Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics

1994 Nonlinearity 7 633

(http://iopscience.iop.org/0951-7715/7/3/001)

Nonlinearity 7 (1994) 633–696. Printed in the UK

Glauber evolution with Kac potentials: I. Mesoscopic and macroscopic limits, interface dynamics*

A De Masi†§, E Orlandi†||, E Presutti‡¶ and L Triolo‡+

† Dipartimento di Matematica, Università di L'Aquila, Coppito, 67100 L'Aquila, Italy
‡ Dipartimento di Matematica, Università di Roma 'Tor Vergata', Via della Ricerca *Scientifica, 00133 Roma, Italy*

Received 23 September 1993, in final form 25 January 1994
Recommended by Y Sinai

Abstract. This is the first of three papers on the Glauber evolution of Ising spin systems with Kac potentials. We begin with the analysis of the mesoscopic limit, where space scales like the diverging range, γ^{-1}, of the interaction while time is kept finite: we prove that in this limit the magnetization density converges to the solution of a deterministic, nonlinear, nonlocal evolution equation. We also show that the long time behaviour of this equation describes correctly the evolution of the spin system till times which diverge as $\gamma \to 0$ but are small in units $\log \gamma^{-1}$. In this time regime we can give a very precise description of the evolution and a sharp characterization of the spin trajectories. As an application of the general theory, we then prove that for ferromagnetic interactions, in the absence of external magnetic fields and below the critical temperature, on a suitable macroscopic limit, an interface between two stable phases moves by mean curvature. All the proofs are consequence of sharp estimates on special correlation functions, the v-functions, whose analysis is reminiscent of the cluster expansion in equilibrium statistical mechanics.

AMS classification scheme numbers: 60K35, 82A05

1. Introduction

The van der Waals theory of phase transitions describes systems with forces which are repulsive at short distances and have long attractive tails. By scaling the attractive part of the interaction, it is possible to construct a family of models where the above condition is satisfied arbitrarily well. This idea was proposed by M Kac in the context of equilibrium statistical mechanics, where the limit case reproduces exactly the van der Waals phase diagram, as proven in a wide variety of systems, [17, 20]. Also the metastable effects predicted by the van der Waals theory are to some extent recovered by this approach [22].

Non equilibrium properties for systems with Kac potentials have been studied in [5, 6, 19] and, more recently, [24]. Inspired by these works, we begin in this paper a systematic analysis of the Glauber dynamics in Ising spin systems with Kac potentials, which will be

* The research has been partially supported by CNR, GNFM and by grant SC1-CT91-0695 of the Commission of European Communities

§ E-mail: demasi@ vxscaq.aquila.infn.it
|| E-mail: orlandi@ vxscaq.aquila.infn.it
¶ E-mail: presutti@ irmtvm51.bitnet; presutti@mat.utovrm.it
+ E-mail: triolo@ irmtvm51.bitnet; triolo@mat.utovrm.it

0951-7715/94/030633+64$19.50

further developed in two successive papers, [II] and [III], we will refer to this one as [I]. (An unpublished version of [I] and [II] has appeared a year ago, [IV]: it contains the analysis of the system with different rates of the Glauber dynamics. It also gives a short survey of the physical backgrounds and a list of open problems). As we shall see, the model has a surprisingly rich structure and it exhibits a great variety of physically interesting effects. Phase separation, development of interfaces and interface dynamics are the main issues of our analysis. In particular in [III] we study the phase separation after quenching down, below the critical value, a high temperature state: we observe the development of the interfaces and characterize their structure. The successive interface dynamics is investigated here, in the simpler case of a single interface. Fluctuations theory and critical phenomena also in relation to stochastic quantization are discussed in [IV] and will be the main object of [II].

We denote by γ the scaling parameter of the Kac interaction, that will eventually go to 0. The characteristic feature of a Kac potential is that its range diverges like γ^{-1}, while the total interaction energy of any single spin with all the others is kept finite. We first study the limit when $\gamma \to 0$, scaling the space by the same γ: we will prove that the limiting magnetization density $m(r, t)$ solves the deterministic, nonlocal evolution equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta(J \star m + h)\} \quad \big(J \star m\big)(r) = \int \mathrm{d}r' J([r - r'])m(r'). \tag{1.1}$$

Times in this limit are not rescaled, so that each individual spin in a time unit undergoes only a finite, random, number of flips. The deterministic behaviour described by (1.1) is a mean field effect due to the scaling of the interaction: many spins (infinitely many in the limit $\gamma \to 0$) feel essentially the same potential and while each of them, individually, has a random behaviour, the collectivity evolves deterministically, due to a law of large numbers which dampens the fluctuations. The full effect of the interaction only arises at longer times, when each spin, after many flips, reaches a (local) equilibrium distribution.

We call the above limit 'mesoscopic', with space scaled by γ and time kept finite, to distinguish it from other, macroscopic, limits, where also the time is scaled with γ. The equation (1.1) obtained in the mesoscopic limit will be called the mesoscopic equation. As explained before, we expect the more interesting effects to appear at longer times. The real question then is whether they are correctly predicted by the mesoscopic equation. There is no answer valid for all the cases: we have indeed situations where the limits can be interchanged, thus finding the true behaviour of the Glauber dynamics at times which diverge as $\gamma \to 0$ by investigating the long time behaviour of (1.1). This happens for instance in the development of the interfaces (at temperatures below the critical one and when the magnetization profile is away from its critical value, except on regular surfaces where the interfaces develop) and also in the successive stage of interface dynamics, at least when the curvature of the interface is not too small. In section 5 we will study the latter case.

It may also happen that the long time predictions of the mesoscopic equation are wrong: the spinodal decomposition after quenching a state from high temperature down below the critical value and the macroscopic fluctuations at the critical temperature are intrinsically random. The stochastic forces responsible for these effects are absent in (1.1), having disappeared in the mesoscopic limit. These aspects will be studied in [II] and [III] while this paper covers cases where the fluctuations are not relevant, as in the mesoscopic limit that we study first. This is easy to investigate, as it is essentially a mean field limit, but we are interested in the behaviour of the system past the mesoscopic times and for that we need sharper estimates. As an outcome of our analysis, we obtain a characterization of most of the individual trajectories of the spin process, both in space and time, proving that with large probability they only slightly deviate from what is predicted by the mesoscopic equation, at least when times are small in units $\log \gamma^{-1}$.

The small deviations mentioned above are not only consequence of errors and approximations in our estimates, but they have an intrinsic origin, related to the buildup of correlations as time grows and γ is kept finite. This issue is treated in [II], in the context of the fluctuation theory for Glauber dynamics with Kac potentials. Here we ignore these effects considering the deviations of the spins distribution from the product measure as errors. We prove, with the help of cluster expansion techniques, that these deviations are small and that, if the evolution described by the mesoscopic equation is sufficiently stable, then they do not have significant effects. As an application, we study in section 5 the case of ferromagnetic interactions without any external magnetic field and below the critical temperature. We fix an initial magnetization profile which decribes an interface between the two stable phases and prove that, on a suitable macroscopic limit, the interface moves by mean curvature.

We would like to draw the attention of the reader also to the techniques used in proving the above results as they are based on powerful methods which may be useful also in other cases. Kinetic theory and cluster expansion play here an important role. The small parameter of the latter, which is the inverse temperature in statistical mechanics, is here played simultaneously by γ and by the time, which, at first, is supposed vanishingly small, as $\gamma \to 0$. To obtain estimates at finite and longer times we then work out an iterative procedure based on special (truncated) correlation functions, the v-functions. This part of the analysis is common to several other models, see [13] for a survey on the method, which, in particular, has been applied to stochastic, discrete-velocity models of the Boltzmann equation, hence the relation with kinetic theory.

The paper is organized as follows. Section 2 contains the main definitions, and the results concerning the derivation of the mesoscopic equation, the interface dynamics and the bounds on the v-functions. The 'short time' bounds are proven in section 3 and extended, in section 4, to longer times. In section 5 we apply the previous considerations to prove that the interface dynamics is ruled by the motion by mean curvature. In an appendix we prove proposition 4.8 of section 4.

2. Main definitions and results

This section is divided in three subsections. In the first one, section 2.1, we define the model and state the theorem on the convergence to the mesoscopic equation. In section 2.2 we consider a class of initial states which describe, in a macroscopic limit, magnetic profiles with an interface. We then have a theorem which states that in the limiting macroscopic evolution the interfaces move by mean curvature. Finally in section 2.3 we present results on the propagation of chaos, that is proven in a very strong form and till times which diverge as $\gamma \to 0$ but that are small in units $\log \gamma^{-1}$. In this subsection we define the v-functions and state bounds on these functions which provide the main technical tools for studying the transition from discrete to continuum.

2.1. The mesoscopic limit

We consider an Ising spin system and start by recalling the main notation and definitions. A spin configuration is a specification of the values of the spins at all the lattice sites, it is therefore a function $\sigma \colon \mathbb{Z}^d \to \{-1, 1\}$, that is an element of $\{-1, 1\}^{\mathbb{Z}^d}$. The value $\sigma(x)$ of the spin at x is thus a function of the configuration σ, thus a random variable on $\{-1,1\}^{\mathbb{Z}^d}$, the space of all the spin configurations. The restriction to $\Delta \subset \mathbb{Z}^d$ of a configuration σ, is denoted by σ_Δ, which is therefore a function on Δ with values $\{-1,1\}$.

We next recall the definition of Kac potentials.

Definition 2.1.1. *A Kac potential (in this paper) is a function* $J_\gamma : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, *which depends on a (scaling) parameter* γ *and has the form*

$$J_\gamma(x,y) = \gamma^d J(\gamma[x-y]). \tag{2.1}$$

We suppose that γ *varies in the set* $\{2^{-n}, n \in \mathbb{Z}_+\}$. *We assume that* $J(r)$ *depends on* $|r|$, $J(r) = 0$ *for all* $|r| > 1$ *and that* $J(r) \in C^3(\mathbb{R}^d)$.

Given (a magnetic field) $h \in \mathbb{R}$, *we define the energy of the spin configuration* σ_Δ *as*

$$H_\gamma(\sigma_\Delta) = -h\sum_{x\in\Delta}\sigma(x) - \frac{1}{2}\sum_{x\neq y\in\Delta} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{2.2}$$

while its energy inclusive of the interaction with the spins in the complement, Δ^c, *of* Δ, *is*

$$H_\gamma(\sigma_\Delta|\sigma_{\Delta^c}) = H_\gamma(\sigma_\Delta) - \sum_{x\in\Delta, y\notin\Delta} J_\gamma(x,y)\sigma(x)\sigma(y). \tag{2.3}$$

The class of the Kac potentials is more general than in definition 2.1.1, the only requirement on J being that it is in $L^1(dr, \mathbb{R}^d)$, see [17, 20]. The restriction on the values of γ is only made with the purpose of simplifying notation when discussing the block spin variables and could be easily lifted.

Definition 2.1.2. *Given (the 'inverse temperature')* $\beta > 0$ *and* $\gamma > 0$, *we denote by Glauber dynamics the unique Markov process on* $\{-1,1\}^{\mathbb{Z}^d}$ *whose pregenerator is the operator* L_γ *with domain the set of all the cylinder functions* f *on which it acts as*

$$L_\gamma f(\sigma) = \sum_{x\in\mathbb{Z}^d} c_\gamma(x,\sigma)[f(\sigma^x) - f(\sigma)]. \tag{2.4}$$

In (2.4) σ^x *is the configuration obtained from* σ *by flipping the spin at* x, *i.e.*

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(x) & \text{if } y = x \end{cases} \tag{2.5}$$

The 'flip rate' $c_\gamma(x,\sigma)$ *of the spin at* x *in the configuration* σ *is*

$$c_\gamma(x,\sigma) = \frac{e^{-\beta h_\gamma(x)\sigma(x)}}{e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}} \tag{2.6a}$$

$$h_\gamma(x) = h + (J_\gamma \circ \sigma)(x) \qquad (J_\gamma\circ\sigma)(x) = \sum_{y\neq x} J_\gamma(x,y)\sigma(y) \tag{2.6b}$$

The proof of the existence and uniqueness of the Markov process, used in the above definition, may be found in [21]. The space of realizations of the Glauber dynamics is $D\big(\mathbb{R}_+, \{-1,1\}^{\mathbb{Z}^d}\big)$, the Skorohod space of cadlag trajectories, (continuous from the right and with limits from the left). The value of the process at time t, that is the spin configuration at time t, is denoted by σ_t, the value of the spin in x at time t is $\sigma(x,t)$ which is thus a random variable on $D\big(\mathbb{R}_+, \{-1,1\}^{\mathbb{Z}^d}\big)$.

Notice that

$$c_\gamma(x,\sigma) = Z_\gamma(\sigma_{x^c})^{-1} e^{-\beta/2\Delta_x H_\gamma(\sigma)}$$

where $\Delta_x H_\gamma(\sigma)$ is the change of energy due to the spin flip at x, namely

$$\Delta_x H_\gamma(\sigma) = H_\gamma\big((\sigma^x)_\Lambda\big) - H_\gamma\big(\sigma_\Lambda\big)$$

where Λ is any set which contains x and such that the spin at x does not interact with those in Λ^c. $Z_\gamma(\sigma_{x^c})^{-1}$ is the denominator in (2.6a), but, for what we say below, it may be any other function, provided it is independent of $\sigma(x)$, as implied by the notation. In fact the important point about the rates is that they verify the 'detailed balance' condition

$$\frac{c_\gamma(x,\sigma^x)}{c_\gamma(x,\sigma)} = e^{-\beta\Delta_x H_\gamma(\sigma)}. \tag{2.6c}$$

The Glauber dynamics is thus intimately related to the notion of:

Definition 2.1.3. *The Gibbs measure $\mu_{\beta,h,\gamma}$ is any probability on $\{-1,1\}^{\mathbb{Z}^d}$ which satisfies the DLR equations: namely, such that for any $x \in \mathbb{Z}^d$ and any σ,*

$$\mu_{\beta,h,\gamma}\Big(\sigma(x) = \pm 1 \Big| \{\sigma(y), y \neq x\}\Big) = \frac{e^{\pm\beta h_\gamma(x)}}{e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}} \qquad \mu_{\beta,h,\gamma} \text{ almost surely} \tag{2.7 }$$

where the left-hand side is the probability that $\sigma(x) = \pm 1$ conditioned on the σ-algebra generated by all the spins $\sigma(y)$, $y \neq x$.

β in definition 2.1.3 has the physical meaning of an inverse temperature and h of an external magnetic field, J_γ of the spin–spin interaction strength. Notice that the left- hand side of (2.7) is a function of $\sigma(x)$ and all $\sigma(y)$, $y \neq x$, it is thus a function of the whole spin configuration σ. Then, from (2.6c) and (2.7), it follows that

$$\mu_{\beta,h,\gamma}\big(\sigma(x)|\{\sigma(y),\ y\neq x\}\big)c_\gamma(x,\sigma) = \mu_{\beta,h,\gamma}\big(\sigma^x(x)|\{\sigma(y),\ y\neq x\}\big)c_\gamma(x,\sigma^x)$$

so that the operator $L_\gamma^{(x)}$ defined by (2.4) after setting $c_\gamma(y,\sigma) = 0$ for all $y \neq x$, is self-adjoint in $L_2\big(\{-1,1\}^{\mathbb{Z}^d}, \mu_{\beta,h,\gamma}\big)$. It then follows that also the full generator of the Glauber dynamics is selfadjoint and that $\mu_{\beta,h,\gamma}$ is stationary, the Glauber dynamics then being a reversible process. We will not exploit this feature of the dynamics and the associated theory of Dirichlet forms, which in many instances has been proven to be a very useful and powerful method. We will use though the existence of the Gibbs measures for a physical interpretation of the results. Many aspects of our analysis are common to other models, as for instance the Glauber + Kawasaki dynamics, which is a very well studied model for reaction-diffusion equations, [8], and phase separation, [13, 12, 15, 16]. The advantage in our case is the explicit connection with equilibrium statistical mechanics, so that we can properly talk of thermodynamic phases, surface tension, critical temperature and so forth. We finally mention that (2.6c) does not depend on the choice of Z_γ, which appears in the definition of c_γ, thus different choices of Z_γ define other, equally acceptable, reversible evolutions. The choice (2.6a) gives rise to a simpler limiting mesoscopic equation.

We have so far discussed the Glauber dynamics at the microscopic level, we next turn to the mesoscopic one. The 'scale separation' between the two levels is specified by γ: in the transition micro-mesoscopic

$$(x,t) \to (r,t) = (\gamma x, t).$$

Time is thus unchanged while space is shrunk by γ. The microscopic points $x \in \mathbb{Z}^d$ are represented in the mesoscopic space $\mathbb{R}^d$ by the lattice $\gamma\mathbb{Z}^d$. It is thus convenient to partition $\mathbb{R}^d$ into the 'elementary squares' $\{r : [r]_\gamma = \gamma x\}$, with $x \in \mathbb{Z}^d$ and, denoting by $r = (r_1, .., r_d)$, $x = (x_1, .., x_d)$,

$$[r]_\gamma = \gamma x \text{ if } x \in \mathbb{Z}^d \text{ and } \gamma x_i \leqslant r_i < \gamma(x_i + 1) \quad \text{for all } i = 1, .., d \tag{2.8a}$$

Definition 2.1.4. *We denote by $\mathcal{M}(X)$, X a measurable space, the space of all the real valued, measurable functions on X. We then define $\Gamma_\gamma : \mathcal{M}(\mathbb{Z}^d) \to \mathcal{M}(\mathbb{R}^d)$ as*

$$\big(\Gamma_\gamma(f)\big)(r) = f(x) \qquad x = \gamma^{-1}[r]_\gamma \text{ and } f \in \mathcal{M}(\mathbb{Z}^d) \tag{2.8b}$$

where $[r]_\gamma$ is defined in (2.8a).

In particular we denote by

$$\sigma_\gamma = \Gamma_\gamma(\sigma) \qquad \sigma_{\gamma,t} = \Gamma_\gamma(\sigma_t) \tag{2.8c}$$

$\sigma_\gamma(r)$ is thus the image of the spin configuration σ in the mesoscopic representation. Our first theorem proves that for small γ's, the Glauber dynamics in the mesoscopic representation is almost deterministic. The statement refers to a smoothened version of σ_γ, defined in terms of the block spin transformation: this amounts to replace the value $f(r)$ of a function in $\mathcal{M}(\mathbb{R}^d)$ by its average in a region containing r, made of elementary squares and whose size vanishes as $\gamma \to 0$.

Definition 2.1.5. *We define for any $0 < \alpha < 1$ and γ as in 2.1.1 the block spin transformation $f \to f^{(\alpha,\gamma)}$, f and $f^{(\alpha,\gamma)}$ both in $\mathcal{M}(\mathbb{R}^d)$, as*

$$f^{(\alpha,\gamma)}(r) = N_\gamma^{-1} \int \mathrm{d}r' \mathbf{1}\big(\{\big|[r]_\gamma - [r']_\gamma\big| \leqslant \gamma^{1-\alpha}\}\big) f(r') \tag{2.9a}$$

$$N_\gamma = \int \mathrm{d}r' \mathbf{1}\big(\{\big|[r]_\gamma - [r']_\gamma\big| \leqslant \gamma^{1-\alpha}\}\big). \tag{2.9b}$$

We may sometimes use the shorthand notation

$$\sigma_\gamma^{(\alpha)} := \big(\sigma_\gamma\big)^{(\alpha,\gamma)} \qquad \sigma_{\gamma,t}^{(\alpha)} := \big(\sigma_{\gamma,t}\big)^{(\alpha,\gamma)} \tag{2.9c}$$

to avoid redundancy in the formulae.

The more familiar form of the block spin transformation is recovered when we apply the transformation to a function $g = \Gamma_\gamma(f)$, $f \in \mathcal{M}(\mathbb{Z}^d)$. In that case $g^{(\alpha,\gamma)}(r)$, $[r]_\gamma =: \gamma x$, is given by

$$g^{(\alpha,\gamma)}(r) = \mathcal{A}_{\gamma^{-\alpha},x}(f) := \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{y \in B_{\gamma^{-\alpha},x}} f(y) \tag{2.9d}$$

where

$$B_{\gamma^{-\alpha},x} = \{|y - x| \leqslant \gamma^{-\alpha}\} \qquad |B_{\gamma^{-\alpha}}| = \text{ cardinality of } B_{\gamma^{-\alpha},x}. \tag{2.9e}$$

We will use extensively the above notation in section 4, where we work in the microscopic rather than in the mesoscopic representation. We will prove there the following result.

Theorem 2.1.6. *For any $\alpha \in (0, 1)$ and $\zeta > 0$, there are a and b positive and for any n and any $k^* \geqslant 2$, there is c so that the following holds. For all γ small enough and, given γ, for all $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$ and $m \in \mathcal{M}(\mathbb{Z}^d)$, $\|m\|_\infty \leqslant 1$, for which (see (2.9a–c) for notation)*

$$\sup_{|r| \leqslant k^*\gamma^{-1}} \left|(\sigma_\gamma)^{(\alpha,\gamma)}(r) - m^{(\alpha,\gamma)}(r)\right| \leqslant \gamma^\zeta \tag{2.10a}$$

we have that (see again (2.9c) for notation)

$$\mathbb{P}^\gamma_\sigma\Big(\sup_{t \leqslant a \log \gamma^{-1}} \sup_{|r| \leqslant (k^*-1)\gamma^{-1}} \left|(\sigma_{\gamma,t})^{(\alpha,\gamma)}(r) - m^{(\alpha,\gamma)}(r,t)\right| > \gamma^b\Big) \leqslant c\gamma^n. \tag{2.10b}$$

$\mathbb{P}^\gamma_\sigma$ is the law of the Glauber dynamics when the process starts at time 0 from σ and

$$m^{(\alpha,\gamma)}(r,t) \equiv \Big(m(\cdot,t)\Big)^{(\alpha,\gamma)}(r) \tag{2.11}$$

$m(\cdot, t)$ being the unique solution of the Cauchy problem (1.1) with intial datum m.

Observe that, given σ, the choice $m(r) = \sigma_\gamma(r)$ automatically satisfies (2.10*a*). The proof of theorem 2.1.6 is essentially the same if we replace, in the sup over r, γ^{-1} by γ^{-k}, no matter how large is k. γ^{-1} is however sufficient for our needs here, as it corresponds to regions unbounded both in mesoscopic and macroscopic units, when the latter are defined as in section 2.2 below.

Theorem 2.1.6 expresses in a very strong form the deterministic nature of the Glauber dynamics in the mesoscopic limit. Notice however that theorem 2.1.6 does not say that $m_\gamma(r,t)$ has a limit when $\gamma \to 0$, in fact we are not imposing that the limit exists at time 0. There are many interesting situations, on the other hand, where there is convergence at time 0, and this will be the main issue in the sequel of the subsection.

We start with a definition:

Definition 2.1.7. *A mesoscopic profile is a function $m_0 \in \mathcal{M}(\mathbb{R}^d)$, such that its sup norm $\|m_0\|_\infty \leqslant 1$. We call initial mesoscopic state a family (m_0, μ^γ), where m_0 is a mesoscopic profile and, for each γ, μ^γ is any product probability measure on $\{-1, 1\}^{\mathbb{Z}^d}$ such that, setting*

$$m_{\gamma,0} := \Gamma_\gamma\Big(\mathbb{E}_{\mu^\gamma}\big(\sigma(\cdot)\big)\Big) \tag{2.12a}$$

$$\lim_{\gamma \to 0} m_{\gamma,0} = m_0 \qquad \textit{Lebesgue almost everywhere} \tag{2.12b}$$

Remarks. There are two typical examples of families (m_0, μ^γ). In the first one, that we call the 'standard initial state', $m_0 \in C(\mathbb{R}^d)$ and

$$\mathbb{E}^\gamma_{\mu^\gamma}\big(\sigma(x)\big) = m_0(\gamma x) \tag{2.12c}$$

In the second example, m_0 is an arbitrary mesoscopic profile and the function $m_{\gamma,0}$ that defines μ^γ via (2.12*a*) is obtained by averaging m_0 over each elementary square, see (2.8*a*). Then, by the martingale convergence theorem, $m_{\gamma,0}$ converges Lebesgue almost everywhere to m_0. Observe that this latter example proves that there is an initial state, in the sense of definition 2.1.7, for any given mesoscopic state m_0, so that the definition 2.1.7 is well posed.

Theorem 2.1.8. *Let (m_0, μ^γ) be an initial mesoscopic state in the sense of definition 2.1.7. Let α, ζ, a, b, n, k^* and c as in theorem 2.1.6. Then*

$$\mu^\gamma\Big(\sup_{|r|\leqslant k^*\gamma^{-1}} \big|(\sigma_\gamma)^{(\alpha,\gamma)}(r) - m^{(\alpha,\gamma)}_{\gamma,0}(r)\big| \leqslant \gamma^\zeta\Big) \geqslant 1 - c\gamma^n \tag{2.13a}$$

with $m^{(\alpha,\gamma)}_{\gamma,0}$ the block spin transform of $m_{\gamma,0}$, defined in (2.12a).

Observe then that as a consequence of (2.13a) and (2.10b),

$$\mu^\gamma\bigg(\Big\{\sigma_0 : \mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t\leqslant a\log\gamma^{-1}}\ \sup_{|r|\leqslant(k^*-1)\gamma^{-1}} \Big|(\sigma_{\gamma,t})^{(\alpha,\gamma)}(r) - m^{(\alpha,\gamma)}_\gamma(r,t)\Big| > \gamma^b\Big) \leqslant c\gamma^n\Big\}\bigg) \geqslant 1 - c\gamma^n \tag{2.13b}$$

$m^{(\alpha,\gamma)}_\gamma(\cdot, t)$ being the block spin transformation of $m_\gamma(\cdot, t)$ this latter solving (1.1) with initial condition $m_\gamma(r, 0) = m_{\gamma,0}(r)$.

Furthermore, for any t, $m_\gamma(r,t) \to m(r,t)$, Lebesgue almost everywhere, where $m(r,t)$ is the solution of (1.1) with initial condition m_0. Finally, let (m_0, μ^γ) be the standard initial state defined in the remark following definition 2.1.7. Suppose that $m_0 \in C^1(\mathbb{R}^d)$, with bounded derivative, then $m^{(\alpha,\gamma)}_\gamma(r,t)$ in (2.13b) may be replaced by $m(r,t)$.

As a straight corollary of (2.13*b*):

$$\mathbb{P}^\gamma_{\mu^\gamma}\Big(\sup_{t\leqslant a\log\gamma^{-1}}\ \sup_{|r|\leqslant(k^*-1)\gamma^{-1}} \Big|(\sigma_{\gamma,t})^{(\alpha,\gamma)}(r) - m^{(\alpha,\gamma)}_\gamma(r,t)\Big| > \gamma^b\Big) \leqslant c\gamma^n \tag{2.13c}$$

where $\mathbb{P}^\gamma_{\mu^\gamma}$ is the law of the Glauber dynamics starting from μ^γ. Moreover, if (m_0, μ^γ) is a standard initial state with $m_0 \in C^1(\mathbb{R}^d)$ having a bounded derivative, then

$$\mathbb{P}^\gamma_{\mu^\gamma}\Big(\sup_{t\leqslant a\log\gamma^{-1}}\ \sup_{|r|\leqslant(k^*-1)\gamma^{-1}} \Big|(\sigma_{\gamma,t})^{(\alpha,\gamma)}(r) - m(r,t)\Big| > \gamma^b\Big) \leqslant c\gamma^n. \tag{2.13d}$$

Further results are stated in theorem 4.9. The proofs of all the statements so far are given in sections 3 and 4. Proposition 4.8 and the appendix are not needed in this part of the analysis.

2.2. Interface dynamics, motion by mean curvature

In this subsection we restrict to $J \geqslant 0$, $h = 0$ and $\beta > 1$ after having imposed the normalization condition $\int \mathrm{d}r\, J(r) = 1$. These assumptions are essential as we expect a completely different behaviour in the other cases.

$\beta = 1$ is the inverse critical temperature in the Lebowitz–Penrose limi2.20 t, [20]: we are thus considering temperatures below the critical one. We denote by m_β the strictly positive solution of

$$m_\beta = \tanh\{\beta_{m_\beta}\} \qquad (\text{ recall that } \int \mathrm{d}r\, J(r) = 1). \tag{2.14}$$

The values $\pm m_\beta$ are the magnetizations of the two extremal Gibbs states in the limit $\gamma \to 0$ and are the thermodynamic values of the magnetization, [20]. Thus the function identically equally to m_β (or to $-m_\beta$), which is a stationary solution of (1.1), is interpreted as the m_β (respectively the $-m_\beta$) pure phase.

The interfaces are then the regions which separate the two thermodinamically pure phases. We are interested here in the dynamical problem, namely the evolution of an initial state where the two phases coexist and are separated by an interface. We study the case when the phase m_β occupies a 'large, but bounded region', while the outside is filled up by the other phase $-m_\beta$. If the region is large and the interface sufficiently flat, then we will show that the evolution is simply described by the motion of the interface, namely with the phase m_β in the region enclosed by the moving interface and the phase $-m_\beta$ outside. The dynamics of the interface, as we shall see, obeys the law of motion by mean curvature.

Definition 2.2.1. *Let Λ_0 be a compact domain whose boundary, Σ_0, is a C^∞ connected surface in $\mathbb{R}^d$. Let*

$$\epsilon = \lambda\gamma \qquad \lambda = \frac{1}{\sqrt{\log \gamma^{-1}}} \tag{2.15}$$

We denote hereafter by μ^ϵ the product measure on $\{-1, 1\}^{\mathbb{Z}^d}$ with $\mathbb{E}^\gamma_{\mu^\epsilon}(\sigma(x)) = m_\beta$ for all x in $\epsilon^{-1}\Lambda_0$ and $= -m_\beta$ in the complement.

The above state does not describe the interface Σ_0 in the sense of definition 2.1.7. In fact the function

$$m_{\gamma,0} := \Gamma_\gamma\big(\mathbb{E}_{\mu^\epsilon}(\sigma(\cdot))\big)$$

converges pointwise to the function constantly equal to m_β which is thus the corresponding mesoscopic profile. The size of the cluster of the phase m_β becomes infinite in mesoscopic variables, which are thus inadequate for describing the interface.

Definition 2.2.2. *We denote by ξ and τ the macroscopic space and time coordinates which are related to the microscopic ones by*

$$\xi = \epsilon x \qquad \tau = \lambda^2 t \tag{2.16}$$

and to the mesoscopic ones (r, t) by

$$\xi = \lambda r \qquad \tau = \lambda^2 t. \tag{2.17}$$

We are denoting by λ^{-1} the parameter which separates the mesoscopic and the macroscopic scales. Times are then separated by the factor λ^{-2}: this is the right scale to observe the motion of the interface, in other phenomena the relevant macroscopic scaling may be different.

The choice (2.15), $\lambda = (\log\gamma^{-1})^{-1/2}$, is motivated by [III] where it is proven that the clusters size after phase separation scales as $(\log\gamma^{-1})^{-1/2}$. Our analysis therefore describes the interface dynamics right after the phase separation, with the further simplifying assumption of considering a single interface. The extension to many interfaces, as they appear in the analysis of [III], should not bring in serious difficulties.

The initial state μ^ϵ in definition 2.2.1 describes a profile with the pure phases m_β and $-m_\beta$ respectively inside and outside of $\epsilon^{-1}\Lambda_0$. Σ_0 is thus the sharp macroscopic interface between the two pure phases. It is sharp in macroscopic coordinates, as it should, but also in the mesoscopic ones, which is less realistic. The interface, after the phase separation, in fact has a different shape, with a smooth profile, in mesoscopic variables, connecting the values $\pm m_\beta$, III. As clear from the proofs in section 5, our analysis applies as well to this and to any other choice of the initial state, provided its interface becomes sharp in macroscopic coordinates. The choice 2.2.1 is dictated by the simplicity of its presentation.

Definition 2.2.3. *We say that the surface Σ_τ evolves according to the classical motion by mean curvature with parameter $\theta > 0$ in the time interval $0 \leqslant \tau \leqslant \tau^*$, if Σ_τ, for any such τ, is the connected boundary of a compact region $\Lambda_\tau \subset \mathbb{R}^d$. We also require that there is a C^∞, $d-1$ dimensional, compact manifold S_0 and a $C^\infty([0,\tau^*]\times S_0)$ function $\xi = \xi(\tau,\xi_0)$, with values in $\mathbb{R}^d$, such that Σ_τ is equal to the set $\{\xi = \xi(\tau,\xi_0),\ \xi_0 \in S_0\}$ and*

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \theta\kappa\nu \tag{2.18}$$

where ν is the unit vector normal to Σ_τ at ξ and pointing toward the interior of Σ_τ; κ is $d-1$ times the (signed) mean curvature of Σ_τ at ξ.

If Σ_0 is a sphere of radius R_0, then Σ_τ is the sphere of radius R_τ, with

$$\frac{\mathrm{d}R_\tau}{\mathrm{d}\tau} = -\theta(d-1)\frac{1}{R_\tau}$$

There is a local existence and uniqueness theorem regarding the motion by mean curvature, in the context of definition 2.2.3, see [2] and references therein, which follows from general results on parabolic equations. It is known that in $d > 2$ singularities may develop after a finite time, while in $d = 2$ the only singularity which may arise is the disappearence of a cluster. More recent results describe what happens after the appearence of singularities yielding global existence theorems for the evolution, see [14] and references therein. Our results only cover the classical case.

Theorem 2.2.4. *Let Σ_τ be as in definition 2.2.3, with θ as in (5.2a) and μ^ϵ as in definition 2.2.1. Then there is $\zeta > 0$ and for any $\tau < \tau^*$ and any $n \geqslant 1$ there is c, so that, for all γ small enough,*

$$\mathbb{P}^\gamma_{\mu^\epsilon}\Bigg(\sup_{t\leqslant\lambda^{-2}\tau}\ \sup_{\substack{|r|\leqslant\gamma^{-1}\\ |d(r,\lambda^{-1}\Sigma_{\lambda^2 t})|\geqslant\lambda^{-1+\zeta}}} \big|(\sigma_{\gamma,t})^{(\alpha,\gamma)}(r) \mp m_\beta\big| \leqslant \lambda^{3/2}\Bigg) \geqslant 1 - c\gamma^n \tag{2.19a}$$

where $|d(r,\lambda^{-1}\Sigma_\tau)|$ denotes the distance of r from the surface $\lambda^{-1}\Sigma_\tau$; the $-$ sign in (2.19a) is for r inside $\lambda^{-1}\Sigma_\tau$ and the $+$ sign for r outside.

More expressively, in macroscopic coordinates the set in (2.19*a*) is

$$\left\{ \sup_{\tau' \leqslant \tau} \sup_{\substack{|\xi| \leqslant \gamma^{-1}\lambda \\ |d(\xi, \Sigma_{\tau'})| \geqslant \lambda^\zeta}} \left| \left(\sigma_{\gamma,t'}\right)^{(\alpha,\gamma)}(r) \mp m_\beta \right| \leqslant \lambda^{3/2} \right\} \tag{2.19b}$$

with $t' \equiv \lambda^{-2}\tau'$ and $r \equiv \lambda^{-1}\xi$. Therefore 'inside' and 'outside' Σ_τ we see respectively the phases m_β and $-m_\beta$. The quotation marks recall that the statement does not cover the neighbourhood of Σ_τ of the points which are at distance less than λ^ζ from Σ_τ. As the distance vanishes as γ (and λ) go to 0, theorem 2.2.4 gives a rather complete description of the evolution in the macroscopic representation. The result is however completely unsatisfactory in the mesoscopic description where space is magnified by a factor λ^{-1}. Then the size of the strip around the interface that is not covered by theorem 2.2.4 diverges as $\gamma \to 0$ and since this is where the interface has developed, our result misses entirely the structure of the interface, which, in macroscopic coordinates, was simply the sharp discontinuity that separates the two phases.

The proof of theorem 2.2.4 is given in section 5, it does not use the results stated in proposition 4.8 and in the appendix, so that it avoids the more refined and certainly more complex arguments involved in the proof of the statements in section 2.3.

As already mentioned, theorem 2.2.4 describes the evolution of the system right after the phases separate, (under the assumption that the different interfaces move independently). The structure of the system much after phase separation should look essentially similar, if distances are measured in units of $t^{1/2}$, which is thus the typical size of the clusters of the two phases. These later stages may be studied, to a first approximation, in the same context of definition 2.1.1, but with a different choice of λ. The regime $\lambda = \gamma^\zeta$, with $\zeta > 0$ small, can be treated, we believe, with techniques similar to those employed here. When the condition that ζ is small is relaxed, the analysis becomes more and more complex. As discussed in [IV], the fluctuations of the interface in a time t and in the simpler case d=2 are of the order of $\gamma t^{1/4}$. They should therefore produce finite displacements of the surface (in mesoscopic units!) on the time scale $t \approx \gamma^{-4}$. The space correlations of the displacements have order $t^{1/2} = \gamma^{-2}$, which thus produce local changes of the curvature of the order of γ^4. By this argument, at $\zeta = 4$ there is a competition between fluctuations and the motion by curvature. When $\zeta > 4$ the leading contribution on a first time regime is purely due to stochastic effects, only later the effect of the initial curvature will influence the evolution, but the way this happens is still not clear to us. The real challenge is when λ is independent of γ and we take first $\lambda \to 0$ and then $\gamma \to 0$: this is the 'true hydrodynamic limit'. The interface is still expected to move by mean curvature and, we believe, with the same parameter. But the mechanism of convergence will be drastically different, because the stochastic effects to take into account will include events that in our analysis here are large deviations and have negligibly small probability.

We conclude this subsection with a few bibliographical remarks. As clear from its proof, given in section 5, theorem 2.2.4 is essentially a corollary of theorem 2.1.6 and of a result in [9], where the motion by mean curvature is derived by scaling (1.1) according to (2.17). Results on convergence to the motion by mean curvature have been obtained for other models: for the Glauber dynamics in Ising spin systems in $d = 2$, with nearest neighbour, ferromagnetic, interactions at 0 temperature, [23]; for the Glauber+Kawasaki dynamics in [1] and [18]. In particular the analysis in [1] allows to characterize the magnetization pattern also at the interface. In [18], the convergence is that of theorem 2.2.4 (i.e. not as sharp as required for determining the interface), but the result is proven even past the appearence of singularities. The limiting evolution, in this case, is the generalized motion by mean curvature.

The value of the coefficient θ in the motion by mean curvature should be related, according to [23], to the mobility of the interface and to the surface tension by an Einstein relation. The validity of such a relation in the present context has been proven in [3].

2.3. Propagation of chaos and bounds on the v-functions

In this subsection we state theorems on the factorization properties of the spin distribution at any given time t showing that the distribution is close (and converges as $\gamma \to 0$) to a product measure with means given by the solution of the mesoscopic equation (1.1). We consider the process starting from an arbitrary product measure, in particular from single configurations. We remain in the lattice, without going to the mesoscopic representation, and introduce the lattice analogue of (1.1) as follows.

Definition 2.3.1. *For any $\gamma > 0$ the discretized evolution equation is*

$$\frac{\mathrm{d}m_\gamma(x,t)}{\mathrm{d}t} = -m_\gamma(x,t) + \tanh\{\beta[(J_\gamma \circ m_\gamma)(x,t) + h]\} \qquad (2.20a)$$

where $(f \circ g)$ denotes the discrete convolution of f and g, as in (2.6b). We then call $m_\gamma(x,t|\sigma)$ the solution of (2.20a) with initial condition

$$m_\gamma(x,0|\sigma) = \sigma(x) \qquad \textit{for all } x \in \mathbb{Z}^d \textit{ and } \sigma \in \{-1,1\}^{\mathbb{Z}^d} \qquad (2.20b)$$

If μ is a measure, $m_\gamma(x,t|\mu)$ denotes the solution of (2.20a) with initial condition $m_\gamma(x,0|\mu) = \mathbb{E}_\mu(\sigma(x))$, for all $x \in \mathbb{Z}^d$.

In section 4, see proposition 4.7.2, we discuss the relation between (2.20a) and (1.1). Equation (2.20a) and the actual spin flip dynamics are related because there is c such that, for all x,

$$\left|\frac{\mathrm{d}m_\gamma(x,t)}{\mathrm{d}t} - \mathbb{E}^\gamma_{\nu_t}\big(L_\gamma\sigma(x)\big)\right| \leqslant c\gamma^d \qquad (2.21a)$$

if ν_t is the product measure on $\{-1,1\}^{\mathbb{Z}^d}$ with means

$$\mathbb{E}^\gamma_{\nu_t}\big(\sigma(x)\big) = m_\gamma(x,t) \qquad \text{for all } x \text{ in } \mathbb{Z}^d \qquad (2.21b)$$

In an ideal case where at all times the measure is a product measure, (2.21a) allows to compute the evolution of the expectations of the spins, hence to determine completely the distribution of the process at any single time. This property is called propagation of chaos, because in product measures there are no correlations between the spins, hence no 'order' is present. We say 'propagation' because factorization is supposed to hold initially. One can easily check, though, that in our case, and in general, propagation of chaos does not hold (the condition is verified for the independent particles, see for instance chapter II of [13], but this is 'the typical exception').

If some weak form of propagation of chaos holds, with the measure at any time t suitably close to a product measure, then, conceivably, the average values of the spins are also close to the solution of (2.20). We thus introduce 'a distance' between the actual measure at time t and the product measure ν^γ_t, defined so that its averages are $m_\gamma(x,t|\sigma)$, and we prove that this distance vanishes when $\gamma \to 0$. The distance is defined in terms of the v-functions, i.e. special linear combinations of the correlation functions and the function m_γ defined in 2.3.1. We will see that the v-functions satisfy an integral equation, whose solution can be characterized quite explicitly yielding the desired results. A similar strategy has already been used in several other models, see [13] for a survey on the method. A refinement of the v-functions, which leads to the introduction of the ω functions, is developed in [II], see [IV], to characterize the behaviour of the fluctuation fields and to evidentiate the effects of the random forces which, though infinitesimal as $\gamma \to 0$, are nonetheless present. When there are instabilities, their effect is amplified and becomes macroscopic, as we shall see in [III] in the case of the spinodal decomposition after quenching from a high temperature pure phase.

Behind the definition of the v and of the ω-functions there are algebraic considerations classical in statistical mechanics, that will be made explicit in [II]. Before the definition of the v-functions we introduce some notation:

Notation 2.3.2. *For any positive integer n we denote by* $\mathbb{Z}^{dn}_{\neq}$ *the collection of all the sets* $\underline{x} = (x_1, \ldots, x_n)$ *in* $\mathbb{Z}^d$ *with n distinct elements and write* $|\underline{x}| = n$. *S is the union of* $\mathbb{Z}^{dn}_{\neq}$ *over all* $n \geqslant 0$, *i.e. it is the collection of all the finite subsets of* $\mathbb{Z}^d$, *while* S_∞ *includes also the subsets of* $\mathbb{Z}^d$ *with infinite cardinality.*

Definition 2.3.3. *When the Glauber dynamics starts from a single configuration* σ, *the v functions are*

$$v^\gamma(\underline{x}, t|\sigma) = \mathbb{E}^\gamma_\sigma\Big(\prod_{x\in\underline{x}}[\sigma(x,t) - m_\gamma(x,t|\sigma)]\Big) \qquad \underline{x} \in S, \geqslant 0 \tag{2.22}$$

Analogous expression defines $v^\gamma(\cdot, t|\mu^\gamma)$, *when the process starts from a general product measure* μ^γ.

Theorem 2.3.4.
(i) There are $a > 0$ *and* C' *and for any n c so that, for all* $t \leqslant a \log \gamma^{-1}$,

$$\sup_\sigma \sup_{|\underline{x}|=n} |v^\gamma_n(\underline{x}, t|\sigma)| \leqslant c e^{C'nt} \gamma^{dn/2} \tag{2.23}$$

(ii) The same bound holds when the process starts from a product measure μ^γ *and with* $m_\gamma(x, t|\sigma)$ *(in the espression defining* v^γ*) replaced either by* $m_\gamma(x, t|\mu^\gamma)$ *or by* $m_\gamma(r, t)$, $r = \gamma x$, *with* m_γ *solution of (1.1) with initial datum* $m_{\gamma,0}$, *as defined in (2.12a).*
(iii) If (m_0, μ^γ) *is a 'standard initial state' (see the remarks after definition 2.1.7) and* $m_0 \in C^1(\mathbb{R}^d)$ *with bounded derivative, then*

$$\lim_{\gamma\to 0} \sup_{0\leqslant t\leqslant a\log\gamma^{-1}} \sup_{|\underline{x}|=n} \left| \mathbb{E}^\gamma_{\mu^\gamma}\Big(\prod_{i=1}^n \sigma(x_i, t)\Big) - \prod_{i=1}^n m(\gamma x_i, t)\right| = 0 \tag{2.24}$$

where $m(r, t)$ *solves (1.1) with initial datum* $m_0(r)$.

Observe that from (2.23) it follows that given any $\zeta > 0$ there is $a > 0$ and, for any n, c so that

$$\sup_\sigma \sup_{|\underline{x}|=n} \sup_{t\leqslant a\log\gamma^{-1}} |v^\gamma_n(\underline{x}, t|\sigma)| \leqslant c\gamma^{n(d/2-\zeta)} \tag{2.25}$$

More precise estimates when $t = \tau \log \gamma^{-1}$, $0 < \tau \leqslant a$ are reported in theorem 4.9.

The mesoscopic limit is defined by letting $\gamma \to 0$ with the mesoscopic position r and the time t kept fixed: by (2.24) it then follows that, for the standard initial state and for any distinct $r_1, .., r_n$ and for any t

$$\lim_{\gamma \to 0} \mathbb{E}^{\gamma}_{\mu^{\gamma}}\Big(\prod_{i=1}^{n} \sigma(\gamma^{-1}[r_i]_{\gamma}, t)\Big) = \prod_{i=1}^{n} m(r_i, t) \tag{2.26}$$

We have thus derived the mesoscopic equation (1.1) also in the sense of (2.26), showing that in the limit the spin distribution factorizes.

A final remark about (2.23) regarding the fact that the bound improves with n, as $\gamma \to 0$. This is to some extent surprising, as one would not expect that the average of a product of n spins converges to the product of the averages faster than its rate when $n = 1$. This is indeed so, the whole point here is that we are not taking the average of the product of n spins, but rather a special combination of averages involving all the products of $k \leqslant n$ spins. It is just this special combination which makes the convergence faster; notice however that if we keep γ fixed and let $n \to \infty$, then the dependence of the coefficient c on n will spoil (at least in our estimates), the decay rate $\gamma^{dn/2}$.

In [II] we will see that there are other combinations of correlation functions which describe in a more accurate way the factorization properties of the correlation functions.

3. Short time estimates

In this section we prove theorem 3.7, a weaker version of (2.23), where we impose the restriction $t \leqslant \gamma^{\delta}$ with δ any positive number independent of γ.

We use the following notation: for $\underline{x} \in S$ (the set of all the finite subsets of $\mathbb{Z}^d$) we shorthand

$$\tilde{\sigma}(\underline{x}, t) = \prod_{x \in \underline{x}} [\sigma(x, t) - m_{\gamma}(x, t|\sigma)]. \tag{3.1}$$

Recalling that $\mathbb{E}^{\gamma}_{\sigma}$ is the expectation of the process which starts at time 0 from the configuration σ, we have

$$\mathbb{E}^{\gamma}_{\sigma}\Big(\tilde{\sigma}(\underline{x}, t)\Big) = \mathbb{E}^{\gamma}_{\sigma}\Big(\tilde{\sigma}(\underline{x}, 0)\Big) + \int_0^t \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} \mathbb{E}_{\sigma}\Big(\tilde{\sigma}(\underline{x}, s)\Big). \tag{3.2}$$

Since

$$\tilde{\sigma}(\underline{x}, 0) = 0 \text{because, forall} x, m_{\gamma}(x, 0|\sigma) = \sigma(x, 0) \tag{3.3}$$

(3.2) becomes

$$\mathbb{E}^{\gamma}_{\sigma}\Big(\tilde{\sigma}(\underline{x}, t)\Big) = \int_0^t \mathrm{d}s \mathbb{E}^{\gamma}_{\sigma}\Big(\sum_{x} D^{(x)} \tilde{\sigma}(\underline{x}, s)\Big) \tag{3.4a}$$

where, for any function $g(\sigma, m)$, $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$, $m \in [-1, 1]^{\mathbb{Z}^d}$,

$$D^{(x)} g(\sigma, m) = \Big\{L^{(x)}_{\gamma} + \frac{\mathrm{d}m_{\gamma}(x, s|\sigma)}{\mathrm{d}s} \frac{\partial}{\partial m_{\gamma}(x, s|\sigma)}\Big\} g(\sigma, m). \tag{3.4b}$$

$L_\gamma^{(x)}$ is the generator of the process where only the spin at x flips, namely:

$$L_\gamma^{(x)} f(\sigma) = c_\gamma(x,\sigma)[f(\sigma^x) - f(\sigma)] \tag{3.5}$$

We now restrict ourselves to $t \leqslant \gamma^\delta$. Since the integrand in (3.4a) is bounded uniformly in γ, we see right away that $|\mathbb{E}_\sigma^\gamma(\tilde\sigma(\underline{x},t))| \leqslant c\gamma^\delta$, for a suitable constant c. The bound that we want to prove, however, is much better than that and its proof requires more work. Our strategy is based on an iterative analysis of (3.4). Since each integral which appears in the iteration is necessarily extended to an interval smaller than γ^δ, the terms with N integrals are then bounded proportionally to $\gamma^{N\delta}$ and, if $\delta N > nd/2$, n the cardinality of $\underline{x}$, we then obtain the desired bound. Unfortunately the argument is not as simple, the trouble is that the expression on the right hand side of (3.4a) is neither a $\tilde\sigma(\underline{y})$, nor a linear combination of them. Therefore when we write the integral equation for such a term, we obtain a more complex expression with a non zero contribution, in general coming from its value computed at time 0, unlike in (3.3)–(3.4a). We thus need a good characterization of the class of functions obtained after the action of products of the operators $D^{(x_i)}$ on $\tilde\sigma(\underline{x})$. This is accomplished in proposition 3.6 below, but we preliminarily need extra notation and definitions. They are slightly more complex than what really needed in this section, but then we have the right setup also for the more delicate case of section 4, where we will extend the proof to finite and longer times.

As we often switch from σ to m, it is convenient to think of them both in the same space $[-1,1]^{\mathbb{Z}^d}$, even though σ belongs to the restricted ensemble $\{-1,1\}^{\mathbb{Z}^d}$. The generic element of $[-1,1]^{\mathbb{Z}^d}$ is denoted by λ, not to be confused with the λ of (2.15), which will appear again only in section 5.

Definition 3.1. *We first define the operators $\hat a_y^\pm$, $y \in \mathbb{Z}^d$, which map $[-1,1]^{\mathbb{Z}^d}$ into itself as*

$$\hat a_y^\pm \lambda(x) = \begin{cases} \lambda(x) & \text{if } x \neq y \\ \pm 1 & \text{if } x = y \end{cases} \tag{3.6}$$

and then the operators $a_y^\pm$ mapping $\mathcal{M}([-1,1]^{\mathbb{Z}^d})$ into itself as

$$a_y^\pm f(\lambda) = f(\hat a_y^\pm \lambda) \tag{3.7a}$$

We also define, for $y \in \mathbb{Z}^d$ and $\underline{y} \in S$,

$$\partial_y = \gamma^{-d}\frac{a_y^+ - a_y^-}{2} \qquad a_{\underline{y}}^\pm = \prod_{y\in\underline{y}} a_y^\pm \qquad \partial_{\underline{y}} = \prod_{y\in\underline{y}} \partial_y. \tag{3.7b}$$

Moreover, recalling the notation (2.9d) and setting $y^\star = (y, y') \in \mathbb{Z}^d \times \mathbb{Z}^d$, $y' \in B_{\gamma^{-\alpha},y} - y$, $\alpha \in (0,1)$, we define

$$\delta_{y^\star} = \frac{1}{2\gamma^{d+1-\alpha}}[a_y^+ a_{y'}^- - a_{y'}^+ a_y^-] \qquad \delta_{\underline{y}^\star} = \prod_{y^\star\in\underline{y}^\star} \delta_{y^\star} \tag{3.7c}$$

where $\underline{y}^\star = (y_1^\star, .., y_\ell^\star)$ is a finite collection of elements $y_i^\star$, such that the entries y_i, y_i', of $y_i^\star$, $i = 1, .., \ell$, are all distinct. Their union is denoted by $\underline{\underline{y}}^\star$. We call S^2 the set of all $\underline{y}^\star$.

We will also use the convention that when a set is empty the corresponding operator is the identity.

The operators ∂_y and δ_{y^*} play the role of derivatives and indeed they are essentially derivatives when acting on functions which depend 'weakly' on $\lambda(y)$, as shown in lemma 3.3 below.

Definition 3.2. *We denote by Ξ the set of all $\xi = (x, \underline{y}_1, \underline{y}_2, \underline{y}_3, \underline{y}^\star)$, $x \in \mathbb{Z}^d$, $\underline{y}_i \in S$, $i = 1, 2, 3$, $\underline{y}^\star \in S^2$, with the condition that all the sites in $\underline{y}_1$, $\underline{y}_2$, $\underline{y}_3$ and $\underline{\underline{y}}^\star$ are different from each other. Their union is denoted by V_ξ.*

Ξ^0 is the subset of Ξ with $\underline{y}^\star = \emptyset$, in that case we simply write $\xi = (x, \underline{y}_1, \underline{y}_2, \underline{y}_3)$. We sometimes use the convention of dropping the entries of ξ which are emptysets, thus writing x for $\xi = (x, \emptyset, \emptyset, \emptyset, \emptyset)$ as in the definition:

$$f_x(\lambda) = \tanh\{\beta[\sum_{y \neq x} J_\gamma(x, y)\lambda(y) + h]\}. \tag{3.8}$$

We also define for $\xi = (x, \underline{y}_1, \underline{y}_2, \underline{y}_3, \underline{y}^\star)$,

$$f_\xi(\lambda) = a^+_{\underline{y}_1} a^-_{\underline{y}_2} \partial_{\underline{y}_3} \delta_{\underline{y}^\star} f_x(\lambda). \tag{3.9a}$$

In particular if $\xi \in \Xi^0$, (3.9a) becomes

$$f_\xi(\lambda) = a^+_{\underline{y}_1} a^-_{\underline{y}_2} \partial_{\underline{y}_3} f_x(\lambda). \tag{3.9b}$$

Finally we call

$$d(\xi) = |\underline{y}_3| + |\underline{y}^\star| \tag{3.10a}$$

the 'number of derivatives in ξ'.

The last notation is justified by the following lemma:

Lemma 3.3. *Let $\xi \in \Xi$, then*

$$f_\xi(\lambda) = \int \mathrm{d}\epsilon\, \lambda_{(\gamma,\xi)}(\epsilon) \tanh^{(d(\xi))}\big(\psi_\xi(\lambda) + \epsilon\big) \tag{3.10b}$$

where $\tanh^{(k)}(\cdot)$ is the kth derivative of $\tanh(\cdot)$;

$$\psi_\xi(\lambda) = \beta \sum_{z \notin V_\xi} J_\gamma(x, z)\lambda(z). \tag{3.10c}$$

$\lambda_{(\gamma,\xi)}(\epsilon)$ is a function supported in an interval $I_{(\gamma,\xi)}$ of $\mathbb{R}$ containing the point

$$\beta\{[\sum_{z \in \underline{y}_1} - \sum_{z \in \underline{y}_2}] J_\gamma(x, z) + h]\} \tag{3.10d}$$

and furthermore there is c so that if $\underline{y}_3 \neq \emptyset$

$$\|\lambda_{(\gamma,\xi)}\|_\infty \leqslant c\gamma^{-d} \qquad |I_{(\gamma,\xi)}| \leqslant c\gamma^d \tag{3.11a}$$

while if $\underline{y}_3 = \emptyset$ and $\underline{y}^\star \neq \emptyset$

$$\|\lambda_{(\gamma,\xi)}\|_\infty \leqslant c\gamma^{-d-1+\alpha} \qquad |I_{(\gamma,\xi)}| \leqslant c\gamma^{d+1-\alpha}. \tag{3.11b}$$

If both $\underline{y}_3 = \underline{y}^\star = \emptyset$ then the density in (3.10b) is a delta function at $\epsilon = 0$.

Proof. As all the sites in $\underline{y}_1, \ldots, \underline{y}^\star$ are distinct, by definition, the operators that define f_ξ in (3.9) commute. We thus start by computing $\partial_{\underline{y}_3} f_x(\lambda)$. We write $\underline{y}_3 = (y_{3,1}, .., y_{3,\ell})$ and

$$\partial_{\underline{y}_3} f_x(\lambda) = \partial_{y_{3,\ell}} \ldots \partial_{y_{3,1}} f_x(\lambda).$$

We set, for $j = 1, .., \ell$,

$$F_j(\lambda) = \partial_{y_{3,j}} \ldots \partial_{y_{3,1}} f_x(\lambda). \tag{3.12}$$

We claim that

$$F_j(\lambda) = \int \mathrm{d}\epsilon \lambda_{\gamma,j}(\epsilon) \tanh^{(j)} \left\{ \beta \sum_{y \in Y^{(j)}} J_\gamma(x, y)\lambda(y) + \beta h + \epsilon \right\} \tag{3.13a}$$

where

$$Y^{(j)} = \text{ the complement of } \{y_1, \ldots, y_j\} \text{in} \mathbb{Z}^d.$$

$\lambda_{\gamma,j}(\epsilon)$ is supported by an interval $I_{\gamma,j}$ and there is c_j so that

$$\|\lambda_{\gamma,j}\|_\infty \leqslant c_j \gamma^{-d} \qquad |I_{\gamma,j}| \leqslant c_j \gamma^d. \tag{3.13b}$$

We will prove the claim (3.13) by induction on j. We thus assume that (3.13) holds for $j < \ell$ and, writing

$$\psi_n := \beta\{ \sum_{y \in Y^{(n)}} J_\gamma(x, y)\lambda(y) + h\} \qquad z = y_{3,j+1}$$

we have

$$\begin{aligned} \partial_z \tanh^{(j)}(\psi_j + \epsilon) &= \frac{1}{2\gamma^d} \Big\{ \tanh^{(j)}\big(\psi_{j+1} + \epsilon + \beta J_\gamma(x, z)\big) \\ &\quad - \tanh^{(j)}\big(\psi_{j+1} + \epsilon - \beta J_\gamma(x, z)\big) \Big\} \\ &= \int_{-\beta J_\gamma(x,z)}^{\beta J_\gamma(x,z)} \frac{\mathrm{d}\epsilon'}{2\gamma^d} \tanh^{(j+1)}(\psi_{j+1} + \epsilon + \epsilon'). \end{aligned} \tag{3.14}$$

Hence

$$\lambda_{\gamma,j+1}(\epsilon) = \int_{-\beta J_\gamma(x,z)}^{\beta J_\gamma(x,z)} \frac{\mathrm{d}\epsilon'}{2\gamma^d} \lambda_{\gamma,j}(\epsilon - \epsilon') \tag{3.15}$$

so that

$$\|\lambda_{\gamma,j+1}\|_\infty \leqslant 2\beta \frac{|J_\gamma(x, z)|}{2\gamma^d} \|\lambda_{\gamma,j}\|_\infty \qquad |I_{\gamma,j+1}| \leqslant |I_{\gamma,j}| + 2\beta |J_\gamma(x, z)| \tag{3.16}$$

hence $c_{j+1} = (1 + 2\beta \|J\|_\infty) c_j$.

The same argument shows that

$$\lambda_{\gamma,1}(\epsilon) = \frac{1}{2\gamma^d} \mathbf{1}\big(|\epsilon| \leqslant \beta |J_\gamma(x, y_{3,1})|\big) \quad I_{\gamma,1} = [-\beta J_\gamma(x, y_{3,1}), \beta J_\gamma(x, y_{3,1})] \tag{3.17}$$

so that the claim (3.13) is proven.

The same argument applies to $\delta_{\underline{y}^\star}$. The action of $\delta_{y^\star}$, $y^\star = (y, y')$, involves a change in the argument of the function $\tanh^{(k)}$ from $-[J_\gamma(x, y) - J_\gamma(x, y')]$ to its opposite. The difference is bounded proportionally to $\gamma^{d+1-\alpha}$, recalling that, by definition $|y - y'| \leqslant \gamma^{-\alpha}$. Since $a^+_{\underline{y}_1}$ simply amounts to setting $\lambda(y) = 1$ for all $y \in \underline{y}_1$, and analogously for $a^-_{\underline{y}_2}$, the lemma follows, we omit the details. □

Remark. *As is clear from its proof, lemma 3.3 extends to the case when f_x is replaced by any function $g \in C^\infty$ of the argument $\sum_y J_\gamma(x,y)\lambda(y)+h$.*

Definition 3.4. *We call H the collection of all $\eta = (\underline{x}, \underline{y}, \xi_1, .., \xi_p)$, with $\underline{x}$ and $\underline{y}$ in S, $\xi_1,..,\xi_p$ in Ξ and H^0 the subset of all $\eta = (\underline{x}, \emptyset, \xi_1, .., \xi_p)$, with $\xi_i \in \Xi^0$. In this case we simply write $\eta = (\underline{x}, \xi_1, .., \xi_p)$.*

We define 'the order of η' as

$$|\eta| = |\underline{x}| + |\underline{y}| + p. \tag{3.18}$$

We also set

$$g(\eta) = |\underline{y}| + p \tag{3.19}$$

and call it the g-order of η and, finally, recalling (3.10a),

$$d(\eta) = \sum_{i=1}^{p} d(\xi_i) \tag{3.20}$$

We need a final definition:

Definition 3.5. *Given $\eta \in H$, we define the function $\rho_\eta \in \mathcal{M}(\{-1,1\}^{\mathbb{Z}^d} \times [-1,1]^{\mathbb{Z}^d})$ as*

$$\rho_\eta(\sigma, m) = \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) \prod_{i=1}^{p} [f_{\xi_i}(\sigma) - f_{\xi_i}(m)] \tag{3.21}$$

where, if $\ell := |\underline{y}|$, $\underline{z} \in \mathbb{Z}^{d\ell}_{\neq}$,

$$\chi_{\underline{x},\underline{y}}(\underline{z}) = \prod_{i=1}^{\ell} \left\{ \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{z_i} \mathbf{1}(z_i \in B_{\gamma^{-\alpha}, y_i} - \underline{x}) \right\} \tag{3.22}$$

In particular, if $\eta \in H^0$, we restrict to this case in the sequel of this section,

$$\rho_\eta(\sigma, m) = \tilde{\sigma}(\underline{x}) \prod_{i=1}^{p} [f_{\xi_i}(\sigma) - f_{\xi_i}(m)] \tag{3.23}$$

Finally if all the entries of η are the emptyset, we write $\eta = \emptyset$ and define $\rho_\emptyset = 1$.

Proposition 3.6. *There is $M_\gamma(\eta, \eta', m)$, η, η' in H^0, $m \in [-1,1]^{\mathbb{Z}^d}$, so that, for all γ, η, m and $\sigma \in \{-1,1\}^{\mathbb{Z}^d}$,*

$$\sum_{x \in \mathbb{Z}^d} D^{(x)} \rho_\eta(\sigma, m) = \sum_{\eta' \in H^0} M_\gamma(\eta, \eta', m) \rho_{\eta'}(\sigma, m) \tag{3.24}$$

with $M_\gamma(\eta, \eta', m) = 0$ if $|\eta'| > |\eta| + 1$ and if $d(\eta') > d(\eta) + |\eta|$. Furthermore there are coefficients $c(n,q)$, $n > 0$, $q \geqslant 0$, (independent of γ) such that for all γ and all $m \in [-1,1]^{\mathbb{Z}^d}$

$$\sup_{\substack{|\eta|=n \\ d(\eta)=q}} \sum_{|\eta'|=p} |M_\gamma(\eta, \eta', m)| \leqslant \begin{cases} c(n,q)\gamma^{d(n-p)/2} & \text{if } p \leqslant n \\ c(n,q) & \text{if } p = n, n+1 \end{cases} \tag{3.25}$$

Proof. We start by computing the action of $D^{(x)}$ on a single spin. By (2.20*a*),

$$D^{(x)}\tilde{\sigma}(x) = L_{\gamma}^{(x)}\sigma(x) - \Big(-m(x) + \tanh\{\beta[J_\gamma \circ m)(x) + h]\}\Big)$$
$$= -\tilde{\sigma}(x) + [f_x(\sigma) - f_x(m)]. \tag{3.26}$$

Notice that if $\underline{x}$ reduces to x with all $\xi_i = \emptyset$, then $\rho_\eta = \tilde{\sigma}(x)$, hence (3.24) and (3.25) are verified. To study the general case we first write:

$$\begin{aligned} f_\xi(\sigma) &= \{\frac{1+\sigma(x)}{2}a_x^+ + \frac{1-\sigma(x)}{2}a_x^-\} f_\xi(\sigma) \\ &= \frac{a_x^+ + a_x^-}{2} f_\xi(\sigma) + \sigma(x)\gamma^d \partial_x f_\xi(\sigma) \\ &= \{\frac{a_x^+ + a_x^-}{2}[f_\xi(\sigma) - f_\xi(m)] + \frac{a_x^+ + a_x^-}{2} f_\xi(m)\} \\ &\quad + \{\tilde{\sigma}(x)\gamma^d \partial_x [f_\xi(\sigma) - f_\xi(m)] + \tilde{\sigma}(x)\gamma^d \partial_x f_\xi(m) \\ &\quad + m(x)\gamma^d \partial_x f_\xi(m) + m(x)\gamma^d \partial_x [f_\xi(\sigma) - f_\xi(m)]\}. \end{aligned} \tag{3.27}$$

Therefore

$$f_\xi(\sigma) - f_\xi(m) = \Gamma_{1,x} + m(x)\gamma^d \Gamma_{2,x} + \tilde{\sigma}(x)\gamma^d[\Gamma_{3,x} + \Gamma_{4,x}] + \gamma^{2d}\Gamma_{5,x} \tag{3.28a}$$

$$\begin{aligned} &\Gamma_{1,x} = \frac{a_x^+ + a_x^-}{2}[f_\xi(\sigma) - f_\xi(m)] \qquad \Gamma_{2,x} = \partial_x[f_\xi(\sigma) - f_\xi(m)] \\ &\Gamma_{3,x} = \partial_x[f_\xi(\sigma) - f_\xi(m)] \qquad \Gamma_{4,x} = \partial_x f_\xi(m) \\ &\Gamma_{5,x} = \gamma^{-2d}\{\frac{a_x^+ + a_x^-}{2} f_\xi(m) + m(x)\gamma^d \partial_x f_\xi(m) - f_\xi(m)\}. \end{aligned} \tag{3.28b}$$

The equations (3.27), (3.28) hold for any function $f(\cdot)$, however if $f = f_\xi$ there is a constant c so that

$$\left|\frac{\partial \Gamma_{5,x}}{\partial m(x)}\right| \leqslant c \qquad |\Gamma_{5,x}| \leqslant c \qquad \text{uniformly in } \gamma, m \text{ and } x \tag{3.28c}$$

as we are going to prove.

First observe that if $x \in V_\xi$ then $\Gamma_{5,x} = 0$. We then suppose that $x \notin V_\xi$, then, by lemma 3.3 there is an interval $I_{(\gamma,\xi)}$ and a function $\lambda_{(\gamma,\xi)}$ supported on $I_{(\gamma,\xi)}$ and such that

$$\rho_\xi(m) = \int_{I_{(\gamma,\xi)}} \mathrm{d}\epsilon \lambda_{(\gamma,\xi)}(\epsilon) \tanh^{(d(\xi))}\big(\psi + \epsilon + J_\gamma(\hat{x}, x)m(x)\big)$$

where

$$\xi = (\hat{x}, \underline{\hat{y}}_1, \underline{\hat{y}}_2, \underline{\hat{y}}_3) \qquad \psi = \beta \sum_{y \notin V_\xi} J_\gamma(\hat{x}, y)m(y) + \beta h$$

and, for a suitable constant c

$$\|\lambda_{(\gamma,\xi)}\|_\infty \leqslant c\gamma^{-d} \qquad |I_{(\gamma,\xi)}| \leqslant c\gamma^d.$$

We then have:

$$\Gamma_{5,x} = \int_{I_{(\gamma,\xi)}} \mathrm{d}\epsilon \lambda_{(\gamma,\xi)}(\epsilon)\gamma^{-2d}\Phi(\epsilon) \tag{3.29a}$$

$$\begin{aligned}\Phi(\epsilon) = {} & \frac{1}{2}\tanh^{(d(\xi))}\big(\psi + \epsilon + J_\gamma(\hat{x}, x)\big) + \frac{1}{2}\tanh^{(d(\xi))}\big(\psi + \epsilon - J_\gamma(\hat{x}, x)\big) \\ & + m(x)\left\{\frac{1}{2}\tanh^{(d(\xi))}\big(\psi + \epsilon + J_\gamma(\hat{x}, x)\big) - \frac{1}{2}\tanh^{(d(\xi))}\big(\psi + \epsilon - J_\gamma(\hat{x}, x)\big)\right\} \\ & - \tanh^{(d(\xi))}\big(\psi + \epsilon + J_\gamma(\hat{x}, x)m(x)\big).\end{aligned} \tag{3.29b}$$

By expanding Φ in powers to second order in the small parameter $J_\gamma(\hat{x}, x)$, we readily see that (3.28*c*) holds.

Notation and remarks. *We denote by* $\Gamma^j_{i,x}$, $i = 1, .., 5$ *and* $1 \leqslant j \leqslant p$, *the term* $\Gamma_{i,x}$ *in (3.28b) when* $\xi = \xi_j$.

We use the notation $A + B$ *to denote the union of the sets A and B and* $A - B$ *for the set theoretical difference of A and B.*

$\Gamma^j_{i,x}$, $i = 1, .., 5$ *does not depend on* $\sigma(x)$; $\Gamma^j_{i,x}$, $i \leqslant 4$, *does not depend on* $m(x)$, *and the bounds (3.28c) hold for* $\Gamma^j_{5,x}$.

We hereafter set $\eta = (\underline{x}, \xi_1, .., \xi_p)$, $\xi_i = (\hat{x}_i, \underline{\hat{y}}_1, \underline{\hat{y}}_2, \underline{\hat{y}}_3)$.

We write

$$\begin{aligned}\sum_{x\in\mathbb{Z}} D^{(x)}\rho_\eta = {} & \sum_{x\in\underline{x}} \tilde{\sigma}(\underline{x} - x)\Big[\prod_{j=1}^{p} \Gamma^j_{1,x}\Big]D^{(x)}\tilde{\sigma}(x) \\ & + \gamma^d \sum_{x\notin\underline{x}} \tilde{\sigma}(\underline{x}) \sum_{i=1}^{p}\Big[\prod_{j\neq i} \Gamma^j_{1,x}\Big](\Gamma^i_{3,x} + \Gamma^i_{4,x})D^{(x)}\tilde{\sigma}(x) + S_1 + S_2\end{aligned} \tag{3.30a}$$

where

$$S_1 = \sum_{x\in\underline{x}} \tilde{\sigma}(\underline{x} - x) \sum_{\substack{\underline{i}_1,...,\underline{i}_5 \\ |\underline{i}_1|<p}} \Delta_{\underline{i}_1,...,\underline{i}_5,\underline{x},x} \tag{3.30b}$$

$$S_2 = \sum_{x\notin\underline{x}} \tilde{\sigma}(\underline{x}) \sum_{\substack{\underline{i}_1,...,\underline{i}_5 \\ |\underline{i}_1|<p-1}} \Delta_{\underline{i}_1,...,\underline{i}_5,\underline{x},x} \tag{3.30c}$$

the sum is over all the partitions of $1, .., p$ into five atoms, $\underline{i}_1,.., \underline{i}_5$; $|\underline{i}_j|$ denotes the number of elements in $\underline{i}_j$

$$\Delta_{\underline{i}_1,...,\underline{i}_5,\underline{x},x} = \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}\Gamma^{\underline{i}_1}_{1,x}\cdots\Gamma^{\underline{i}_4}_{4,x}D^{(x)}\{\Gamma^{\underline{i}_5}_{5,x}m(x)^{|\underline{i}_2|}\tilde{\sigma}(x)^k\} \tag{3.30d}$$

where

$$\Gamma^{\underline{i}_j}_{j,x} = \prod_{i\in\underline{i}_j} \Gamma^i_{j,x}$$

and

$$k = k(x, \underline{x}, \underline{i}_3 + \underline{i}_4) = \begin{cases} 1 + |\underline{i}_3| + |\underline{i}_4| & \text{if } x \in \underline{x} \\ |\underline{i}_3| + |\underline{i}_4| & \text{if } x \notin \underline{x} \end{cases}. \tag{3.30e}$$

By using (3.26), the first term on the right-hand side of (3.30a) can be written as

$$\sum_{\eta'} M_{0,1}(\eta, \eta', m)\rho_{\eta'} \tag{3.31a}$$

where the sum is restricted to $|\eta'| = |\eta|$ and

$$\sum_{\eta'} |M_{0,1}(\eta, \eta', m)| \leqslant 2|\underline{x}| \tag{3.31b}$$

In fact the sum in (3.30a) has $|\underline{x}|$ terms and, using (3.26), each gives rise to two ρ_η functions.

The second term in (3.30a) is also of the form

$$\sum_{\eta'} M_{0,2}(\eta, \eta', m)\rho_{\eta'} \tag{3.31c}$$

with $|\eta'| = |\eta| + 1$ and $|\eta'| = |\eta|$, respectively for $\Gamma^i_{3,x}$ and $\Gamma^i_{4,x}$. We also have, for a suitable constant c,

$$\sum_{\eta'} |M_{0,2}(\eta, \eta', m)| \leqslant cp \tag{3.31d}$$

because $\Gamma^i_{3,x} = \Gamma^i_{4,x} = 0$ unless $|x - \hat{x}_i| \leqslant \gamma^{-1}$: in fact if $|x - \hat{x}_i| > \gamma^{-1}$, then $f_{\xi_i}(\lambda)$ does not depend on $\lambda(x)$ and $\partial_x f_{\xi_i} = 0$. We then have (3.31d) since by lemma 3.3, $\partial_x f_{\xi_i}(m)$ is bounded.

We are going to prove that also S_1 and S_2 can be decomposed as the two terms above with coefficients which satisfy (3.25), and this will prove proposition 3.6.

We start with some easy algebraic computation: let $k \geqslant 1$, then

$$\begin{aligned} \tilde{\sigma}(x)^k &= \{\frac{1+\sigma(x)}{2}[1 - m(x)]^k + \frac{1-\sigma(x)}{2}[-1 - m(x)]^k \\ &= \tilde{\sigma}(x)\left\{\frac{[1 - m(x)]^k}{2} - \frac{[-1 - m(x)]^k}{2}\right\} \\ &\quad + \frac{1+m(x)}{2}[1 - m(x)]^k + \frac{1-m(x)}{2}[-1 - m(x)]^k \\ &= \tilde{\sigma}(x)a(k, m(x)) + b(k, m(x)) \end{aligned} \tag{3.32}$$

with a and b defined by the last equality. Thus $a(1, m(x)) = 1$, $b(1, m(x)) = 0$ and there are constants b_k so that $|a(k, m(x))| \leqslant b_k$ and $|b(k, m(x))| \leqslant b_k$.

Recalling that $k = 1 + |\underline{i}_3| + |\underline{i}_4|$ if $x \in \underline{x}$ and $k = |\underline{i}_3| + |\underline{i}_4|$ if $x \notin \underline{x}$, we have

$$\begin{aligned} \Delta_{\underline{i}_1,\ldots,\underline{i}_5,\underline{x},x} = \sum_{\underline{i}_1,\ldots,\underline{i}_5} &\Big\{m(x)^{|\underline{i}_2|}a(k, m(x))\Gamma^{\underline{i}_5}_{5,x}D^{(x)}\tilde{\sigma}(x) \\ &+ \tilde{\sigma}(x)[m(x)^{|\underline{i}_2|}a(k, m(x))\Gamma^{\underline{i}_5}_{5,x}]' + [m(x)^{|\underline{i}_2|}b\big(k, m(x)\big)\Gamma^{\underline{i}_5}_{5,x}]'\Big\} \\ &\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}\tilde{\sigma}(\underline{x} - x)\Gamma^{\underline{i}_1}_{1,x}\ldots\Gamma^{\underline{i}_4}_{4,x} \end{aligned} \tag{3.33}$$

where given h, h' denotes its derivative with respect to $m(x)$ times $\dot{m}(x)$.

We write

$$S_1 = S_{1,1} + S_{1,2} \tag{3.34}$$

where the decomposition arises from taking the term with and respectively without $\tilde{\sigma}(x)$ on the right-hand side of (3.33). Namely

$$S_{1,1} = \sum_{x\in\underline{x}} \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5\\ |\underline{i}_1|<p}} \left\{ m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x} D^{(x)} \tilde{\sigma}(x) + \tilde{\sigma}(x) [m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}]' \right\}$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \tilde{\sigma}(\underline{x}-x) \Gamma^{\underline{i}_1}_{1,x} \dots \Gamma^{\underline{i}_4}_{4,x}. \tag{3.35a}$$

By (3.26),

$$S_{1,1} = \sum_{\eta'} M_{1,1}(\eta, \eta', m) \rho_{\eta'} \tag{3.35b}$$

with $|\eta'| = |\eta| - (|\underline{i}_4| + |\underline{i}_5|)$, and, by (3.28$c$), for $q \geqslant 0$,

$$\sum_{|\eta'|=|\eta|-q} |M_{1,1}(\eta, \eta', m)| \leqslant c|\underline{x}| \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5\\ |\underline{i}_1|<p}} \mathbf{1}(|\underline{i}_4| + |\underline{i}_5| = q) \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \leqslant c'|\underline{x}|\gamma^{dq/2}\gamma^{d/2} \tag{3.35c}$$

with c and c' suitable constants.

We next consider $S_{1,2}$, which, with the shorthand notation $k = |\underline{i}_3| + |\underline{i}_4| + 1$, is

$$S_{1,2} = \sum_{x\in\underline{x}} \sum_{\underline{i}_1,\dots,\underline{i}_5} [m(x)^{|\underline{i}_2|} b(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}]' \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \tilde{\sigma}(\underline{x}-x) \Gamma^{\underline{i}_1}_{1,x} \dots \Gamma^{\underline{i}_4}_{4,x} \tag{3.36a}$$

which also has the form

$$S_{1,2} = \sum_{\eta'} M_{1,2}(\eta, \eta', m) \rho_{\eta'} \tag{3.36b}$$

with $|\eta'| = |\eta| - (|\underline{i}_4| + |\underline{i}_5| + 1)$. These terms are only present when $k = |\underline{i}_3| + |\underline{i}_4| + 1 \geqslant 2$, because $b(1, m(x)) = 0$. We have, for $q \geqslant 1$,

$$\sum_{|\eta'|=|\eta|-q} |M_{1,2}(\eta, \eta', m)| \leqslant c|\underline{x}| \sum_{\underline{i}_1,\dots,\underline{i}_5} \mathbf{1}(|\underline{i}_3| + |\underline{i}_4| \geqslant 1; |\underline{i}_4| + |\underline{i}_5| + 1 = q)$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \leqslant c'|\underline{x}|\gamma^{d(q-1)/2}\gamma^{d/2} = c'|\underline{x}|\gamma^{dq/2} \tag{3.36c}$$

with c and c' suitable constants.

Splitting S_2 as we did for S_1, we get $S_2 = S_{2,1} + S_{2,2}$. Recalling that in this case $k = |\underline{i}_3| + |\underline{i}_4|$, we have

$$S_{2,1} = \sum_{x\notin\underline{x}} \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5\\ |\underline{i}_1|\leqslant p-2}} \left\{ m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x} D^{(x)} \tilde{\sigma}(x) + \tilde{\sigma}(x) [m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}]' \right\}$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \tilde{\sigma}(\underline{x}-x) \Gamma^{\underline{i}_1}_{1,x} \dots \Gamma^{\underline{i}_4}_{4,x}. \tag{3.37a}$$

The only contribution to (3.37*a*) comes from the terms with $|\underline{i}_3| + |\underline{i}_4| \geqslant 1$ because $a(0, m(x)) = 0$ and $k = |\underline{i}_3| + |\underline{i}_4|$. If $i \in \underline{i}_3$, $|x - \hat{x}_i| \leqslant \gamma^{-1}$, because of the operator ∂_x in $\Gamma_{i,3}$ and $\Gamma_{i,4}$. Therefore the sum is 'only' over $(2\gamma^{-1} + 1)^d$ elements, at most.

$S_{2,1}$ can be written as

$$S_{2,1} = \sum_{\eta'} M_{2,1}(\eta, \eta', m)\rho_{\eta'} \tag{3.37b}$$

with $\eta' = |\eta| - (|\underline{i}_4| + |\underline{i}_5| - 1)$. Recalling that

$$|\underline{i}_2| + |\underline{i}_3| + |\underline{i}_4| + |\underline{i}_5| \geqslant 2$$

because $|\underline{i}_1| \leqslant p - 2$, we have

$$\sum_{|\eta'|=|\eta|+1} |M_{2,1}(\eta, \eta', m)| \leqslant \{(2\gamma^{-1} + 1)^d p\} \sum_{\substack{|\underline{i}_4|+|\underline{i}_5|=0 \\ |\underline{i}_2|+|\underline{i}_3|\geqslant 2}} c\gamma^{d|\underline{i}_2|+|\underline{i}_3|} \leqslant c'\gamma^d. \tag{3.37c}$$

Analogously, for $q \geqslant 0$:

$$\sum_{|\eta'|=|\eta|-q} |M_{2,1}(\eta, \eta', m)| \leqslant \{(2\gamma^{-1} + 1)^d p\} \sum_{\substack{\underline{i}_1 \cdots \underline{i}_5 \\ |\underline{i}_1| \leqslant p-2}} \mathbf{1}(|\underline{i}_4| + |\underline{i}_5| = q + 1)$$
$$\times c\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \leqslant c'\gamma^{d(q+1)/2} \tag{3.37d}$$

with c and c' suitable constants.

The last term $S_{2,2}$ can be written as $S_{2,2} = S_{2,2,1} + S_{2,2,2}$, where

$$S_{2,2,1} = \sum_{x \notin \underline{x}} \tilde{\sigma}(x) \sum_{\substack{\underline{i}_1, \underline{i}_2, \underline{i}_5 \\ |\underline{i}_1| \leqslant p-2}} [m(x)^{|\underline{i}_2|} \Gamma_{5,x}^{\underline{i}_5}]' \gamma^{d(|\underline{i}_2|+2|\underline{i}_5|)} \Gamma_{1,x}^{\underline{i}_1} \Gamma_{2,x}^{\underline{i}_2} \tag{3.38a}$$

takes into account the sum of the terms with $k = |\underline{i}_3| + |\underline{i}_4| = 0$. Since $b(1, m(x)) = 0$ the remaining terms have $k \geqslant 2$, hence

$$S_{2,2,2} = \sum_{x \notin \underline{x}} \tilde{\sigma}(x) \sum_{\substack{\underline{i}_1, \dots, \underline{i}_5 \\ |\underline{i}_3|+|\underline{i}_4|\geqslant 2}} [m(x)^{|\underline{i}_2|} b\big(k, m(x)\big) \Gamma_{5,x}^{\underline{i}_5}]'$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \Gamma_{1,x}^{\underline{i}_1} \dots \Gamma_{4,x}^{\underline{i}_4}. \tag{3.38b}$$

We have

$$S_{2,2,1} = \sum_{\eta'} M_{2,2,1}(\eta, \eta', m)\rho_{\eta'} \tag{3.38c}$$

with $|\eta'| = |\eta| - |\underline{i}_5|$. Thus for $q \geqslant 0$ and suitable constants c and c'

$$\sum_{|\eta'|=|\eta|-q} |M_{2,2,1}(\eta, \eta', m)| \leqslant \{p(2\gamma^{-1} + 1)^d\}$$
$$\times \sum_{\underline{i}_1, \underline{i}_2, \underline{i}_5 |\underline{i}_2|+|\underline{i}_5|\geqslant 2} 1(|\underline{i}_5| = q) c\gamma^{d(|\underline{i}_2|+2|\underline{i}_5|)} \leqslant c'\gamma^d\gamma^{qd}. \tag{3.38d}$$

Finally,

$$S_{2,2,2} = \sum_{\eta'} M_{2,2,2}(\eta, \eta', m)\rho_{\eta'} \tag{3.38e}$$

with $|\eta'| = |\eta| - |\underline{i}_4| + |\underline{i}_5|$ and, for $q \geqslant 0$, with suitable constants c and c'

$$\sum_{|\eta'|=|\eta|-q} |M_{2,2,2}(\eta, \eta', m)| \leqslant \{p(2\gamma^{-1} + 1)^d\} \sum_{\substack{\underline{i}_1, \dots, \underline{i}_5 \\ |\underline{i}_3|+|\underline{i}_4|\geqslant 2}} 1(|\underline{i}_4| + |\underline{i}_5| = q)$$
$$\times c\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \leqslant c'\gamma^{qd/2}. \tag{3.38f}$$

The proposition is thus proven. □

Theorem 3.7. *For any $\delta > 0$ and $n \geqslant 1$ there is c so that for all σ:*

$$\sup_{\sigma} \sup_{|\underline{x}|=n} \sup_{t \leqslant \gamma^{\delta}} |v_n^{\gamma}(\underline{x}, t|\sigma)| \leqslant c\gamma^{dn/2}. \tag{3.39}$$

Proof. By proposition 3.6, for $\eta \neq \emptyset$

$$\mathbb{E}_{\sigma}^{\gamma}\Big(\rho_{\eta}\big(\sigma_t, m_{\gamma}(\cdot, t|\sigma)\big)\Big) = \int_0^t \mathrm{d}s \sum_{\eta'} M_{\gamma}\big(\eta, \eta', m_{\gamma}(\cdot, s|\sigma)\big)\mathbb{E}_{\sigma}^{\gamma}\Big(\rho_{\eta'}\big(\sigma_s, m_{\gamma}(\cdot, s)\big)\Big) \tag{3.40a}$$

because $\rho_{\eta}(\sigma, m_{\gamma}(\cdot, 0|\sigma)) = 0$, since $\sigma = m(\cdot, 0|\sigma)$. We then write

$$M_{\gamma}(\eta, \eta', m) = M'_{\gamma}(\eta, \eta', m) + M''_{\gamma}(\eta, \eta', m) \tag{3.40b}$$

where $M'_{\gamma}(\eta, \eta', m) = 0$ if $\eta' = \emptyset$ and $M''_{\gamma}(\eta, \eta', m)$ if $\eta' \neq \emptyset$. We then define

$$\hat{M}'_{\gamma}(\eta, \eta') = \sup_m |M'_{\gamma}(\eta, \eta', m)| \qquad \hat{M}''_{\gamma}(\eta, \eta') = \sup_m |M''_{\gamma}(\eta, \eta', m)| \tag{3.40c}$$

We call $\eta_0 = (\underline{x}, \emptyset, \emptyset, \emptyset)$, $\underline{x} \in S$, $|\underline{x}| = n$ and set

$$N:\ \delta N > \frac{dn}{2}. \tag{3.41}$$

By iterating (3.40a) N-times we get:

$$\Big|\mathbb{E}_{\sigma}^{\gamma}\Big(\rho_{\eta_0}\big(\sigma_t, m(\cdot, t|\sigma)\big)\Big)\Big| \leqslant \sum_{j=0}^{N-1} \frac{\gamma^{\delta j}}{(j+1)!} \sum_{\eta'} (\hat{M}'_{\gamma})^j(\eta_0, \eta')\hat{M}''_{\gamma}(\eta', \emptyset)$$
$$+ \frac{\gamma^{\delta N}}{N!} \sum_{\eta'} (\hat{M}'_{\gamma})^N(\eta_0, \eta') \sup_{\sigma, m} \big|\rho_{\eta'}(\sigma, m)\big|. \tag{3.42}$$

By (3.25) there is $C(n, q)$, $(n, q) \in \mathbb{Z}_+^2$, such that

$$\gamma^{-|\eta|d/2} \sum_{\eta'} \hat{M}_{\gamma}(\eta, \eta')\gamma^{|\eta'|d/2} \leqslant C\big(|\eta|, d(\eta)\big) \tag{3.43}$$

with $\hat{M}_{\gamma} := \hat{M}'_{\gamma} + \hat{M}''_{\gamma}$. Calling

$$C^* = \max_{\substack{p \leqslant n+N \\ q \leqslant (n+N)N}} C(p, q) \tag{3.44}$$

we get from (3.43), after telescopic cancellations:

$$\gamma^{-|\eta_0|d/2} \sum_{\eta'} \hat{M}_{\gamma}^j(\eta, \eta')\hat{M}''_{\gamma}(\eta', \emptyset) \leqslant C^{*j+1} \tag{3.45}$$

so that the sum over j on the right-hand side of (3.42) is bounded proportionally to $\gamma^{dn/2}$.

The last term on the right-hand side of (3.42) is bounded by

$$\frac{\gamma^{\delta N}}{N!} \sum_{\eta'} (\hat{M}'_{\gamma})^N(\eta_0, \eta') \sup_{\substack{|\eta| \leqslant n+N \\ d(\eta) \leqslant N(n+N)}} \sup_{\sigma, m} \big|\rho_{\eta}(\sigma, m)\big|. \tag{3.46}$$

By lemma 3.3

$$\sup_{\lambda} \sup_{d(\xi) \leqslant N(n+N)} \big|f_{\xi}(\lambda)\big| \leqslant c \tag{3.47}$$

so that the sup in (3.46) is bounded independently of γ. By (3.25)

$$\sum_{\eta'} (\hat{M}'_{\gamma})^N(\eta_0, \eta') \leqslant c'$$

independently of γ, hence by (3.41) the expression in (3.46) is bounded proportionally to $\gamma^{dn/2}$. The theorem is therefore proven. □

4. Extension to longer times

In this section we prove the theorems stated in subsections 2.1 and 2.3 of section 2, extending the estimates on the v functions to times $t \leqslant a \log \gamma^{-1}$ with $a > 0$ suitably small.

We postpone to proposition 4.7 the proof of the existence and of the uniqueness of the Cauchy problems for (1.1) and (2.20*a*) and of several other properties of these equations, in particular that the sup norms of the solutions of (1.1) and (2.20*a*) are bounded by 1, if that is so for the initial datum. First we introduce the basic notion of 'quasi-solutions' of (2.20*a*).

Definition 4.1. *Given a configuration $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$ and $s \geqslant 0$ we denote by*

$$m_{\gamma,s}(x, t|\sigma) \qquad t \geqslant s \tag{4.1}$$

the solution of (2.20a) for $t \geqslant s$, which starts from σ at time s. We set

$$t_k = k\gamma^{\delta} \qquad k \in \mathbb{Z}_+, \delta > 0 \tag{4.2}$$

and given any sequence $\sigma_{t_k} \in \{-1, 1\}^{\mathbb{Z}^d}$, $k \in \mathbb{Z}_+$, we define, for $t \geqslant 0$,

$$m_{\gamma}\big(x, t|\{\sigma_{t_k}\}\big) = m_{\gamma,t_k}\big(x, t|\sigma_{t_k}\big),\ \ t_k \leqslant t < t_{k+1}. \tag{4.3}$$

The function $m_{\gamma}\big(x, t|\{\sigma_{t_k}\}\big)$ is called a 'quasi-solution' of (2.20a).

Strategy of proof. We study the Glauber dynamics by successively conditioning the process at the times t_k. The conditioning at t_k fixes a configuration σ_{t_k} and the evolution in the next time step, $[t_k, t_{k+1})$, is well approximated in terms of $m_{\gamma,t_k}\big(x, t|\sigma_{t_k}\big)$, as it follows from the analysis of the previous section. At the end of this time interval, i.e. at time t_{k+1}, we replace $m_{\gamma,t_k}\big(x, t_{k+1}|\sigma_{t_k}\big)$ by one of the true configurations which appear in the conditioned process, say $\sigma_{t_{k+1}}$. By iterating this procedure we thus construct a quasi solution of (2.20*a*). See chapter V of [13] for more comments on this approach.

Clearly the method will be effective if two conditions are satisfied: first, for each k, $\sigma_{t_{k+1}}$ and $m_{\gamma,t_k}\big(x, t_{k+1}|\sigma_{t_k}\big)$ should be close (with large probability and in a sense to be specified). If this holds we will say that $m_{\gamma}\big(x, t|\{\sigma_{t_k}\}\big)$ has 'small discontinuities'. The second condition to prove is that a quasi solution with small discontinuities is close to the true solution, $m_{\gamma}(x, t|\sigma_0)$. We start from the latter condition.

Definition 4.2. *Given $\alpha \in (0, 1)$ and $k^* \geqslant 2$, we define the seminorms $\|f\|^{+}_{k^*,\alpha}$ and $\|f\|_{k^*,\alpha}$ of $f \in \mathcal{M}\big(\{-1, 1\}^{\mathbb{Z}^d}\big)$ as*

$$\|f\|^{+}_{k^*,\alpha} = \sup_{|x| \leqslant k^*\gamma^{-2}} \Big|\mathcal{A}_{\gamma^{-\alpha},x}(f)\Big| \qquad \|f\|_{k^*,\alpha} = \sup_{|x| \leqslant (k^*-1)\gamma^{-2}} \Big|\mathcal{A}_{\gamma^{-\alpha},x}(f)\Big| \tag{4.4a}$$

where, recalling (2.9d),

$$\mathcal{A}_{\gamma^{-\alpha},x}(f) = \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{y \in B_{\gamma^{-\alpha},x}} f(y). \tag{4.4b}$$

Thus, given $\zeta > 0$, we say that a quasi-solution $m_{\gamma}\big(x, t|\{\sigma_{t_k}\}\big)$ is ζ-accurate till time T and with respect to the seminorm $\|\cdot\|^{+}_{k^*,\alpha}$, if, for all γ small enough,

$$\|m_{\gamma,t_k}(\cdot, t_{k+1}|\sigma_{t_k}) - \sigma_{t_{k+1}}\|^{+}_{k^*,\alpha} \leqslant \gamma^{\zeta} \qquad \text{for all } t_k \leqslant T. \tag{4.5a}$$

We also say that it is ζ-accurate relative to the initial condition $m_{\gamma}(\cdot, 0)$ if

$$\|m_{\gamma}(\cdot, 0) - \sigma_0\|^{+}_{k^*,\alpha} \leqslant \gamma^{\zeta} \tag{4.5b}$$

Proposition 4.3. *Let* $0 < \alpha < 1$, $0 < \delta < \zeta$, $0 < b_2 < \min\{\zeta - \delta, 1 - \alpha\}$, $\hat{c}$ *as in (4.18) below and*

$$0 < a < b_2/\hat{c}, \quad 0 \le b < b_2 - a\hat{c} \tag{4.6}$$

Then for any $k^* \geqslant 2$, *any* $m_\gamma(\cdot, 0) \in \mathcal{M}(\mathbb{Z}^d)$, $|m_\gamma(\cdot, 0)| \leqslant 1$, *and any quasi-solution* $\{m_\gamma(\cdot, \cdot|\{\sigma_{t_k}\})$ *which is* ζ*-accurate till time* $a \log \gamma^{-1}$ *with respect to the seminorm* $\|\cdot\|^+_{k^*,\alpha}$, *and relative to the initial condition* $m_\gamma(\cdot, 0)$, *the following bound holds*

$$\sup_{t \leqslant a \log \gamma^{-1}} \|m_\gamma(\cdot, t|\{\sigma_{t_k}\}) - m_\gamma(\cdot, t)\|_{k^*,\alpha} \leqslant \gamma^b \tag{4.7}$$

for all γ *small enough.* $m_\gamma(\cdot, t)$ *in (4.7) is the solution of (2.20a) with initial condition* $m_\gamma(\cdot, 0)$.

Proof. For $t \geqslant t_k$ we define

$$\ell_{\gamma,t_k}(x, t) = m_{\gamma,t_k}(x, t|\sigma_{t_k}) - m_\gamma(x, t). \tag{4.8a}$$

We then have

$$\ell_{\gamma,t_k}(x, t) = [\sigma_{t_k}(x) - m_\gamma(x, t_k)] + \int_{t_k}^t \mathrm{d}s \left[F_{\gamma,t_k}(x, s) - \ell_{\gamma,t_k}(x, s)\right] \tag{4.8b}$$

where

$$F_{\gamma,t_k}(x, s) = \tanh\{\beta[(J_\gamma \circ m_{\gamma,t_k})(x, s) + h]\} - \tanh\{\beta[(J_\gamma \circ m_\gamma)(x, s) + h]\}. \tag{4.8c}$$

Given any $x \in \mathbb{Z}^d$, and $t \geqslant t_k$, we denote by

$$L_{\gamma,t_k}(x, t) = \left|\mathcal{A}_{\gamma^{-\alpha},x}\big(\ell_{\gamma,t_k}(\cdot, t)\big)\right|. \tag{4.9}$$

By adding and subtracting $m_{\gamma,t_{k-1}}(x, t_k|\sigma_{t_{k-1}})$ to the right-hand side of (4.8b), we get

$$\begin{aligned} L_{\gamma,t_k}(x, t) \leqslant\ & \left|\mathcal{A}_{\gamma^{-\alpha},x}\Big(\sigma_{t_k} - m_{\gamma,t_{k-1}}(\cdot, t_k|\sigma_{t_{k-1}})\Big)\right| + L_{\gamma,t_{k-1}}(x, t_k) \\ & + \int_{t_k}^t \mathrm{d}s\Big(\left|\mathcal{A}_{\gamma^{-\alpha},x}\big(F_{\gamma,t_k}(\cdot, s)\big)\right| + L_{\gamma,t_k}(x, s)\Big). \end{aligned} \tag{4.10}$$

We are going to prove that there are c_1 and c_2 so that for all x and s

$$|F_{\gamma,t_k}(x, s)| \leqslant c_1 \sum_y |J_\gamma(x, y)| L_{\gamma,t_k}(y, t) + c_2\gamma^{1-\alpha}. \tag{4.11}$$

We have, in fact,

$$|F_{\gamma,t_k}(x, s)| \leqslant c_1' \left|\big(J_\gamma \circ \ell_{\gamma,t_k}\big)(x, s)\right| \tag{4.12a}$$

and, for any function $f(y)$,

$$\begin{aligned} \Big|\sum_y J_\gamma(x, y) f(y)\Big| &= \Big|\sum_y J_\gamma(x, y)\{f(y) - \mathcal{A}_{\gamma^{-\alpha},y}(f) + \mathcal{A}_{\gamma^{-\alpha},y}(f)\}\Big| \\ &\leqslant \sum_y |J_\gamma(x, y)||\mathcal{A}_{\gamma^{-\alpha},y}(f)| + \|f\|_\infty \sum_y \left|J_\gamma(x, y) - \mathcal{A}_{\gamma^{-\alpha},y}\big(J_\gamma(x, \cdot)\big)\right|. \end{aligned} \tag{4.12b}$$

Using (4.12) we then obtain (4.11).

Going back to (4.10), using the assumption that $m_\gamma(\cdot, \cdot|\{\sigma_{t_k}\})$ is ζ-accurate, by (4.11), we get, for $|x| \leqslant k^\star\gamma^{-2}$ and all γ small enough,

$$L_{\gamma,t_k}(x,t) \leqslant \gamma^\zeta + L_{\gamma,t_{k-1}}(x,t_k) + c_2\gamma^{\delta+1-\alpha} + \int_{t_k}^t \mathrm{d}s \sum_y K_\gamma(x,y) L_{\gamma,t_k}(y,s) \tag{4.13a}$$

where

$$K_\gamma(x,y) = \mathbf{1}_{\{x=y\}} + c_1 \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{z\in B_{\gamma^{-\alpha},x}} |J_\gamma(z,y)|. \tag{4.13b}$$

We define

$$L_\gamma(x,t) = L_{\gamma,t_k}(x,t) \qquad \text{if } t_k \leqslant t < t_{k+1}. \tag{4.14a}$$

We bound $L_{\gamma,t_{k-1}}(x,t_k)$ in (4.13a) using the same (4.13a), then, iterating this procedure, we get for any $t \leqslant a \log\gamma^{-1}$

$$L_\gamma(x,t) \leqslant \gamma^{-\delta} a \log\gamma^{-1}[\gamma^\zeta + c_2\gamma^{\delta+1-\alpha}] + \int_0^t \mathrm{d}s \sum_z K_\gamma(x,z) L_\gamma(z,s). \tag{4.14b}$$

Let b_2 be as in the text of the proposition, then for all γ small enough and all $|x| \leqslant k^\star\gamma^{-2}$

$$L_\gamma(x,t) \leqslant \gamma^{b_2} + \int_0^t \mathrm{d}s \sum_z K_\gamma(x,z)\, L_\gamma(z,s). \tag{4.15}$$

We now restrict ourselves to $|x| \leqslant (k^\star - 1)\gamma^{-2}$. We can then iterate (4.15) $n = [\gamma^{-1}/2]$ times. In fact $K_\gamma(x,z) = 0$ if $|x-z| > 2\gamma^{-1}$ so that all sites z reached in the iteration are such that

$$|z| \leqslant |x| + 2\gamma^{-1}n \leqslant (k^\star - 1)\gamma^{-2} + \gamma^{-2} \leqslant k^\star\gamma^{-2}. \tag{4.16}$$

We then get, for $|x| \leqslant (k^\star - 1)\gamma^{-2}$,

$$\begin{aligned} L_\gamma(x,t) \leqslant \sum_{j=1}^{n-1} \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \dots \int_0^{s_{j-1}} \mathrm{d}s_j \sum_{z_1,\dots,z_j} K_\gamma(x,z_1)\dots K_\gamma(z_{j-1},z_j)\gamma^{b_2} \\ + \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{n-1}} \mathrm{d}s_n \sum_{z_1,\dots,z_n} K_\gamma(x,z_1)\dots K_\gamma(z_{n-1},z_n) L_\gamma(z,s_n). \end{aligned} \tag{4.17}$$

We set

$$\hat{c} := \sup_\gamma \sum_x K_\gamma(0,x). \tag{4.18}$$

Since $L_\gamma(x,t) \leqslant 2$, see proposition 4.7.1 below, we finally have, for all γ small enough,

$$L_\gamma(x,t) \leqslant \gamma^{b_2} \mathrm{e}^{\hat{c}t} + 2\frac{(\hat{c}t)^n}{n!} \leqslant \gamma^b \tag{4.19}$$

having used the definition of a and b and observed that for any $b_3 > 0$ there is c so that

$$\frac{(\hat{c}\, a \log\gamma^{-1})^n}{n!} \leqslant \gamma^b \qquad n = [\gamma^{-1}/2] \tag{4.19b}$$

The proposition is therefore proven. □

The following is a corollary of the proof of proposition 4.3:

Proposition 4.4. Using the same parameters as in proposition 4.3, let $m_\gamma(x,t)$ and $\tilde{m}_\gamma(x,t)$, $t \geqslant 0$, be two solutions of (2.20a) such that

$$\|m_\gamma(\cdot,0) - \tilde{m}_\gamma(\cdot,0)\|^+_{k^*,\alpha} < \gamma^\zeta. \tag{4.20a}$$

Then

$$\sup_{t \leqslant a \log \gamma^{-1}} \|m_\gamma(\cdot,t) - \tilde{m}_\gamma(\cdot,t)\|_{k^*,\alpha} < \gamma^b. \tag{4.20b}$$

Proof. Replace in the proof of proposition 4.3 $m_\gamma(x,t|\{\sigma_{t_k}\})$ by $\tilde{m}_\gamma(x,t)$ and σ_{t_k} by $\tilde{m}_\gamma(x,t_k)$, for all $t_k > 0$. We then obtain (4.20b) from (4.7). □

We next prove that with large probability the trajectories of the Glauber dynamics give rise to ζ-accurate quasi solutions:

Proposition 4.5. For any $k^* \geqslant 2$, $0 < \alpha < 1$, $\delta > 0$, $a > 0$, $n \geqslant 1$ and

$$\zeta < \alpha d/2 \tag{4.21}$$

there is c so that, for all σ_0,

$$\mathbb{P}_{\sigma_0}\Big(\sup_{t_k \leqslant a \log \gamma^{-1}} \|\sigma_{t_{k+1}} - m_{\gamma,t_k}(\cdot, t_{k+1}|\sigma_{t_k})\|^+_{k^*,\alpha} \geqslant \gamma^\zeta\Big) \leqslant c\gamma^n \tag{4.22a}$$

and for any product measure μ^γ

$$\mu^\gamma\Big(\|\sigma - m_\gamma(\cdot,0)\|^+_{k^*,\alpha} \geqslant \gamma^\zeta\Big) \leqslant c\gamma^n \qquad m_\gamma(x,0) := \mathbb{E}_{\mu^\gamma}\big(\sigma(x)\big). \tag{4.22b}$$

Proof. We first write

$$\mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t_k \leqslant a \log \gamma^{-1}} \|\sigma_{t_{k+1}} - m_{\gamma,t_k}(\cdot, t_{k+1}|\sigma_{t_k})\|^+_{k^*,\alpha} > \gamma^\zeta\Big)$$
$$\leqslant \gamma^{-\delta} a \log \gamma^{-1} [4k^*\gamma^{-2} + 1]^d \sup_{x,\sigma} \mathbb{P}_\sigma\Big(\big|\Lambda_{\gamma,x,\sigma}\big| > \gamma^\zeta\Big) \tag{4.23a}$$

$$\Lambda_{\gamma,x,\sigma} := \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{y \in B_{\gamma^{-\alpha},x}} [\sigma(y,t_1) - m_\gamma(y,t_1|\sigma)]. \tag{4.23b}$$

Then, using the Chebyshev inequality with power $2n$,

$$P_\sigma\Big(\big|\Lambda_{\gamma,x,\sigma}\big| > \gamma^\zeta\Big) \leqslant \gamma^{-2n\zeta} \frac{1}{|B_{\gamma^{-\alpha}}|^{2n}} \sum_{\substack{x_1,\dots,x_{2n} \\ x_i \in B_{\gamma^{-\alpha},x}}} \mathbb{E}^\gamma_\sigma\Big(\prod_{i=1}^{2n} [\sigma(x_i,t_1) - m_\gamma(x_i,t_1|\sigma)]\Big).$$

If all the x_i are distinct, the expectation is a v-function, so that, by (3.39), it is bounded by γ^{dn}. If the sites are not all distinct we use (3.32). We observe that for each sum which is missing we gain a factor $\gamma^{d\alpha}$ from the normalization, hence we obtain that the last expression is bounded by:

$$c_n \gamma^{-2n\zeta} \max(\gamma^{dn}, \gamma^{\alpha dn}).$$

Since $\alpha < 1$, the max is achieved by the second term, which gives the bound $\gamma^{n(d\alpha - 2\zeta)}$, and this proves (4.22$a$). A completely similar argument gives (4.22b), the proposition is thus proven. □

Collecting the above results we have:

Proposition 4.6. *Let k^*, α, δ, ζ, a and b as in proposition 4.3 and such that (4.21) holds. Then for any n there is c so that for all σ_0:*

$$\mathbb{P}^{\gamma}_{\sigma_0}\Big(\sup_{t\leqslant a\log\gamma^{-1}} \|\sigma_t(\cdot) - m_\gamma(\cdot, t|\sigma_0)\|_{k^*,\alpha} > \gamma^b\Big) \leqslant c\gamma^n. \tag{4.24}$$

Furthermore let μ^γ be a product measure, $m_\gamma(x,0)$ its means, $m_\gamma(x,t)$ the solution of (2.20a) with initial condition $m_\gamma(\cdot,0)$, then

$$\mathbb{P}^{\gamma}_{\mu^\gamma}\Big(\sup_{t\leqslant a\log\gamma^{-1}} \|\sigma_t(\cdot) - m_\gamma(\cdot, t)\|_{k^*,\alpha} > \gamma^b\Big) \leqslant c\gamma^n \tag{4.25a}$$

$$\mathbb{P}^{\gamma}_{\mu^\gamma}\Big(\Big\{\sigma_0 : \mathbb{P}^{\gamma}_{\sigma_0}\Big(\sup_{t\leqslant a\log\gamma^{-1}} \|\sigma_t(\cdot) - m_\gamma(\cdot, t)\|_{k^*,\alpha} > \gamma^b\Big) \leqslant c\gamma^n\Big\}\Big) \geqslant 1 - c\gamma^n \tag{4.25b}$$

Proof. The left-hand side of (4.24) is bounded by

$$\gamma^{-\delta} a\log\gamma^{-1} \sup_{t_k\leqslant a\log\gamma^{-1}} \mathbb{P}_{\sigma_0}\Big(\sup_{t_k\leqslant t\leqslant t_{k+1}} \|\sigma_t(\cdot) - m_\gamma(\cdot, t|\sigma_0)\|_{k^*,\alpha} > \gamma^b\Big). \tag{4.26}$$

We add and subtract $m_{\gamma,t_k}(\cdot, t|\sigma_{t_k})$ so that the probability in (4.26) is bounded by

$$P_{\sigma_0}\Big(\sup_{t_k\leqslant t\leqslant t_{k+1}} \|m_\gamma(\cdot, t|\sigma_0) - m_{\gamma,t_k}(\cdot, t|\sigma_{t_k})\|_{k^*,\alpha} > \frac{1}{2}\gamma^b\Big)$$
$$+ \sup_\sigma P_\sigma\Big(\sup_{0\leqslant t\leqslant t_1} \|\sigma_t(\cdot) - m_\gamma(\cdot, t|\sigma)\|_{k^*,\alpha} > \frac{1}{2}\gamma^b\Big). \tag{4.27}$$

We use propositions 4.3 and 4.5 to conclude that the first term vanishes faster than any power of γ. The factor $\gamma^b/2$ can be easily taken into account by observing that $\gamma^b/2 > \gamma^{b'}$ for $b' > b$ and all γ small enough. It is then enough to consider (4.7) with b' in the place of b. For the second term in (4.27) we proceed as follows. With a proof similar to that of proposition 4.5, we have that for $b < \alpha d/2$ and for any n there is c so that

$$\sup_{0\leqslant t\leqslant t_1} \sup_\sigma P_\sigma\Big(\|\sigma_t(\cdot) - m_\gamma(\cdot, t|\sigma)\|_{k^*,\alpha} > \frac{1}{4}\gamma^b\Big) \leqslant c\gamma^n. \tag{4.28}$$

We then split the time interval $[0, t_1]$ in N subintervals of length ϵ, $\epsilon N = t_1$. Let $\mathcal{O}$ be the set of orbits σ_s, $0 \leqslant s \leqslant t_1$, for which at least two spins flip in the same subinterval and at sites within distance $k^*\gamma^{-2}$ from the origin. More precisely, $\mathcal{O}$ is the set of orbits such that there is $n \leqslant N-1$ and two times, s_1 and s_2, both in $(n\epsilon, (n+1)\epsilon)$ and two sites x_1 and x_2, $|x_i| \leqslant k^*\gamma^{-2}$ such that $\sigma(x_i, s_i^+) = -\sigma(x_i, s_i^-)$, $i = 1, 2$. We denote by

$$\mathcal{B} = \bigcap_{n\leqslant N-1} \Big\{\|\sigma_{n\epsilon}(\cdot) - m_\gamma(\cdot, n\epsilon|\sigma)\|_{k^*,\alpha} < \frac{1}{4}\gamma^b\Big\} \tag{4.29a}$$

and by $\mathcal{B}^c$ its complement.

We claim that for ϵ small enough

$$\left\{\sup_{0\leqslant t\leqslant t_1}\|\sigma_t(\cdot)-m_\gamma(\cdot,t|\sigma)\|_{k^*,\alpha}>\frac{1}{2}\gamma^b\right\}\subset B^c\cup BO. \tag{4.29b}$$

To prove the claim we will show that BO^c does not intersect the set on the left-hand side of (4.29b). To this end we observe that $m_\gamma(x,t|\sigma)$ has a bounded time derivative so that in a time interval ϵ it varies at most by $c\epsilon$, for some c independent of σ, γ and t. In O^c there is at most one spin which flips in a time interval $(n\epsilon,(n+1)\epsilon)$ within distance $k^*\gamma^{-2}$ from the origin, so that if $n\epsilon<t<(n+1)\epsilon$,

$$\begin{aligned}\|\sigma_t(\cdot)-m_\gamma(\cdot,t|\sigma)\|_{k^*,\alpha}&\leqslant\|\sigma_{n\epsilon}(\cdot)-m_\gamma(\cdot,n\epsilon|\sigma)\|_{k^*,\alpha}+\frac{2}{|B_{\gamma^{-\alpha}}|}+c\epsilon\\&\leqslant\frac{1}{4}\gamma^b+c\epsilon+c'\gamma^{\alpha d}\leqslant\frac{1}{2}\gamma^b\end{aligned}$$

if b and ϵ are small enough. (4.29b) is thus proven.

From (4.29b) and (4.28) it follows that

$$\mathbb{P}^\gamma_\sigma\Big(\{\|\sigma_t(\cdot)-m_\gamma(\cdot,n\epsilon|\sigma)\|_{k^*,\alpha}>\frac{1}{2}\gamma^b\}\Big)\leqslant Nc\gamma^n+\mathbb{P}_\sigma(O). \tag{4.29c}$$

Since $c_\gamma(x,\sigma)\leqslant 1$, for a suitable constant c',

$$P_\sigma(O)\leqslant Nc'[(4k^*\gamma^{-2}+1)^d\epsilon t_1]^2.$$

Thus (4.29c) is bounded by

$$\epsilon^{-1}t_1c\gamma^n+\epsilon^{-1}t_1c'[(4k^*\gamma^{-2}+1)^d\epsilon t_1]^2.$$

By taking $\epsilon=\gamma^k$, with k sufficiently large, we make the second term vanish as any desired power of γ. Given that k, we then choose n so large that also the first term vanishes as fast as desired. We have thus shown that also the second term on the right- hand side of (4.27) vanishes faster than any power of γ, hence (4.24) is proven.

Proof of (4.25b). We write

$$\begin{aligned}\left\{\sup_{t\leqslant a\log\gamma^{-1}}\|m_\gamma(\cdot,t)-\sigma_t\|_{k^*,\alpha}>\gamma^b\right\}&\subset\left\{\sup_{t\leqslant a\log\gamma^{-1}}\|m_\gamma(\cdot,t|\sigma_0)-\sigma_t\|_{k^*,\alpha}>\frac{\gamma^b}{2}\right\}\\&\bigcup\left\{\sup_{t\leqslant a\log\gamma^{-1}}\|m_\gamma(\cdot,t)-m_\gamma(\cdot,t|\sigma_0)\|_{k^*,\alpha}>\frac{\gamma^b}{2}\right\}.\end{aligned}$$

By (4.24) we then get for all γ small enough

$$\mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t\leqslant a\log\gamma^{-1}}\|m_\gamma(\cdot,t|\sigma_0)-\sigma_t\|_{k^*,\alpha}>\frac{\gamma^b}{2}\Big)\leqslant c\gamma^n.$$

Furthermore, if

$$\|\sigma_0-m_\gamma(\cdot,0)\|^+_{k^*,\alpha}<\gamma^\zeta \tag{4.30}$$

by proposition 4.4, for all γ small enough,

$$\sup_{t\leqslant a\log\gamma^{-1}}\|m_\gamma(\cdot,t)-m_\gamma(\cdot,t|\sigma_0)\|_{k^*,\alpha}\leqslant\frac{\gamma^b}{2}.$$

(4.25b) then follows from (4.22b). Since (4.25a) follows immediately from (4.25b) the proposition is proven. □

Proposition 4.6 'almost' proves (2.10b) and (2.13b). The 'almost' is because $m_\gamma(\cdot, t|\sigma_0)$ and $m_\gamma(\cdot, t)$ solve (2.20a) and not (1.1), as required in (2.10b) and (2.13b). We fill this gap in the next proposition, where we also prove several properties of (1.1) and (2.20a). For ease of reference we split the proposition in several distinct statements.

Proposition 4.7.1. *For any $\gamma > 0$ and any $m \in \mathcal{M}(\mathbb{Z}^d)$, $\|m\|_\infty \leqslant 1$, there is a unique bounded function $m_\gamma(x, t)$ on $\mathbb{Z}^d \times \mathbb{R}_+$ which is differentiable in t in sup-norm, solves (2.20a) for all x and all $t \geqslant 0$ and satisfies the initial condition $m_\gamma(x, 0) = m(x)$, for all x. Moreover $\|m_\gamma(\cdot, t)\|_\infty \leqslant 1$ for all $t \geqslant 0$.*

Analogously, for any $m \in \mathcal{M}(\mathbb{R}^d)$, $\|m\|_\infty \leqslant 1$, there is a unique bounded function $m(r, t)$, which is differentiable in t in sup-norm, solves (1.1) and satisfies the initial condition $m(\cdot, 0) = m(\cdot)$. Moreover $\|m(r, t)\|_\infty \leqslant 1$ for all $t \geqslant 0$.

Proof. We consider the statement relative to (1.1), the argument for (2.20a) is completely analogous and omitted.

We fix $T > 0$ and for $0 \leqslant t \leqslant T$ we write the integral version of (1.1)

$$m(r, t) = \mathrm{e}^{-t} m(r, 0) + \int_0^t \mathrm{d}s \mathrm{e}^{-(t-s)} \tanh\Big(\beta[J \star m(r, s) + h]\Big) \tag{4.31a}$$

which can be thought of as a fixed point problem for the map

$$K : \Big\{u \in \mathcal{M}(\mathbb{R}^d \times [0, T]) : \|u\|_\infty < \infty\Big\} \to \Big\{u \in \mathcal{M}(\mathbb{R}^d \times [0, T]) : \|u\|_\infty \leqslant 1\Big\}$$

defined as

$$Ku(r, t) = \mathrm{e}^{-t} m(r, 0) + \int_0^t \mathrm{d}s \mathrm{e}^{-(t-s)} \tanh\Big(\beta[J \star u(r, s) + h]\Big). \tag{4.31b}$$

For T small enough K is a strict contraction in the sup norm, with therefore a unique fixed point, $m(r, t)$. Since $Ku(r, t)$ is a differentiable function of t in sup-norm, the fixed point $m(\cdot, t)$ is differentiable in t, hence $m(r, t)$ solves (1.1) for $t \leqslant T$. Observe also that $m(r, t) - \mathrm{e}^{-t} m(r, 0)$ is in $C^1\big(\mathbb{R}^d \times [0, T]\big)$.

As the solution at time T is in the same class as the datum at time 0, by iteration we have global existence. Uniqueness follows because any solution of (1.1) solves its integral version, which is a contraction. □

Proposition 4.7.2. *There are c_2' and c_2 so that the following holds. Let $u_\gamma(x, t)$ be for each γ a solution of (2.20a) and denote by $u^\star_\gamma(r, t)$ the Γ_γ image of $u_\gamma(x, t)$. Let $m_\gamma(r, t)$ solve (1.1) with initial datum $m_\gamma(r, 0) = u^\star_\gamma(r, 0)$ and suppose $\|u^\star_\gamma(\cdot, 0)\|_\infty \leqslant 1$. Then*

$$\|m_\gamma(\cdot, t) - u^\star_\gamma(\cdot, t)\|_\infty \leqslant c_2' \mathrm{e}^{c_2 t} \gamma. \tag{4.32}$$

Proof. Observe that $u^\star_\gamma(r, t)$ solve (1.1) with $J(r - r')$ replaced by

$$\Delta_\gamma(r, r') := J([r]_\gamma - [r']_\gamma).$$

Let

$$L_\gamma(r, t) := |u^\star_\gamma(r, t) - m_\gamma(r, t)|$$

then

$$L_\gamma(r, t) \leqslant \int_0^t \mathrm{d}s \Big\{\beta[|J| \star L_\gamma(r, s) + 2c\gamma]\Big\}$$

because, for a suitable c,

$$|J(r - r') - \Delta_\gamma(r, r')| \leqslant c\gamma$$

and, by proposition 4.7.1, $\|u^\star_\gamma\|_\infty \leqslant 1$, $\|m_\gamma\|_\infty \leqslant 1$. Hence (4.32). □

Proposition 4.7.3. *Let $m_\gamma(r,t)$ solve (1.1), $\|m_\gamma(\cdot,0)\|_\infty \leqslant 1$, $m_\gamma(r,0) \to m(r,0)$ Lebesgue almost everywhere and let $\|m(\cdot,0)\|_\infty \leqslant 1$. Then, for any t, $m_\gamma(r,t) \to m(r,t)$ Lebesgue almost everywhere, $m(r,t)$ being the solution of (1.1) with initial condition $m(r,0)$.*

Proof. By proposition 4.7.1

$$L_\gamma(r,t) := |m_\gamma(r,t) - m(r,t)| \leqslant 2.$$

Then

$$\begin{aligned} L_\gamma(r,t) &\leqslant e^{-t} L_\gamma(r,0) + \beta \int_0^t ds e^{-(t-s)} |J| \star L_\gamma(r,s) \\ &\leqslant e^{-t} L_\gamma(r,0) + e^{-t} \sum_{n=1}^{N-1} \frac{(\beta t)^n}{n!} [J]^n \star L_\gamma(r,0) + 2 \frac{(\beta \|J\|_\infty t)^N}{N!} \\ &\leqslant e^{-t} L_\gamma(r,0) + e^{-t} \sum_{n=1}^{N-1} \frac{(\beta t \|J\|_\infty)^n}{n!} \int_{|r-r'| \leqslant N-1} dr' L_\gamma(r',0) + 2 \frac{(\beta \|J\|_\infty t)^N}{N!}. \end{aligned} \tag{4.33}$$

Given any $\epsilon > 0$, we choose N so large that the last term is less than ϵ. On the other hand, by the Lebesgue theorem, for any bounded region Λ,

$$\lim_{\gamma \to 0} \int_\Lambda dr\, L_\gamma(r,0) = 0.$$

It then follows that for all γ small enough, also the second term on the right-hand side of (4.33) is less than ϵ. By the arbitrarity of ϵ we then conclude that $L_\gamma(r,t) \to 0$ for all r for which this happens at $t = 0$. The proposition is thus proven. □

Proposition 4.7.4. *There is c_4 so that for any $V > 3\beta \|J\|_\infty$, any $T > 0$ and any pair $m(r,t)$, $\tilde{m}(r,t)$ of solutions of (1.1) such that $m(r,0) = \tilde{m}(r,0)$ for all $|r| \leqslant VT$ and $\|m(\cdot,0)\|_\infty \leqslant 1$, $\|\tilde{m}(\cdot,0)\|_\infty \leqslant 1$,*

$$\left| m(0,t) - \tilde{m}(0,t) \right| \leqslant c_4 e^{-TV \log[V/(3\beta\|J\|_\infty)]} \qquad \textit{for all } t \leqslant T \tag{4.34}$$

Proof. Define $L(r,t) = \left| m(r,t) - \tilde{m}(r,t) \right|$, then $L(r,t)$ satisfies the same inequality (4.33) as $L_\gamma(r,t)$. We write (4.33) with $r = 0$ and $N = 1 + [VT]$ ([VT] the integer part of VT). Then only the last term on the right-hand side of the last inequality of (4.33) survives. Thus for the Stirling formula, there is c_4 so that for any $t \leqslant T$

$$L(0,t) \leqslant c_4 \exp\Big(-N[\log \frac{N}{\beta \|J\|_\infty t} - 1] \Big) \leqslant c_4 \exp\Big(-N \log \frac{N}{3\beta \|J\|_\infty T} \Big).$$

Since $N \geqslant VT$ and $V > 3\beta\|J\|_\infty$ the log is positive and we have an upper bound if we replace N by VT. The proposition is then proved. □

Proposition 4.7.5. *There is c_5 so that if $m(r,t)$ and $\tilde{m}(r,t)$ solve (1.1), $\|m(\cdot,0)\|_\infty \leqslant 1$, $\|\tilde{m}(\cdot,0)\|_\infty \leqslant 1$ and $\|m(\cdot,0) - \tilde{m}(\cdot,0)\|_\infty < \epsilon$, then,*

$$\|m(\cdot,t) - \tilde{m}(\cdot,t)\|_\infty \leqslant e^{c_5 t} \epsilon \qquad \textit{for all } t \geqslant 0.$$

Proof. Defining $L(r, t)$ as in the proof of proposition 4.7.4, we obtain the result from the first inequality in (4.33). $\square$

Proposition 4.7.6. *There are c_6' and c_6 so that the following holds. Let $m(r,t)$ and $\tilde{m}(r,t)$ solve (1.1), $\|m(\cdot,0)\|_\infty \leqslant 1$, $\|\tilde{m}(\cdot,0)\|_\infty \leqslant 1$, and suppose that there are $0 < \alpha < 1$, $b > 0$, $u > 0$ and c' so that*

$$\sup_{r\in\mathbb{R}^d} \Big| D \cap \{r' : |r - r'| \leqslant 1\} \Big| \leqslant c'\gamma^u$$

where

$$D := \Big\{ r \in \mathbb{R}^d : \|m^{(\alpha,\gamma)}(r,0) - \tilde{m}^{(\alpha,\gamma)}(r,0)\|_\infty > \gamma^b \Big\}$$

(see 2.1.5 for notation). Then

$$\sup_{r\notin D} |m^{(\alpha,\gamma)}(r,t) - \tilde{m}^{(\alpha,\gamma)}(r,t)| \leqslant c_6' e^{c_6 t}[\gamma^b + \gamma^{1-\alpha} + c'\gamma^u].$$

Proof. We have

$$\begin{aligned}
&\Big| m(r,t) - \tilde{m}(r,t) - e^{-t}[m(r,0) - \tilde{m}(r,0)] \Big| \\
&\leqslant \int_0^t ds e^{-(t-s)} \beta \Big| \big(J \star m(\cdot,s)\big)(r) - \big(J \star \tilde{m}(\cdot,s)\big)(r) \Big|.
\end{aligned}$$

We fix r and, setting $j(r') \equiv J(r - r')$, we write the first convolution as

$$\begin{aligned}
\int dr' j(r') m(r',s) &= \int dr' j(r') \Big\{ m^{(\alpha,\gamma)}(r',s) + [m(r',s) - m^{(\alpha,\gamma)}(r',s)] \Big\} \\
&= \int dr' j(r') m^{(\alpha,\gamma)}(r',s) + \int dr' m(r',s)[j(r') - j^{(\alpha,\gamma)}(r')].
\end{aligned}$$

An analogous expression holds for $\tilde{m}$. We then set $L_\gamma(r,t) = |m^{(\alpha,\gamma)}(r,t) - \tilde{m}^{(\alpha,\gamma)}(r,t)|$, and, for $r \notin D$,

$$L_\gamma(r,t) \leqslant e^{-t}\gamma^b + \int_0^t ds e^{-(t-s)} \beta \Big\{ \|J\|_\infty \Big[2c'\gamma^u + \sup_{r'\notin D} |L_\gamma(r',s)| \Big] + 2\|J'\|_\infty \gamma^{1-\alpha} \Big\}$$

hence the proposition is proven. $\square$

Proposition 4.7.7. *There is c_7 so that the following holds. Let $m(\cdot,0) \in C^1(\mathbb{R}^d)$, assume that $\|m(\cdot,0)\|_\infty \leqslant 1$ and that $\|m^{(1)}(\cdot,0)\|_\infty < \infty$, $m^{(1)}$ being the derivative with respect to r of m. Let $m(r,t)$ be the solution of (1.1) with initial condition $m(r,0)$, then $m(\cdot,t) \in C^1(\mathbb{R}^d)$ and*

$$\|m^{(1)}(\cdot,t)\|_\infty \leqslant e^{-t}\|m^{(1)}(\cdot,0)\|_\infty + c_7. \tag{4.35}$$

Proof. The statement follows by differentiating (4.31a) with respect to r and using that $\|J^{(1)}\|_\infty < \infty$ and that $\|m(\cdot,t)\|_\infty \leqslant 1$. We omit the details. □

Proposition 4.7.8. *There is a constant* c_8 *so that for any solution* $m(r,t)$ *of (1.1) with* $\|m(\cdot,0)\|_\infty \leqslant 1$, *for any* $r \in \mathbb{R}^d$ *and any* $\ell \in \mathbb{R}^d$:

$$|m(r,t) - m(r+\ell,t)| \leqslant c_8|\ell| + 2\mathrm{e}^{-t}. \tag{4.36}$$

Proof. Let

$$D(\ell,t) = \sup_r |m(r,t) - m(r+\ell,t)|.$$

Then, for a suitable c,

$$D(\ell,t) \leqslant \mathrm{e}^{-t}D(\ell,0) + \int_0^t \mathrm{d}s\mathrm{e}^{-(t-s)}c|\ell|$$

because

$$\beta|J\star m(r,s) - J\star m(r+\ell,s)| \leqslant \beta\int \mathrm{d}r'|J(r-r') - J(r+\ell-r')| \leqslant c|\ell|.$$

Recalling that $|D(\ell,0)| \leqslant 2$, we then obtain (4.36). □

Proof of theorem 2.1.6. By (4.24) and using the notation (4.4a) and (2.9d), we have

$$\mathbb{P}^\gamma_\sigma\Big(\sup_{t\leqslant a\log\gamma^{-1}}\ \sup_{|r|\leqslant(k^*-1)\gamma^{-1}} \big|\sigma^{(\alpha,\gamma)}_{\gamma,t}(r) - \big(\Gamma_\gamma(m_\gamma(\cdot,t|\sigma))\big)^{(\alpha,\gamma)}(r)\big| \leqslant \gamma^b\Big) \geqslant 1 - c_n\gamma^n. \tag{4.37a}$$

Recall that $m_\gamma(x,t|\sigma)$ solves (2.20a) whit initial condition $\sigma(\cdot)$. We call $\psi_\gamma(r,t)$ the solution of (1.1) with initial datum $\sigma_\gamma = \Gamma_\gamma\sigma$.

By (4.32)

$$\|\Gamma_\gamma(m_\gamma(\cdot;t|\sigma)) - \psi_\gamma(\cdot,t)\|_\infty \leqslant c_2'\mathrm{e}^{c_2t}\gamma. \tag{4.37b}$$

We need to relate $\psi_\gamma(r,t)$ to $m(r,t)$, the latter being as in the statement of theorem 2.1.6. To this end we define $\phi_\gamma(r,t)$ as the solution of (1.1) with initial datum equal to $\sigma_\gamma(r)$ for $|r| \leqslant k^*\gamma^{-1}$ and equal to $m(r)$ for $|r| > k^*\gamma^{-1}$. By proposition 4.7.4 with $T = a\log\gamma^{-1}$ and $VT = \gamma^{-1}/2$ we get that for γ small enough

$$\sup_{t\leqslant a\log\gamma^{-1}}\ \sup_{|r|\leqslant(k^*-1/2)\gamma^{-1}} |\psi_\gamma(r,t) - \phi_\gamma(r,t)| \leqslant c_4\mathrm{e}^{-\gamma^{-1}}. \tag{4.37c}$$

We next compare $\big(\phi_\gamma(\cdot,t)\big)^{(\alpha,\gamma)}$ and $\big(m(\cdot,t)\big)^{(\alpha,\gamma)}$ using proposition 4.7.6. In this case

$$D \subset \{k^*\gamma^{-1} - \gamma^{1-\alpha} \leqslant |r| \leqslant k^*\gamma^{-1} + \gamma^{1-\alpha}\}.$$

In fact, for $|r| \leqslant k^*\gamma^{-1} - \gamma^{1-\alpha}$

$$|\phi_\gamma^{(\alpha,\gamma)}(r) - m^{(\alpha,\gamma)}(r)| = |\psi_\gamma^{(\alpha,\gamma)}(r) - m^{(\alpha,\gamma)}(r)| \leqslant \gamma^\zeta$$

because the block spin transformation involves values of ϕ_γ and m in $|r| \leqslant k^*\gamma^{-1}$ and for such r, $\psi_\gamma(r) = \phi_\gamma(r)$. The bound then follows using (2.10*a*) recalling that $\psi_\gamma(r) = \sigma_\gamma(r)$. Outside D and for $|r| > k^*\gamma^{-1} + \gamma^{1-\alpha}$, by the same argument $\phi_\gamma^{(\alpha,\gamma)}(r) = m^{(\alpha,\gamma)}(r)$. We can therefore apply proposition 4.7.6 with $b \to \zeta$ and $u \to 1-\alpha$. We then get, recalling the notation (2.11),

$$\sup_{r \notin D} |\phi_\gamma^{(\alpha,\gamma)}(r,t) - m^{(\alpha,\gamma)}(r,t)| \leqslant c_6' e^{c_6 t}\big[\gamma^\zeta + \gamma^{1-\alpha}(1+c')\big]. \tag{4.37d}$$

Then, by (4.37*c*),

$$\sup_{t \leqslant a \log \gamma^{-1}} \sup_{|r| \leqslant (k^*-1)\gamma^{-1}} |\psi_\gamma^{(\alpha,\gamma)}(r,t) - m^{(\alpha,\gamma)}(r,t)| \leqslant c_4 e^{-\gamma^{-1}} + c_6'\gamma^{-c_6 a}\big[\gamma^\zeta + \gamma^{1-\alpha}(1+c')\big]. \tag{4.37e}$$

We finally use (4.37*b*) to recover, from (4.37*a*), the bound

$$\mathbb{P}_\sigma^\gamma\Big(\sup_{t \leqslant a \log \gamma^{-1}} \sup_{|r| \leqslant (k^*-1)\gamma^{-1}} \big|\sigma_{\gamma,t}^{(\alpha,\gamma)}(r) - \big(\Gamma_\gamma(m_\gamma(\cdot,t|\sigma))\big)^{(\alpha,\gamma)}(r)\big| \leqslant A_\gamma\Big) \geqslant 1 - c_n\gamma^n \tag{4.37f}$$

where

$$A_\gamma = \gamma^b + c_2'\gamma^{-c_2 a}\gamma + c_4 e^{-\gamma^{-1}} + c_6'\gamma^{-c_6 a}\big[\gamma^\zeta + \gamma^{1-\alpha}(1+c')\big].$$

By chosing a small enough we have $A_\gamma \leqslant \gamma^{b'}$, for all γ small enough and a suitable $b' > 0$. We have thus proven theorem 2.1.6 with b in the statement of the theorem taken equal to b'. □

Proof of theorem 2.1.8. (2.13*a*) is proven by (4.25*a*) and proposition 4.7.2. (2.13*b*) is a straight consequence of (2.10*b*) and (2.13*a*). $m_\gamma(r,t)$ converges Lebesgue almost everywhere to $m(r,t)$ by proposition 4.7.3. The statements in theorem 2.1.8 relative to $m_0 \in C^1(\mathbb{R}^d)$ are proved using that $|m_\gamma(\cdot,0) - m(\cdot,0)|_\infty \leqslant C\gamma$. Then, by proposition 4.7.5,

$$\|m_\gamma(\cdot,t) - m(\cdot,t)\|_\infty \leqslant e^{c_5 t} C\gamma \qquad \|m_\gamma^{(\alpha,\gamma)}(\cdot,t) - m^{(\alpha,\gamma)}(\cdot,t)\|_\infty \leqslant e^{c_5 t} C\gamma$$

hence (2.13*a*) with $m^{(\alpha,\gamma)}$. Then, by proposition 4.7.7, the sup-norm of $m - m^{(\alpha,\gamma)}$ vanishes as γ^b, for some $b > 0$. The proof of theorem 2.1.8 is thus concluded. □

We next extend the bounds on the v-functions proven in section 3 to finite times and to times $t \leqslant a \log \gamma^{-1}$, when a is sufficiently small, proving theorem 2.3.4.

There are two key observations: first, by using proposition 4.6, it follows that the average of $v_n^\gamma(\underline{x}, t|\sigma)$ when each x_i independently varies in the box $B_{\gamma^{-\alpha}, y_i}$ is bounded, for any given $\underline{y} = (y_1, \ldots, y_n)$, by $c\gamma^{bn}$. Second observation is that while the v-functions do not obey a closed set of equations, the new terms appear as averages or, more precisely, can be reduced to averages. The right algebra to evidentiate this finer structure is that introduced, but not fully used, in section 3, to which we refer also for the notation that we will be using in the sequel. The main conclusions of our analysis are summarized in a generalization of proposition 3.6:

Proposition 4.8. *There are real valued coefficients $M_\gamma(\eta,\eta',m)$ and $M_\gamma^{dg}(\eta,\eta',m)$, η and η' in H, (see definition 3.4), $m\in[-1,1]^{\mathbb{Z}^d}$, such that for all γ, η, m and $\sigma\in\{-1,1\}^{\mathbb{Z}^d}$*

$$\sum_{x\in\mathbb{Z}^d} D^{(x)}\rho_\eta(\sigma,m)=\sum_{\eta'\in H}\left\{M_\gamma(\eta,\eta',m)+M_\gamma^{dg}(\eta,\eta',m)\right\}\rho_{\eta'}(\sigma,m) \tag{4.38}$$

(ρ_η as in definition 3.5). The coefficients M_γ and M_γ^{dg} satisfy the following support properties and bounds.

Support properties.

$M_\gamma^{dg}(\eta,\eta',m)=0$ unless $|\eta|=|\eta'|$, $g(\eta)=g(\eta')$, $d(\eta)=d(\eta')$ and

$$\max_{\xi'\in\eta'}|V_{\xi'}|\leqslant\max_{\xi\in\eta}|V_\xi| \tag{4.39a}$$

with V_ξ as in definition 3.2; $\xi\in\eta$ meaning that ξ is an entry of η.
$M_\gamma(\eta,\eta',m)=0$ unless $|\eta'|\leqslant|\eta|+2$, $d(\eta')-d(\eta)\leqslant|\eta|$ and

$$\max_{\xi'\in\eta'}|V_{\xi'}|\leqslant\max_{\xi\in\eta}|V_\xi|+2. \tag{4.39b}$$

Bounds.

There is C so that for all $n\geqslant 1$:

$$\sup_{m\in[-1,1]^{\mathbb{Z}^d}}\sup_{|\eta|=n}\sum_{\eta'}\left|M_\gamma^{dg}(\eta,\eta',m)\right|\leqslant Cn. \tag{4.40a}$$

There is $c_0(n,n')$, $(n,n')\in\mathbb{Z}_+^2$, so that

$$\sup_{m\in[-1,1]^{\mathbb{Z}^d}}\sup_{|\eta|=n,d(\eta)=n'}\sum_{\eta'}\left|M_\gamma(\eta,\eta',m)\right|\leqslant c_0(n,n'). \tag{4.40b}$$

Moreover for any $b>0$ there are coefficients $c(n,n')$, $(n,n')\in\mathbb{Z}_+^2$, and positive parameters $b_2^\star$, $b_3^\star$, $\zeta^\star$ such that $b>b_2^\star>b_3^\star>\zeta^\star>0$ and the following holds. Let (b_1,b_2,b_3,ζ) be either equal to $(d/2,b_2^\star,b_3^\star,\zeta^\star)$ or equal to $(0,0,0,0)$, then

$$\sup_{m\in[-1,1]^{\mathbb{Z}^d}}\sup_{|\eta|=n,d(\eta)=n'}N_\gamma(\eta)\sum_{\eta'}|M_\gamma(\eta,\eta',m)|N_\gamma(\eta')^{-1}\leqslant c(n,n')\gamma^\zeta \tag{4.40c}$$

$$N_\gamma(\eta):=\gamma^{-(d/2-b_1)|\eta|-b_2 g(\eta)+b_3 d(\eta)}. \tag{4.40d}$$

The proposition is proven in the appendix by computing the left-hand side of (4.38) with a very careful classification of all the terms which are involved. There are no mathematically sophysticated argument to use, it is only necessary to find out the precise combination of the terms which leads to the desidered cancellations necessary for proving proposition 4.8. To make the proof really readable we have found no other way than to report it in great details. For its considerable length it is shifted to the appendix.

We proceed by proving (2.23), the proof will make clear the need for the many different bounds in the statement of proposition 4.8.

Proof of (2.23). Let $\eta_0 \neq \emptyset$, then, using (4.38) and denoting by $m_t = m_\gamma(\cdot, t|\sigma)$,

$$\mathbb{E}^\gamma_\sigma\Big(\rho_{\eta_0}(\sigma_t, m_t)\Big) = \int_0^t \mathrm{d}s \sum_{\eta'} [M_\gamma(\eta, \eta', m_s) + M^{dg}_\gamma(\eta, \eta', m_s)]\mathbb{E}^\gamma_\sigma\Big(\rho_{\eta'}(\sigma_s, m_s)\Big). \tag{4.41}$$

The term at $t = 0$ is missing because $m_0 = \sigma$, hence $\rho_{\eta_0}(\sigma, m) = 0$ since $\eta_0 \neq \emptyset$.

As our bounds on the coefficients M_γ and M^{dg}_γ do not distinguish η's with the same values of $\big(|\eta|, g(\eta), d(\eta)\big)$, we introduce the following notation:

$$\underline{n} = (n, m, \ell) \tag{4.42a}$$

$$\psi_{\underline{n}}(t) = \sup_{\substack{|\eta|=n, d(\eta)=m\\ g(\eta)=\ell}} \Big|\mathbb{E}^\gamma_\sigma\Big(\rho_{\eta_0}(\sigma_t, m_t)\Big)\Big| \tag{4.42b}$$

$$K_\gamma(\underline{n}, \underline{n}') = \sup_{\substack{|\eta|=n, d(\eta)=m\\ g(\eta)=\ell}} \sum_{\substack{|\eta'|=n', d(\eta')=m'\\ g(\eta')=\ell'}} \sup_{m(\cdot)\in[-1,1]^{\mathbb{Z}^d}} \big|M_\gamma(\eta, \eta', m)\big|. \tag{4.43}$$

Then, using (4.40*a*), we get from (4.41)

$$\Big|\mathbb{E}^\gamma_\sigma\Big(\rho_{\eta_0}(\sigma_t, m_t)\Big)\Big| \leqslant \int_0^t \mathrm{d}s[Cn_0\psi_{\underline{n}_0}(s) + \sum_{\underline{n}_1} K_\gamma(\underline{n}_0, \underline{n}_1)\psi_{\underline{n}_1}(s) \tag{4.44a}$$

having denoted by $\underline{n}_0 = (n_0, m_0, \ell_0)$ the triple $(|\eta_0|, d(\eta_0), g(\eta_0))$.

From (4.44*a*) we get

$$\psi_{\underline{n}_0}(t) \leqslant \int_0^t \mathrm{d}s e^{Cn_0(t-s)} \sum_{\underline{n}_1} K_\gamma(\underline{n}_0, \underline{n}_1)\psi_{\underline{n}_1}(s). \tag{4.44b}$$

As in the proof of theorem 3.6, we split $K_\gamma = K'_\gamma + K''_\gamma$, where $K'_\gamma(\underline{n}, \underline{n}') = 0$ if $\underline{n}' = (0, 0, 0) \equiv \emptyset$ and $K''_\gamma(\underline{n}, \underline{n}') = 0$ if $\underline{n}' \neq \emptyset$. We then iterate (4.44*b*) N times, N as in (4.53) below, and get:

$$\psi_{\underline{n}_0}(t) \leqslant \sum_{p=1}^N I_p + L \tag{4.45a}$$

where I_p is a shorthand for

$$I_p = \int_0^t \mathrm{d}s_1 \ldots \int_0^{s_{p-1}} \mathrm{d}s_p \sum_{\underline{n}_1 \cdots \underline{n}_{p-1}} \exp\left\{C\sum_{i=1}^p n_{i-1}(s_{i-1} - s_i)\right\}\{\prod_{i=1}^{p-1} K'_\gamma(\underline{n}_{i-1}, \underline{n}_i)\}K''_\gamma(\underline{n}_{p-1}, \emptyset) \tag{4.45b}$$

where $s_0 \equiv t$ and

$$L = \int_0^t \mathrm{d}s_1 \ldots \int_0^{s_{N-1}} \mathrm{d}s_N \sum_{\underline{n}_1 \cdots \underline{n}_N} \exp\left\{C\sum_{i=1}^N n_{i-1}(s_{i-1} - s_i)\right\}\{\prod_{i=1}^{N} K'_\gamma(\underline{n}_{i-1}, \underline{n}_i)\}\psi_{\underline{n}_N}(s_N). \tag{4.45c}$$

Observe that

$$n_i \leqslant \bar{n}_i \equiv n_0 + 2i \qquad m_i \leqslant \bar{m}_i \equiv m_0 + i\bar{n}_i \qquad \ell_i \leqslant \bar{\ell}_i \equiv n_i \tag{4.46}$$

where the first two inequalities follow from the support properties of $M_\gamma(\eta, \eta', m)$ (see proposition 4.8), the third one is true by definition, as $g(\eta) \leqslant |\eta|$.

We start by bounding I_p. We divide the terms in the sum over $\underline{n}_1..\underline{n}_{p-1}$ according to the maximum value u attained by the first entry n_i of $\underline{n}_i$ and to j, the label where it is attained, i.e. $n_j = u$, and j is the first label where this happens. Then

$$I_p \leqslant \sum_{u=n_0}^{n_0+2p} \sum_{j=0}^{p-1} \mathrm{e}^{Cut} \frac{t^p}{p!} \sum_{\underline{n}_j, \underline{n}_{p-1}} \mathbf{1}_{n_j=u} (K'_\gamma)^j (\underline{n}_0, \underline{n}_j) (K'_\gamma)^{p-1-j} (\underline{n}_j, \underline{n}_{p-1})\, K''_\gamma(\underline{n}_{p-1}, \emptyset). \tag{4.47a}$$

For $(K'_\gamma)^j(\underline{n}_0, \underline{n}_j)$ we use the bound (4.40*b*), for $(K')^{p-1-j}$ we use (4.40*c*). We have

$$\sum_{\underline{n}_{p-1}} (K'_\gamma)^{p-1-j} (\underline{n}_j, \underline{n}_{p-1})\, K''_\gamma(\underline{n}_{p-1}, \emptyset) = N_\gamma(\underline{n}_j)^{-1} N_\gamma(\emptyset)$$
$$\times \left\{ N_\gamma(\underline{n}_j) \sum_{\underline{n}_{p-1}} (K'_\gamma)^{p-1-j} (\underline{n}_j, \underline{n}_{p-1})\, K''_\gamma(\underline{n}_{p-1}, \emptyset) N_\gamma(\emptyset)^{-1} \right\}$$

having noticed that N_γ, defined in (4.40*d*), depends on η only via the triple $\underline{n}$ associated to η. Since $N_\gamma(\emptyset) = 1$, recalling that by (4.46) $(n_j, m_j, \ell_j) \leqslant (\bar{n}_j, \bar{m}_j, \bar{\ell}_j)$, we have from (4.40*c*) with $b_1 = b_2 = b_3 = 0$

$$\leqslant c^\star(u, m_0 + pu)^{p-j} N_\gamma(\underline{n}_j)^{-1} = c^\star(u, m_0 + pu)^{p-j} \gamma^{du/2} \tag{4.47b}$$

where we have used the notation

$$c^\star(n, m) = \max_{n' \leqslant n, m' \leqslant m} c(n', m') \tag{4.48a}$$

with $c(n, m)$ as in (4.40*c*). We also set

$$c_0^\star(n, m) = \max_{n' \leqslant n, m' \leqslant m} c_0(n', m') \tag{4.48b}$$

with $c_0(n, m)$ as in (4.40*b*) and have

$$I_p \leqslant \sum_{u=n_0}^{n_0+2p} \sum_{j=0}^{p-1} \mathrm{e}^{Cut} \frac{t^p}{p!} c_0^\star(u, m_0 + ju)^j c^\star(u, m_0 + pu)^{p-j} \gamma^{du/2}. \tag{4.49}$$

The leading term in (4.49), as $\gamma \to 0$, is the one with $u = n_0$. There is therefore a constant $c(N, \underline{n}_0)$ such that

$$\sum_{p=1}^{N} I_p \leqslant c(N, \underline{n}_0) \mathrm{e}^{Cn_0t} t^N \gamma^{dn_0/2}. \tag{4.50}$$

We suppose that $aC < d/2$ (and as usual $t \leqslant a \log \gamma^{-1}$), then the right-hand side vanishes satisfying the bound (2.23).

The bound of L, in (4.45a), is analogous. We use (4.40c), in the version with $b_1 = d/2$ and (4.46) to get

$$L \leqslant e^{C(n_0+2N)t} \int_0^t ds_1 ... \int_0^{s_{N-1}} ds_N N_\gamma(\underline{n}_0)^{-1} c^\star\big(n_0 + 2N, m_0 + N(n_0+2N)\big)^N \times \gamma^{\zeta^\star N} \sup_{\underline{n}_N}{}^\star \{N_\gamma(\underline{n}_N)\psi_{\underline{n}_N}(s_N)\}$$

where the $\sup^\star$ is over all those $\underline{n}_N$ whose entries are bounded as in (4.46).

We will prove afterwards that there is $c_{\underline{n}_0,N}$ such that

$$\sup_{t \leqslant a \log \gamma^{-1}} \sup_{\underline{n}_N}{}^\star N_\gamma(\underline{n}_N)\psi_{\underline{n}_N}(t) \leqslant c_{\underline{n}_0,N}. \tag{4.51}$$

Then by (4.51)

$$L \leqslant e^{C(n_0+2N)t} \frac{t^N}{N!} c^\bullet\big(n_0+2N, m_0 + N(n_0+2N)\big)^N c_{\underline{n}_0,N} N_\gamma(\underline{n}_0)^{-1} \gamma^{\zeta^\star N}. \tag{4.52}$$

We then restrict ourselves to $\underline{n}_0 = (n, 0, 0)$, as we are interested in the case $\rho_{\eta_0} = \tilde\sigma(\underline{x}, t)$, $\underline{x} = n$. With this choice $N_\gamma(\underline{n}_0) = 1$ (recall that we are considering (4.40d) with $b_1 = d/2$, $b_2 = b_2^\star$ and $b_3 = b_3^\star$). Then with a such that $2aC < \zeta^\star$, we can finally specify the choice of N, namely such that

$$\zeta^\star N > aC(n+2N) + dn/2. \tag{4.53}$$

Then, from (4.52).

$$L \leqslant c\gamma^{dn/2}$$

with c a suitable constant. This bound together with (4.50) proves (2.23) which is therefore proven modulo (4.51).

Proof of (4.51). Let $\eta = (\underline{x}, \underline{y}, \xi_1, .., \xi_p)$, then

$$\big|\rho_\eta(\sigma, m)\big| \leqslant 2^{|\underline{x}|}\Big\{\prod_{i=1}^p \big|f_{\xi_i}(\sigma) - f_{\xi_i}(m)\big|\Big\}\Big\{\prod_{i=1}^{|\underline{y}|}\Big(\big|\mathcal{A}_{\gamma^{-\alpha}, y_i}(\tilde\sigma(\cdot))\big| + 2|\eta|\gamma^{d\alpha}\Big)\Big\}. \tag{4.54a}$$

(4.54a) is straightforward consequence of the definition (3.21), with the last factor obtained as follows.

We write

$$\underline{z} = (z_1, .., z_\ell);\ \underline{y} = (y_1, .., y_\ell);\ \underline{y}' = (y_1, .., y_{\ell-1});\ \underline{z}' = (z_1, .., z_{\ell-1}).$$

Then, recalling (3.22),

$$\chi_{\underline{x},\underline{y}}(\underline{z}) = \chi_{\underline{x},\underline{y}'}(\underline{z}')\chi_{\underline{x}+\underline{z}', y_\ell}(z_\ell)$$

so that

$$\sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) = \sum_{\underline{z}'} \chi_{\underline{x},\underline{y}'}(\underline{z}')\tilde\sigma(\underline{z}') \sum_z \chi_{\underline{x}+\underline{z}', y_\ell}(z)\sigma(z)$$

and

$$\Big|\sum_z \chi_{\underline{x}+\underline{z}',y_\ell}(z)\tilde\sigma(z)\Big| \leqslant \big|\mathcal{A}_{\gamma^{-\alpha},y_\ell}(\tilde\sigma(\cdot))\big| + (|\underline{x}|+\ell)2\gamma^{d\alpha}$$

hence (4.54a).

We next show that the differences $|f_{\xi_i}(\sigma) - f_{\xi_i}(m)|$ can be also bounded in terms of the averages $\mathcal{A}_{\gamma^{-\alpha},\cdot}(\tilde\sigma(\cdot))$.

By lemma 3.3 there is $c(d(\xi))$ so that, for any ξ, $\xi = (x, \underline{y}_1, \underline{y}_2, \underline{y}_3, \underline{y}^\star)$,

$$|f_\xi(\sigma) - f_\xi(m)| \leqslant c(d(\xi)) \sup_{\epsilon\in\mathbb{R}} \Big|\tanh^{(d(\xi))}(\psi_\sigma+\epsilon) - \tanh^{(d(\xi))}(\psi_m+\epsilon)\Big| \qquad (4.54b)$$

where

$$\psi_\lambda = \beta\sum_{y\notin V_\xi} J_\gamma(x,y)\lambda(y).$$

Since in (4.51) we have a sup*, we may restrict in (4.54b) to $d(\xi) \leqslant \bar m_N = N(n_0+2N)$, by (4.46), and to $|V_\xi| \leqslant 2N$, because in η_0 no ξ is present and, by proposition 4.8, the transitions $\eta \to \eta'$ in $M_\gamma^{dg}(\eta,\eta',m)$ do not increase the max of $|V_\xi|$, see (4.39a), while those due to M_γ make the max of $|V_\xi|$ increase at most by 2, see (4.39b).

There is therefore a constant c, which depends on n_0 and N, such that

$$|f_\xi(\sigma) - f_\xi(m)| \leqslant c\Big|\sum_{y\notin V_\xi} J_\gamma(x,y)\tilde\sigma(y,t)\Big| \leqslant c\gamma^d[2|V_\xi|\|J\|_\infty] + c\Big|\sum_y J_\gamma(x,y)\tilde\sigma(y,t)\Big|$$

and

$$\Big|\sum_y J_\gamma(x,y)\tilde\sigma(y,t)\Big| \leqslant \Big|\sum_y\Big\{\sum_{z\in B_{\gamma^{-\alpha},y}} \frac{1}{|B_{\gamma^{-\alpha}}|} J_\gamma(x,z)\Big\}\tilde\sigma(y,t)\Big| + 2\gamma^{1-\alpha}\|J'\|_\infty.$$

The first term on the right-hand side is equal to

$$\Big|\sum_z J_\gamma(x,z)\mathcal{A}_{\gamma^{-\alpha},z}(\tilde\sigma(\cdot,t))\Big| \leqslant c\gamma^b \qquad (4.54c)$$

and the inequality, by proposition 4.6, holds for all $t \leqslant a\log\gamma^{-1}$, with probability that vanishes faster than any power of γ, as $\gamma \to 0$. Observe that we are not supposing that $|x| \leqslant (k^\star-1)\gamma^{-2}$, however the processes

$$\Big\{\sum_z J_\gamma(x,z)\mathcal{A}_{\gamma^{-\alpha},z}(\tilde\sigma(\cdot,t)),\ t\geqslant 0\Big\}$$

starting from σ_0 and

$$\Big\{\sum_z J_\gamma(0,z)\mathcal{A}_{\gamma^{-\alpha},z}(\tilde\sigma(\cdot,t)),\ t\geqslant 0\Big\}$$

starting from σ_0', $\sigma_0'(y) = \sigma_0(y+x)$, have the same law. For the latter we can apply proposition 4.6. The same argument applies to the other averages in (4.54a), we thus conclude that with probability that vanishes faster than any power of γ, $\rho_\eta(\sigma_t, m_t)$ is bounded proportionally to $\gamma^{bg(\eta)}$ with a proportionality constant which depends, via (4.46), only on n_0 and N. The factor $N_\gamma(\eta)$ in (4.51) has an exponential factor $\gamma^{-b_2^\star g(\eta)}$, so that by choosing $b_2^\star$ equal to b in (4.54c) we obtain a bound independent of γ, in the set where (4.54c) holds. The complement of this set has a probability which vanishes faster than any power of γ, hence the proof of (4.51) is completed. □

Proof of theorem 2.3.4. We have already proven (2.23), so that (i) is proven.

Proof of (ii). the same proof applies to the case when the process starts from a product measure μ^γ and $m_\gamma(x,t|\sigma)$ (that enters in the definition of the v-function) is replaced by $m_\gamma(x,t|\mu^\gamma)$, see definition 2.3.1 for notation.

If $m_\gamma(x,t|\sigma)$ is replaced by $m_\gamma(\gamma x,t)$, as in the statement (ii), we use proposition 4.7.2 to prove that

$$\sup_{t\leqslant a\log\gamma^{-1}}\sup_x\left|m_\gamma(x,t|\mu^\gamma)-m_\gamma(\gamma x,t)\right|\leqslant c_2'\gamma^{-ac_2+1}.$$

Hence for a small enough (2.23) remains valid even with $m_\gamma(\gamma x,t)$ in the definition of the v-functions. The proof of (ii) is therefore completed.

Proof of (iii). We have just proved the validity of (2.23) with $m_\gamma(\gamma x,t)$, $m_\gamma(r,t)$ being the solution of (1.1) with initial datum $m_\gamma(r,0)=m_0(\gamma r)$, see (2.12$a$) and the definition of the standard initial state, (remarks after definition 2.1.7). Since $m_0\in C^1(\mathbb{R}^d)$, by assumption, and $\|m_0'\|_\infty<\infty$, we have that for some c

$$\|m_\gamma(\cdot,0)-m_0(\cdot)\|_\infty\leqslant c\gamma. \tag{4.55a}$$

Then by proposition 4.7.5

$$\|m_\gamma(\cdot,t)-m(\cdot,t))\|_\infty\leqslant c\mathrm{e}^{c_5t}\gamma \tag{4.55b}$$

so that if $t\leqslant a\log\gamma^{-1}$ and $a>0$ is small enough, (2.23) holds with $m(\gamma x,t)$ replacing $m_\gamma(x,t|\mu^\gamma)$ in the definition of the v-functions. We write

$$\prod_{i=1}^n\sigma(x_i,t)=\prod_{i=1}^n\left\{[\sigma(x_i,t)-m(\gamma x_i,t)]+m(\gamma x_i,t)\right\} \tag{4.56a}$$

so that

$$\prod_{i=1}^n\sigma(x_i,t)-\prod_{i=1}^n m(\gamma x_i,t)=\sum_{\substack{I\subset\{1,\ldots,n\}\\ I\neq\emptyset}}\Big\{\prod_{i\in I}[\sigma(x_i,t)-m(\gamma x_i,t)]\Big\}\Big\{\prod_{i\notin I}m(\gamma x_i,t)\Big\}. \tag{4.56b}$$

By taking the expectation and using the version of (2.23) proved with $m(\gamma x,t)$, we then obtain (2.24) so that theorem 2.3.4 is proved. □

We actually have stronger results:

Theorem 4.9. *Let (m_0,μ^γ) be an initial state in the sense of definition 2.1.7 and assume that there are $0<\alpha<1$ and $b'>0$ such that, for all γ small enough*

$$\|m_{\gamma,0}^{(\alpha,\gamma)}-m_0^{(\alpha,\gamma)}\|_\infty\leqslant\gamma^{b'} \tag{4.57}$$

where $m_{\gamma,0}^{(\alpha,\gamma)}$ is the block spin transform of $m_{\gamma,0}$, the latter being as in (2.12a), while $m_0^{(\alpha,\gamma)}$ is the block spin image of m_0.

Then there are $a>0$ and given any $\tau>0$ there is $b>0$ so that the following holds. For any n and k^* there is c so that

$$\mathbb{P}_{\mu^\gamma}^\gamma\left(\left\{\sigma_0:\mathbb{P}_{\sigma_0}^\gamma\Big(\sup_{|r|\leqslant k^*\gamma^{-1}}\left|\sigma_{\gamma,\tau\log\gamma^{-1}}^{(\alpha)}(r)-m(r,\tau\log\gamma^{-1})\right|>\gamma^b\Big)\leqslant c\gamma^n\right\}\right)\geqslant 1-c\gamma^n \tag{4.58}$$

where $m(r,t)$ is the solution of (1.1) starting from m_0.

Remark. the two examples mentioned in the remarks following definition 2.1.7 fulfil the conditions of theorem 4.9.

Proof. By (2.13*b*), (4.58) holds with $m(\tau \log \gamma^{-1})$ replaced by $m_\gamma^{(\alpha,\gamma)}(r, \tau \log \gamma^{-1})$ which is the block spin transform of $m_\gamma(\cdot, \tau \log \gamma^{-1})$, solution of (1.1) starting from $m_{\gamma,0}$. Denoting by $m^{(\alpha,\gamma)}(r, t)$ the block spin transform of $m(\cdot, t)$, by proposition 4.7.6 and (4.57) we can then replace $m_\gamma^{(\alpha,\gamma)}$ by $m^{(\alpha,\gamma)}(r, \tau \log \gamma^{-1})$, their difference being bounded by $c_6' e^{c_6 \tau \log \gamma^{-1}} c\gamma$, where $c\gamma$ bounds their difference at time 0. By proposition 4.7.8, $m^{(\alpha,\gamma)}(r, \tau \log \gamma^{-1})$ is, for any given τ, $\gamma^{b''}$, $b'' > 0$, close to $m(r, \tau \log \gamma^{-1})$ (if $\tau < a$ and a is suitably small). theorem 4.9 is therefore proven. □

5. Motion by mean curvature

In this section we restrict ourselves to ferromagnetic interactions, $J \geqslant 0$, in the absence of magnetic fields, $h = 0$, and below the critical temperature, $\beta > 1$, having imposed the normalization condition $\int \mathrm{d}r\, J = 1$.

The proof of theorem 2.2.4 is based on an iterative procedure similar to that used in [18] to prove convergence to the motion by mean curvature in the Glauber + Kawasaki spin dynamics. The single steps of the iteration exploit the results in [9] and the previous estimates on the Glauber dynamics valid for $t \leqslant a \log \gamma^{-1}$.

We start by recalling some definitions and results in [9] that we will use in the following.

Definition 5.1. *The instanton* $\bar{m} : \mathbb{R} \to [-1, 1]$ *is an antisymmetric, strictly increasing function such that* $m^{(d)}(r) := \bar{m}(r_1)$ (r_1 *the first coordinate of* r) *is a stationary solution of (1.1). The instanton solution exists and it is unique, see [7, 9, 10] and [11], furthermore*

$$\lim_{r_1 \to \pm\infty} \bar{m}(r_1) = \pm m_\beta, \quad \textit{exponentially fast} \tag{5.1}$$

with m_β *as in (2.14).*

In [9] it is proven that under the macroscopic scaling defined in (2.17) the mesoscopic equation (1.1) gives rise to the motion by mean curvature of definition 2.2.3 with parameter

$$\theta = N \int \mathrm{d}r_1 \bar{m}'(r_1)^2 \beta \int \mathrm{d}r_1' dr_\perp J\big(|(r_1' - r_1)^2 + r_\perp^2|^{1/2}\big) \frac{\bar{m}'(r_1')}{\bar{m}'(r_1)} r_\perp^2 / 2 \tag{5.2a}$$

$r_\perp$ denoting the vector in the plane perpendicular to the first coordinate axis; N is the normalization constant:

$$N^{-1} = \int_{\mathbb{R}} \mathrm{d}r_1 \frac{\bar{m}'(r_1)^2}{1 - \bar{m}(r_1)^2}. \tag{5.2b}$$

In [9] as well as in the analysis of the Allen–Cahn equation, see for instance [4], a fundamental role is played by the ferromagnetic inequalities. At the level of the mesoscopic equation (1.1), they say that if $\tilde{m}(r, t)$ and $m(r, t)$ solve (1.1) and $\tilde{m}(r, 0) \geqslant m(r, 0)$ for all r, then $\tilde{m}(r, t) \geqslant m(r, t)$ for all r and t. Here it is essential the assumption that $J \geqslant 0$. A variant of the above inequality is based on the notion of super- and subsolutions of (1.1), namely functions which satisfy (1.1) with $\geqslant$ instead of $=$, for the supersolutions, and with $\leqslant$ for the subsolutions. Such functions are then respectively upper and lower bounds for the true solution, if they are so at time 0. The whole game is then to find super- and subsolutions which, in the limit $\gamma \to 0$, squeeze (in macroscopic coordinates) the true solution toward a function which describes the interface moving by mean curvature. This is what done in [9], here we describe the supersolution, as the subsolution is defined symmetrically.

Definition 5.2.1. *Let Σ denote a C^∞ surface which is the boundary of a compact, connected region $\Lambda \subset \mathbb{R}^d$. Let $d(\xi, \Sigma)$, $\xi \in \mathbb{R}^d$ be the distance of ξ from Σ, if $\xi \in \Lambda$, and minus the distance, if $\xi \notin \Lambda$. The signed distance in mesoscopic coordinates is then $d(r, \lambda^{-1}\Sigma) = \lambda^{-1} d(\xi, \Sigma)$, where $\xi = \lambda r$.*

Definition 5.2.2. *For each λ we set*

$$m^{+}_{(\Sigma,\lambda)}(r) = \bar{m}\big(d(r, \lambda^{-1}\Sigma)\big) \qquad \text{for } |d(r, \lambda^{-1}\Sigma)| \leqslant R_0^\star \log \lambda^{-1} \tag{5.3a}$$

with $R_0^\star > 0$ the constant R_0 defined in (34a) of [9]. The definition of $m^{+}_{(\Sigma,\lambda)}(r)$ is then completed by setting

$$m^{+}_{(\Sigma,\lambda)}(r) = \pm m_\beta + \lambda^{3/2} \qquad \text{for } d(r, \lambda^{-1}\Sigma) \gtrless \pm R_0^\star \log \lambda^{-1} \tag{5.3b}$$

The definition of $m^{-}_{(\Sigma,\lambda)}(r)$ differs from the above one only in (5.3*b*), where we subtract, instead of adding, $\lambda^{3/2}$. $m^{\pm}_{(\Sigma_\tau,\lambda)}(r)$ are then super- and subsolutions provided Σ_τ moves according to the following:

Definition 5.3. *Given λ and $\bar\tau > 0$, we consider the two biased motions by mean curvature $\Sigma^{\pm}_{(\tau,\lambda)}$, in the time interval $0 \leqslant \tau \leqslant \bar\tau$, $\bar\tau > 0$, defined as follows. For each τ, $\Sigma^{\pm}_{(\tau,\lambda)}$ is a connected C^∞ surface which has a C^∞ parametrization*

$$\xi^{\pm} = \xi^{\pm}(\tau, \xi_0, \lambda),\ \xi_0 \in S_0,\ S_0\ a C^\infty,\ d-1 \text{ dimensional, compact manifold}$$

and

$$\frac{\mathrm{d}\xi^{\pm}}{\mathrm{d}\tau} = \theta\kappa\nu \mp h\nu. \tag{5.4}$$

We hereafter fix θ as in (5.2) and $h = \lambda^{\delta/2}$, where $\delta > 0$ is the same as in section 8 of [9]. We next state a classical result on parabolic equations, see [2] and references therein:

Theorem 5.4. *Let Σ_τ, $0 \leqslant \tau \leqslant \tau^\star$, be a motion by mean curvature as in definition 2.2.3 and let $\xi = \xi(\tau, \xi_0)$, $\xi_0 \in S_0$, be the corresponding parametrization. Then, given any $R_0 > 0$ (we shall use the result with $R_0^\star$ as in 5.2.2), for all λ small enough,*

$$\Sigma^{\pm}_{(0,\lambda)} := \{\xi : d(\xi, \Sigma_0) = \mp 2R_0 \lambda \log \lambda^{-1}\}$$

is also a C^∞ surface. Moreover, for any $\bar\tau < \tau^\star$, for all h small enough and for all λ small enough, there is a h-biased motion by curvature $\Sigma^{\pm}_{(\tau,\lambda)}$, $0 \leqslant \tau \leqslant \bar\tau$, in the sense of definition 5.3, starting at $\tau = 0$ from $\Sigma^{\pm}_{(0,\lambda)}$ and with parametrization $\xi^{\pm}(\tau, \xi_0, \lambda)$, where $\xi_0 \in S_0$ and S_0 is the same manifold used for the parametrization of Σ_τ. Furthermore there is c, that depends only on $\bar\tau$ and Σ_0, so that

$$\left|\xi^{\pm}(\xi_0, \tau, \lambda) - \xi(\xi_0, \tau)\right| \leqslant ch \qquad 0 \leqslant \tau \leqslant \bar\tau. \tag{5.5}$$

We next recall a result proven in [9]:

Theorem 5.5. *Let Σ_τ, $0 \leqslant \tau \leqslant \tau^*$, as in theorem 5.4 and δ as below definition 5.3. Then there is $\omega > 0$ and c so that for all $u \in [1, 2]$ the following holds.*

Recalling definition 5.2.2, let $\bar{\tau}$ be as in theorem 5.4 and for all λ small enough and all $0 \leqslant s \leqslant \lambda^{-2}\bar{\tau}$, let

$$m_\lambda^\pm(r, s) := m_{(\Sigma,\lambda)}^\pm(r) \qquad \Sigma \equiv \Sigma_{(\lambda^2 s,\lambda)}^\pm \tag{5.6a}$$

where $\Sigma_{(\tau,\lambda)}^\pm$ are the motions by curvature with bias $h = \lambda^{\delta/2}$ starting from $\Sigma_{(0,\lambda)}^\pm$ as in theorem 5.4. If $m_\lambda(r, t)$ solves (1.1) for $t \geqslant s$ and

$$m_\lambda^-(r, s) \leqslant m_\lambda(r, s) \leqslant m_\lambda^+(r, s) \qquad \text{for all } r \in \mathbb{R}^d \tag{5.6b}$$

then

$$m_\lambda^-(r, s + u\lambda^{-\delta}) + \lambda^\omega \leqslant m_\lambda(r, s + u\lambda^{-\delta}) \leqslant m_\lambda^+(r, s + u\lambda^{-\delta}) - \lambda^\omega. \tag{5.7a}$$

By repeated use of theorem 5.5 with suitably different values of u we also have that for all t such that $s + \lambda^{-\delta} \leqslant t \leqslant \lambda^{-2}\bar{\tau}$,

$$m_\lambda^-(r, t) + \lambda^\omega \leqslant m_\lambda(r, t) \leqslant m_\lambda^+(r, t) - \lambda^\omega. \tag{5.7b}$$

We have now all the ingredients for proving theorem 2.2.4. The idea of the proof is actually rather simple. We know already that for times $a \log \gamma^{-1}$, a small, the spins and the solution of (1.1) remain close (with large probability) if they are so initially. As the error at the final time $a \log \gamma^{-1}$ is of the order of γ^b, $b \geqslant 0$, we can then exploit the (much larger) extra term $\lambda^\omega \equiv (\log \gamma^{-1})^\omega$ in (5.7) to bound it and to conclude that the spin configuration at this time is squeezed between the sub- and supersolutions $m_\lambda^\pm$, if it was so initially. This property is obviously preserved under finitely many iterations and since the sub- and supersolutions disagree only by $\lambda^{3/2}$ outside of a macroscopically infinitesimal strip around the moving interface Σ_τ, we then obtain the result stated in theorem 2.2.4.

When carrying out this strategy of proof we meet two kinds od complications, that we have already met in section 4. The first one is an ultraviolet problem, and it is dealt with by using averages, as we do not have a control in sup-norm of the difference between the spins and the solution of (1.1). The second one is an infrared problem, we do not have closeness of the averages everywhere so we need to control the propagation of the errors unavoidably present at large distances.

Let $\tau > 0$ be as in theorem 2.2.4. We then take $\bar{\tau} = \tau$ in theorem 5.5. Let α, ζ, a and b as in theorem 2.1.6, with $a > 0$ so small that (5.21) below holds and such that $\tau/a =: N$ is a positive integer. Setting

$$s_n = na \log \gamma^{-1} \qquad R_n = (N + 1 - n)\gamma^{-1} \qquad n \text{ a non} - \text{negative integer} \tag{5.8}$$

we first define for any $n \geqslant 0$ and any spin trajectory $(\sigma_t)_{t \geqslant 0}$ $m_{(n)}(r, t)$ to be the solution of (1.1) for $t \geqslant s_n$ and such that $m_{(n)}(r, s_n) = \sigma_{\gamma, s_n}(r)$, for all $r \in \mathbb{R}^d$, see (2.8c) for notation. Then we introduce the set

$$\mathcal{G}_n = \left\{ (\sigma_t)_{t \geqslant 0} : \left| \sigma_{\gamma,t}^{(\alpha)}(r) - m_{(n)}^{(\alpha,\gamma)}(r, t) \right| \leqslant \gamma^b, \quad s_n \leqslant t \leqslant s_{n+1}, \quad |r| \leqslant R_n \right\} \tag{5.9a}$$

with $m_{(n)}^{(\alpha,\gamma)}$ as in definition 2.1.5. We finally set

$$\mathcal{G} = \left\{\bigcap_{n=0}^{N-1} \mathcal{G}_n\right\} \bigcap \left\{\left|\sigma_{\gamma,0}^{(\alpha)}(r) - m_{\gamma,0}^{(\alpha,\gamma)}(r)\right| \leqslant \gamma^b, \quad |r| \leqslant R_0\right\} \tag{5.9b}$$

with $m_{\gamma,0}$ as in (2.12a).

By theorem 2.1.6, for any k there is c so that

$$\mathbb{P}^{\gamma}_{\mu^{\epsilon}}(\mathcal{G}) \geqslant 1 - c\gamma^k. \tag{5.10}$$

We will prove theorem 2.2.4 by showing that for all γ small enough, any spin trajectory in $\mathcal{G}$ is in the set appearing on the right-hand side of (2.19a).

We first prove a weaker version with the sup over t in (2.19a) restricted to $t \notin (s_n, s_n + \lambda^{-\delta})$, for any $0 \leqslant n \leqslant N-1$. For $n \geqslant 1$ we define

$$\psi_{(n)}(r) = \begin{cases} m_{(n-1)}(r, s_n) & \text{if } |r| \leqslant R_n \\ -m_\beta & \text{otherwise} \end{cases} \tag{5.11}$$

and $\psi_{(n)}(r,t)$, $t \geqslant s_n$, as the solution of (1.1) such that $\psi_{(n)}(r, s_n) = \psi_{(n)}(r)$, for all $r \in \mathbb{R}^d$. We also define $\psi_{(0)}(r) = m_{\gamma,0}(r)$ for all $r \in \mathbb{R}^d$ and $\psi_{(0)}(r,t)$ the solution of (1.1) for $t \geqslant 0$ with initial datum $\psi_{(0)}(\cdot)$.

Observe that the definition of the $\psi_{(n)}$'s depends upon the trajectory $(\sigma_t)_{t\geqslant 0}$. We start by proving that in $\mathcal{G}$, for all $0 \leqslant n \leqslant N-1$,

$$m_\lambda^-(\cdot, s_n) \leqslant \psi_{(n)}(\cdot) \leqslant m_\lambda^+(\cdot, s_n). \tag{5.12}$$

We prove (5.12) by induction. Since it is evidently true for $n = 0$, we need only show that if (5.12) holds for $n < N-1$, then it also holds for $n+1$. By (5.7b)

$$m_\lambda^-(r, s_{n+1}) + \lambda^\omega \leqslant \psi_{(n)}(r, s_{n+1}) \leqslant m_\lambda^+(r, s_{n+1}) - \lambda^\omega. \tag{5.13}$$

Then (5.12) with $n+1$ follows from (5.13) and the following bound, that we will prove next. There are $\vartheta > 0$ and c so that

$$\left|m_{(n)}(r, s_{n+1}) - \psi_{(n)}(r, s_{n+1})\right| \leqslant c\gamma^\vartheta \qquad \text{for all } |r| \leqslant R_{n+1}. \tag{5.14}$$

Analogously to (5.11) we define:

$$z_{(n)}(r) = \begin{cases} \sigma_{\gamma,s_n}(r) & \text{if } |r| \leqslant R_n \\ -m_\beta & \text{otherwise} \end{cases} \tag{5.15}$$

and $z_{(n)}(r,t)$, $t \geqslant s_n$, as the solution of (1.1) with $z_{(n)}(\cdot, s_n) = z_{(n)}(\cdot)$. Then, by proposition 4.7.4, for any k there is c so that

$$\left|z_{(n)}(r,t) - m_{(n)}(r,t)\right| \leqslant c\gamma^k \qquad |r| \leqslant R_{n+1}, \quad s_n \leqslant t \leqslant s_{n+1} \tag{5.16a}$$

$$\left|z_{(n)}^{(\alpha,\gamma)}(r,t) - m_{(n)}^{(\alpha,\gamma)}(r,t)\right| \leqslant c\gamma^k \qquad |r| \leqslant R_{n+1}, \quad s_n \leqslant t \leqslant s_{n+1}. \tag{5.16b}$$

By the definition of $\mathcal{G}$,

$$\left|z_{(n)}^{(\alpha,\gamma)}(r)-\psi_{(n)}^{(\alpha,\gamma)}(r)\right|\leqslant\gamma^{b}\qquad r\notin D,\ \ D=\{r:R_n-\gamma^{1-\alpha}\leqslant|r|\leqslant R_n+\gamma^{1-\alpha}\}. \tag{5.17}$$

We then apply proposition 4.7.6 to $z_{(n)}$ and $\psi_{(n)}$, with α and b and D as above, $u=1-\alpha$ and c' a suitable constant. Therefore

$$\left|z_{(n)}^{(\alpha,\gamma)}(r,t)-\psi_{(n)}^{(\alpha,\gamma)}(r,t)\right|\leqslant c_6'\mathrm{e}^{c_6(t-s_n)}[\gamma^b+\gamma^{1-\alpha}+c'\gamma^{1-\alpha}]$$
$$|r|\leqslant R_{n+1},\quad s_n\leqslant t\leqslant s_{n+1}. \tag{5.18}$$

On the other hand, by proposition 4.7.8, for all $r\in\mathbb{R}^d$,

$$\left|z_{(n)}^{(\alpha,\gamma)}(r,s_{n+1})-z_{(n)}(r,s_{n+1})\right|\leqslant c_8\gamma^{1-\alpha}+2\gamma^{a}. \tag{5.19}$$

The analogous inequality holds for $\psi_{(n)}$ as well. Hence, from (5.18), (5.19) and (5.16*a*)

$$\left|m_{(n)}(r,s_{n+1})-\psi_{(n)}(r,s_{n+1})\right|\leqslant c\gamma^k+2[c_8\gamma^{1-\alpha}+2\gamma^a]$$
$$+c_6'\mathrm{e}^{c_6a\log\gamma^{-1}}[\gamma^b+\gamma^{1-\alpha}+c'\gamma^{1-\alpha}]\qquad|r|\leqslant R_{n+1}. \tag{5.20}$$

We choose a as

$$a\leqslant c_6^{-1}\min\{1-\alpha,b\}. \tag{5.21}$$

Then (5.14) holds with $\vartheta>0$ and with $c\gamma^{\vartheta}$ an upper bound for the right-hand side of (5.20). We have thus completed the proof of the induction and of (5.12).

By (5.16) and (5.18), we have for any $0\leqslant n\leqslant N-1$,

$$\left|m_{(n)}^{(\alpha,\gamma)}(r,t)-\psi_{(n)}^{(\alpha,\gamma)}(r,t)\right|\leqslant c\gamma^k+c_6'\mathrm{e}^{c_6(t-s_n)}[\gamma^b+\gamma^{1-\alpha}+c'\gamma^{1-\alpha}]$$
$$|r|\leqslant R_{n+1},\quad s_n\leqslant t\leqslant s_{n+1}. \tag{5.22}$$

By (5.12) and (5.7*b*)

$$m_\lambda^-(r,t)+\lambda^\omega\leqslant\psi_{(n)}(r,t)\leqslant m_\lambda^+(r,t)-\lambda^\omega\qquad s_n+\lambda^{-\delta}\leqslant t\leqslant s_{n+1} \tag{5.23}$$

Hence, by (5.22) and (5.23), there are c' and $b'>0$ so that

$$\left(m_\lambda^-\right)^{(\alpha,\gamma)}(r,t)+\lambda^\omega-c'\gamma^{b'}\leqslant m_{(n)}^{(\alpha,\gamma)}(r,t)\leqslant\left(m_\lambda^+\right)^{(\alpha,\gamma)}(r,t)-\lambda^\omega+c'\gamma^{b'}$$
$$s_n+\lambda^{-\delta}\leqslant t\leqslant s_{n+1},\quad|r|\leqslant R_{n+1} \tag{5.24a}$$

and, by (5.9),

$$\left(m_\lambda^-\right)^{(\alpha,\gamma)}(r,t)+\lambda^\omega-c'\gamma^{b'}-\gamma^b\leqslant\sigma_{\gamma,t}^{(\alpha)}(r)\leqslant\left(m_\lambda^+\right)^{(\alpha,\gamma)}(r,t)-\lambda^\omega+c'\gamma^{b'}+\gamma^b$$
$$s_n+\lambda^{-\delta}\leqslant t\leqslant s_{n+1},\quad|r|\leqslant R_{n+1}. \tag{5.24b}$$

For γ small enough $\lambda^{\omega} \geqslant c'\gamma^{b'} + \gamma^{b}$. Then for $t \leqslant \lambda^{-2}\bar{\tau}$, $\left|d\left(r, \lambda^{-1}\Sigma^{-}_{(\lambda^2 t,\lambda)}\right)\right| > 2R_0^* \log\lambda^{-1}$ and for all γ small enough

$$\left(m_{\lambda}^{-}\right)^{(\alpha,\gamma)}(r,t) + \lambda^{\omega} - c'\gamma^{b'} - \gamma^{b} \geqslant \pm m_{\beta} - \lambda^{3/2} \tag{5.25}$$

for r inside, respectively outside, $\lambda^{-1}\Sigma^{-}_{(\lambda^2 t,\lambda)}$.

By (5.5), expressed in mesoscopic coordinates,

$$\left|d\left(r, \lambda^{-1}\Sigma^{-}_{\lambda^2 t,\lambda}\right)\right| \geqslant \left|d\left(r, \lambda^{-1}\Sigma_t\right)\right| - c\lambda^{-1}h. \tag{5.26}$$

Hence (5.25) holds for all r such that

$$\left|d\left(r, \lambda^{-1}\Sigma_t\right)\right| \geqslant 2R_0^* \log\lambda^{-1} + c\lambda^{-1}h \geqslant 2c\lambda^{-1+\delta/2} \tag{5.27}$$

(for all γ small enough). Recall that $h = \lambda^{\delta/2}$.

By (5.24*b*), for all r as in (5.27) and all γ small enough we prove the lower bound in the following inequality

$$\pm m_{\beta} - \lambda^{3/2} \leqslant \sigma^{(\alpha)}_{\gamma,t}(r) \leqslant \pm m_{\beta} + \lambda^{3/2} \tag{5.28}$$

The upper bound is proven by similar arguments. Recall that (5.28) holds for all t as in (5.23). We choose $\zeta < \delta/2$ in (2.19*a*) so that we have the desired estimate, but only at the times considered in (5.23). We can however repeat the previous proof with suitably different values of a, we derive the bound (5.28) for all $\lambda^{-\delta} \leqslant t \leqslant \lambda^{-2}\bar{\tau}$. In the 'short' time interval $[0, \lambda^{-\delta})$ we can afford a very rough estimate: by proposition 4.7.4, in fact, for any k there is c, so that ($\psi_{(0)}(r,t)$ being defined below (5.11))

$$\left|\psi_{(0)}(r,t) \pm m_{\beta}\right| \leqslant c\gamma^{k} \qquad t \leqslant \lambda^{-\delta},\ \ d(r, \lambda^{-1}\Sigma_0) \geqslant \lambda^{-2\delta} \tag{5.29}$$

for r respectively outside and inside $\lambda^{-1}\Sigma_0$. By theorem 2.1.6 and choosing ζ in (2.19*a*) so that $1 - \zeta > 2\delta$ we then complete the proof of theorem 2.2.4. □

Acknowledgments

Two of us, A.DM and E.P, acknowledge very kind hospitality at the Isaac Newton Institute for Mathematical Sciences, University of Cambridge.

Notes added. Katsoulakis and Souganidis in a recent paper [25] have generalized our results of subsection 2.2 by proving that the interface dynamics in the limit is ruled at all times by the 'generalized motion by mean curvature'. The macroscopic scaling parameter λ, involved in this result, has the form $\lambda = \gamma^{\zeta}$, $\zeta > 0$ and small enough.

The global (in time) convergence to a motion by mean curvature in $d = 2$ by scaling (1.1) has been proven earlier by Buttà, [26].

In this case the only singularities which develop from an initially regular surface are due to its disappearence.

Appendix

In this appendix we prove proposition 4.8. The notation is taken from section 3 and the strategy of proof is similar to that of proposition 3.6 but considerably more complex.

When we compute $\sum_x D^{(x)}\rho_\eta(\sigma, m)$ we find many terms, we first group them together according to the values of x in 3 classes giving rise to an expression of the form $\sum_i T_i$, with $i = 1, 2, 3$. The terms in each T_i are still too many to be estimate conveniently and after some manipulation we write $T_i = \sum_j T_{(i,j)}$. We keep doing this kind of operations till we have a decomposition where each T_κ, κ a multi-index, is such that

$$T_\kappa = \sum_{\eta'} M_\kappa^{dg}(\eta, \eta', m)\rho_{\eta'} \tag{A.1a}$$

or

$$T_\kappa = \sum_{\eta'} M_\kappa(\eta, \eta', m)\rho_{\eta'} \tag{A.1b}$$

with M_κ^{dg} and M_κ having the right support properties as in proposition 4.8 and satisfying the bounds (4.40). Since the number of subcases necessary to reach the situations (A.1a) and (A.1b) are finite, we will then have proved proposition 4.8.

The above classification has a branching structure, each branch ends when we reach an expression compatible with (A.1). Observe that we are not making explicit the dependence on γ which is however present in M_κ^{dg} and M_κ.

Notation. *We write*

$$\eta = (\underline{x}, \underline{y}, \xi_1, .., \xi_p); \quad \xi_i = \left(\hat{x}_i, \underline{\hat{y}}_{i,1}, \underline{\hat{y}}_{i,2}, \underline{\hat{y}}_{i,3}, \underline{\hat{y}}_i^*\right) \tag{A.2}$$

and denote by V_{ξ_i} the collection of all the sites in ξ_i, see definition 3.2.

The set $\underline{y}$, which is the second entry in η, will be written as $\underline{y} = (y_1, .., y_\ell)$; we also denote $\underline{z} = (z_1, .., z_\ell)$ and recall that

$$\chi_{\underline{x},\underline{y}}(\underline{z}) = \prod_{i=1}^{\ell} \left\{ \frac{1}{|B_{\gamma^{-\alpha}}|} 1(z_i \in B_{\gamma^{-\alpha}, y_i} - \underline{x}) \right\} \tag{A.3}$$

with $A \pm B$ being the union, respectively the difference of the sets A and B.

We will also use the shorthand notation

$$F_i := f_{\xi_i}(\sigma) - f_{\xi_i}(\bar{m}). \tag{A.4}$$

As already mentioned, we start with

$$\sum_x D^{(x)}\rho_\eta(\sigma, m) = \sum_{i=1}^{3} T_{(i)} \tag{A.5a}$$

where the terms $T_{(i)}$ are identified by the following identity with the labelling corresponding to the order of appearance:

$$\sum_x D^{(x)}\rho_\eta = \sum_{x\in\underline{x}} D^{(x)}\rho_\eta + \tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\sum_{x\in\underline{z}} D^{(x)}[\tilde\sigma(\underline{z})\prod_{i=1}^p F_i]$$
$$+\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})\sum_{x\notin\underline{x}+\underline{z}} D^{(x)}\prod_{i=1}^p F_i. \tag{A.5b}$$

<u>*Analysis of $T_{(1)}$.*</u> We have

$$T_{(1)} = T_{(1,1)} + T_{(1,2)} \tag{A.6a}$$

where, recalling (3.28) and (3.30*d*)-(3.30*e*)

$$T_{(1,1)} = \sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\sum_{x\in\underline{x}}[\prod_{i=1}^p\Gamma^i_{1,x}]\tilde\sigma(\underline{x}+\underline{z}-x)D^{(x)}\tilde\sigma(x) \tag{A.6b}$$

$$T_{(1,2)} = \sum_{x\in\underline{x}}\left\{\tilde\sigma(\underline{x}-x)\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})\sum_{\substack{\underline{i}_1\ldots\underline{i}_5\\ |\underline{i}_1|<p}}\Delta_{\underline{i}_1,\ldots\underline{i}_5,\underline{x},x}\right\}. \tag{A.6c}$$

We start from $T_{(1,1)}$. We write $D^{(x)}\sigma(x)$ using (3.26) and we get

$$T_{(1,1)} =: T_{(1,1,1)} + T_{(1,1,2)} \tag{A.7a}$$

$$T_{(1,1,1)} = -\sum_{x\in\underline{x}}\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{i=1}^p\Gamma^i_{1,x}]. \tag{A.7b}$$

$T_{(1,1,2)}$ is defined in (A.8) below. The right-hand side of (A.7*b*) can already be written as

$$T_{(1,1,1)} = \sum_{\eta'} M^\star_{(1,1,1)}(\eta,\eta',m)\rho_{\eta'}.$$

The sum is restricted to $|\eta'| = |\eta|$, $g(\eta') = g(\eta)$ and $d(\eta') = d(\eta)$. Moreover

$$\sum_{\eta'}|M^\star_{(1,1,1)}(\eta,\eta',m)| \leqslant |\underline{x}|$$

but

$$\eta' = (\underline{x},\underline{y},\xi'_1,..\xi'_p) \qquad \textit{with } \xi'_i = \Gamma^i_{1,x}.$$

Hence, according to the value of x, it may be that $|V_{\xi'_i}| = |V_{\xi_i}| + 1$. Thus $M^\star$ is neither of the form M_κ (4.40*c*) is not satisfied with $\zeta = \zeta^\star$) nor M^{dg}_κ, ((4.39*a*) does not hold).

To solve this problem we go back to (3.28*a*) that we read as a relation which expresses $\Gamma^i_{1,x}$ in terms of F_i and $\Gamma^i_{2,x}$,..,$\Gamma^i_{5,x}$. We then distinguish the term with F_i for all i and the remaining ones, thus writing

$$T_{(1,1,1)} = T_{(1,1,1,1)} + T_{(1,1,1,2)} \qquad T_{(1,1,1,1)} = -|x|\rho_\eta \tag{A.7d}$$

$$T_{(1,1,1,2)} = \sum_{x\in\underline{x}} \Big\{ \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|+1} \times \gamma^{d(|\underline{i}_2|+\dots+2|\underline{i}_5|)} \Big[\prod_{i\in\underline{i}_1} F_i \Big] \Gamma_{2,x}^{\underline{i}_2} \dots, \Gamma_{5,x}^{\underline{i}_5} \tilde\sigma(x)^k \Big] \Big\} \tag{A.7e}$$

with $k = |i_3| + |i_4| + 1$.

We rewrite $\tilde\sigma(x)^k$ using (3.32) and distinguish the terms with and without $\tilde\sigma(x)$, thus obtaining $T_{(1,1,1,2)} = T_{(1,1,1,2,1)} + T_{(1,1,1,2,2)}$ with

$$T_{(1,1,1,2,1)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|+1} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \times a\big(k, m(x)\big) \Big[\prod_{j\in\underline{i}_1} F_j \Big] \Gamma_{2,x}^{\underline{i}_2} \dots \Gamma_{5,x}^{\underline{i}_5} \tag{A.7f}$$

$$T_{(1,1,1,2,2)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|+1} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \times b\big(k, m(x)\big) \Big[\prod_{j\in\underline{i}_1} F_j \Big] \Gamma_{2,x}^{\underline{i}_2} \dots \Gamma_{5,x}^{\underline{i}_5}. \tag{A.7g}$$

By (A.7d) $T_{(1,1,1,1)}$ is like in (A.1a) with

$$M^{dg}_{(1,1,1,1)} = -|\underline{x}|\delta_{\eta,\eta'}$$

and it thus satisfies the conditions of proposition 4.8.

The two other terms $T_{(1,1,1,2,1)}$ and $T_{(1,1,1,2,2)}$ will be examined together with some of the terms arising from $T_{(1,2)}$.

We next consider

$$T_{(1,1,2)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) \Big[\prod_{i=1}^{p} \Gamma_{1,x}^{i} \Big] [f_x(\sigma) - f_x(m)]. \tag{A.8}$$

This term has the 'wrong' x-function because in (A.8) there is $\tilde\sigma(\underline{x}-x)$ instead of $\tilde\sigma(\underline{x})$. We thus add and subtract $\chi_{\underline{x}-x,\underline{y}}(\underline{z})$, getting

$$T_{(1,1,2)} = T_{(1,1,2,1)} + T_{(1,1,2,2)} \tag{A.9}$$

with $T_{(1,1,2,2)}$ as in (A.11b) below and

$$T_{(1,1,2,1)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \tilde\sigma(\underline{z}) \chi_{\underline{x}-x,\underline{y}}(\underline{z}) \Big[\prod_{i=1}^{p} \Gamma_{1,x}^{i} \Big] [f_x(\sigma) - f_x(m)] \tag{A.10a}$$

which has therefore the form (A.1b) and

$$\sum_{\eta'} |M_{(1,1,2,1)}(\eta,\eta',m)| \leqslant |\underline{x}|. \tag{A.10b}$$

Moreover, since $|\eta| = |\eta'|$, $d(\eta) = d(\eta')$ and $g(\eta') = g(\eta) + 1$,

$$N_\gamma(\eta) \sum_{\eta'} |M_{(1,1,2,1)}(\eta, \eta', m)| N_\gamma(\eta')^{-1} \leqslant |\underline{x}|\gamma^{b_2} \leqslant |\underline{x}|\gamma^{\zeta}. \tag{A.10c}$$

Thus $M_{(1,1,2,1)}$ satisfies the bounds (4.40) and the support properties mentioned in proposition 4.8.

Recalling that $\ell = |\underline{y}|$, writing $\underline{z}^i := \underline{z} - z_i$, $\underline{y}^i := \underline{y} - y_i$

$$\chi_{\underline{x},\underline{y}}(\underline{z}) - \chi_{\underline{x}-x,\underline{y}}(\underline{z}) = -\sum_{i=1}^{\ell} \mathbf{1}(z_i = x) \frac{1}{|B_{\gamma^{-\alpha}}|} \mathbf{1}\big(x \in B_{\gamma^{-\alpha},y_i}\big) \chi_{\underline{x},\underline{y}^i}(\underline{z}^i) \tag{A.11a}$$

we then have

$$T_{(1,1,2,2)} = -\sum_{x\in\underline{x}} \sum_{i=1}^{\ell} \frac{1}{|B_{\gamma^{-\alpha}}|} \mathbf{1}\big(x \in B_{\gamma^{-\alpha},y_i}\big) \tilde{\sigma}(\underline{x}) \sum_{\underline{z}^i} \chi_{\underline{x},\underline{y}^i}(\underline{z}^i) \tilde{\sigma}(\underline{z}^i) [\prod_{i=1}^{p} \Gamma^{i}_{1,x}][f_x(\sigma) - f_x(m)] \tag{A.11b}$$

which is also of the form (A.1b). For a suitable constant c, (proportional to $|\underline{x}||\underline{y}|$, this is why these terms are not M^{dg} terms),

$$N_\gamma(\eta) \sum_{\eta'} |M_{(1,1,2,2)}(\eta, \eta', m)| N_\gamma(\eta')^{-1} \leqslant c\gamma^{d\alpha} \leqslant c\gamma^{\zeta^*}. \tag{A.11c}$$

Observe that $|\eta| = |\eta'|$, $g(\eta) = g(\eta')$, $d(\eta) = d(\eta')$, $N_\gamma(\eta)N_\gamma(\eta')^{-1} = 1$, hence $M_{(1,1,2,2)}$ satisfies both (4.40b) and (4.40c). For the last inequality to hold we must choose $b < d\alpha$, which implies $\zeta^* < d\alpha$.

We next examine $T_{(1,2)}$ together with the remaining terms $T_{(1,1,1,2,1)}$ and $T_{(1,1,1,2,2)}$, thus completing the analysis of $T_{(1)}$. By comparing (A.6c) and (3.30b),

$$T_{(1,2)} = \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) S_1. \tag{A.12a}$$

The decomposition (3.34) gives rise to

$$T_{(1,2)} = T_{(1,2,1)} + T_{(1,2,2)} \tag{A.12b}$$

where $T_{(1,2,1)}$, given in (A.12c)-(A.12d) below, and $T_{(1,2,2)}$, see (A.17) below, are obtained from (A.12a) with S_1 replaced respectively by $S_{1,1}$, see (3.35a), and by $S_{1,2}$, see (3.36a). With respect to the analysis of section 3 we need to distinguish the two terms that arise using (3.26) for $D^{(x)}\tilde{\sigma}(x)$ in (3.35a). We obtain $T_{(1,2,1)} = T_{(1,2,1,1)} + T_{(1,2,1,2)}$, with

$$T_{(1,2,1,1)} = \sum_{x\in\underline{x}} \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} \phi_{x,\dots} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \Gamma^{\underline{i}_1}_{1,x} \dots \Gamma^{\underline{i}_4}_{4,x} \tag{A.12c}$$

$$T_{(1,2,1,2)} = \sum_{x\in\underline{x}} \tilde{\sigma}(\underline{x} - x) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} \psi_{x,\dots} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \times \Gamma^{\underline{i}_1}_{1,x} \dots \Gamma^{\underline{i}_4}_{4,x} [f_x(\sigma) - f_x(m)] \tag{A.12d}$$

where, recalling that in the following expressions $k = |\underline{i}_3| + |\underline{i}_4| + 1$,

$$\phi_{x,\ldots} = -m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x} + [m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}]' \tag{A.12e}$$

$$\psi_{x,\ldots} = m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}. \tag{A.12f}$$

By changing

$$\Gamma^{i}_{1,x} \to F_i \qquad \phi_{x,\ldots} \to (-1)^{p-|\underline{i}_1|+1} a\big(k, m(x)\big) \Gamma^{\underline{i}_5}_{5,x}$$

we obtain $T_{(1,1,1,2,1)}$ from $T_{(1,2,1,1)}$, see (A.7*f*). The bounds for these terms are identical, and we will only consider $T_{(1,2,1,1)}$ in the following.

The right-hand side of (A.12*c*) has the form (A.1*b*) with

$$|\eta'| = |\eta| - (|\underline{i}_4| + |\underline{i}_5|) \qquad g(\eta') = g(\eta) - (|\underline{i}_4| + |\underline{i}_5|) \qquad d(\eta') = d(\eta) + (|\underline{i}_2| + |\underline{i}_3|)$$

and

$$V(\eta') \leqslant V(\eta) + (|\underline{i}_1| + |\underline{i}_2| + |\underline{i}_3|)$$

so that the support properties stated in proposition 4.8 are satisfied by $M_{(1,2,1,1)}$. Moreover

$$\sum_{\eta'} |M_{(1,2,1,1)}(\eta, \eta', m)| \leqslant c|\underline{x}| \tag{A.13a}$$

with c a consant dependent on η, and

$$N_\gamma(\eta) \sum_{\eta'} |M_{(1,2,1,1)}(\eta, \eta', m)| N_\gamma(\eta')^{-1} \leqslant c \sum_{\substack{\underline{i}_1 \ldots \underline{i}_5 \\ |\underline{i}_1| < p}} \gamma^{A(\underline{i}_2, \ldots, \underline{i}_5)} \tag{A.13b}$$

where

$$\begin{aligned} A(\underline{i}_2, .., \underline{i}_5) &= -(d/2 - b_1)(|\underline{i}_4| + |\underline{i}_5|) - b_2(|\underline{i}_4| + |\underline{i}_5|) \\ &\quad - b_3(|\underline{i}_2| + |\underline{i}_3|) + d(|\underline{i}_2| + |\underline{i}_3| + |\underline{i}_4| + 2|\underline{i}_5|) \\ &= (d/2 + b_1 - b_2)(|\underline{i}_4| + |\underline{i}_5|) + (d - b_3)(|\underline{i}_2| + |\underline{i}_3|) + d|\underline{i}_5|. \end{aligned} \tag{A.13c}$$

If $b_1 = b_2 = b_3 = 0$, then $A \geqslant d/2(|\underline{i}_4| + |\underline{i}_5|)$ and (4.40*c*) is satisfied. If on the other hand $b_1 = d/2$, $b_2 = b_2^*$, $b_3 = b_3^*$, by the first equality in (A.13*c*) we have (4.40*c*) satisfied, choosing $\zeta^* < d - b_2^* - b_3^*$: the left-hand side of (A.13*b*) is then bounded by $c\gamma^{\zeta^*}$, having recalled that $|\underline{i}_2| + \ldots + |\underline{i}_5| \geqslant 1$ (because $|\underline{i}_1| \leqslant p$).

Proceeding as in (A.11), we decompose $T_{(1,2,1,2)}$ into the sum of the following two terms:

$$\begin{aligned} T_{(1,2,1,2,1)} &= \sum_{x \in \underline{x}} \tilde{\sigma}(\underline{x} - x) \sum_{\underline{z}} \chi_{\underline{x}-x,\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) \\ &\quad \times \sum_{\substack{\underline{i}_1 \ldots \underline{i}_5 \\ |\underline{i}_1| < p}} \psi_{x,\ldots} \gamma^{d(|\underline{i}_2| + |\underline{i}_3| + |\underline{i}_4| + 2|\underline{i}_5|)} \Gamma^{\underline{i}_1}_{1,x} \ldots \Gamma^{\underline{i}_4}_{4,x} [f_x(\sigma) - f_x(m)] \end{aligned} \tag{A.14a}$$

$$T_{(1,2,1,2,2)} = -\sum_{x\in\underline{x}}\sum_{i=1}^{\ell}\frac{1}{|B_{\gamma^{-\alpha}}|}\mathbf{1}(x\in B_{\gamma^{-\alpha},y_i})\tilde\sigma(\underline{x})\sum_{\underline{z}^i}\chi_{\underline{x},\underline{y}^i}(\underline{z}^i)\tilde\sigma(\underline{z}^i)$$
$$\times\sum_{\substack{\underline{i}_1,\dots,\underline{i}_5\\|\underline{i}_1|<p}}\psi_{x,\dots}\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}\Gamma_{1,x}^{\underline{i}_1}\dots\Gamma_{4,x}^{\underline{i}_4}[f_x(\sigma)-f_x(m)] \tag{A.14b}$$

(recall that $\underline{z}^i := \underline{z} - z_i$ and $\underline{y}^i := \underline{y} - y_i$). The representation (A.1b) holds as well for both terms, the coefficients M_κ satisfying the support properties of proposition 4.8 and the bound (4.40b). Moreover, recalling (A.13c),

$$N_\gamma(\eta)\sum_{\eta'}|M_{(1,2,1,2,1)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\leqslant c\sum_{\substack{\underline{i}_1\dots\underline{i}_5\\ r|\underline{i}_1|<p}}\gamma^{A(\underline{i}_2,\dots,\underline{i}_5)+b_2}\leqslant c\gamma^{\zeta^*}. \tag{A.15}$$

We have, in a similar way,

$$N_\gamma(\eta)\sum_{\eta'}|M_{(1,2,1,2,2)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\leqslant c\sum_{\substack{\underline{i}_1\dots\underline{i}_5\\|\underline{i}_1|<p}}\gamma^{A(\underline{i}_2,\dots,\underline{i}_5)+d\alpha}\leqslant c\gamma^{\zeta^*}. \tag{A.16}$$

The term

$$T_{(1,2,2)} = \sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\sum_{x\in\underline{x}}\tilde\sigma(\underline{x}-x)\tilde\sigma(\underline{z})\sum_{\substack{\underline{i}_1,\dots,\underline{i}_5\\|\underline{i}_1|<p}}[m(x)^{|\underline{i}_2|}b(k,m(x))\Gamma_{5,x}^{\underline{i}_5}]'$$
$$\times\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}\Gamma_{1,x}^{\underline{i}_1}\dots\Gamma_{4,x}^{\underline{i}_4} \tag{A.17}$$

is obtained from (A.12a) *with* S_1 *replaced by* $S_{1,2}$, *the latter as in* (3.36a).

By changing in (A.17)

$$\Gamma_{1,x}^{i}\to F_i \qquad [m(x)^{\underline{i}_2}b(k,m(x))\Gamma_{5,x}^{\underline{i}_5}]'\to(-1)^{p-|\underline{i}_1|+1}b(k,m(x))\Gamma_{5,x}^{\underline{i}_5}$$

we obtain $T_{(1,1,1,2,2)}$ so that the bound for this term is reduced to that for $T_{(1,2,2)}$.

By using again the decomposition (A.11a), we get $T_{(1,2,2)} = T_{(1,2,2,1)} + T_{(1,2,2,2)}$ which both have the representation (A.1b). The support properties and (4.40b) are verified and since $b(1,m(x)) = 0$ and $k = |\underline{i}_3| + |\underline{i}_4| + 1$,

$$N_\gamma(\eta)\sum_{\eta'}|M_{(1,2,2,1)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\leqslant c\sum_{|\underline{i}_3|+|\underline{i}_4|\geqslant 1}\gamma^{A(\underline{i}_2,\dots,\underline{i}_5)-(d/2-b_1)}. \tag{A.18a}$$

Recalling (A.13c) *we have that in each term of the sum there is* γ *raised to the power*

$$d/2(|\underline{i}_3|+|\underline{i}_4|-1)+(b_1-b_2)|\underline{i}_4|+(3d/2-b_2)|\underline{i}_5|+(d/2-b_3)|\underline{i}_3|$$
$$+(d-b_3)|\underline{i}_2|+b_1(1+|\underline{i}_5|). \tag{A.18b}$$

Since $|\underline{i}_3|+|\underline{i}_4|\geqslant 1$, (A.18$a$) is bounded proportionally to γ^{ζ^*} if $b_1 = d/2$ and to a constant if $b_1 = 0$ (recall that if $b_1 = 0$ then also $b_2 = b_3 = \zeta = 0$). This is due to the term $|\underline{i}_4| = 1$, $|\underline{i}_2| = |\underline{i}_3| = |\underline{i}_5| = 0$; all the other terms have a factor γ raised to some positive power.

With similar arguments, we also get

$$N_\gamma(\eta)\sum_{\eta'}|M_{(1,2,2,2)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\leqslant c\sum_{|\underline{i}_3|+|\underline{i}_4|\geqslant 1}\gamma^{A(\underline{i}_2,\dots,\underline{i}_5)-(d/2-b_1)+d\alpha-b_2} \tag{A.18c}$$

which is bounded proportionally to $\gamma^{d\alpha-b_2^*}$, because we have already seen that the right-hand side of (A.18a) is bounded by a constant.

Analysis of $T_{(2)}$. We have

$$T_{(2)} = T_{(2,1)} + T_{(2,2)} \tag{A.19}$$

where

$$T_{(2,1)} = \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \sum_{i=1}^{\ell} [\prod_{j=1}^{p} \Gamma^{j}_{1,z_i}] \tilde{\sigma}(\underline{z}^i) D^{(z_i)} \tilde{\sigma}(z_i) \tag{A.20a}$$

$$T_{(2,2)} = \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \sum_{i=1}^{\ell} \tilde{\sigma}(\underline{z}^i) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} \Delta_{\underline{i}_1 \dots \underline{i}_5, \underline{z}, z_i} \tag{A.20b}$$

with Δ as in (3.30d–e) and $\underline{z}^i = \underline{z} - z_i$.

We start with $T_{(2,1)}$. Like in (A.7d) we write Γ^{j}_{1,z_i} in terms of F_j and Γ^{j}_{ℓ,z_i}, $\ell = 2, \dots, 5$:

$$T_{(2,1)} = T_{(2,1,1)} + T_{(2,1,2)} \tag{A.21}$$

$$T_{(2,1,1)} = \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \sum_{i=1}^{\ell} \tilde{\sigma}(\underline{z}^i) [\prod_{j=1}^{p} F_j] D^{(z_i)} \tilde{\sigma}(z_i) \tag{A.22a}$$

and calling $x := z_i$

$$T_{(2,1,2)} = \sum_{i=1}^{\ell} \sum_{x} \chi_{\underline{x},y_i}(x) \Big\{ \tilde{\sigma}(\underline{x}) \sum_{\underline{z}^i} \chi_{\underline{x}+x,\underline{y}^i}(\underline{z}^i) \tilde{\sigma}(\underline{z}^i) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|}$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} [\prod_{i\in\underline{i}_1} F_i] \Gamma^{\underline{i}_2}_{2,x} \dots \Gamma^{\underline{i}_5}_{5,x} \tilde{\sigma}(x)^{|\underline{i}_3|+|\underline{i}_4|} D^{(x)} \tilde{\sigma}(x) \Big\}. \tag{A.22b}$$

By (3.26), $T_{(2,1,2)} = T_{(2,1,2,1)} + T_{(2,1,2,2)}$ where

$$T_{(2,1,2,1)} = \sum_{i=1}^{\ell} \sum_{x} \chi_{\underline{x},y_i}(x) \Big\{ \tilde{\sigma}(\underline{x}) \sum_{\underline{z}^i} \chi_{\underline{x}+x,\underline{y}^i}(\underline{z}^i) \tilde{\sigma}(\underline{z}^i) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|+1}$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} [\prod_{i\in\underline{i}_1} F_i] \Gamma^{\underline{i}_2}_{2,x} \dots \Gamma^{\underline{i}_5}_{5,x} \tilde{\sigma}(x)^{|\underline{i}_3|+|\underline{i}_4|+1} \Big\} \tag{A.22c}$$

$$T_{(2,1,2,1)} = \sum_{i=1}^{\ell} \sum_{x} \chi_{\underline{x},y_i}(x) \Big\{ \tilde{\sigma}(\underline{x}) \sum_{\underline{z}^i} \chi_{\underline{x}+x,\underline{y}^i}(\underline{z}^i) \tilde{\sigma}(\underline{z}^i) \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|}$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} [\prod_{i\in\underline{i}_1} F_i] \Gamma^{\underline{i}_2}_{2,x} \dots \Gamma^{\underline{i}_5}_{5,x} \tilde{\sigma}(x)^{|\underline{i}_3|+|\underline{i}_4|} (f_x(\sigma) - f_x(m)) \Big\}. \tag{A.22d}$$

We use (3.26) to write

$$T_{(2,1,1)} = T_{(2,1,1,1)} + T_{(2,1,1,2)} \tag{A.23a}$$

$$T_{(2,1,1,1)} = -\ell\rho_\eta, \qquad M^{dg}_{(2,1,1,1)}(\eta,\eta',m) = -\ell\delta_{\eta,\eta'}. \tag{A.23b}$$

As in (A.8), $T_{(2,1,1,2)}$ has the wrong χ function, then, as in (A.11),

$$T_{(2,1,1,2)} = T_{(2,1,1,2,1)} + T_{(2,1,1,2,2)} \tag{A.24a}$$

where

$$T_{(2,1,1,2,1)} = \sum_{i=1}^{\ell}\sum_x \chi_{\underline{x},y_i}(x)\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}^i}\chi_{\underline{x},\underline{y}^i}(\underline{z}^i)\tilde\sigma(\underline{z}^i)[\prod_{j=1}^{p}F_j][f_x(\sigma)-f_x(m)]\Big\} \tag{A.24b}$$

$$T_{(2,1,1,2,2)} = \sum_{i=1}^{\ell}\sum_x \chi_{\underline{x},y_i}(x)\sum_{j\neq i}\chi_{\underline{x},y_j}(x)\big\{\tilde\sigma(\underline{x}+x)\sum_{\underline{z}^{i,j}}\chi_{\underline{x}+x,\underline{y}^{i,j}}(\underline{z}^{i,j})$$
$$\times[\prod_{j=1}^{p}F_j][f_x(\sigma)-f_x(m)]\big\} \tag{A.24c}$$

where $\underline{z}^{i,j} = \underline{z} - z_i - z_j$ and, analogously, $\underline{y}^{i,j} = \underline{y} - y_i - y_j$. Then (A.1$a$) holds with $\kappa = (2,1,1,2,1)$ and

$$\eta' = (\underline{x}, \underline{y}^i, \xi_1, \ldots, \xi_p, \xi_{p+1}) \qquad \xi_{p+1} = (z, \emptyset, \emptyset, \emptyset, \emptyset) \tag{A.25a}$$

$$\sum_{\eta'}|M^{dg}_{(2,1,1,2,1)}(\eta,\eta',m)| \leqslant \ell. \tag{A.25b}$$

The representation (A.1b) is valid for $\kappa = (2,1,1,2,2)$ and we have

$$\sum_{\eta'}|M_{(2,1,1,2,2)}(\eta,\eta',m)| \leqslant \ell^2\gamma^{d\alpha} \tag{A.26a}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(2,1,1,2,2)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leqslant \ell^2\gamma^{-b_2+d\alpha} \tag{A.26b}$$

bounded by $c\gamma^{\zeta^*}$ having supposed $d\alpha > \zeta^* + b_2^*$.

We next consider the term $T_{(2,1,2,2)}$, see (A.22d). We use (3.32) and get $T_{(2,1,2,2)} = T_{(2,1,2,2,1)} + T_{(2,1,2,2,2)}$, where

$$T_{(2,1,2,2,1)} = \sum_{i=1}^{\ell}\sum_x \chi_{\underline{x},y_i}(x)\Big\{\tilde\sigma(\underline{x}+x)\sum_{\underline{z}^i}\chi_{\underline{x}+x,\underline{y}^i}(\underline{z}^i)\tilde\sigma(\underline{z}^i)\sum_{\substack{\underline{i}_1,\ldots,\underline{i}_5\\ |\underline{i}_1|<p}}(-1)^{p-|\underline{i}_1|}$$
$$\times\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}a\big(|\underline{i}_3|+|\underline{i}_4|, m(x)\big)[\prod_{i\in\underline{i}_1}F_i]\Gamma^{\underline{i}_2}_{2,x}\ldots\Gamma^{\underline{i}_5}_{5,x}(f_x(\sigma)-f_x(m)\Big\}. \tag{A.26c}$$

For the first term the representation (A.1b) holds, (4.40b) is satisfied and, recalling (A.12c)-(A.13b), for a suitable constant c

$$N_\gamma(\eta)\sum_{\eta'}|M_{(2,1,2,2,1)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leqslant c\sum_{\substack{\underline{i}_1,\ldots,\underline{i}_5\\ |\underline{i}_1|<p}}\gamma^{A(\underline{i}_2,\ldots,\underline{i}_5)+(d/2-b_1)}. \tag{A.26d}$$

Thus, using (A.13c), we prove that for this term (4.40c) holds. We then consider $T_{(2,1,2,2,2)}$ that has the same expression (A.26c) except for $\tilde\sigma(\underline{x}+x) \to \tilde\sigma(x)$ and $a\big(|\underline{i}_3|+|\underline{i}_4|, m(x)\big) \to b\big(|\underline{i}_3|+|\underline{i}_4|, m(x)\big)$. Since now there is just $\tilde\sigma(x)$, the χ function is the wrong one and we need to proceed like in (A.11). We then write $T_{(2,1,2,2,2)} = T_{(2,1,2,2,2,1)} + T_{(2,1,2,2,2,2)}$ with

$$T_{(2,1,2,2,2,1)} = \sum_{i=1}^{\ell}\sum_{x}\chi_{\underline{x},y_i}(x)\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}^i}\chi_{\underline{x},\underline{y}^i}(\underline{z}^i)\tilde\sigma(\underline{z}^i)\sum_{\substack{\underline{i}_1\ldots\underline{i}_5\\ |\underline{i}_1|<p}}(-1)^{p-|\underline{i}_1|} \times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}b\big(|\underline{i}_3|+|\underline{i}_4|, m(x)\big) \times [\prod_{i\in\underline{i}_1}F_i]\Gamma^{\underline{i}_2}_{2,x}..\Gamma^{\underline{i}_5}_{5,x}(f_x(\sigma)-f_x(m)\Big\} \tag{A.27a}$$

$$T_{(2,1,2,2,2,2)} = \sum_{i=1}^{\ell}\sum_{j\neq i}\sum_{x}\chi_{\underline{x},y_i}(x)\chi_{\underline{x},y_j}(x)\Big\{\tilde\sigma(\underline{x}+x)\sum_{\underline{z}^{i,j}}\chi_{\underline{x}+x,\underline{y}^{i,j}}(\underline{z}^{i,j})\tilde\sigma(\underline{z}^{i,j}) \times \sum_{\substack{\underline{i}_1\ldots\underline{i}_5\\ |\underline{i}_1|<p}}(-1)^{p-|\underline{i}_1|-1}\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}b\big(|\underline{i}_3|+|\underline{i}_4|, m(x)\big) \times [\prod_{i\in\underline{i}_1}F_i]\Gamma^{\underline{i}_2}_{2,x}..\Gamma^{\underline{i}_5}_{5,x}(f_x(\sigma)-f_x(m)\Big\}. \tag{A.27b}$$

For the last two terms the representation (A.1b) holds, (4.40b) is satisfied and, for a suitable constant c,

$$N_\gamma(\eta)\sum_{\eta'}|M_{(2,1,2,2,2,1)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leqslant c\sum_{\substack{\underline{i}_1\ldots\underline{i}_5\\ |\underline{i}_1|<p}}\gamma^{A(\underline{i}_2,\ldots,\underline{i}_5)} \tag{A.27c}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(2,1,2,2,2,2)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leqslant c\sum_{\substack{\underline{i}_1\ldots\underline{i}_5\\ |\underline{i}_1|<p}}\gamma^{A(\underline{i}_2,\ldots,\underline{i}_5)+d\alpha-b_2}. \tag{A.27d}$$

The term $T_{(2,1,2,1)}$, see (A.22c), will be studied together with the term $T_{(2,2)}$, see (A.20b). This can be written as

$$T_{(2,2)} = \sum_{i=1}^{\ell}\sum_{x}\chi_{\underline{x},y_i}(x)\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}^i}\chi_{\underline{x}+x,\underline{y}^i}(\underline{z}^i)\tilde\sigma(\underline{z}^i)\sum_{\substack{\underline{i}_1\ldots\underline{i}_5\\ |\underline{i}_1|<p}}\Delta_{\underline{i}_1,\ldots\underline{i}_5,\underline{x}+x,x}\Big\}. \tag{A.27e}$$

For any fixed i in (A.27e) we write

$$\underline{x}' = \underline{x}+x \qquad \underline{y}' = \underline{y}^i \qquad \eta' = (\underline{x}', \underline{y}', \xi_1, \ldots, \xi_p). \tag{A.28}$$

Then the curly bracket in (A.27e) is equal to the curly bracket in (A.6c), when the latter is defined starting from η' instead of η. The same remark applies to $T_{(2,1,2,1)}$ which, via (A.28), is identified to the term (A.7e), its estimate is omitted and we will only consider the latter. Proceeding as in (A.12b)-(A.18) we have

$$T_{(2,2)} = T_{(2,2,1,1)} + T_{(2,2,1,2,1)} + T_{(2,2,1,2,2)} + T_{(2,2,2)} \tag{A.29}$$

All these terms have the form (A.1b), the bound (4.40b) holds for each of them and, with $A = A(\underline{i}_2, \ldots, \underline{i}_5)$ as in (A.13c),

$$N_\gamma(\eta) \sum_{\eta'} |M_{(\kappa)}(\eta, \eta', m)| N_\gamma(\eta)^{-1}$$

$$\leqslant c\gamma^{-b_2} \begin{cases} \sum \gamma^A & \text{if } \kappa = (2,2,1,1), \text{ see (A.13}b\text{)} \\ \sum \gamma^{A+b_2} & \text{if } \kappa = (2,2,1,2,1), \text{ see (A.15)} \\ \sum \gamma^{A+d\alpha} & \text{if } \kappa = (2,2,1,2,2), \text{ see (A.16).} \end{cases} \tag{A.30}$$

For $\kappa = (2,2,2)$ we get γ^{-b_2} times the bound in (A.18c).

We have thus completed the analysis of $T_{(2)}$.

Analysis of $T_{(3)}$. We write

$$T_{(3)} = T_{(3,1)} + T_{(3,2)} \tag{A.31a}$$

$$T_{(3,1)} = \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) \sum_{x \notin (\underline{x}+\underline{z})} \sum_{i=1}^{p} [\prod_{j \neq i} \Gamma^j_{1,x}] \gamma^d (\Gamma^i_{3,x} + \Gamma^i_{4,x}) D^{(x)} \tilde\sigma(x) \tag{A.31b}$$

$$T_{(3,2)} = \sum_{z} \chi_{\underline{x},\underline{y}}(\underline{z}) \Big\{ \sum_{x \notin \underline{x}+\underline{z}} \tilde\sigma(\underline{x}+\underline{z}) \sum_{\substack{\underline{i}_1 \ldots \underline{i}_5 \\ |\underline{i}_1| < p-1}} \Delta_{\underline{i}_1, \ldots \underline{i}_5, \underline{x}+\underline{z}, x} \Big\}. \tag{A.31c}$$

The ananlysis of $T_{(3,2)}$ is postponed to the end of this appendix and it is similar to that of the other terms already considered. The analysis of $T_{(3,1)}$ instead brings new difficulties not already met before, as we are going to see. We write

$$T_{(3,1)} = T_{(3,1,1)} + T_{(3,1,2)} \tag{A.32}$$

$$T_{(3,1,1)} = \sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) [\prod_{j \neq i} F_j] (\Gamma^i_{3,x} + \Gamma^i_{4,x}) D^{(x)} \tilde\sigma(x) \tag{A.32a}$$

$$T_{(3,1,2)} = \sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde\sigma(\underline{z})$$
$$\times \sum_{\substack{|\underline{i}_3|+|\underline{i}_4| \geq 1 \\ |\underline{i}_1| < p-1}} (-1)^{p+1-|\underline{i}_1|} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} [\prod_{i \in \underline{i}_1} F_i] \Gamma^{\underline{i}_2}_{2,x} \ldots \Gamma^{\underline{i}_5}_{5,x} D^{(x)} \tilde\sigma(x). \tag{A.32b}$$

By (3.26) we write

$$T_{(3,1,1)} = T_{(3,1,1,1)} + T_{(3,1,1,2)} \tag{A.33}$$

where

$$T_{(3,1,1,1)} = -\sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \tilde\sigma(\underline{x}+x) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) [\prod_{j \neq i} F_j] (\Gamma^i_{3,x} + \Gamma^i_{4,x}) \tag{A.33a}$$

$$T_{(3,1,1,2)} = -\sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) [\prod_{j \neq i} F_j] (\Gamma^i_{3,x} + \Gamma^i_{4,x}) [f_x(\sigma) - f_x(m)]. \tag{A.33b}$$

By (3.28*b*), $\Gamma_{3,x} + \Gamma_{4,x} = \partial_x f_\xi(\sigma)$. We will use this in the analysis of (A.33*a*) together with the identity:

$$\partial_x f_{\xi_i}(\sigma) = \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x}} \partial_u f_{\xi_i}(\sigma) + \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x}} \left\{ \partial_x f_{\xi_i}(\sigma) - \partial_u f_{\xi_i}(\sigma) \right\}. \tag{A.34}$$

We then have

$$T_{(3,1,1,1)} = T_{(3,1,1,1,1)} + T_{(3,1,1,1,2)} + T_{(3,1,1,1,3)} \tag{A.35a}$$

where

$$T_{(3,1,1,1,1)} = -\sum_{i=1}^{p} \gamma^d \sum_{u} \left\{ \tilde{\sigma}(\underline{x}) \sum_{\underline{z},x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x) \tilde{\sigma}(\underline{z}+x) [\prod_{j \neq i} F_j] \partial_u f_{\xi_i}(\sigma) \right\} \tag{A.35b}$$

and, recalling the definition of V_ξ in definition 3.2,

$$T_{(3,1,1,1,2)} = -\sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x} - V_i} \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) \times [\prod_{j \neq i} F_j] \tilde{\sigma}(x) [\partial_x - \partial_u] f_{\xi_i}(\sigma) \tag{A.35c}$$

$$T_{(3,1,1,1,3)} = -\sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x} \cap V_i} \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) \times [\prod_{j \neq i} F_j] \tilde{\sigma}(x) \partial_x f_{\xi_i}(\sigma) \tag{A.35d}$$

because $\partial_u f_{\xi_i} = 0$ if $u \in V_i$.

We write $T_{(3,1,1,1,1)} = T_{(3,1,1,1,1,1)} + T_{(3,1,1,1,1,2)}$, by adding and subtracting $f_{\xi_i}(m)$ to $f_{\xi_i}(\sigma)$, obtaining

$$T_{(3,1,1,1,1,1)} = -\sum_{i=1}^{p} \gamma^d \sum_{u} \tilde{\sigma}(\underline{x}) \sum_{\underline{z}+x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x) \tilde{\sigma}(\underline{z}+x) [\prod_{j \neq i} F_j] \partial_u F_i \tag{A.36a}$$

$$T_{(3,1,1,1,1,2)} = -\sum_{i=1}^{p} \gamma^d \sum_{u} \tilde{\sigma}(\underline{x}) \sum_{\underline{z}+x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x) \tilde{\sigma}(\underline{z}+x) [\prod_{j \neq i} F_j] \partial_u f_{\xi_i}(m). \tag{A.36b}$$

Then (A.1*b*) holds with $\kappa = (3, 1, 1, 1, 1, 1)$ and, for a suitable constant c, which depends on η,

$$\sum_{\eta'} |M_{(3,1,1,1,1,1)}(\eta, \eta', m)| \leqslant c \tag{A.36c}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(3,1,1,1,1,1)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\leqslant c\gamma^{(d/2-b_1)+b_2-b_3} \tag{A.36d}$$

which, for $b_1 = d/2$ is bounded by $c\gamma^{\zeta^*}$, provided $b_2 - b_3 > \zeta^*$.

We next write $T_{(3,1,1,1,1,2)} = T_{(3,1,1,1,1,2,1)} + T_{(3,1,1,1,1,2,2)}$, collecting in the first term the labels i such that $d(\xi_i) = 0$ (see (3.10a) for notation) and in the latter the other values of i:

$$T_{(3,1,1,1,1,2,1)} = -\sum_{i:d(\xi_i)=0}\gamma^d\sum_u \partial_u f_{\xi_i}(m)\tilde\sigma(\underline{x})\sum_{\underline{z}+x}\chi_{\underline{x},\underline{y}+u}(\underline{z}+x)\tilde\sigma(\underline{z}+x)[\prod_{j\neq i}F_j] \tag{A.37}$$

$$T_{(3,1,1,1,1,2,2)} = -\sum_{i:d(\xi_i)>0}\gamma^d\sum_u \partial_u f_{\xi_i}(m)\tilde\sigma(\underline{x})\sum_{\underline{z}+x}\chi_{\underline{x},\underline{y}+u}(\underline{z}+x)\tilde\sigma(\underline{z}+x)[\prod_{j\neq i}F_j]. \tag{A.38}$$

Thus (A.1a) holds with $\kappa = (3,1,1,1,1,2,1)$ and (A.1b) with $\kappa = (3,1,1,1,1,2,2)$ We have

$$\sum_{\eta'}|M^{dg}_{(3,1,1,1,1,2,1)}(\eta,\eta',m)|\leqslant(2\gamma^{-1}+1)^d\gamma^d pC(0) \tag{A.39a}$$

where $C(k)$ is such that, for any ξ,

$$\sup_{m\in[-1,1]^{\mathbb{Z}^d}}\ \sup_u|\partial_u f_\xi(m)|\leqslant C\big(d(\xi)\big) \tag{A.39b}$$

$$\sum_{\eta'}|M_{(3,1,1,1,1,2,2)}(\eta,\eta',m)|\leqslant(2\gamma^{-1}+1)^d\gamma^d p\max_{i=1,p}C(d(\xi_i)) \tag{A.39c}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(3,1,1,1,1,2,2)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\leqslant\gamma^{b_3}[(2\gamma^{-1}+1)^d\gamma^d p\max_{i=1,p}C(d(\xi_i))]. \tag{A.39d}$$

To estimate $T_{(3,1,1,1,2)}$, we write

$$\begin{aligned}\partial_x-\partial_u &= \partial_x\Big\{\frac{1+\sigma(u)}{2}a_u^+ + \frac{1-\sigma(u)}{2}a_u^-\Big\} - \partial_u\Big\{\frac{1+\sigma(x)}{2}a_x^+ + \frac{1-\sigma(x)}{2}a_x^-\Big\}\\ &= \partial_x\frac{a_u^+ + a_u^-}{2} - \partial_u\frac{a_x^+ + a_x^-}{2} + \gamma^d\sigma(u)\partial_u\partial_x - \gamma^d\sigma(x)\partial_x\partial_u\\ &= \frac{1}{2\gamma^d}[a_x^+a_u^- - a_u^+a_x^-] + \gamma^d[\sigma(u)-\sigma(x)]\partial_u\partial_x\\ &= \gamma^{1-\alpha}\delta_{x,u} + \gamma^d[\sigma(u)-\sigma(x)]\partial_u\partial_x\end{aligned} \tag{A.40a}$$

see (3.7c) for notation. Thus

$$\tilde\sigma(x)[\partial_x-\partial_u] = \tilde\sigma(x)\gamma^{1-\alpha}\delta_{x,u} + \gamma^d\big\{\tilde\sigma(x)\tilde\sigma(u)+\tilde\sigma(x)2m(x)-1+m(x)^2\big\}\partial_u\partial_x. \tag{A.40b}$$

We then have

$$T_{(3,1,1,1,2)} = \sum_{i=1}^4 T_{(3,1,1,1,2,i)} \tag{A.41}$$

where i labels the four terms on the right-hand side of (A.40b), in the order of appearence. By adding and subtracting $f_{\xi_i}(m)$ to $f_{\xi_i}(\sigma)$, we get

$$T_{(3,1,1,1,2,i)} = T_{(3,1,1,1,2,i,1)} + T_{(3,1,1,1,2,i,2)} \tag{A.42a}$$

the first term being the one with F_i.

Again the decomposition (A.1b) holds with $\kappa = (3,1,1,1,2,i,j)$, $i = 1,..,3$ and $j = 1,2$. For all such κ and for a suitable constant c:

$$\sum_{\eta'} |M_{(3,1,1,1,2,i,j)}(\eta,\eta',m)| \leqslant c \tag{A.42b}$$

and

$$N_\gamma(\eta)\sum_{\eta'} |M_{(3,1,1,1,2,i,1)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leqslant \begin{cases} c\gamma^{1-\alpha+(d/2-b_1)-b_3} & \text{if } i=1 \\ c\gamma^{d+2(d/2-b_1)-2b_3} & \text{if } i=2 \\ c\gamma^{d+(d/2-b_1)-2b_3} & \text{if } i=3. \end{cases} \tag{A.42c}$$

For $\kappa = (3,1,1,1,2,i,2)$ we obtain the previous bounds times the factor

$$\gamma^{-(d/2-b_1)-b_2+b_3} \qquad \text{for } i=1 \text{ and } \gamma^{-(d/2-b_1)-b_2+2b_3} \qquad \text{for } i=2,3. \tag{A.42d}$$

When $i=4$, we have $\tilde\sigma(\underline{x})$ instead of $\tilde\sigma(\underline{x}+x)$, hence the wrong χ function. We write, as in (A.11),

$$\chi_{\underline{x}+x,\underline{y}}(\underline{z}) = \chi_{\underline{x},\underline{y}}(\underline{z}) - \frac{1}{|B_{\gamma^{-\alpha}}|}\sum_{j=1}^{\ell} \mathbf{1}(z_j = x)\chi_{\underline{x}+x,\underline{y}^j}(\underline{z}^j) \tag{A.43}$$

and we correspondingly write

$$T_{(3,1,1,1,2,4,j)} = T_{(3,1,1,1,2,4,j,1)} + T_{(3,1,1,1,2,4,j,2)} \tag{A.44}$$

where

$$T_{(3,1,1,1,2,4,1,1)} = \sum_{i=1}^{p}\gamma^d \sum_{x\notin\underline{x}} \frac{1}{|B_{\gamma^{-\alpha}}|}\sum_{u\in B_{\gamma^{-\alpha},x}} \tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) \\ \times[\prod_{j\neq i}F_j][\gamma^d(m(x)^2-1)\partial_u\partial_x F_i]. \tag{A.45a}$$

$$T_{(3,1,1,1,2,4,1,2)} = \sum_{i=1}^{p}\sum_{j=1}^{\ell}\gamma^d\sum_x \chi_{\underline{x},y_i}(x)\frac{1}{|B_{\gamma^{-\alpha}}|}\sum_{u\in B_{\gamma^{-\alpha},x}}\tilde\sigma(\underline{x}+x) \\ \times\sum_{\underline{z}^i}\chi_{\underline{x}+x,\underline{y}^i}(\underline{z}^i)\tilde\sigma(\underline{z}^i)[\prod_{j\neq i}F_j][\gamma^d(m(x)^2-1)\partial_u\partial_x F_i]. \tag{A.45b}$$

The terms $T_{(3,1,1,1,2,4,2,i)}$ are those in (A.45) with F_i replaced by $f_{\xi_i}(m)$. Thus we have that (A.1b) holds for $\kappa = (3,1,1,1,2,4,i,j)$ and, for a suitable constant c,

$$\sum_{\eta'} |M_{(3,1,1,1,2,4,i,j)}(\eta,\eta',m)| \leqslant c \tag{A.46a}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(3,1,1,1,2,4,i,j)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}$$
$$\leqslant \begin{cases} c\gamma^{d-2b_3} & \text{if } i=1,\ j=1 \\ c\gamma^{d-b_2-2b_3} & \text{if } i=2,\ j=1 \\ c\gamma^{d-(d/2-b_1)} & \text{if } i=1,\ j=2 \\ c\gamma^{d-(d/2-b_1)-b_2} & \text{if } i=2,\ j=2. \end{cases} \tag{A.46b}$$

We next consider $T_{(3,1,1,1,3)}$, see (A.35d), that we rewrite as the sum of $T_{(3,1,1,1,3,1)} + T_{(3,1,1,1,3,2)}$, by adding and subtracting $f_{\xi_i}(m)$, ($T_{(3,1,1,1,3,1)}$ corresponds to F_i). The representation (A.1b) then holds for $\kappa = (3,1,1,1,3,i)$, and, for a suitable constant c, and for $i=1,2$,

$$\sum_{\eta'}|M_{(3,1,1,1,3,i)}(\eta,\eta',m)| \leqslant c \tag{A.47a}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(3,1,1,1,3,i)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leqslant \begin{cases} c\gamma^{d/2-b_1-b_3+d\alpha} & \text{if } i=1 \\ c\gamma^{-b_2+d\alpha} & \text{if } i=2. \end{cases} \tag{A.47b}$$

We now go back to $T_{(3,1,1,2)}$, see (A.33b), that we write as

$$T_{(3,1,1,2)} = \sum_{h=1}^{2}\sum_{j=1}^{2} T_{(3,1,1,2,h,j)} \tag{A.48}$$

where $h=1$ comes from selecting $\Gamma^i_{3,x}$ in (A.33b), $h=2$ from $\Gamma^i_{4,x}$; $j=1$ corresponds to the first term on the right-hand side of (A.43) and $j=2$ to the second one. Therefore

$$T_{(3,1,1,2,1,1)} = \sum_{i=1}^{p}\gamma^d\sum_{x\notin\underline{x}}\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i}F_j]\Gamma^i_{3,x}[f_x(\sigma)-f_x(m)]\Big\} \tag{A.49a}$$

$$T_{(3,1,1,2,1,2)} = -\sum_{i=1}^{p}\sum_{j=1}^{\ell}\gamma^d\sum_{x}\chi_{\underline{x},y_j}(x)\Big\{\tilde\sigma(\underline{x}+x)\sum_{\underline{z}^j}\chi_{\underline{x}+x,\underline{y}^j}(\underline{z}^j)\tilde\sigma(\underline{z}^j)$$
$$\times[\prod_{j\neq i}F_j]\Gamma^i_{3,x}[f_x(\sigma)-f_x(m)]\Big\} \tag{A.49b}$$

$$T_{(3,1,1,2,2,1)} = \sum_{i=1}^{p}\gamma^d\sum_{x\notin\underline{x}}\Gamma^i_{4,x}\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i}F_j][f_x(\sigma)-f_x(m)]\Big\} \tag{A.49c}$$

$$T_{(3,1,1,2,2,2)} = -\sum_{i=1}^{p}\sum_{j=1}^{\ell}\gamma^d\sum_{x}\chi_{\underline{x},y_j}(x)\Gamma^i_{4,x}\Big\{\tilde\sigma(\underline{x}+x)\sum_{\underline{z}^j}\chi_{\underline{x}+x,\underline{y}^j}(\underline{z}^j)\tilde\sigma(\underline{z}^j)$$
$$\times[\prod_{j\neq i}F_j][f_x(\sigma)-f_x(m)]\Big\} \tag{A.49d}$$

The representation (A.1b) holds for $\kappa = (3, 1, 1, 2, 1, i)$, $(3, 1, 1, 2, 2, 2)$ and

$$\sum_{\eta'} |M_\kappa(\eta, \eta', m)| \leqslant c \tag{A.50}$$

for a suitable constant c: moreover

$$N_\gamma(\eta) \sum_{\eta'} |M_\kappa(\eta, \eta', m)| N_\gamma(\eta')^{-1}$$

$$\leqslant c \begin{cases} \gamma^{(d/2-b_1)+b_2-b_3} & \text{when } \kappa = (3,1,1,2,1,1) \\ \gamma^{(d/2-b_1)-b_3+d} & \text{when } \kappa = (3,1,1,2,1,2) \\ \gamma^{-b_2+d} & \text{when } \kappa = (3,1,1,2,2,2). \end{cases} \tag{A.51}$$

We write

$$T_{(3,1,1,2,2,1)} = T_{(3,1,1,2,2,1,1)} + T_{(3,1,1,2,2,1,2)} \tag{A.52a}$$

$$T_{(3,1,1,2,2,1,1)} = \sum_{d(\xi_i)=0} \gamma^d \sum_{x \notin \underline{x}} \Gamma^i_{4,x} \Big\{ \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) [\prod_{j \neq i} F_j][f_x(\sigma) - f_x(m)] \Big\} \tag{A.52b}$$

$$T_{(3,1,1,2,2,1,1)} = \sum_{d(\xi_i)>0} \gamma^d \sum_{x \notin \underline{x}} \Big\{ \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) [\prod_{j \neq i} F_j] \Gamma^i_{4,x} [f_x(\sigma) - f_x(m)] \Big\} \tag{A.52c}$$

Therefore (A.1a) holds with $\kappa = (3, 1, 1, 2, 2, 1, 1)$ and also for this term the bound (4.40a) holds. The representation (A.1b) holds for $\kappa = (3, 1, 1, 2, 2, 1, 2)$, (A.40$b$) is satisfied and

$$N_\gamma(\eta) \sum_{\eta'} |M_{(3,1,1,2,2,1,2)}(\eta, \eta', m)| N_\gamma(\eta)^{-1} \leqslant c\gamma^{b_3} \leqslant c\gamma^{\zeta^*}. \tag{A.53}$$

The only terms which still need to be estimated are $T_{(3,2)}$ and $T_{(3,1,2)}$. Except for constant factors, the terms in $T_{(3,1,2)}$ are identified with terms which are present in $T_{(3,2)}$, in analogy with similar identifications in the analysis of $T_{(1)}$ and $T_{(2)}$, we omit the details and just consider $T_{(3,2)}$.

The curly bracket term in (A.31c) is the term S_2 in (3.30c) with $\underline{x} \to \underline{x} + \underline{z}$. The decomposition of S_2 in section 3, see(3.37), (3.38), will be repeated here, but the special role of $\underline{z}$ will require a finer classification of the terms.

We write

$$T_{(3,2)} = T_{(3,2,1)} + T_{(3,2,2)} + T_{(3,2,3)} \tag{A.54}$$

according to the presence of $\tilde\sigma(x)$ (the first one), the presence of $[f_x(\sigma) - f_x(m)]$ (the last one), or the absence of both (the second one):

$$T_{(3,2,1)} = \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \Big\{ \sum_{x \notin \underline{x}+\underline{z}} \tilde\sigma(\underline{x} + \underline{z} + x) \sum_{\substack{|\underline{i}_1|,\dots,|\underline{i}_5| \\ |\underline{i}_1| \leqslant p-2}} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} G_{x\dots} \Gamma^{\underline{i}_1}_{1,x} \cdots \Gamma^{\underline{i}_4}_{4,x} \Big\} \tag{A.55a}$$

$$T_{(3,2,2)} = \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \Big\{ \sum_{x \notin \underline{x}+\underline{z}} \tilde\sigma(\underline{x} + \underline{z}) \sum_{\substack{|\underline{i}_1|,\dots,|\underline{i}_5| \\ |\underline{i}_1| \leqslant p-2}} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} H_{x\dots} \Gamma^{\underline{i}_1}_{1,x} \cdots \Gamma^{\underline{i}_4}_{4,x} \Big\} \tag{A.55b}$$

$$T_{(3,2,3)} = \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \Big\{ \sum_{x \notin \underline{x}+\underline{z}} \tilde{\sigma}(\underline{x}+\underline{z}) \sum_{\substack{|\underline{i}_1|,\dots,|\underline{i}_5| \\ |\underline{i}_1| \leqslant p-2}} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \times L_{x\dots} \Gamma^{\underline{i}_1}_{1,x} .. \Gamma^{\underline{i}_4}_{4,x} [f_x(\sigma) - f_x(m)] \Big\} \tag{A.55c}$$

where, see (3.37a)

$$G_{x\dots} = \left(m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}\right)' - \left(m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}\right) \tag{A.56a}$$

with $k = |\underline{i}_3| + |\underline{i}_4|$,

$$H_{x\dots} = \left(m(x)^{|\underline{i}_2|} b(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}\right)' \tag{A.56b}$$

$$L_{x\dots} = m(x)^{|\underline{i}_2|} a(k, m(x)) \Gamma^{\underline{i}_5}_{5,x}. \tag{A.56c}$$

$T_{(3,2,1)}$ is already of the form (A.1b) and, for a suitable c,

$$\sum_{\eta'} |M_{(3,2,1)}(\eta, \eta', m)| \leqslant c(2\gamma^{-1}+1)^d \gamma^{2d} \tag{A.57a}$$

because $|\underline{i}_2|+..+2|\underline{i}_5| \geqslant 2$. We also have

$$N_\gamma(\eta) \sum_{\eta'} |M_{(3,2,1)}(\eta, \eta', m)| N_\gamma(\eta')^{-1} \leqslant c(2\gamma^{-1}+1)^d \sum_{\substack{|\underline{i}_1|,\dots,|\underline{i}_5| \\ |\underline{i}_1| \leqslant p-2}} \gamma^{A(\underline{i}_2,\dots,\underline{i}_5)+(d/2-b_1)} \tag{A.57b}$$

with $A(\cdot)$ as in (A.13c). $A(\cdot)$ can be also written as

$$A(\cdot) = [d/2 - b_3](|\underline{i}_2| + |\underline{i}_3|) + [b_1 - b_2](|\underline{i}_4| + |\underline{i}_5|) + (|\underline{i}_2| + |\underline{i}_3| + |\underline{i}_4| + 2|\underline{i}_5|)d/2 \tag{A.57c}$$

and, since $|\underline{i}_2| + \cdots + |\underline{i}_5| \geqslant 2$, we have that

$$A(\cdot) - d/2 - b_1 \geqslant d/2 - b_1 + 2\min\{[d/2 - b_3], [b_1 - b_2]\}. \tag{A.57d}$$

The bound (4.40c) thus holds for $M_{(3,2,1)}$.

Both $T_{(3,2,2)}$ and $T_{(3,2,3)}$ have the wrong χ-function. As before we write

$$\tilde{\sigma}(\underline{x}) \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) = \tilde{\sigma}(\underline{x}) \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) - \sum_{i=1}^{\ell} \chi_{\underline{x},y_i}(x) \sigma(\underline{x}+x) \chi_{\underline{x}+x,\underline{y}'}(\underline{z}^i) \tag{A.58}$$

and using this in the expressions for $T_{(3,2,2)}$ and $T_{(3,2,3)}$ we obtain $T_{(3,2,2,i)}$ and $T_{(3,2,3,i)}$, $i = 1, 2$, for the first and the second term in (A.58). These are finally of the form (A.1b), the bound (A.57a) holds for these terms as well and

$$N_\gamma(\eta) \sum_{\eta'} |M_{(3,2,j,h)}(\eta, \eta', m)| N_\gamma(\eta')^{-1} \leqslant c \sum_{\substack{|\underline{i}_1|,\dots,|\underline{i}_5| \\ |\underline{i}_1| \leqslant p-2}} \gamma^{A(\underline{i}_2,\dots,\underline{i}_5)} \begin{cases} 1 & \text{if } j=2, h=1 \\ \gamma^{(d\alpha - b_2)} & \text{if } j=2, h=2 \\ \gamma^{(d/2-b_1)+b_2} & \text{if } j=3, h=1 \\ \gamma^{(d/2-b_1)+d\alpha} & \text{if } j=3, h=2 \end{cases} \tag{A.59}$$

that, recalling (A.57c)-(A.57d), proves the validity of (4.40c) also for these terms.

Proposition 4.8 is therefore proven. □

References

[II] De Masi A, Orlandi E, Presutti E and Triolo L 1994 Glauber evolution with Kac potentials II. Fluctuation theory, in preparation

[III] De Masi A, Orlandi E, Presutti E and Triolo L 1994 Glauber evolution with Kac potentials.III. Spinodal decomposition, in preparation

[IV] De Masi A, Orlandi E, Presutti E and Triolo L 1992 Glauber evolution for Kac potentials in the Lebowitz-Penrose limit: I. Macroscopic equation and fluctuation theory *CARR preprints 9/92*

[1] Bonaventura L 1994 Motion by curvature in an interacting spin system *Ann. Inst. H Poincaré : Nonlinear Analysis Theory Methods and Applications* in press

[2] Bracke K A 1978 *The Motion of a Surface by its Mean Curvature* (Princeton, NJ: Princeton University Press)

[3] Buttà P 1993 On the validity of an Einstein relation in models of interface dynamics *J. Stat. Phys.* 72 1401–6

[4] Chen X 1992 Generation and propagation of interfaces in reaction diffusion systems *J. Diff. Eq.* 96 116–41

[5] Comets F 1987 Nucleation for a long range magnetic model *Ann. Inst. H Poincaré* 23 135–78

[6] Comets F and Eisele Th 1988 Asymptotic dynamics, noncritical and critical fluctuations for a geometric long range interacting model *Commun. Math. Phys.* **118** 531–68

[7] Dal Passo R and de Mottoni P 1991 The heat equation with a nonlocal density dependent advection term *preprint* 1991

[8] De Masi A, Ferrari P A and Lebowitz J L 1985 Rigorous derivation of reaction diffusion equations with fluctuations *Phys. Rev. Lett.* 55 1947–9; 1986 Reaction diffusion equations for interacting particle systems *J. Stat. Phys.* 44 589–644

[9] De Masi A, Orlandi E, Presutti E and Triolo L 1993 Motion by curvature by scaling nonlocal evolution equations *J. Stat. Phys.* 73 543–70

[10] De Masi A, Orlandi E, Presutti E and Triolo L Stability of the interface in a model of phase separation *Proc. R. Soc. Edinburgh A* in press

[11] De Masi A, Orlandi E, Presutti E and Triolo L 1993 Uniqueness of the instanton profile and global stability in nonlocal evolution equations *CARR report n. 10/93*

[12] De Masi A, Pellegrinotti A, Presutti E and Vares M E 1994 Spatial patterns when phases separate in an interacting particle system *Ann. Prob.* in press

[13] De Masi A and Presutti E 1991 Mathematical methods for hydrodynamic limits *Lectures Notes in Mathematics* **1501** (Berlin: Springer)

[14] Evans L C and Spruck J 1991 Motion of level sets by mean curvature *J. Diff. Geom.* 33 635–81; 1992 Motion of level sets by mean curvature. II. *Trans. Am. Math. Soc.* **330** 321–32; 1992 Motion of level sets by mean curvature. III. *J. Geom. Anal.* 2 121-50

[15] Giacomin G 1992 Phase separation and random domain paterns in a stochastic particle model *Preprint* Rutgers

[16] Giacomin G 1992 Interface formation and global spatial structure in a reaction-diffusion model *Preprint* Rutgers

[17] Kac M, Uhlenbeck G and Hemmer P C 1963 On the van der Waals theory of vapor–liquid equilibrium. I. Discussion of a one dimensional model *J. Math. Phys.* 4 216–28; 1963 II. Discussion of the distribution functions *J. Math. Phys.* 4 229–47; 1964 III. Discussion of the critical region *J. Math. Phys.* 5 60–74

[18] Katsoulakis M A and Souganidis E 1992 Interacting particle systems and generalized motion by mean curvature *preprint*

[19] Lebowitz J L, Orlandi E and Presutti E 1991 A particle model for spinodal decomposition *J. Stat. Phys.* 63 933–74

[20] Lebowitz J and Penrose O 1966 Rigorous treatment of the van der Waals Maxwell theory of the liquid vapour transition *J. Math. Phys.* 7 98

[21] Liggett T M 1985 *Interacting Particle Systems* (New York: Springer)

[22] Penrose O and Lebowitz J 1971 Rigorous treatment of metastable states in the van der Waals Maxwell theory *J. Stat. Phys.* 3 211–36

[23] Spohn H 1993 Interface motion in models with stochastic dynamics *J. Stat. Phys.* **71** 1081–132

[24] Yau H T 1993 Metastability of Ginzburg–Landau model with a conservation law *preprint*

[25] Katsoulakis M A and Souganidis P E 1993 Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range dynamics and Glauber dynamics *preprint*

[26] Buttà P 1993 Motion by mean curvature by scaling a nonlocal equation: convergence at all times in the $d = 2$ case. *Lett. Math. Phys.* in press