbb{R})$, putting $g(t, dx) = \rho_t(x)dx$,

$$\partial_t \int \phi(x)g(t, dx) = \int [\phi(0) - \phi(x)]f(x)g(t, dx) + \int \phi'(x)[\lambda\bar\rho_t + p_t - \lambda x]g(t, dx)dx,$$

$$g(0, dx) = \psi_0(x)dx, \ p_t = \int f(x)g(t, dx), \ \bar\rho_t = \int xg(t, dx). \tag{7.66}$$

Proposition 7 *$\rho_t(x)dx$ is the unique solution of (7.66) solving the initial condition $\rho_0(x) = \psi_0(x)$ for all x and*

$$1 = \int_0^\infty \rho_t(x)dx, \ \ p_t = \int_0^\infty f(x)\rho_t(x)dx, \ \ \bar\rho_t = \int x\rho_t(x)dx.$$

Proof We address the uniqueness of the solution. Any law $g(t, dx)$ solving (7.66) is the law of the Markov process $U(t), t \geq 0$, which is solution of the non-linear SDE

$$dU(t) = (-\lambda U(t) + \lambda E(U(t)) + E(f(U(t))))dt - U(t-) \int_{\mathbb{R}_+} 1_{\left\{z \leq f(U(t-))\right\}} N(dt, dz). \tag{7.67}$$

Here, $N(ds, dz)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ having intensity $dsdz$. It suffices to show the existence of a unique strong solution of the above non-linear SDE on a fixed time interval $[0, T]$.

Let U and V be two solutions starting from $U(0) = V(0)$, and write for short $a_t = E(\lambda U(t) + f(U(t)))$, and a'_t for the corresponding quantity for V.

We start by giving a priori bounds on $U(t)$ and $V(t)$. It follows directly from (7.67) that

$$E(U(t)) \le E(U(0)) + f^* t \le R_0 + f^* T,$$

for all $t \le T$. But clearly

$$U(t) \le U(0) + \int_0^t E(\lambda U(s) + f(U(s)))ds \le U(0) + f^* T + \lambda \int_0^t E(U(s))ds \le C_T,$$

for all $t \le T$, where the constant C_T depends only $U(0)$, R_0, f^* and λ and where we recall that R_0 is the support of ψ_0. The same upper bound holds obviously for $V(t)$.

Coupling U and V such that they have the most common jumps possible, we obtain

$$\begin{aligned}
&\frac{d}{dt}E|U(t) - V(t)| \\
&= -E(f(U(t)) \wedge f(V(t))|U(t) - V(t)| + |f(U(t)) - f(V(t))|(U(t) \wedge V(t) \\
&\quad - |U(t) - V(t)|) \\
&\quad - \lambda E(sign(U(t) - V(t))(U(t) - V(t)) \\
&\quad + \lambda E(sign(U(t) - V(t))(a_t - a'_t)).
\end{aligned}$$

Since f is non-decreasing, the first line is equal to

$$E(U(t) \wedge V(t))|f(U(t)) - f(V(t))| \le CC_T E|(U(t) - V(t)|,$$

since $U(t) \wedge V(t) \le C_T$. Moreover, it is evident that the second and third line are bounded from above by

$$CE|U(t) - V(t)|.$$

Hence,

$$\frac{d}{dt}E|U(t) - V(t)| \le CE|U(t) - V(t)|,$$

for all $t \le T$, implying that $U(t) = V(t)$ almost surely, for all $t \le T$. $\square$

We shall now prove that the true process converges to $\rho_t(x)dx$ in the hydrodynamic limit. Call $\mathcal{P}^N$ the law of the measure valued process $\mu_{U^N(t)}, t \in [0, T]$. By the tightness proved in Proposition 2, we have convergence by subsequences $\mathcal{P}^{N_i}$ to a measure valued process $\mathcal{P}$. We will show that any such limit measure $\mathcal{P}$ is given by the Dirac measure supported by the single deterministic trajectory $\rho_t(x)dx, t \in [0, T]$, where $\rho_t(x)$ is the limit of $\rho_t^{(\delta)}(x)$ found above.

First of all we state the following support property.

Proposition 8 *Any weak limit $\mathcal{P}$ of $\mathcal{P}^N$ satisfies*

$$\mathcal{P}(C([0, T], \mathcal{S}')) = 1,$$

where $C([0, T], \mathcal{S}')$ is the space of all continuous trajectories $[0, T] \to \mathcal{S}'$.

Proof The proof is analogous to the proof of Theorem 2.7.8 in De Masi and Presutti (1991).
                                                                                                     $\square$

Let us denote the elements of $C([0, T], \mathcal{S}')$ by $\omega = (\omega_t, t \in [0, T])$ and let $t \in [0, T]$. Suppose $\mathcal{P}$ is the weak limit of $\mathcal{P}^{N_i}$. We shall prove that $\mathcal{P}$ is supported by $\{\omega : \omega_t = \rho_t(x)dx\}$. Thus $\mathcal{P}$ coincides with $\rho_t(x)dx$ on the rationals of $[0, T]$ and by continuity on all $t \in [0, T]$ and therefore any weak limit of $\mathcal{P}^N$ is supported by $\rho_t dx$.

The marginal of $\mathcal{P}$ at time t is determined by the expectations

$$\int h\big(\omega_t(a_1), \dots, \omega_t(a_k)\big)d\mathcal{P} =: \mathcal{P}_t(h)$$

where, as in Definition 3, h is a smooth function on $\mathbb{R}^k$, $k \geq 1$, and a_i are smooth functions on $\mathbb{R}_+$ with compact support. We need to show that

$$\mathcal{P}_t(h) = h\Big(\int a_1(x)\rho_t(x)dx, \dots, \int a_k(x)\rho_t(x)dx\Big). \tag{7.68}$$

In the sequel, $t \in \mathcal{T}$ and $\delta \in \{2^{-n}T, n \geq 1\}$. For any $\varepsilon > 0$ there exists n_0 such that for all $n \geq n_0$,

$$\left|h\Big(\int a_1\rho_t dx, \dots, \int a_k\rho_t dx\Big) - h\Big(\int a_1\rho_t^{(2^{-n}T)}dx, \dots, \int a_k\rho_t^{(2^{-n}T)}dx\Big)\right| \leq \varepsilon.$$

Moreover there exists N^* so that for all $N_i \geq N^*$,

$$\left|\mathcal{P}_t^{N_i}(h) - \mathcal{P}_t(h)\right| \leq \varepsilon,$$

where $\mathcal{P}_t^{N_i}(h) := P_x^{(N_i,\lambda)}(h(\mu_{U(t)}))$, see (5.35). By (5.35) for δ small enough and N_i large enough

$$|P_x^{(N_i,\lambda)}(h(\mu_{U(t)})) - S_x^{(\delta,N_i,\lambda)}(h(\mu_{Y^{(\delta)}(t)}))| \leq \varepsilon.$$

Applying Corollary 2 for N_i large enough,

$$\left|S_x^{(\delta,N_i,\lambda)}(h(\mu_{Y^{(\delta)}(t)})) - h\Big(\int \rho_t^{(\delta)}a_1, \dots, \int \rho_t^{(\delta)}a_k\Big)\right| \leq \varepsilon.$$

Collecting the above estimates and by the arbitrariness of ε we then get (7.68). This finishes the proof of Theorem 2.

Finally, to prove Theorem 3 we need to show that

$$\lim_{x \nearrow x_t^*} \rho_t(x) = \psi_0\left(\varphi_{0,t}^{-1}(x_t^*)\right) \exp\left\{-\int_0^t [f - \lambda](\varphi_{s,t}^{-1}(x_t^*))ds\right\} \tag{7.69}$$

where $x_t^* = \varphi_{0,t}(0)$. For $x < x_t^*$ we use (2.18)

$$\rho_t(x) = \frac{p_s}{p_s + \lambda\bar{\rho}_s} \exp\left\{-\int_s^t [f(\varphi_{s,u}(0)) - \lambda]du\right\}$$

with s such that $\varphi_{s,t}(0) = x$. By continuity

$$\lim_{s \to 0} \varphi_{s,u}(0) = \varphi_{0,u}(0) = \varphi_{u,t}^{-1}(x_t^*).$$

Moreover, since we have proved earlier the continuity of p_s and $\bar{\rho}_s$

$$\lim_{s \to 0} \frac{p_s}{p_s + \lambda\bar{\rho}_s} = \frac{p_0}{p_0 + \lambda\bar{\rho}_0}$$

so that (7.69) follows from (2.19).

Appendix 1: Proof of Theorem 1

We are working at fixed N and therefore drop the superscript N from U^N. Recall that the average potential of configuration $U(t)$ is given by $\bar{U}_N(t) = \frac{1}{N} \sum_{i=1}^N U_i(t)$ and let

$$K(t) = \sum_{i=1}^N \int_0^t 1_{\{U_i(s-) \le 2\}} dN_i(s) \qquad (8.70)$$

be the total number of fires in $[0, t]$ when $U_i \le 2$. Recall (2.3). The key element of our proof is the following lemma which is due to discussions with Nicolas Fournier.

Lemma 2 *We have*

$$\bar{U}_N(t) \le \bar{U}_N(0) + \frac{K(t)}{N}, \quad \frac{N(t)}{N} \le \bar{U}_N(0) + 2\frac{K(t)}{N} \qquad (8.71)$$

and

$$\|U(t)\| \le 2\|U(0)\| + 2\frac{K(t)}{N}. \qquad (8.72)$$

Proof Suppose U_i fires at time t, then

$$\bar{U}_N(t) = \frac{1}{N} \sum_{j \ne i} \left(U_j(t-) + \frac{1}{N} \right) = \bar{U}_N(t-) + \frac{N-1}{N^2} - \frac{U_i(t-)}{N}.$$

Thus the average potential decreases if $U_i(t-) \ge 1$ (and a fortiori if $U_i(t-) \ge 2$) which implies the first assertion of (8.71). Concerning the second assertion of (8.71), we start with

$$\bar{U}_N(t) = \bar{U}_N(0) + \frac{1}{N} \sum_{i=1}^N \int_0^t \left(\frac{N-1}{N} - U_i(s-) \right) dN_i(s),$$

which implies, since $\bar{U}_N(t) \ge 0$ and $\frac{N-1}{N} \le 1$, that

$$\frac{1}{N} \sum_{i=1}^N \int_0^t (U_i(s-) - 1) dN_i(s) \le \bar{U}_N(0).$$

We use that $x - 1 \ge \frac{x}{2} 1_{\{x \ge 2\}} - 1_{\{x \le 1\}}$ and obtain from this that

$$\frac{1}{N} \sum_{i=1}^N \int_0^t \frac{U_i(s-)}{2} 1_{\{U_i(s-) \ge 2\}} dN_i(s) \le \bar{U}_N(0) + \frac{1}{N} \sum_{i=1}^N \int_0^t 1_{\{U_i(s-) \le 1\}} dN_i(s)$$

$$\le \bar{U}_N(0) + \frac{K(t)}{N}.$$

Observing that $1 \leq \frac{x}{2} 1_{\{x \geq 2\}} + 1_{\{x \leq 2\}}$, we deduce from the above that

$$\frac{N(t)}{N} = \frac{1}{N} \sum_{i=1}^{N} \int_0^t dN_i(s) \leq \bar{U}_N(0) + \frac{K(t)}{N} + \frac{K(t)}{N},$$

implying the second assertion of (8.71).

Since between successive jumps the largest $U_i(t)$ is attracted towards the average potential we can upper bound its position by neglecting the action of the gap junction, implying that

$$\|U(t)\| \leq \|U(0)\| + \frac{N(t)}{N},$$

which, together with (8.71), gives (8.72), since $\bar{U}_N(0) \leq \|U(0)\|$.   □

Proof of Theorem 1 By (8.72) we have

$$\|U(t)\| \leq 2\|U(0\| + 2\frac{K(T)}{N},$$

for all $t \leq T$. But the process $K(t)$ is stochastically bounded by a Poisson process with intensity $Nf(2)$. Therefore, there exists a constant K such that

$$P\Big[|K(T)| \leq KN\Big] \geq 1 - e^{-CNT}.$$

This implies (2.4). Finally, notice that the above arguments give implicitly the proof of the existence of the process $U(t)$, since the process can be constructed explicitly, by piecing together trajectories of the deterministic flow between successive jump times, once we know that the number of jumps of the process is finite almost surely on any finite time interval. □

Appendix 2:Proof of Theorem 4

In what follows, C is a constant which may change from one appearance to another.

The Stopped Process

A technical difficulty in the proof of Theorem 4 comes from the possible occurrence of an anomalously large number of fires in one of the time steps $[(n-1)\delta, n\delta]$. To avoid the problem we stop the process as soon as this happens and prove the theorem for such a stopped process. We then conclude by a large deviation estimate for the probability that the process is stopped before reaching the final time T.

Recalling from Proposition 1 and Proposition 3 that the number of fires in an interval $[(n-1)\delta, n\delta]$ in either one of the two processes is stochastically bounded by a Poisson variable of intensity $f^* \delta N$ we stop the algorithm defining the coupled process as soon as the number of firings in either one of the two processes exceeds $2f^* \delta N$ in one of the time steps $[(n-1)\delta, n\delta]$. We call E the event when the process is stopped before reaching the final time. Then uniformly in the initial datum $Y^{(\delta)}(0) = U(0) = x$,

$$P_x(E) \leq 2\frac{T}{\delta} e^{-CN\delta}. \tag{8.73}$$

By an abuse of notation we denote by the same symbol the stopped processes and in the sequel, unless otherwise stated, we refer to the stopped process. We fix arbitrarily $A > 0$ and consider the process starting from $Y^{(\delta)}(0) = U(0) = x$ with $\|x\| \leq A$.

Writing $B^* := B + A + 2f^*T$, we have by (3.21) for all $t \leq T$ and all $n\delta \leq T$

$$\|U(t)\| \leq B^*, \quad \|Y^{(\delta)}(n\delta)\| \leq B^* \quad \text{for the stopped processes.} \quad (8.74)$$

It follows that the same bounds hold for the unstopped process with probability $\geq 1 - e^{-CN\delta}$.
Thus by restricting to the stopped process we have

- the firing rate of each neuron is $\leq f^*$ and the number of fires of all neurons in any of the steps $[(n-1)\delta, n\delta]$ is $\leq 2f^*\delta N$.
- The bounds (8.74) are verified and as a consequence the average potentials in the U and $Y^{(\delta)}$ processes are $\leq B^*$ so that the gap-junction drift on each neuron is $\leq \lambda B^*$.

Bounds on the Increments of M_n

We write $M_n = M_{n-1} + |A_n \cap \mathcal{G}_{n-1}| + |B_n \cap \mathcal{G}_{n-1}| \leq M_{n-1} + |A_n| + |B_n \cap \mathcal{G}_{n-1}|$ where recalling the algorithm given in Sect. 5.2 and Definition 2

- A_n is the set of all labels i for which a clock associated to label i rings at least twice during $[(n-1)\delta, n\delta]$.
- B_n is the set of all labels i for which a clock associated to label i rings only once during $[(n-1)\delta, n\delta]$, and it is the clock τ_i^2.

We shall prove that (for the stopped processes)

$$P\left[|A_n| > N(\delta f^*)^2\right] \leq e^{-CN\delta^2}, \quad (8.75)$$

$$P\left[|B_n \cap \mathcal{G}_{n-1}| > 2CN\delta\left[\theta_{n-1} + \delta\right]\right] \leq e^{-CN\delta^2} \quad (8.76)$$

(recall C is a constant whose value may change at each appearance).

It will then follow that with probability $\geq 1 - 2e^{-CN\delta^2}$

$$M_n \leq M_{n-1} + N(\delta f^*)^2 + 2CN\delta\left[\theta_{n-1} + \delta\right] \leq M_{n-1} + CN\delta\left[\theta_{n-1} + \delta\right]. \quad (8.77)$$

Iterating the upper bound and using that $n\delta \leq T$, we will then conclude that with probability $\geq 1 - 2ne^{-CN\delta^2} \geq 1 - \frac{C}{\delta}e^{-CN\delta^2}$, where C depends on T,

$$\frac{M_n}{N} \leq C\delta\sum_{k=1}^{n-1}\theta_k + C\delta \leq C(\theta_{n-1} + \delta) \quad \text{for all } n \leq \frac{T}{\delta}, \quad (8.78)$$

having used that, by definition, $\theta_k \leq \theta_{n-1}$.

Proof of (8.75).

$|A_n|$ is stochastically upper bounded by $S^* := \sum_{i=1}^N 1_{\{N_i^* \geq 2\}}$, where $N_1^*, \ldots N_N^*$ are independent Poisson variables of parameter $f^*\delta$, $f^* = \|f\|_\infty$. We write $p^* = P(N_i^* \geq 2)$ and have

$$e^{-\delta f^*}\frac{1}{2}\delta^2(f^*)^2 \leq p^* \leq \frac{1}{2}(\delta f^*)^2, \quad p^* \approx \frac{1}{2}(\delta f^*)^2 \text{ as } \delta \to 0.$$

S^* is the sum of N Bernoulli variables, each with average p^*. Then by the Hoeffding's inequality, we get (8.75).

Proof of (8.76).

We shall prove that the random variable $|B_n \cap \mathcal{G}_{n-1}|$ (for the stopped process) is stochastically upper bounded by $\sum_{i=1}^N \mathbf{1}_{\{\bar{N}_i \geq 1\}}$, where $\bar{N}_i, i = 1, \ldots, N$, are independent Poisson variables of parameter $C(\theta_{n-1} + \delta)\delta$. (8.76) will then follow straightly.

We shorthand

$$y := Y^{(\delta)}((n-1)\delta), \ x := U((n-1)\delta), \ y(\delta) := Y^{(\delta)}(n\delta),$$
$$x(t) := U((n-1)\delta + t), t \in [0, \delta],$$

and introduce independent random times τ_i^2, $i = 1, \ldots, N$, of intensity $|f(y_i) - f(x_i(t))|$, $t \in [0, \delta[$. Then $|B_n|$ is stochastically bounded by $\sum_{i=1}^{N} \mathbf{1}_{\{\tau_i^2 < \delta\}}$ because we are neglecting some of the conditions for being in B_n. We also obviously have

$$|B_n \cap \mathcal{G}_{n-1}| \leq \sum_{i=1}^{N} \mathbf{1}_{\{\tau_i^2 < \delta, i \in \mathcal{G}_{n-1}\}} \quad \text{stochastically.}$$

To control the right hand side we bound

$$|f(x_i(t)) - f(y_i)| \leq \|f\|_{Lip} |x_i(t) - y_i|,$$

$\|f\|_{Lip}$ the Lipschitz constant of the function f. Denote by $N_j(s, t)$ the number of spikes of $U_j(\cdot)$ in the time interval $[s, t]$, then analogously to (5.26),

$$|x_i(t) - y_i| \leq |x_i - y_i| + \int_0^t \lambda e^{-\lambda(t-s)} |\bar{x}(s) - x_i| ds + \frac{1}{N} \sum_{j \neq i} N_j([(n-1)\delta, (n-1)\delta + t]).$$

We have $|\bar{x}(s) - x_i| \leq B^*$ and $\sum_{j \neq i} N_j([(n-1)\delta, (n-1)\delta + t]) \leq 2f^*\delta N$ because we are considering the stopped process. Then if $i \in \mathcal{G}_{n-1}$,

$$|x_i(t) - y_i| \leq \theta_{n-1} + B^*\delta + 2f^*\delta$$

and therefore

$$|f(x_i(t)) - f(y_i)| \leq \|f\|_{Lip} \left(\theta_{n-1} + B^*\delta + 2f^*\delta\right) \leq C(\theta_{n-1} + \delta)$$

so that

$$\sum_{i=1}^{N} \mathbf{1}_{\{\sigma_i < \delta, i \in \mathcal{G}_{n-1}\}} \leq \sum_{i=1}^{N} \mathbf{1}_{\{\bar{N}_i \geq 1\}} \quad \text{stochastically,}$$

where the $\bar{N}_i$ are independent Poisson random variables of intensity $C(\theta_{n-1} + \delta)$. This proves (8.76).

Bounds on θ_n

The final bound on θ_n is reported in (8.87) at the end of this subsection. We start by characterizing the elements $i \in \mathcal{G}_n$ as $i \in \mathcal{G}_{n-1} \cap (C_n \cup F_n)$ where:

1. C_n is the set of all labels i for which a clock associated to label i rings only once during $[(n-1)\delta, n\delta]$, and it is a clock τ_i^1.
2. F_n is the set of all labels i which do not have any jump during $[(n-1)\delta, n\delta]$.

In other words, we study labels i which are good at time $(n-1)\delta$ and which stay good at time $n\delta$ as well. We shall use in the proofs the following formula for the potential $U_i(t)$ of a neuron which does not fire in the interval $[t_0, t]$:

$$U_i(t) = e^{-\lambda(t-t_0)} U_i(t_0) + \int_{t_0}^t \lambda e^{-\lambda(t-s)} \{\bar{U}(s)ds + \frac{1}{\lambda N} dN(s)\}, \tag{8.79}$$

$N(t)$ denoting the total number of fires till time t. For the $Y^{(\delta)}$ process we shall instead use (5.26) and the expressions thereafter.

- Labels $i \in C_n \cap \mathcal{G}_{n-1}$.

For such labels i there is a random time $t \in [(n-1)\delta, n\delta[$ at which a τ_i^1 event happens. Then by (8.79)

$$U_i(n\delta) = \int_t^\delta \lambda e^{-\lambda(\delta-s)} \bar{U}_N(s)ds + e^{-\lambda\delta} \frac{1}{N} \int_t^\delta e^{\lambda s} dN(s),$$

because $U_i(t^+) = 0$. Since we are considering the stopped process, $\bar{U}(\cdot) \leq B^*$ and $N(n\delta) - N((n-1)\delta) \leq 2f^*\delta N$ so that $U_i(n\delta) \leq C\delta$. In the same way, $Y_i^{(\delta)}(n\delta) \leq C\delta$, and therefore

$$D_i(n) = |U_i(n\delta) - Y_i^{(\delta)}(n\delta)| \leq C\delta, \quad \text{for the stopped process.} \tag{8.80}$$

Notice that the bound does not depend on $D_i(n-1)$.

- Labels $i \in F_n \cap \mathcal{G}_{n-1}$.

This means that i is good at time $(n-1)\delta$ and does not jump, neither in the U nor in the $Y^{(\delta)}$ process. Let $U((n-1)\delta) = x$ and $Y^{(\delta)}((n-1)\delta) = y$. By (8.79) and (5.27) $|U_i(n\delta) - Y_i^{(\delta)}(n\delta)| = D_i(n)$ is bounded by

$$D_i(n) \leq e^{-\lambda\delta}|x_i - y_i| + (1 - e^{-\lambda\delta})|\bar{x} - \bar{y}| + \int_{(n-1)\delta}^{n\delta} \lambda e^{-\lambda(n\delta-t)} |\bar{U}_N(t) - \bar{U}_N(0)] dt$$

$$+ \frac{1}{N}| \int_{(n-1)\delta}^{n\delta} e^{-\lambda(n\delta-t)} dN(t) - Nq| \tag{8.81}$$

where Nq is the total number of fires in the process $Y^{(\delta)}$ in the step from $(n-1)\delta$ to $n\delta$.

We bound the right hand side of (8.81) as follows. We have $e^{-\lambda\delta}|x_i - y_i| \leq \theta_{n-1}$. Moreover,

$$(1 - e^{-\lambda\delta})|\bar{x} - \bar{y}| \leq \lambda\delta\left(\theta_{n-1} + B^* \frac{M_{n-1}}{N}\right), \quad B^* \text{ as in (7.74),}$$

and

$$\int_{(n-1)\delta}^{n\delta} \lambda e^{-\lambda(n\delta-t)} |\bar{U}_N(t) - \bar{U}_N(0)] dt \leq \lambda\delta \frac{1}{N} N((n-1)\delta, n\delta),$$

where $N((n-1)\delta, n\delta) = N(n\delta) - N((n-1)\delta)$. Writing

$$\int_{(n-1)\delta}^{n\delta} e^{-\lambda(n\delta-t)} dN(t) = N((n-1)\delta, n\delta) + \int_{(n-1)\delta}^{n\delta} \{e^{-\lambda(n\delta-t)} - 1\} dN(t),$$

we bound the last term on the right hand side of (8.81) as

$$\frac{1}{N}| \int_{(n-1)\delta}^{n\delta} e^{-\lambda(n\delta-t)} dN(t) - Nq| \leq \frac{1}{N}\left(|N((n-1)\delta, n\delta) - Nq| + \lambda\delta N((n-1)\delta, n\delta)\right).$$

Collecting all these bounds we then get

$$D_i(n) \leq \theta_{n-1}(1 + \lambda\delta) + \lambda\delta B^* \frac{M_{n-1}}{N} + 2\lambda\delta \frac{1}{N} N((n-1)\delta, n\delta)$$

$$+ \frac{1}{N}|N((n-1)\delta, n\delta) - Nq|,$$

and since we are considering the stopped process

$$D_i(n) \leq \theta_{n-1}(1 + \lambda\delta) + \lambda\delta B^* \frac{M_{n-1}}{N} + 2\lambda\delta 2 f^* \delta + \frac{1}{N} |N((n-1)\delta, n\delta) - Nq|. \quad (8.82)$$

By the definition of the sets $A_n, \ldots, F_n$ we have

$$|N((n-1)\delta, n\delta) - Nq| \leq \sum_{j \in A_n} N_j((n-1)\delta, n\delta) + |B_n|. \quad (8.83)$$

What follows is devoted to the control of the rhs of (8.83). We start with $|B_n|$. With probability $\geq 1 - e^{-C\delta^2 N}$

$$|B_n| \leq |B_n \cap \mathcal{G}_{n-1}| + |B_n \cap M_{n-1}| \leq 2CN\delta^2 + 2CN\delta\theta_{n-1} + |M_{n-1}|2f^*\delta, \quad (8.84)$$

having used (8.76) and that the number of neurons among those in M_{n-1} which fire in a time δ is bounded by a Poisson variable of intensity $f^*\delta|M_{n-1}|$. Moreover,

$$P\left[\sum_{j \in A_n} N_j((n-1)\delta, n\delta) \geq 4(f^*\delta)^2 N \right] \leq P\left[\sum_{j \in A_n} N_j((n-1)\delta, n\delta) \right.$$
$$\left. \geq 4(f^*\delta)^2 N; |A_n| \leq (f^*\delta)^2 N \right] + P\left[|A_n| > (f^*\delta)^2 N \right]. \quad (8.85)$$

The last term is bounded using (8.75).

We are now going to bound the first term in (8.83). For that sake, let $A \subset \{1, \ldots, N\}$, $|A| \leq (f^*\delta)^2 N$, then

$$P\left[\sum_{j \in A_n} N_j((n-1)\delta, n\delta) \geq 4(f^*\delta)^2 N \mid A_n = A \right] \leq P^*\left[\sum_{j \in A} (N_j^* - 2) \geq 2(f^*\delta)^2 N \right],$$

where P^* is the law of independent Poisson variables N_j^*, $j \in A$, each one of parameter $f^*\delta$ and conditioned on being $N_j^* \geq 2$. Thus the probability that $N_j^* - 2 = k$ is

$$P^*[N_j^* - 2 = k] = Z_\xi^{-1} \frac{\xi^k}{(k+2)!}, \quad Z_\xi = \xi^{-2}\left(e^\xi - 1 - \xi \right), \quad \xi = f^*\delta.$$

Denote by X_j independent Poisson variables of parameter ξ. Then it is easy to see that $N_j^* - 2 \leq X_j$ stochastically for ξ small enough, hence for δ small enough. Notice that $X = \sum_{j \in A} X_j$ is a Poisson variable of parameter $|A|\xi \leq (f^*\delta)^2 N f^*\delta$ having expectation $E^*(X) \leq (f^*\delta)^2 N$ for δ small. As a consequence we may conclude that

$$P^*\left[\sum_{j \in A} (N_j^* - 2) \geq 2(f^*\delta)^2 N \right] \leq P^*\left[X \geq 2(f^*\delta)^2 N \right] \leq e^{-CN\delta^2}.$$

In conclusion for $i \in F_n \cap \mathcal{G}_{n-1}$:

$$D_i(n) \leq \theta_{n-1}(1 + C\delta) + C\delta \frac{M_{n-1}}{N} + C\delta^2 \quad (8.86)$$

with probability $\geq 1 - e^{-C\delta^2 N}$. Together with (8.80) this proves that with probability $\geq 1 - e^{-C\delta^2 N}$

$$\theta_n \leq \max\left\{ C\delta; \theta_{n-1}(1 + C\delta) + C\delta \frac{M_{n-1}}{N} + C\delta^2 \right\}. \quad (8.87)$$

Iteration of the Inequalities

By (8.78), $\frac{M_n}{N} \leq C(\theta_{n-1} + \delta)$ for all $n\delta \leq T$ with probability $\geq 1 - \frac{T}{\delta}e^{-CN\delta^2}$. By (8.87), with probability $\geq 1 - \frac{T}{\delta}e^{-C\delta^2 N}$ we have

$$\theta_n \leq \max\left\{C\delta; \theta_{n-1}(1 + C\delta) + C\delta\frac{M_{n-1}}{N} + C\delta^2\right\}.$$

Thus

$$\theta_n \leq \max\left(C\delta, [1 + C\delta]\theta_{n-1} + C\delta^2\right),$$

since $\theta_{n-2} \leq \theta_{n-1}$. Iterating this inequality we obtain

$$\theta_n \leq C\sum_{k=0}^{n-1}[1 + C\delta]^k \delta^2 + (1 + C\delta)^n C\delta = C\frac{[1 + C\delta]^n - 1}{C\delta}\delta^2 + (1 + C\delta)^n C\delta$$

$$\leq Ce^{CT}\delta,$$

where we have used once more that $n\delta \leq T$. Hence

$$\theta_n \leq C\delta$$

for all $\delta \leq \delta_0$, with probability $\geq 1 - \frac{C}{\delta^2}e^{-CN\delta^2}$. This finishes the proof of Theorem 4.

Appendix 3: Proof of Theorem 5

The proof is by induction on n. Firstly, (6.61) holds for $n = 0$, since $B_0 = 0$ and $x_1^N, \ldots, x_N^N$ i.i.d. according to $\psi_0(x)dx$. We then suppose that (6.61) holds for all $j \leq n$. We condition on $\mathcal{F}_n$ and introduce $G_n = \bigcap_{j\leq n}\{d_j(Y^{(\delta)}, \rho^{(\delta)}) \leq \kappa_j\}$. Then

$$S_x^{(\delta,N,\lambda)}\left[d_{n+1}(Y^{(\delta)}, \rho^{(\delta)}) > \kappa_{n+1}\right]$$
$$\leq S_x^{(\delta,N,\lambda)}\left(\mathbf{1}_{G_n}S_x^{(\delta,N,\lambda)}\left[d_{n+1}(Y^{(\delta)}, \rho^{(\delta)}) > \kappa_{n+1} \mid \mathcal{F}_n\right]\right)$$
$$+ nc_1(n)e^{-c_2 N^\gamma}. \tag{8.88}$$

Therefore, we need to prove that for some constant c

$$S_x^{(\delta,N,\lambda)}\left[d_{n+1}(Y^{(\delta)}, \rho^{(\delta)}) > \kappa_{n+1} \mid \mathcal{F}_n\right] \leq ce^{-c_2 N^\gamma}, \quad on\ G_n. \tag{8.89}$$

From the conditioning we know that $d_j(Y^{(\delta)}, \rho^{(\delta)}) \leq \kappa_j$ for all $j \leq n$; we know also the value of $Y^{(\delta)}(n\delta)$, say $Y^{(\delta)}(n\delta) = y$, we know the location of the edges R'_j, $j \leq n$, and we know which are the good and the bad intervals at time $n\delta$.

Consequences of Being in G_n

The condition to be in G_n does not only allow to control the quantities directly involved in the definition of d_n but also several other quantities. The first one is the difference $|R'_n - R_n|$.

Indeed, recalling (6.43) and (6.52),

$$
\begin{aligned}
|R'_n - R_n| &\le e^{-\lambda\delta}|R'_{n-1} - R_{n-1}| + |V((n-1)\delta) - p^{(\delta)}_{(n-1)\delta}|\delta \\
&\quad + (1 - e^{-\lambda\delta})|\bar{Y}^{(\delta)}((n-1)\delta) - \bar{\rho}^{(\delta)}_{(n-1)\delta}|. \\
&\le e^{-\lambda\delta}|R'_{n-1} - R_{n-1}| + \kappa_n\ell + \lambda\delta\kappa_{n-1}\ell.
\end{aligned} \tag{8.90}
$$

Iterating this argument yields

$$
|R'_n - R_n| \le \Big(\sum_{j=1}^{n} \kappa_j\Big)(1 + \lambda\delta)\ell.
$$

Writing $I_{i,n} = [a_{i,n}, b_{i,n}]$ and $I'_{i,n} = [a'_{i,n}, b'_{i,n}]$, we obtain in particular

$$
|a_{i,n} - a'_{i,n}| = |b'_{i,n} - b_{i,n}| \le \Big(\sum_{j=1}^{n} \kappa_j\Big)(1 + \lambda\delta)\ell. \tag{8.91}
$$

We also get a bound on the increments of the number of bad intervals. Recalling (6.59) for notation we have indeed

$$
B_n \le B_{n-1} + 1 + \frac{|R_n - R'_n|}{\ell}. \tag{8.92}
$$

We finally have bounds on $N'_{i,n}$. Firstly we suppose that $I_{i,n}$ is a good interval such that $I_{i,0} \subset \mathbb{R}_+$. Then $N'_{i,n} \le N'_{i,0}$, whence for N large enough, since $N\mathrm{w}_i = N_{i,0}$,

$$
N'_{i,n} \le N_{i,0} + \kappa_0\mathrm{w}_i N\ell \le (1 + \kappa_0\ell)N\mathrm{w}_i \le 2N\mathrm{w}_i. \tag{8.93}
$$

We also have a lower bound. By (6.46), $N_{i,n} \ge N_{i,0}e^{-f^*\delta n}$, hence

$$
N'_{i,n} \ge N_{i,n} - \kappa_n\mathrm{w}_i N\ell \ge \mathrm{w}_i N\Big(e^{-f^*T} - \kappa_n\ell\Big) \ge c\mathrm{w}_i N \ge c\ell^3 N = cr^3 N^{1-3\alpha}, \tag{8.94}
$$

for N large enough.

Now consider a good interval $I_{i,n} \subset \mathbb{R}_+$ such that $I_{i,0} \subset \mathbb{R}_-$. Then there exists $k \le n$ such that $I_{i,k-1} \subset \mathbb{R}_-$ and $I_{i,k} \subset \mathbb{R}_+$. Recalling the definition of d_k in (5.29) and the definition (5.30) we notice that $d_k \ge 1/N$, if $N\delta V(k\delta) \ge 2$, i.e. in case that at least two neurons spike. At step k, the number of neurons falling into the interval $I_{i,k}$ is upper bounded by $\frac{\ell}{d_k} + 1$, if $N\delta V(k\delta) \ge 2$, otherwise, there is at most one neuron falling into it. In both cases, this yields the upper bound $\ell N + 1$ for the number of neurons falling into the interval. After time k, neurons originally in $I_{i,k}$ can only disappear due to spiking. Hence,

$$
N'_{i,n} \le N'_{i,k} \le N(\ell + N^{-1}) \le CN\mathrm{w}_i, \tag{8.95}
$$

by definition of w_i. In order to obtain a lower bound, we first use that

$$
N'_{i,n} \ge N_{i,n} - \kappa_n\mathrm{w}_i N\ell.
$$

Since $I_{i,k-1} \subset \mathbb{R}_-$, we have that $\rho^{(\delta)}_{k\delta} \equiv \rho^{(\delta)}_{k\delta}(0)$ on $I_{i,k}$, hence $N_{i,k} = N\rho^{(\delta)}_{k\delta}(0)\ell e^{-\lambda\delta k}$. Using Proposition 6 and (6.46),

$$
N_{i,n} \ge N_{i,k}e^{-f^*\delta(n-k)} \ge N\rho^{(\delta)}_{k\delta}(0)\ell e^{-\lambda T}e^{-f^*T} \ge CN\psi_0(0)\ell = CN\mathrm{w}_i,
$$

where C depends on T, which allows to conclude as above.

In case that $I_{i,n}$ is a bad interval it is easy to see that the upper bound

$$N'_{i,n} + N_{i,n} \leq CN\ell \tag{8.96}$$

holds.

Expected Fires in Good Intervals

Recall that we are working on G_n and conditionally on $Y^{(\delta)}(n\delta) = y$. Using (5.25) and (6.51), we write

$$V(n\delta) = \frac{1}{\delta N} \sum_i \Delta_i, \text{ where } \Delta_i = \sum_{j:y_j \in I'_{i,n}} \Phi_j(n), \tag{8.97}$$

and call $\langle \Delta_i \rangle$ its conditional expectation (given $\mathcal{F}_n$, and hence given that $Y^{(\delta)}(n\delta) = y$). Then

$$\langle \Delta_i \rangle = \sum_{j:y_j \in I'_{i,n}} \left(1 - e^{-\delta f(y_j)}\right).$$

Write $I'_{i,n} = [a'_{i,n}, b'_{i,n}]$. Since f is non decreasing, we have

$$\langle \Delta_i \rangle \leq N'_{i,n}(1 - e^{-\delta f(b'_{i,n})}) \leq N'_{i,n}(1 - e^{-\delta f(a'_{i,n})}) + N'_{i,n} \left| e^{-\delta f(b'_{i,n})} - e^{-\delta f(a'_{i,n})} \right|.$$

Moreover, $f(b'_{i,n}) \leq f(a'_{i,n}) + \|f\|_{Lip} \ell$, which implies that

$$\left| e^{-\delta f(b'_{i,n})} - e^{-\delta f(a'_{i,n})} \right| \leq \delta C\ell.$$

Suppose first that $I'_{i,n}$ is good so that $|N'_{i,n} - N_{i,n}| \leq \kappa_n w_i N\ell$. Then by (8.93) and (8.95),

$$\langle \Delta_i \rangle \leq \{N_{i,n} + \kappa_n w_i N\ell\}(1 - e^{-\delta f(a'_{i,n})}) + \delta C\ell N w_i$$
$$\leq N_{i,n}(1 - e^{-\delta f(a'_{i,n})}) + C(\kappa_n + 1)\delta w_i N\ell.$$

Write $I_{i,n} = [a_{i,n}, b_{i,n}]$, so that by (8.91), $|a'_{i,n} - a_{i,n}| \leq K_n\ell$, where $K_n = (\sum_{j=1}^n \kappa_n)(1 + \lambda\delta)$. Then

$$\langle \Delta_i \rangle \leq N_{i,n}(1 - e^{-\delta f(a_i)}) + (K_n + C(1 + \kappa_n))\delta w_i N\ell.$$

Since f is non decreasing and $N_{i,n} = N \int_{I_{i,n}} \rho_{n\delta}^{(\delta)}(x)\, dx$,

$$\langle \Delta_i \rangle \leq N \int_{I_{i,n}} \rho_{n\delta}^{(\delta)}(x) \left(1 - e^{-\delta f(x)}\right) dx + (K_n + C(1 + \kappa_n))\delta w_i N\ell. \tag{8.98}$$

An analogous argument gives

$$\langle \Delta_i \rangle \geq N \int_{I_{i,n}} \rho_{n\delta}^{(\delta)}(x) \left(1 - e^{-\delta f(x)}\right) dx - (K_n + C(1 + \kappa_n))\delta w_i N\ell. \tag{8.99}$$

Fires Fluctuations in Good Intervals

Hoeffding's inequality implies that for any $b > 0$,

$$P\left[|\Delta_i - \langle \Delta_i \rangle| \geq (N'_{i,n})^{b+\frac{1}{2}}\right] \leq 2e^{-2(N'_{i,n})^{2b}}. \qquad (8.100)$$

We introduce the contribution of $I_{i,n}$ to $p_{n\delta}^{(\delta)}$

$$p_{i,n\delta}^{(\delta)} = \frac{N}{\delta} \int_{I_{i,n}} \rho_{n\delta}^{(\delta)}(x)(1 - e^{-\delta f(x)})dx.$$

We then use (8.93), (8.94) and (8.95) together with (8.98) and (8.100) to get

$$P\left[\bigcap_{I_{i,n} \text{ good}} \left\{\Delta_i \leq \delta p_{i,n\delta}^{(\delta)} + \left(K_n + C(1 + \kappa_n)\right)\delta w_i N\ell + \left(CN w_i\right)^{\frac{1}{2}+b}\right\}\right]$$
$$\geq 1 - 2m_n e^{-2(cr^3 N^{1-3\alpha})^{2b}}, \qquad (8.101)$$

where m_n is an upper bound on the number of good intervals which can be upper bounded by

$$m_n \leq \left(\frac{R_n}{e^{-\lambda n\delta}\ell} \vee \frac{R'_n}{e^{-\lambda n\delta}\ell}\right) + 1 \leq C\left(\frac{R_n}{\ell} + \left[\sum_{j=1}^{n}\kappa_j\right]\right) + 1 \leq CN^{\alpha},$$

because $R_n \leq R_0 + c^* T$ and $n\delta \leq T$. As a consequence, the right hand side of (8.101) can be lower bounded by $1 - CN^{\alpha}e^{-C(N^{1-3\alpha})^{2b}}$. By an analogous argument

$$P\left[\bigcap_{I_{i,n} \text{ good}} \{\Delta_i \geq \delta p_{i,n\delta}^{(\delta)} - \left(K_n + C(1 + \kappa_n)\right)\delta w_i N\ell - \left(2N w_i\right)^{\frac{1}{2}+b}\}\right]$$
$$\geq 1 - CN^{\alpha}e^{-C(N^{1-3\alpha})^{2b}}. \qquad (8.102)$$

Now we choose b and α sufficiently small such that for N large enough, $(CN w_i)^{\frac{1}{2}+b} \leq C\delta(1 + \kappa_n))w_i N\ell$. Then

$$P\left[\bigcap_{I_{i,n} \text{ good}} \left\{|\Delta_i - \delta p_{i,n\delta}^{(\delta)}| \leq \left(K_n + 2C(1 + \kappa_n)\right)\delta w_i N\ell\right\}\right]$$
$$\geq 1 - CN^{\alpha}e^{-C(N^{1-3\alpha})^{2b}} \geq 1 - Ce^{-CN^{\gamma}}, \qquad (8.103)$$

where $\gamma = (1 - 3\alpha)2b$ and C a suitable constant.

The Bounds on $V(n\delta)$ and on B_{n+1}

By (8.96) and (8.103), with probability $\geq 1 - Ce^{-CN^{\gamma}}$,

$$|\delta V(n\delta) - \delta p_{n\delta}^{(\delta)}| \leq \left[\sum_{I_{i,n} \text{ good}} (K_n + 2C(1 + \kappa_n))\delta w_i\ell\right] + \frac{CN\ell}{N}B_n \leq \kappa'_{n+1}\ell. \qquad (8.104)$$

Hence, we have proven the desired assertion for V_n at time $(n + 1)\delta$.

To bound B_{n+1}, the number of bad intervals at time $(n+1)\delta$, we use the first inequality in (8.90) with $n \to n+1$ together with (8.104), so that by (8.92)

$$B_{n+1} \leq B_n + 1 + \frac{|R_{n+1} - R'_{n+1}|}{\ell} \leq B_n + 1 + \left(\sum_{j=1}^{n} \kappa_j + \kappa'_{n+1} + \kappa_n \right)(1 + \lambda\delta),$$

whence the assertion concerning B_{n+1}.

Bounds on $|N'_{i,n+1} - N_{i,n+1}|$

Let $I_{i,n}$ be a good interval at time $n\delta$ which is contained in $\mathbb{R}_+$. Then it is good also at time $(n+1)\delta$ and we have

$$N'_{i,n+1} = \sum_{j:y_j \in I'_{n,i}} (1 - \Phi_j(n)) = N'_{i,n} - \Delta_i \text{ and } N_{i,n+1} = N_{i,n} - \delta p^{(\delta)}_{i,n\delta}.$$

Thus

$$\frac{|N'_{i,n+1} - N_{i,n+1}|}{w_i N\ell} \leq \kappa_n + \frac{|\Delta_i - \delta p^{(\delta)}_{i,n\delta}|}{w_i N\ell},$$

and the desired bound follows from (8.103).

It remains to consider a good interval $I'_{i,n+1}$ such that $I'_{i,n} \subset \mathbb{R}_-$ (and hence also $I_{i,n} \subset \mathbb{R}_-$). Thus $I'_{i,n+1}$ consists entirely of "new born" neurons which arise due to firing events where the energies are reset to 0. For such an interval,

$$\frac{N'_{i,n+1}}{N\ell} \in [\frac{1}{Nd_n}e^{-\lambda\delta n}, \frac{1}{Nd_n}e^{-\lambda\delta n} + 1].$$

But, recalling the definition of d_n in (5.29) and of $\rho^{(\delta)}_{(n+1)\delta}(0)$ in (6.41), by continuity of $(u, p) \to \frac{p\delta}{p\delta + (1-e^{-\lambda\delta})u}$, we have

$$\left| \frac{1}{Nd_n} - \rho^{(\delta)}_{(n+1)\delta}(0) \right| \leq C\kappa_n\ell. \tag{8.105}$$

Since $\rho^{\delta}_{(n+1)\delta}(x)1_{I_{i,n+1}}(x) \equiv \rho^{(\delta)}_{(n+1)\delta}(0)$ on this interval, this implies that also for such intervals,

$$\frac{1}{N\ell}\left|N'_{i,n+1} - N_{i,n+1}\right| \leq C\kappa_n\ell = C\kappa_n w_i,$$

by definition of w_i. This concludes the bound of $|N'_{i,n+1} - N_{i,n+1}|$.

The bound on $|\bar{Y}^{(\delta)}((n+1)\delta) - \bar{\rho}^{(\delta)}_{(n+1)\delta}|$ follows from the bounds on $|N'_{i,n+1} - N_{i,n+1}|$ and B_{n+1}; details are omitted. This concludes the proof of Theorem 5.

Appendix 4: Proof of Theorem 2 for General Firing Rates

Let f, T, A, B as in Theorem 1, x^N the initial state of the neurons as in Theorem 2 and such that $\|x^N\| \leq A$. Let ψ be a bounded continuous function on $D([0, T], \mathcal{S}')$. We need to prove that

$$\lim_{N \to \infty} \mathcal{P}^N_{[0,T]}(\psi) = \psi(\rho)$$

where $\mathcal{P}^N_{[0,T]}(\psi)$ is the expected value of ψ under the law of $(\mu_{U^N})_{[0,T]}$ when the process U^N starts from x^N and $\psi(\rho)$ is the value of ψ on the element $\rho := (\rho_t dx)_{t \in [0,T]}$ of $D([0,T], \mathcal{S}')$.

Let $\mathbf{1}_{\mathcal{U}}$ be the characteristic function of the event $\{\|U^N(t)\| \leq B, t \in [0,T]\}$. Then by Theorem 1

$$\lim_{N \to \infty} \left| \mathcal{P}^N_{[0,T]}(\psi) - \mathcal{P}^N_{[0,T]}(\psi \mathbf{1}_{\mathcal{U}}) \right| = 0. \tag{8.106}$$

By an abuse of notation we call $\mathcal{P}^{*,N}_{[0,T]}$ the law of the process with a firing rate $f^*(\cdot)$ which satisfies Assumption 3 and coincides with f for $x \leq B$. Then

$$\mathcal{P}^N_{[0,T]}(\psi \mathbf{1}_{\mathcal{U}}) = \mathcal{P}^{*,N}_{[0,T]}(\psi \mathbf{1}_{\mathcal{U}}). \tag{8.107}$$

Since we have proved Theorem 2 under Assumption 3, we have convergence for the process with rate $f^*(\cdot)$ to a limit density that we call $\rho^* = (\rho^*_t)_{t \in [0,T]}$, so that

$$\lim_{N \to \infty} \mathcal{P}^{*,N}_{[0,T]}(\psi \mathbf{1}_{\mathcal{U}}) = \psi(\rho^* \mathbf{1}_{\mathcal{U}}). \tag{8.108}$$

As a consequence of (8.106) and (8.107),

$$\lim_{N \to \infty} \mathcal{P}^N_{[0,T]}(\psi) = \psi(\rho^* \mathbf{1}_{\mathcal{U}}).$$

By the arbitrariness of ψ, $\rho^* = \rho^* \mathbf{1}_{\mathcal{U}}$. Indeed, taking $\psi(\omega) = \sup\{\omega_t(1), t \leq T\} \wedge 1$, we have $\lim_{N \to \infty} \mathcal{P}^N_{[0,T]}(\psi) \equiv 1$, which implies that ρ^* must have support in $[0, B]$. As a consequence,

$$\lim_{N \to \infty} \mathcal{P}^N_{[0,T]}(\psi) = \psi(\rho^* \mathbf{1}_{\mathcal{U}}) = \psi(\rho^*),$$

and the limit ρ^* is equal to the solution of the equation with the true firing rate f. This concludes the proof of the theorem.

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