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Stability of the interface in a model of phase separation*

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The paper is concerned with the asymptotic behaviour of the solutions to a nonlocal evolution equation which arises in models of phase separation. As in the Allen–Cahn equations, stationary spatially nonhomogeneous solutions exist, which represent the interface profile between stable phases. Local stability of these interface profiles is proved.

1. Introduction

We consider the initial value problem for the following nonlocal differential equation:

$$\partial_t m(x,t) = -m(x,t) + \tanh\left(\beta(J*m)(x,t)\right), \quad (x,t)\in\mathbb{R}+\mathbb{R}^2, \tag{1.1a}$$

$$m(x,0) = m_0(x), \tag{1.1b}$$

where m_0 is a continuous function with sup-norm $\|m_0\|_\infty \leqq 1$;

$$(J*m)(x) = \int J(x-y)m(y)\,dy, \tag{1.2}$$

β is a positive constant such that $\beta\int J > 1$; $J(\cdot)$ is an even, non-negative function

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with compact support. We further assume that $J(\cdot)$ is in $\mathscr{C}^2(\mathbb{R})$ and that it is monotonically nonincreasing on the positive axis. Without loss of generality, we suppose that $\int J = 1$ (correspondingly $\beta > 1$) and that supp J is the interval $[-1, 1]$.

The Cauchy problem (1.1) is well posed and has a global solution in the space $\mathscr{C}^0(\mathbb{R})$ of continuous bounded functions, as the right-hand side in (1.1a) is uniformly Lipschitz in the sup-norm. Observe also that, since $|\tanh z| < 1$, for all $z \in \mathbb{R}$, the set $\{m : \|m\|_\infty \leqq 1\}$, which contains the initial datum in (1.1b), is invariant. It is possible to consider the evolution described in (1.1) in a more general setting but we shall not elaborate on that.

The main purpose of this paper is to study the stability properties of a special stationary, spatially nonhomogeneous solution of (1.1a), which thus satisfies the integral equation:

$$m(x) = \tanh(\beta(J * m)(x)), \quad x \in \mathbb{R}. \tag{1.3}$$

In [2] it is proved that under the above assumptions, there is a unique nonconstant solution $\bar{m}$ of (1.3) in the class of the odd, strictly increasing functions. It is also shown that

$$\lim_{x \to \pm\infty} \bar{m}(x) = \pm m_\beta, \quad \text{where } m_\beta = \tanh \beta m_\beta, \quad m_\beta > 0. \tag{1.4}$$

Observe that $m_\pm(\cdot) \equiv \pm m_\beta$ are also stationary (locally stable) solutions of (1.3), so that $\bar{m}(\cdot)$ can be interpreted as the stationary solution of (1.1a) which interpolates between the homogeneous solutions $m_\pm(\cdot)$. $\bar{m}$ is thus called the "instanton solution" of (1.1). More comments on the physical interpretation of $\bar{m}$ are given at the end of this section.

The translation invariance of (1.3) implies the existence of a whole one-parameter manifold $\bar{\mathscr{M}}$ of solutions of (1.3), all called "instantons", obtained by translations of $\bar{m}$ which are thus parametrised by $\alpha \in \mathbb{R}$, so that $\bar{\mathscr{M}} = \{\bar{m}_\alpha(\cdot) = \bar{m}(\cdot - \alpha), \alpha \in \mathbb{R}\}$.

The main result of this paper is the following: if the initial datum $m_0(\cdot)$ is sufficiently close to the manifold $\bar{\mathscr{M}}$, then the solution is asymptotically attracted by an element of $\bar{\mathscr{M}}$. More precisely, we consider the $L_2(\mathbb{R}, dx)$ metrix to define a distance between profiles, namely if m and $\tilde{m}$ are two measurable functions on $\mathbb{R}$, we set

$$d(m, \tilde{m}) = \begin{cases} \|m - \tilde{m}\|_{L_2}, & \text{if } m - \tilde{m} \in L_2(\mathbb{R}, dx), \\ \infty, & \text{otherwise.} \end{cases}$$

($\|\cdot\|_{L^2}$ denotes the L_2 norm.) We recall also that if G is a set of measurable functions, the distance of m from G is

$$d(m, G) = \inf_{\tilde{m} \in G} d(m, \tilde{m}).$$

We then have the following theorem:

THEOREM 1.1. *There is $\varepsilon > 0$ such that if $d(m_0, \bar{\mathscr{M}}) < \varepsilon$, then there is $\alpha \in \mathbb{R}$ such that*

$$\lim_{t \to \infty} d(m(\cdot, t), \bar{m}_\alpha(\cdot)) = 0, \tag{1.5}$$

where $m(\cdot, \cdot)$ is the solution of (1.1); *the convergence in* (1.5) *is exponential.*

The function $m(\cdot, \cdot)$ in (1.1) can be interpreted as the magnetisation profile for a spin model evolving the Glauber dynamics and a local mean field interaction (Kac potential), in a suitable scaling [**1**, **3**, **4**, **9**]. β has then the meaning of an inverse temperature and $\beta = 1$ is the (inverse) critical temperature, as proved in [**8**], where the equilibrium statistical mechanics properties of the model are established. Thus the assumption $\beta > 1$, see just after (1.3), frames our analysis in the context of the phase transition phenomena. The thermodynamic equilibrium magnetisations are $\pm m_\beta$, see [**8**], thus the equilibrium phases are described by the magnetisation profiles constantly equal to $\pm m_\beta$. In [**4**] it is proved that when the phases separate the interface between the $\pm m_\beta$ phases is described by the function $\bar{m}(\cdot)$. Therefore the stability stated in Theorem 1.1 shows that the shape of the interface is stable. Observe, however, that the instanton itself is not stable, since its location is only marginally stable. As a consequence, a profile initially close to a given instanton will indeed become an instanton, but not necessarily that one.

This result is analogous to that obtained in [**5**], where the evolution is given by the reaction–diffusion (Allen–Cahn) equation

$$\partial_t m = \partial_{xx} m - V'(m), \tag{1.6}$$

where $V(m)$ is a symmetric double-well free energy density with two minima corresponding to the two different phases. The methods of proof are, however, quite different; in particular we miss the parabolic character of (1.6) and the neat characterisation of its stationary solutions. This is reflected in our weaker results, since we prove only local stability, as compared with the result proved in [**5**], where a larger class of initial data is allowed. The extension of this result to (1.1) remains an open problem.

The paper is organised as follows: first we consider some relevant properties of the stationary solutions, then we study the linearised problem in a L_2 setting. Finally, we pass to the full problem exploiting the special structure of the nonlinear part, and improve somewhat the result showing that this behaviour holds in a uniform setting.

2. General properties of the instanton

Recall that $\bar{m}$ is the nondecreasing, odd, nontrivial solution of

$$\bar{m}(x) = \tanh(\beta(J * \bar{m})(x)), \quad x \in \mathbb{R}, \tag{2.1}$$

and that $-m_\beta < \bar{m}(x) < m_\beta$, $x \in \mathbb{R}$. We need some more information on $\bar{m}$, given in the following proposition:

PROPOSITION 2.1. *$\bar{m} \in \mathscr{C}^2(\mathbb{R})$ and its first and second derivatives are bounded. Furthermore, $\bar{m}'(x) > 0$ for all $x \in \mathbb{R}$.*

Proof. Since $\bar{m} = \tanh \beta(J * \bar{m})$, recalling that $\|\bar{m}\|_\infty \leqq m_\beta$ and that J is in C^2, we conclude that $\bar{m}'$ exists and

$$\bar{m}' = (1 - \bar{m}(x)^2)\beta J' * \bar{m}(x), \tag{2.2}$$

hence $\bar{m}'$ is bounded. The same argument shows that also $\bar{m}''$ exists and it is bounded.

We are going to prove by contradiction that $\bar{m}' > 0$. From the above expression for $\bar{m}'$, we have that

$$\bar{m}' = (1 - \bar{m}^2)\beta(J * \bar{m}'). \tag{2.3}$$

Recalling that $\bar{m}'$ *and* J are non-negative and that $\bar{m}'$ is continuous, we have that $\bar{m}'(x)=0$ for all $|x-x_0|<1$. The same argument applies to $x_\pm := x_0 \pm \frac{1}{2}$, hence $\bar{m}'(x)=0$ for all $|x-x_0|<\frac{3}{2}$. By iteration, we then get $\bar{m}'(x)=0$ for all x, which contradicts the fact that $\bar{m}$ is nonconstant. □

Next we show a fast (at least exponential) convergence of $\bar{m}$ to its limits at $\pm\infty$.

PROPOSITION 2.2. *There is* $\eta>0$ *such that*

$$\lim_{x\to\pm\infty} e^{\eta|x|}|\bar{m}(x)\mp m_\beta|=0. \tag{2.4}$$

Proof. Recalling that $\bar{m}$ is an odd function, it is sufficient to prove (2.4) for $x\to\infty$, hence hereafter we only consider $x>0$.

Since $\bar{m}$ is increasing, from the support properties of J we get

$$\bar{m}(x-1)\leqq(J*\bar{m})(x)\leqq\bar{m}(x+1). \tag{2.5}$$

Therefore

$$\tanh\beta\bar{m}(x-1)\leqq\tanh\beta(J*\bar{m})(x)\leqq\tanh\beta\bar{m}(x+1). \tag{2.6}$$

We set

$$\Phi_\beta(s)=\tanh\beta s \tag{2.7a}$$

and observe that

$$\Phi_\beta(s)\leqq\Phi_\beta(s'),\quad s\leqq s', \tag{2.7b}$$

$$\Phi_\beta(s)\geqq s,\quad \text{if } 0\leqq s\leqq m_\beta. \tag{2.7c}$$

From (2.6) and (2.1) we then get

$$\Phi_\beta(\bar{m}(x-1))\leqq\bar{m}(x)\leqq\Phi_\beta(\bar{m}(x+1)), \tag{2.8a}$$

which, by (2.7b), implies that for all integers $k>0$

$$\Phi_\beta^k(\bar{m}(x))\leqq\bar{m}(x+k)\leqq m_\beta. \tag{2.8b}$$

Since $\Phi_\beta'(m_\beta)<1$, by (1.4), there is $\bar{x}>0$ such that $\Phi_\beta'(\bar{m}(\bar{x}))<1$. Let

$$r_k=m_\beta-\Phi_\beta^k(\bar{m}(\bar{x})).$$

Then

$$r_k=\Phi_\beta'(\tilde{m}_k)r_{k-1},\quad \text{where } \tilde{m}_k\in(\Phi_\beta^{k-1}(\bar{m}(\bar{x})),m_\beta).$$

From (2.7c) and (2.8b) we have for all k

$$\bar{m}(\bar{x})\leqq\Phi_\beta^k(\bar{m}(\bar{x}))\leqq\Phi_\beta^{k+1}(\bar{m}(\bar{x}))\leqq m_\beta, \tag{2.9}$$

hence $\tilde{m}_k\geqq\bar{m}(\bar{x})$. We then have

$$r_k\leqq\Phi_\beta'(\bar{m}(\bar{x}))r_{k-1}, \tag{2.10}$$

hence by (2.8b) and (2.10)

$$0<m_\beta-\bar{m}(\bar{x}+k)\leqq m_\beta\Phi_\beta'(\bar{m}(\bar{x}))^k, \tag{2.11}$$

which, by the choice of $\bar{x}$, proves (2.4). □

Before concluding this section, we mention that in [6] one can find explicit expressions for the solutions to (1.3), for a special choice of J: $J(x)=\frac{1}{2}$ for $|x|\leqq 1$, and 0 elsewhere. In this case the instanton is $\bar{m}(x)=m_\beta \tanh(\beta m_\beta x)$.

3. Linear stability in L^2

The linearisation of the evolution equation (1.1a) around $\bar{m}$ is

$$\partial_t v = -v + (1-\bar{m}^2)\beta J * v \equiv \mathscr{L} v. \qquad (3.1)$$

We shall study this equation in $\mathscr{H} = L^2(\mathbb{R}, d\nu(x))$, where

$$d\nu(x) = \frac{dx}{1-\bar{m}^2(x)}$$

is equivalent to the Lebesgue measure. The quadratic form associated to $\mathscr{L}$ has the simple expression

$$(v, \mathscr{L}v) = -\int d\nu(x) v(x)^2 + \int \beta J(x-y) v(x) v(y)\, dx\, dy. \qquad (3.2)$$

Despite the different signs on the right-hand side of (3.2), it is possible to prove that the spectrum of $\mathscr{L}$ lies on the negative axis; more precisely we have:

PROPOSITION 3.1. *Let $\mathscr{L}$ be the operator on $\mathscr{H}$ defined above. Then:*
(i) *$\mathscr{L}$ is a bounded symmetric operator;*
(ii) *$\mathscr{L}$ is negative semidefinite, 0 is a simple eigenvalue and $\bar{m}'$ its eigenfunction;*
(iii) *("gap") property) 0 is an isolated eigenvalue, namely there is $\omega > 0$ such that:*

$$(v^\perp, \mathscr{L}v^\perp) \leqq -\omega (v^\perp, v^\perp), \quad \forall\, v^\perp \perp \operatorname{Ker}(\mathscr{L}), \text{ i.e. } (v^\perp, \bar{m}') = 0.$$

Proof. While (i) is clear by inspection, to see (ii) we find it useful to represent the quadratic form in another way.

By (2.3), and some straightforward algebra, we find

$$(v, \mathscr{L}v) = -\frac{1}{2}\int dx\, dy\, \beta J(x-y) \bar{m}'(x) \bar{m}'(y) \left[\frac{v}{\bar{m}'}(x) - \frac{v}{\bar{m}'}(y)\right]^2 \leqq 0. \qquad (3.3)$$

Hence $\operatorname{Ker}(\mathscr{L})$ is the one-dimensional space generated by $\bar{m}'$.

To prove the gap property, we use the decomposition $\mathscr{L} = \mathscr{L}_0 + K$, where:

$$\mathscr{L}_0 v(x) = -v(x) + (1-m_\beta^2)\beta(J*v)(x), \quad Kv(x) = (m_\beta^2 - \bar{m}^2(x))\beta(J*v)(x). \quad (3.4)$$

The above decomposition of $\mathscr{L}$ foresees the use of Weyl's theorem, since it will be easy to determine the spectrum of $\mathscr{L}_0$, which has constant coefficients, while K is compact on $\mathscr{H}$.

By Fourier analysis, in fact, we easily localise the spectrum of $\mathscr{L}_0$ on the negative real line. Let $f \in \mathscr{H}$ and consider the equation

$$\mathscr{L}_0 \varphi - \lambda \varphi = f, \quad \text{for } \lambda \in \mathbb{C}. \qquad (3.5)$$

We denote as usual by $f(\cdot)$ its Fourier transform. Recall that $\beta(1-\bar{m}_\beta^2) < 1$, $\hat{J}$ is real even and $|\hat{J}(\sigma)| < \hat{J}(0) = 1$ for $\sigma \neq 0$. For λ in the resolvent set of $\mathscr{L}_0$, (3.5) has a solution that in Fourier form is

$$\hat{\varphi}(\sigma)=\frac{\hat{f}(\sigma)}{[\beta(1-\bar{m}_\beta^2)]\hat{J}(\sigma)-1-\lambda}. \tag{3.6}$$

In particular, (3.5) has a solution if λ is such that the denominator in (3.6) is different from zero for all σ. This localises the spectrum of $\mathscr{L}_0$ in the interval: $[-1-\beta(1-\bar{m}_\beta^2), -1+\beta(1-\bar{m}_\beta^2)]\subset(-\infty,0)$.

The bounded operator K is compact because it maps the bounded sets of $\mathscr{H}$ into relatively compact sets in the same space. Namely for any φ such that $\|\varphi\|\leqq 1$ we prove:

(i) for all $\varepsilon>0$ there is $X_\varepsilon>0$ such that

$$\int_{|x|>X}|K\varphi|^2\,d\nu<\varepsilon,\quad \text{if } X>X_\varepsilon;$$

(ii) for all $\varepsilon>0$ there is $h_\varepsilon>0$ such that

$$\int|K\varphi(x+h)-K\varphi(x)|^2\,d\nu(x)<\varepsilon,\quad \text{if } |h|<h_\varepsilon.$$

These properties are easily proved using the regularity of the convolution term and the fact that $(m_\beta^2-\bar{m}^2(\cdot))$, vanishing at ∞, has a bounded derivative.

By Weyl's theorem, [**10**], the essential spectrum is invariant under compact perturbations. We then conclude that the full operator $\mathscr{L}$ has the same essential spectrum as $\mathscr{L}_0$, so that its eigenvalue 0 cannot be a cluster point of the spectrum. □

In this way the gap constant ω defined in Proposition 3.1(iii) is the distance between 0 and the negative part of the spectrum. By exploiting more refined properties of the instanton $\bar{m}$, we could give more explicit bounds on the spectral gap, using the theory developed in [**7**]. It is also possible to prove the existence of a gap for $\mathscr{L}$ as an operator in some weighted L_∞ spaces; but these properties are more sophisticated and far from necessary for proving Theorem 1.1, so we omit precise statements and proofs.

4. Local nonlinear stability

We return to the full nonlinear equation to prove Theorem 1.1, the main result in this paper.

Proof of Theorem 1.1. From now on, we represent the evolving profile $m(\cdot,t)$, solution to (1.1), in terms of a moving instanton and the corresponding variation, by writing

$$m(\cdot,t)=\bar{m}_{\alpha_t}(\cdot)+v(\cdot,t),\quad \bar{m}_\alpha(x)=\bar{m}(x-\alpha), \tag{4.1a}$$

where $\bar{m}_{\alpha_t}\in\bar{\mathscr{M}}$ and the variation part v is orthogonal to $\bar{m}'_{\alpha_t}$, in $L_2(\mathbb{R}, dx/(1-\bar{m}_{\alpha_t}^2(x)))$ space, namely

$$(v(\cdot,t),\bar{m}'_{\alpha_t}(\cdot))_{\alpha_t}=0. \tag{4.1b}$$

In the sequel, for notational simplicity we shall omit the explicit dependence of α_t on t, when confusion will not arise. In the Appendix we prove that any profile m, in a suitably small neighbourhood of $\bar{\mathscr{M}}$, can be uniquely represented as in (4.1). In order to use this representation for the actual solution of (1.1), we shall suppose that the initial profile m_0 is suitably close to $\bar{\mathscr{M}}$, and we prove that at all the later times it remains close to $\bar{\mathscr{M}}$, so that the representation (4.1) is valid at all times.

Whenever the representation (4.1) holds, we can write

$$\begin{aligned}\partial_t(m(\cdot,t)-\bar{m}_{\alpha_t}) = \partial_t v &= -(\bar{m}_{\alpha_t}+v)+\tanh \beta J*(\bar{m}_{\alpha_t}+v)-\dot{\alpha}\partial_\alpha \bar{m}_\alpha \\ &= \mathscr{L}_\alpha v+\dot{\alpha}\bar{m}'_\alpha+R[v],\end{aligned} \tag{4.2}$$

where $\mathscr{L}_\alpha$ is the operator linearised around the instanton $\bar{m}_\alpha$ and the nonlinear term $R[v]$ is defined in (4.3) below. To derive (4.2), we have used that $\bar{m}_\alpha(x)=\bar{m}(x-\alpha)$.

By the Taylor expansion of the function tanh$(\cdot)$, we have

$$|R[v](x)|:=|\tanh \beta J*(\bar{m}_{\alpha_t}+v)(x)-\bar{m}_{\alpha_t}(x)-(1-\bar{m}^2_{\alpha_t}(x))\beta J*v(x)|\leqq(\beta J*v)(x)^2. \tag{4.3}$$

We next prove that for a suitable $c>0$ (L_2 below denotes $L_2(\mathbb{R},dx)$)

$$R[v](\cdot)\in L_2 \quad \text{and} \quad \|R[v]\|_{L_2}\leqq c\|v\|^2_{L_2}, \tag{4.4}$$

since $R[v]$ depends on v through the convolution $J*v$.

Namely since $\|J*v\|_{L_2}\leqq\|J\|_{L_1}\|v\|_{L_2}$ and $\|J*v\|_{L_\infty}\leqq\|J\|_{L_\infty}\|v\|_{L_2}\sqrt{2}$, it follows that, [**11**], $|J*v|\in L_p\cap L_\infty$, for $p\geqq 2$. (We have used that J is bounded and the Schwartz inequality to estimate $\int_{x-1}^{x+1}dy v(y)$.) Therefore (4.4) easily follows, with c depending on the function J.

By taking the time derivative in (4.1b) (the bracket in the scalar product without subscripts in the following refers to $L_2(\mathbb{R},dx)$), we obtain:

$$(\partial_t v,\bar{m}'_\alpha)_\alpha+\dot{\alpha}(v,\Phi_\alpha)=0, \tag{4.5a}$$

where

$$\Phi_\alpha=\partial_\alpha\frac{\bar{m}'_\alpha}{1-\bar{m}^2_\alpha} \tag{4.5b}$$

is continuous, bounded and in L_2.

From (4.5a), using (4.2) and the fact that $\bar{m}'_{\alpha_t}\in\text{Ker}(\mathscr{L}_{\alpha_t})$, we get

$$[\|\bar{m}'_\alpha\|^2_\alpha+(v,\Phi_\alpha)_\alpha]\dot{\alpha}=-(R[v],\bar{m}'_\alpha)_\alpha. \tag{4.6}$$

By taking the scalar product of both sides of (4.2) with $2v$, we get, using (4.1b),

$$2(v,\partial_t v)_\alpha=2(v,\mathscr{L}_\alpha v)_\alpha+2(v,R[v])_\alpha. \tag{4.7}$$

Since α depends on t, the left-hand side of (4.7) is not just equal to the derivative of the squared norm: namely

$$2(v,\partial_t v)_\alpha=\frac{d}{dt}\|v\|^2_\alpha-2\dot{\alpha}(v^2,\Psi_\alpha), \tag{4.8a}$$

where

$$\Psi_\alpha=\frac{\bar{m}_\alpha\bar{m}'_\alpha}{(1-\bar{m}^2_\alpha)^2} \quad \text{is a bounded continuous function.} \tag{4.8b}$$

From (4.6), (4.7), (4.8) and (4.4) we get a system of differential inequalities:

$$\frac{d}{dt}\|v\|_\alpha^2 \leqq -2\omega\|v\|_\alpha^2 + k_1\|v\|_\alpha^3 + |\dot{\alpha}|k_2\|v\|_\alpha^2, \tag{4.9a}$$

$$|\dot{\alpha}|\,|\,\|\bar{m}'_\alpha\|_\alpha^2 - |(v, \Phi_\alpha)|\,| \leqq k_3\|v\|_\alpha^2. \tag{4.9b}$$

We choose the initial datum m_0 so that $v_0 := m_0 - \bar{m}_{\alpha_0}$ satisfies the bound

$$\|v_0\|_{\alpha_0} \leqq \tfrac{1}{4}\|\Phi_{\alpha_0}\|_{\alpha_0}^{-1}\|\bar{m}'_{\alpha_0}\|_{\alpha_0}^2. \tag{4.10a}$$

We denote by

$$t^* = \sup\{t : \|v(\cdot, t)\|_{\alpha_t} \leqq \tfrac{1}{2}\|\Phi_{\alpha_t}\|_{\alpha_t}^{-1}\|\bar{m}'_{\alpha_t}\|_{\alpha_t}^2\}. \tag{4.10b}$$

Then, by (4.9), there are positive constants c_1 and c_2 so that for all $t \leqq t^*$

$$\dot{U} \leqq -2\omega U + c_1 U^{3/2} + c_2 U^2, \quad \text{where } U(t) := \|v(\cdot, t)\|_{\alpha_t}^2. \tag{4.11}$$

From this we get $U \leqq c_0^2 e^{-2\omega t}$, for a suitable c_0, which implies that $t^* = \infty$ and $\|v(\cdot, t)\|_{\alpha_t} \leqq c_0 e^{-\omega t}$ if $\|v_0\|_{\alpha_0}$ is sufficiently small, as in (4.10a). Moreover, from the estimate on $|\dot{\alpha}_t|$ we have that for any $\varepsilon > 0$ there is T_ε so that for all $t' \geqq t'' \geqq T_\varepsilon$,

$$|\alpha_{t'} - \alpha_{t''}| < \varepsilon, \tag{4.12}$$

hence the convergence to the limiting instanton:

$$\lim_{t\to\infty} \alpha_t = \alpha_\infty. \tag{4.13}$$

From (4.9b) and the estimate on $\|v(\cdot, t)\|_{\alpha_t}$, we have an exponential convergence in (4.13). □

5. Appendix

(1) We first check that the claim made at the beginning of the proof of Theorem 1.1 in Section 4 is correct. Let us first change notation slightly, by writing

$$\xi \equiv \alpha - \alpha_0, \quad p \equiv m - \bar{m}_{\alpha_0}, \tag{5.1}$$

and consider the equation defining the new coordinates:

$$F(p, \xi, v) = 0, \tag{5.2}$$

where $F : L_2 \times \mathbb{R} \times L_2 \to \mathbb{R} \times L_2$ is defined as $F(p, \xi, v) = (F_1, F_2)$, where

$$F_1 = \int dx\, \frac{\bar{m}'_{\alpha_0+\xi}(x)}{1 - \bar{m}^2_{\alpha_0+\xi}(x)}\, v(x), \quad F_2 = p + \bar{m}_{\alpha_0} - \bar{m}_{\alpha_0+\xi} - v. \tag{5.3}$$

The properties of F are studied in the following proposition:

Proposition A1. *The map $F : L_2 \times \mathbb{R} \times L_2 \to \mathbb{R} \times L_2$ is $\mathscr{C}^1$ and such that:*
(i) $F(0, 0, 0) = 0$;
(ii) *the partial derivative $F'_{\xi,v}(0, 0, 0)$ is boundedly invertible.*

Proof. The $\mathscr{C}^1$ property and (i) are easily checked by inspection, while (ii) is verified by direct computation of the solution to the linear equation:

$$F'_{\xi,v}(0, 0, 0)(\hat{\xi}, \hat{v}) = (b, w), \tag{5.4a}$$

that is

$$\int dx \hat{v}(x) \frac{\bar{m}'_{\alpha_0}(x)}{1-\bar{m}_{\alpha_0}(x)^2} = b, \quad -\bar{m}'_{\alpha_0}\xi - \hat{v} = w. \tag{5.4b}$$

By taking the scalar product of both sides of the last equality with $\bar{m}'_{\alpha_0}$, we obtain

$$\xi = -\frac{b + (\bar{m}'_{\alpha_0}, w)_{\alpha_0}}{\|\bar{m}'_{\alpha_0}\|^2}, \quad \hat{v} = -w + \bar{m}'_{\alpha_0} \frac{b + (\bar{m}'_{\alpha_0}, w)_{\alpha_0}}{\|\bar{m}'_{\alpha_0}\|^2}. \tag{5.5}$$

As $\|\bar{m}'_{\alpha_0}\|^2$ is a positive constant, we get the result.

Then we can apply the implicit function theorem in Banach spaces (see for instance [**10**]): we get the (local) existence of a $\mathscr{C}^1$ map

$$\alpha = \alpha(p), \quad v = v(p), \quad p \in U_\delta \subset L_2 \tag{5.6}$$

with the desired properties.

(2) We get the uniform convergence by the following argument. If the initial datum is a bounded function in $\mathscr{C}^2$ with bounded derivative, then it is easy to see that the first derivative of the solution is uniformly bounded. Then the claim follows from this and the following estimate, proved in [**5**],

$$\|f\|^3_{L_\infty} \leqq \tfrac{3}{2}\|f'\|_{L_\infty}\|f\|^2_{L_2}. \tag{5.7}$$

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Notes added in proof

We have recently extended the results of this paper proving uniqueness and global stability of the instantons: A. De Masi, E. Orlandi, E. Presutti, L. Triolo, *Uniqueness and global stability of the instanton in non local evolution equations*, preprint (1994).

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