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ASYMPTOTIC EQUIVALENCE OF FLUCTUATION FIELDS FOR REVERSIBLE EXCLUSION PROCESSES WITH SPEED CHANGE[1]

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We consider stationary, reversible exclusion processes with speed change and prove that for sufficiently small interaction the fluctuation fields constructed from local functions become proportional to the density fluctuation field when averaged over suitably large space-time regions. If the exclusion process is of gradient type, this result implies that the density fluctuation field converges to an infinite dimensional Ornstein–Uhlenbeck process.

1. Introduction and results. We consider a stochastic lattice gas (exclusion process with speed change) on a simple hypercubic lattice $\mathbb{Z}^d$ in d dimensions. We follow the usual notations and denote by $\Omega = \{0,1\}^{\mathbb{Z}^d}$ the configuration space of the lattice gas. $\eta \in \Omega$ stands for a configuration, η_Λ for its restriction to $\Lambda \subset \mathbb{Z}^d$. $\eta(x)$ denotes the occupation variable for lattice site $x \in \mathbb{Z}^d$ with $\eta(x) = 1$ (0) corresponding to x occupied (empty). $C(\Omega)$ is the space of bounded continuous functions on Ω and $\mathscr{D}_0 \subset C(\Omega)$ the space of local functions, i.e., functions depending only on finitely many $\eta(x)$'s.

The particles of the lattice gas move by random jumps respecting the hard-core exclusion. The dynamics is completely specified by the jump rates $c(x, y, \eta) \geq 0$ which give the rate of interchanging the occupations at sites x and y when the configuration is η. Clearly, $c(x, y, \eta) = c(y, x, \eta)$ and if $\eta(x) = \eta(y)$ we may set, arbitrarily, $c(x, y, \eta) = 0$. We assume that the rates are translation invariant, i.e., $c(x, y, \eta) = c(x+z, y+z, \tau_z\eta)$ with τ_z the shift by $z \in \mathbb{Z}^d$, and of finite range r, in the sense that $c(x, y, \eta) = 0$ for $|x-y| > r$ and that $c(0, y, \eta)$ depends on η only through $\{\eta(x): |x| \leq r\}$. To avoid degeneracies we also assume that

$$c(x, y, \eta) > 0 \tag{1.1}$$

for $|x-y| = 1$ and $\eta(x) \neq \eta(y)$.

We want to ensure that the exclusion process is reversible. We therefore impose the condition of detailed balance:

$$c(x, y, \eta) = c(x, y, \eta^{xy})\exp\{-\partial_{xy}H(\eta)\}. \tag{1.2}$$

Here η^{xy} denotes the configuration η with $\eta(x)$ and $\eta(y)$ interchanged and for

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$f \in C(\Omega)$ we define

(1.3) $$\partial_{xy} f(\eta) = f(\eta^{xy}) - f(\eta).$$

$H(\eta)$ is the energy of the configuration η, formally given by

(1.4) $$H(\eta) = \sum_{X \subset \mathbb{Z}^d, |X| < \infty} \Phi(X) \prod_{x \in X} \eta(x)$$

with $\{\Phi(X) | X \subset \mathbb{Z}^d, |X| < \infty\}$ a family of potentials. We assume that Φ is translation invariant and of finite range r, i.e., $\Phi(X) = \Phi(X + z)$ and $\Phi(X) = 0$ for $\operatorname{diam}(X) > r$. Then the difference in energy, $\partial_{xy} H$, after and before the jump is always well defined.

The generator of our jump process is given by

(1.5) $$\begin{aligned} Lf(\eta) &= \tfrac{1}{2} \sum_{x, y} c(x, y, \eta)\, \partial_{xy} f(\eta) \\ &= \tfrac{1}{2} \sum_{x, y} L_{xy} f(\eta) \end{aligned}$$

for $f \in \mathscr{D}_0$. Under our assumptions L on $\mathscr{D}_0$ determines uniquely the Markov semigroup $e^{Lt} = T_t$ acting on $C(\Omega)$. In the usual way one constructs then the Markov process η_t with paths in $D([0, \infty), \Omega)$, see Liggett (1977).

Let μ be a Gibbs (DLR) measure associated with Φ [Ruelle (1969)]. Detailed balance together with the DLR equations imply the symmetry of L_{xy} and of L in $\mathscr{D}_0$

(1.6) $$\mu(f L_{xy} g) = \mu(g L_{xy} f),$$

(1.7) $$\mu(f L g) = \mu(g L f),$$

which in turn implies

(1.8) $$\mu(f T_t g) = \mu(g T_t f).$$

T_t extends to a self-adjoint contraction semigroup on $L^2(\Omega, \mu)$. Thus μ is reversible and, in particular, invariant under T_t. η_t with starting measure μ is the stationary, reversible exclusion process with speed change under consideration.

In the detailed balance condition (1.2) the energy difference does not depend on the choice of the chemical potential $\lambda = \Phi(\{0\})$. If we denote by μ_λ a Gibbs measure associated with $\Phi = \{\Phi(\{0\}) = \lambda, \Phi(X), |X| \geq 2\}$, then every μ_λ is a reversible measure for η_t. Conversely, the only translation invariant, reversible measures for η_t are mixtures of the translation invariant μ_λs. For dimensions $d = 1, 2$, the canonical Gibbs measures even exhaust all stationary measures [Vanheuverzwijn (1981)]. We refer to Georgii (1980) for a more detailed account.

In the following we will need strong mixing properties (in space) for μ. This is ensured if either $d = 1$ or Φ is sufficiently small, cf. (3.2). Under these assumptions there is a unique Gibbs measure associated with Φ. We think of Φ, and therefore μ, as fixed. Averages with respect to μ are also denoted by $\langle \cdot \rangle_\lambda$. $\rho = \rho(\lambda) = \langle \eta(0) \rangle_\lambda$ is the average density and $\chi = \sum_x \langle \eta(x)(\eta(0) - \rho) \rangle$ is the compressibility. We will denote by $\lambda(\rho)$ the inverse function of $\rho(\lambda)$ and we will write $\langle \cdot \rangle_{\lambda(\rho)}$ for the averages with respect to the corresponding Gibbs measure.

To explain the goal of our paper we make the following definition.

DEFINITION 1. Let $h \in \mathscr{D}_0$ with $\mu(h) = 0$ and $\phi \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions. Then

$$Y_t^\varepsilon(\phi; h) = \varepsilon^{d/2} \sum_{x \in \mathbb{Z}^d} \phi(\varepsilon x) \tau_x h(\eta_{\varepsilon^{-2}t}) \tag{1.9}$$

is called the scaled fluctuation field for h (indexed by t and ϕ). If $h(\eta) = \eta(0) - \rho$ we set

$$Y_t^\varepsilon(\phi; \eta(0) - \rho) = Y_t^\varepsilon(\phi). \tag{1.10}$$

$\phi(\varepsilon x)$ determines a spatial average over a region with diameter of the order ε^{-1}. The prefactor anticipates normal fluctuations and $\varepsilon^{-2}t$ reflects the scale invariance of Brownian motion.

We are interested in the asymptotic form of the fluctuation fields $Y_t^\varepsilon(\phi; h)$, as $\varepsilon \to 0$. Among these fields the density fluctuation field $Y_t^\varepsilon(\phi)$ plays a special role —as the fluctuation field associated with the locally conserved quantity (particle number). It is therefore expected to vary on the slowest time scale. Furthermore, the other fields should fluctuate "around" the density field i.e., they should be asymptotically a multiple of the density field plus a rapidly varying term. These considerations are formalized in the following theorem.

THEOREM 1. *Let $d = 1$ or $d \geq 2$ and the potential Φ and density ρ satisfy the smallness condition* (3.2). *Let $h \in \mathscr{D}_0$ with $\mu(h) = 0$ and let*

$$a(h) = d/d\rho \, \langle h \rangle_{\lambda(\rho)} = \chi^{-1} \sum_x \langle h\eta(x) \rangle_{\lambda(\rho)}. \tag{1.11}$$

Then

$$\lim_{T \to \infty} \lim_{\varepsilon \to 0} E\left(\frac{1}{2\varepsilon^2 T} \int_{t-\varepsilon^2 T}^{t+\varepsilon^2 T} ds \, Y_s^\varepsilon(\phi; h) - a(h) Y_t^\varepsilon(\phi) \right)^2 = 0. \tag{1.12}$$

The importance of the asymptotic equivalence (1.12) was first realized by Rost (1982), and called "Boltzmann–Gibbs principle" there. It is further explained in De Masi et al. (1984). It has been proved and employed for the stationary, reversible zero-range processes in Brox and Rost (1983), for the nonstationary reversible zero-range process with jump rate one in Ferrari et al. (1983), and for interacting Brownian particles in Spohn (1986).

Let us explain on a heuristic level why (1.12) together with the particular choice of the constant $a(h)$ should hold. Consider a given bounded region Λ. Assume that at time 0 there is a density fluctuation so that the number of particles in Λ is $n(\Lambda)$ while its expected value is $\rho|\Lambda|$. Then due to the good mixing properties of the stochastic time evolution one may hope that there exists some time T which is on one hand so short that the number of particles in Λ at time T is essentially unchanged, i.e., still $n(\Lambda)$, but on the other hand so long that in Λ equilibrium has itself established corresponding to the new density

$n(\Lambda)/|\Lambda|$. Denoting by $\langle \cdot \rangle_{\lambda(n(\Lambda)/|\Lambda|)}$ the expectation with respect to the corresponding Gibbs measure, we have

$$E\left(|\Lambda|^{-1/2} \sum_{x \in \Lambda} \tau_x h(\eta_T) | \mathscr{F}_0\right) \sim |\Lambda|^{-1/2} |\Lambda| \langle h \rangle_{\lambda(n(\Lambda)/|\Lambda|)}, \tag{1.13}$$

where $\mathbb{E}(\cdot|\mathscr{F}_0)$ denotes the conditional expectation having fixed η_0. We expand $\langle h \rangle_{\lambda(n(\Lambda)/|\Lambda|)}$ around $\lambda(\rho)$ to leading order,

$$\langle h \rangle_{\lambda(n(\Lambda)/|\Lambda|)} \sim \langle h \rangle_{\lambda(\rho)} + \frac{d}{d\rho} \langle h \rangle_{\lambda(\rho)} \left(\frac{n(\Lambda)}{|\Lambda|} - \rho \right). \tag{1.14}$$

Since $\langle h \rangle_{\lambda(\rho)} = 0$ and using (1.11),

$$\begin{aligned} E\left(|\Lambda|^{-1/2} \sum_{x \in \Lambda} \tau_x h(\eta_T) | \mathscr{F}_0\right) &\sim a(h) |\Lambda|^{-1/2} (n(\Lambda) - \rho |\Lambda|) \\ &\sim a(h) |\Lambda|^{-1/2} \sum_{x \in \Lambda} (\eta_T(x) - \rho). \end{aligned} \tag{1.15}$$

In Section 5 we apply Theorem 1 to exclusion processes of gradient type and prove that their density field converges as $\varepsilon \to 0$ to an infinite-dimensional Ornstein–Uhlenbeck process.

2. Outline of the proof of Theorem 1. To prove Theorem 1 we square out (1.12). Then we are left to show that for $f, g \in \mathscr{D}_0$ with $\mu(f) = 0 = \mu(g)$

$$\lim_{t \to \infty} \sum_x \langle f \tau_x T_t g \rangle = 1/\chi \left(\sum_x \langle f \eta(x) \rangle \right) \left(\sum_x \langle g \eta(x) \rangle \right). \tag{2.1}$$

[See Brox and Rost (1983).]

To understand how we tackle (2.1) let us first consider the simpler case of showing

$$\lim_{t \to \infty} \langle f T_t g \rangle = \langle f \rangle \langle g \rangle. \tag{2.2}$$

Since T_t is a self-adjoint contraction semigroup on $L^2(\Omega, \mu)$ the limit (2.2) exists. Therefore we only have to show that the T_t invariant subspace of $L^2(\Omega, \mu)$ consists of constant functions.

For bounded subsets $\Lambda \subset \mathbb{Z}^d$ let

$$L_\Lambda = \tfrac{1}{2} \sum_{x, y \in \Lambda} L_{xy} \tag{2.3}$$

and let Γ_Λ be the projection in $L^2(\Omega, \mu)$ defined by the conditional expectation

$$\Gamma_\Lambda g = \mu(g | n(\Lambda), \eta_{\Lambda^c}). \tag{2.4}$$

Here $\mu(\cdot | n(\Lambda), \eta_{\Lambda^c})$ is the canonical Gibbs measure in Λ with boundary conditions η_{Λ^c}, i.e., μ conditioned on the number $n(\Lambda)$ of particles in Λ and the configuration η_{Λ^c} in the complement of Λ. By (1.6) L_Λ is self-adjoint. For fixed η_{Λ^c}, L_Λ generates a finite Markov chain. By (1.1) for fixed $n(\Lambda)$ this chain is irreducible and therefore Γ_Λ is the projection onto the subspace corresponding to the eigenvalue zero of L_Λ. Let $h \in L^2(\Omega, \mu)$ be such that $Lh = 0$. Then

$\langle hL_\Lambda h\rangle = 0$ because in $\langle hL_\Lambda h\rangle + \langle h(L - L_\Lambda)h\rangle = 0$ both terms are nonpositive. Therefore for any $\tilde{g} \in \mathscr{D}_0$

$$\langle hL_\Lambda \tilde{g}\rangle^2 \le \langle hL_\Lambda h\rangle\langle \tilde{g}L_\Lambda \tilde{g}\rangle = 0. \tag{2.5}$$

In particular, the choice $\tilde{g} = L_\Lambda^{-1}(g - \Gamma_\Lambda g)$ implies that

$$\langle h\Gamma_\Lambda g\rangle = \langle hg\rangle. \tag{2.6}$$

Let $\Lambda \uparrow \mathbb{Z}^d$. Then, by the equivalence of ensembles, see e.g., Georgii (1980), $\langle h\Gamma_\Lambda g\rangle \to \langle h\rangle\langle g\rangle$ and (2.6) implies $h =$ constant.

To imitate this approach, in view of (2.1), it is natural to define a Hilbert space $\mathscr{H}$ as the completion of $\mathscr{D}_0$ with the (degenerate) scalar product

$$\langle f|g\rangle = \sum_x (\langle f\tau_x g\rangle - \langle f\rangle\langle g\rangle). \tag{2.7}$$

Clearly, to make (2.7) meaningful we need mixing for μ, cf. Section 3. T_t is still a self-adjoint contraction semigroup in $\mathscr{H}$. Therefore the limit (2.1) exists and we only have to identify the T_t invariant subspace $P\mathscr{H}$ of $\mathscr{H}$. We prove in Section 3 the analogue of (2.6): namely that for every $\psi \in P\mathscr{H}$ and any $g \in \mathscr{D}_0$

$$\langle \psi|\Gamma_\Lambda g\rangle = \langle \psi|g\rangle. \tag{2.8}$$

Let $\Lambda \uparrow \mathbb{Z}^d$ and note that $\langle \eta(0)|\eta(0)\rangle = \chi$ and that in $\mathscr{H}$, $|\Lambda|^{-1}\Sigma_{x\in\Lambda}\eta(x) = \eta(0)$. Then we are left to prove that for any $g \in \mathscr{D}_0$

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \left\{\mu(g|n(\Lambda), \eta_{\Lambda^c}) - a(g)|\Lambda|^{-1} \sum_{x\in\Lambda} (\eta(x) - \rho)\right\} = 0 \tag{2.9}$$

in $\mathscr{H}$, where $a(g)$ is defined in (1.11). (2.9) will be established in Section 4 under a strong mixing condition for μ. (2.9) is the reduction of the dynamical problem (2.1) to a purely static, equilibrium problem.

We summarize our discussion in Proposition 1, which will be proved in the following two sections.

PROPOSITION 1. *Let $P\mathscr{H}$ be the T_t invariant subspace of $\mathscr{H}$. Then P is the one-dimensional projection onto $\chi^{-1/2}\eta(0)$.*

3. Reduction to an equilibrium problem. Let $\lambda = \Phi(\{0\})$ and

$$\|\Phi\| = \sum_{X\ni 0, |X|\ge 2} |\Phi(X)|. \tag{3.1}$$

Then if $d \ge 2$ we require that

$$\exp\lambda \cdot \exp\|\Phi\| \cdot [\exp(\exp\|\Phi\| - 1) - 1] < 0.1. \tag{3.2}$$

Condition (3.2) is sufficient for the convergence of the cluster expansion, Del Grosso (1974). Since the density ρ is an increasing function of λ, (3.2) holds for ρ and $\|\Phi\|$ sufficiently small. Let $\mathscr{F}_\Lambda$ be the set of functions depending only on η_Λ. Then by Del Grosso (1974), cf. also (4.8) in Di Liberto et al. (1973), there exist positive constants A, α depending only on λ and Φ such that for all bounded sets

$\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ (we denote by $\langle \eta_\Lambda \rangle := \langle \prod_{x \in \Lambda} \eta(x) \rangle$)

$$\left| \frac{\langle \eta_{\Lambda_1 \cup \Lambda_2} \rangle}{\langle \eta_{\Lambda_1} \rangle \langle \eta_{\Lambda_2} \rangle} - 1 \right| \le A \min(|\partial \Lambda_1|, |\partial \Lambda_2|) e^{-\alpha d(\Lambda_1, \Lambda_2)}, \tag{3.3a}$$

where $\partial \Lambda_i$ denotes the boundary of Λ_i, $i = 1, 2$, and $d(\Lambda_1, \Lambda_2)$ is the distance between the sets Λ_1 and Λ_2. (3.3a) will be used in the sequel to derive (4.12), (4.25), (4.26), (4.29) as well as in Definition 2 and Lemma 2. Another (easy) consequence of (3.3a) is that for $f_i \in \mathscr{F}_{\Lambda_i}$, $\langle f_i \rangle = 0$, $i = 1, 2$,

$$|\langle f_1 f_2 \rangle| \le \| f_1 \|_2 \| f_2 \|_2 A \min(|\partial \Lambda_1|, |\partial \Lambda_2|) e^{-\alpha d(\Lambda_1, \Lambda_2)}, \tag{3.3b}$$

which will be used to prove Lemma 3, cf. (4.3).

For $d = 1$ (3.3) is proved in Cassandro and Olivieri (1981) and Dobrushin (1973).

The estimates (3.3a) and (3.3b) with $|\partial \Lambda_1|$ and $|\partial \Lambda_2|$ replaced by $|\Lambda_1|$ and $|\Lambda_2|$, respectively, hold for the larger class of potentials defined by Dobrushin's uniqueness criterion (1968). Although this would suffice for the proofs of the lemma mentioned, we need condition (3.2) not only for deriving (3.3) but also for the local central limit theorem estimates of Proposition 3. We do not know whether they keep their validity within the Dobrushin uniqueness region.

In view of (3.3) the following definition is meaningful.

DEFINITION 2. $\mathscr{H}$ is the Hilbert space obtained as the completion of $\mathscr{D}_0$ with scalar product (2.7) modulo $\{ f \mid \langle f \mid f \rangle = 0 \}$.

LEMMA 1. (i) *Let $f, g \in \mathscr{D}_0$. Then there exist positive constants $c_1(t), c_2(t)$ such that*

$$|\langle f \tau_x T_t g \rangle - \langle f \rangle \langle g \rangle| \le c_1(t) e^{-c_2(t)|x|}. \tag{3.4}$$

(ii) *T_t is a strongly continuous, self-adjoint contraction semigroup on $\mathscr{H}$. On $\mathscr{D}_0$ the generator, L, of T_t is given by* (1.5). *$\mathscr{D}_0$ is a domain of essential self-adjointness for L in $\mathscr{H}$.*

PROOF. Let $\mathscr{D}$ be the set of functions $f \in C(\Omega)$ such that there exist positive constants c_1 and c_2 with

$$\sup_{\substack{\eta, \eta': \\ \eta(x) = \eta'(x) \text{ for } |x| \le \ell}} | f(\eta) - f(\eta') | \le c_1 e^{-c_2 \ell} \tag{3.5}$$

for all $\ell \ge 0$. Then $T_t \mathscr{D} \subset \mathscr{D}$. This can be proved by the methods of Holley and Stroock (1976a, b). Part (i) follows then from (3.3) and (3.5), since any function in $\mathscr{D}$ can be approximated uniformly by functions $f_\Lambda \in \mathscr{F}_\Lambda$ such that the error is exponentially small in the diameter of Λ.

To prove part (ii) we note that for $f \in \mathscr{D}_0$ with $\langle f \rangle = 0$

$$\langle f \mid f \rangle = \lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-1} \sum_{x, y \in \Lambda} \langle \tau_x f \tau_y f \rangle. \tag{3.6}$$

Since T_t is a self-adjoint contraction in $L^2(\Lambda, \mu)$, we conclude that for $f \in \mathscr{D}_0$

$$\begin{aligned}\langle T_t f | T_t f \rangle &= \lim_{\Lambda \uparrow \mathbb{Z}^d} \left\langle \left(T_t |\Lambda|^{-1/2} \sum_{x \in \Lambda} \tau_x f \right)^2 \right\rangle \\ &\le \lim_{\Lambda \uparrow \mathbb{Z}^d} \left\langle \left(|\Lambda|^{-1/2} \sum_{x \in \Lambda} \tau_x f \right)^2 \right\rangle = \langle f | f \rangle.\end{aligned} \tag{3.7}$$

The symmetry of T_t follows by the same argument.

$\mathscr{D}_0$ is in the domain of the generator of T_t on $L_2(\Lambda, \mu)$ [Liggett (1977)]. Therefore for $f \in \mathscr{D}_0$

$$t^{-1}(T_t f - f) = t^{-1} \int_0^t ds\, T_s L f \tag{3.8}$$

which tends to Lf as $t \to 0$ in $\mathscr{H}$ by (3.4). By Theorem X.49 of Reed and Simon (1975), $\mathscr{D}$ is a domain of essential self-adjointness for L in $\mathscr{H}$. But for $f \in \mathscr{D}_0$, by (3.3) and (3.5), there exist functions $f_\Lambda \in \mathscr{F}_\Lambda \subset \mathscr{D}_0$ such that $f_\Lambda \to T_t f$ and $Lf_\Lambda \to T_t L f$ in $\mathscr{H}$ as $\Lambda \uparrow \mathbb{Z}^d$. □

From the spectral theorem we conclude that for every f

$$\lim_{t \to \infty} T_t f = \psi \in P\mathscr{H} \tag{3.9}$$

exists.

LEMMA 2. *Let Λ be a hypercube and let Γ_Λ be as defined in* (2.4). *Then for any $\psi \in P\mathscr{H}$ and any $g \in \mathscr{D}_0$*

$$\langle \psi | \Gamma_\Lambda g \rangle = \langle \psi | g \rangle. \tag{3.10}$$

PROOF. Let $f, g \in \mathscr{D}_0$ with $\langle f \rangle = \langle g \rangle = 0$. Then, using detailed balance (1.2) and Schwarz's inequality,

$$\begin{aligned}|\langle f | L_{xy} g \rangle|^2 &= \tfrac{1}{4} \left| \left\langle c(x, y, \eta)\, \partial_{xy} g \left(\sum_z \partial_{xy} \tau_z f \right) \right\rangle \right|^2 \\ &\le \tfrac{1}{4} \left\langle c(x, y, \eta) (\partial_{xy} g)^2 \right\rangle \left\langle c(x, y, \eta) \left(\sum_z \partial_{xy} \tau_z f \right)^2 \right\rangle \\ &\le |\langle g L_{xy} g \rangle \langle f | L f \rangle.\end{aligned} \tag{3.11}$$

In the last step we used the identity

$$\begin{aligned}\langle f | L f \rangle &= \tfrac{1}{2} \sum_{z'} \sum_{z, y} \left\langle (\tau_{z'} f) L_{zy} f \right\rangle \\ &= \tfrac{1}{2} \sum_{z', z, y} \left\langle (\tau_{z'} f) \tau_{x-z} \tau_{z-x} L_{zy} \tau_{x-z} \tau_{z-x} f \right\rangle \\ &= \tfrac{1}{2} \sum_{z', z, y} \left\langle \tau_{z'} f L_{xy} \tau_z f \right\rangle\end{aligned} \tag{3.12}$$

by the translation invariance of μ.

Let L_Λ be defined as in (2.3). Since by Lemma 1, $\mathscr{D}_0$ is a domain of essential self-adjointness for L, for any $f \in \mathscr{H}$ and any $g \in \mathscr{D}_0$

$$|\langle T_t f | L_\Lambda g \rangle|^2 \leq |\Lambda| \langle g L_\Lambda g \rangle \langle T_t f | L T_t f \rangle, \tag{3.13}$$

$t > 0$, and in the limit as $t \to \infty$

$$\langle \psi | L_\Lambda g \rangle = 0 \tag{3.14}$$

for any $\psi \in P\mathscr{H}$. By the argument given in Section 2 $g = L_\Lambda^{-1}(\tilde{g} - \Gamma_\Lambda \tilde{g}) \in \mathscr{D}_0$ for $\tilde{g} \in \mathscr{D}_0$. Inserting this g in (3.14) results then in (3.10). □

4. Equivalence of ensembles and the local limit theorem. In this section we establish the validity of (2.9).

PROPOSITION 2. *Let either Φ satisfy* (3.2) *or $d = 1$. Then in $\mathscr{H}$*

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} [\Gamma_\Lambda g - a(g)((1/|\Lambda|)n(\Lambda) - \rho)] = 0$$

for every $g \in \mathscr{D}_0$.

The proof of Proposition 2 is based on some refined version of the equivalence of ensembles imposed by the necessity of proving convergence in $\mathscr{H}$ rather than in the usual L^2-sense. To get a feeling for the meaning of convergence in $\mathscr{H}$ we state a criterion which will be used extensively in the sequel.

LEMMA 3. *Let ϕ satisfy* (3.2). *For a sequence $\{\Lambda\}$ of cubes centered at the origin let $f_\Lambda \in \mathscr{F}_\Lambda$ with $\langle f_\Lambda \rangle = 0$ and such that*

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda| \langle f_\Lambda^2 \rangle = 0. \tag{4.1}$$

Then

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle f_\Lambda | f_\Lambda \rangle = 0. \tag{4.2}$$

PROOF. Let ℓ be the side of the cube Λ. Then by (3.3)

$$\begin{aligned} \langle f_\Lambda | f_\Lambda \rangle &= \sum_x \langle f_\Lambda \tau_x f_\Lambda \rangle \\ &\quad + \sum_{|x| > 2\ell} \langle f_\Lambda \tau_x f_\Lambda \rangle + \sum_{|x| \leq 2\ell} \langle f_\Lambda \tau_x f_\Lambda \rangle \\ &\leq 2^d |\Lambda| \langle f_\Lambda^2 \rangle + \langle f_\Lambda^2 \rangle A |\partial\Lambda| \sum_{|x| > 2\ell} \exp(-\alpha[|x| - \ell]), \end{aligned} \tag{4.3}$$

which proves (4.2). □

The proof of Proposition 2 is obtained in two steps. In the first one we show that $\Gamma_\Lambda(g) := \mu_\lambda(g | n(\Lambda), \eta_\Lambda)$ and $\mu_{\lambda(n(\Lambda)/|\Lambda|)}(g)$ with $\lambda(\rho)$ implicitly defined by $\mu_{\lambda(\rho)}(\eta(0)) = \rho$ are close, in the sense that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda| \Big\langle \big(\Gamma_\Lambda(g) - \mu_{\lambda(n(\Lambda)/|\Lambda|)}(g)\big)^2 \Big\rangle = 0.$$

By Lemma 3 this implies closeness in $\mathscr{H}$. In the second step, we expand $\mu_{\lambda(n(\Lambda)/|\Lambda|)}(g)$ in powers of $((n(\Lambda)/|\Lambda|) - \rho_0)$. Here ρ_0 is the density of the Gibbs state we are considering and which was previously denoted by ρ. λ_0 is the corresponding chemical potential, i.e., $\lambda(\rho_0) = \lambda_0$. We have [for small $((n(\Lambda)/|\Lambda|) - \rho_0)$, the other values will be neglected because of large deviation estimates],

$$(4.4)\quad \mu_{\lambda(n(\Lambda)/|\Lambda|)}(g) = \mu_{\lambda_0}(g) + \mu'_{\lambda_0}(g)\left(\frac{n(\Lambda)}{|\Lambda|} - \rho\right) + \tfrac{1}{2}\mu''_{\tilde{\lambda}_0}(g)\left(\frac{n(\Lambda)}{|\Lambda|} - \rho\right)^2,$$

where μ'_λ and μ''_λ denote the first and the second derivatives of $\mu_{\lambda(\rho)}(g)$ with respect to ρ and $\tilde{\lambda}_0$ is some value between λ_0 and $\lambda(n(\Lambda)/|\Lambda|)$. By assumption $\mu_{\lambda_0}(g) = 0$ and it will be easy to show that the last term in the r.h.s. of (4.4) vanishes as $\Lambda \uparrow \mathbb{Z}^d$ in $\mathscr{H}$. The above strategy can be straightforwardly implemented when $d = 1$. The crucial point is that the dependence of $\Gamma_\Lambda(g)$ on η_{Λ^c} in one dimension goes like $1/|\Lambda|$ hence can be neglected by Lemma 3. In $d \geq 2$ the same estimates show that $\Gamma_\Lambda(g)$ changes with η_{Λ^c} of the order $|\partial\Lambda|/|\Lambda| \gtrsim |\Lambda|^{-1/2}$. At first sight, one would conclude that $\Gamma_\Lambda(g) - \mu_{\lambda(n(\Lambda)/|\Lambda|)}(g)$ does not tend to zero in the sense of Lemma 3. But we will show that $\Gamma_\Lambda(g)$ can be approximated by $\mu_{\lambda(n(\Lambda)/|\Lambda|, \eta_{\Lambda^c})}(g)$ where $\lambda(n(\Lambda)/|\Lambda|, \eta_{\Lambda^c})$ still depends on η_{Λ^c}, cf. Proposition 3 and (4.7). We expand then $\mu_{\lambda(n(\Lambda)/|\Lambda|, \eta_{\Lambda^c})}(g)$ in powers of $((n(\Lambda)/|\Lambda|) - \rho_0)$ and compare the terms of this expansion with those arising from expanding $\mu_{\lambda(n(\Lambda)/|\Lambda|)}(g)$. Their difference will be shown to be negligible in the sense of Lemma 3.

PROPOSITION 3. *Let Φ satisfy* (3.2). *For every outside configuration η_{Λ^c} we define the function $\rho \mapsto \lambda(\rho, \eta_{\Lambda^c})$, $0 < \rho < 1$, implicitly by*

$$(4.5)\qquad |\Lambda|^{-1}\mu_{\lambda(\rho, \eta_{\Lambda^c})}(n(\Lambda)|\eta_{\Lambda^c}) = \rho.$$

Then for every bounded $\bar{\Lambda} \subset \mathbb{Z}^d$ there exist a bounded set Λ_0, and constants $z, k > 0$, independently of η_{Λ^c}, such that the following holds: for any $\Lambda \supset \Lambda_0$, for any $\rho_\Lambda = n(\Lambda)/|\Lambda|$ such that

$$(4.6)\qquad |\rho_\Lambda - \rho_0| < z,$$

and for any $g \in \mathscr{F}_{\bar{\Lambda}}$, $|g| \leq 1$,

$$(4.7)\qquad \left|\Gamma_\Lambda(g) - \mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g)\right| \leq k|\bar{\Lambda}|/|\Lambda|$$

and $\lambda(\rho_\Lambda, \eta_{\Lambda^c})$ satisfies (3.2).

REMARKS. (i) The dependence of the canonical measure on η_{Λ^c}, given $n(\Lambda)$, is not exponentially small as it is for the corresponding Gibbs measure in the high temperature, low density domain. In fact it behaves as $|\partial\Lambda|/|\Lambda|$: first note that $\lambda(\rho_\Lambda, \eta_{\Lambda^c})$ changes with η_{Λ^c} of the order $|\partial\Lambda|/|\Lambda|$ and then use (4.7) together with the smooth dependence of $\mu_\lambda(g)$ on λ.

(ii) If we keep η_{Λ^c} fixed and vary N by δN, $\mu_\lambda(g|n(\Lambda) = N, \eta_{\Lambda^c})$ changes by order $\delta N \cdot |\Lambda|^{-1}$. Thus for typical fluctuations of the particle number, i.e.,

$\delta N \sim |\Lambda|^{1/2}$, the change is of order $|\Lambda|^{-1/2}$. As a consequence the choice of the Gibbs measure which approximates the canonical measure with error $\sim |\Lambda|^{-1}$ requires a very accurate choice of the chemical potential [the one of (4.5), which cannot be taken independently of η_{Λ^c} and of N].

PROOF OF PROPOSITION 3. The proof uses a local central limit theorem estimate with a sharp control on the error term. The connection with the local theorem comes from the remark [Dobrushin and Tirozzi (1977)] that

$$\frac{\mu_\lambda(\{\eta_{\bar\Lambda} = \xi_{\bar\Lambda}\}|\{n(\Lambda) = N\}, \eta_{\Lambda^c})}{\mu_\lambda(\{\eta_{\bar\Lambda} = \xi_{\bar\Lambda}\}|\eta_{\Lambda^c})} = \frac{\mu_\lambda(\{n(\Lambda) = N\}|\{\eta_{\bar\Lambda} = \xi_{\bar\Lambda}\}, \eta_{\Lambda^c})}{\mu_\lambda(\{n(\Lambda) = N\}|\eta_{\Lambda^c})} \tag{4.8}$$

for a given configuration $\xi_{\bar\Lambda}$ in $\bar\Lambda$ and for $N \le |\Lambda|$.

The local theorem estimate gives for the denominator in the r.h.s. of (4.8)

$$\sim (2\pi\sigma_1|\Lambda|)^{-1/2} \exp\left\{-(N - \bar N)^2/2\sigma_1^2|\Lambda|\right\}(1 + \text{error}),$$

where $\bar N$ is the mean of $n(\Lambda)$.

In order to have the accuracy dictated by (4.7) we can only allow for an error which goes as $|\Lambda|^{-1}$. The usual error bound proven in local theorems is of the order $|\Lambda|^{-1/2}$ and is uniform in N. However, in computing probabilities near the mean ($N = \bar N + O(1)$) the error drops to $|\Lambda|^{-1}$. This result—well known for Bernoulli random variables—extends also to weakly dependent Gibbs random fields [Del Grosso (1974), Iagolnitzer and Souillard (1979), and Pogosian (1979)].

We choose λ for each given value of N and η_{Λ^c} so that $\bar N = N$ [cf. (4.5) with $\rho = N/|\Lambda|$.] If N and η_{Λ^c} in (4.8) are such that the chemical potential $\lambda(N/|\Lambda|, \eta_{\Lambda^c})$ defined in (4.5) falls in the range specified by (3.2), then we have [Del Grosso (1974)], $\Lambda \ne \varnothing$,

$$\left|\mu_{\lambda(N/|\Lambda|, \eta_{\Lambda^c})}(n(\Lambda) = N|\eta_{\Lambda^c}) - \left(2\pi\sigma_1^2|\Lambda|\right)^{-1/2}\right| \le a|\Lambda|^{-3/2}, \tag{4.9a}$$

$$\begin{aligned}&\left|\mu_{\lambda(N/|\Lambda|, \eta_{\Lambda^c})}(n(\Lambda) = N|\eta_{\Lambda^c}, \eta_{\bar\Lambda} = \xi_{\bar\Lambda}) - \left(2\pi\sigma_1^2|\Lambda|\right)^{-1/2}\right| \\ &\qquad\le a|\bar\Lambda|\,|\Lambda|^{-3/2},\end{aligned} \tag{4.9b}$$

$$\sigma_1^2 = \mu_{\lambda(N/|\Lambda|, \eta_{\Lambda^c})}\left(|\Lambda|^{-1}(n(\Lambda) - N)^2|\eta_{\Lambda^c}\right), \tag{4.9c}$$

where a is a constant, the same for all cubes Λ centered at the origin. (4.7) now follows from (4.8) and (4.9).

To conclude the proof of the proposition we must show that $\lambda(n(\Lambda)/|\Lambda|, \eta_{\Lambda^c})$ satisfies (3.2) whenever $n(\Lambda)$ fulfills (4.6). We take z so small that for all $\tilde\rho$ with $|\tilde\rho - \rho| < 2z$, the corresponding $\tilde\lambda$ defined by $\mu_{\tilde\lambda}(\eta(0)) = \tilde\rho$ satisfies (3.2). [Such a z exists because $\mu_\lambda(\eta(0)) = \rho$ and λ satisfies (3.2).] Then for Λ sufficiently large

$$\left\{\tilde\lambda\colon |\mu_{\tilde\lambda}(\rho_\Lambda|\eta_{\Lambda^c}) - \rho| < z\right\} \subset \left\{\tilde\lambda\colon |\mu_{\tilde\lambda} - \rho| < 2z\right\} \tag{4.10}$$

uniformly in η_{Λ^c} since for Λ large

$$|\mu_{\tilde\lambda}(\rho_\Lambda|\eta_{\Lambda^c}) - \mu_{\tilde\lambda}(\eta(0))| \le (\text{const.})|\partial\Lambda|/|\Lambda| < z \tag{4.11}$$

uniformly in $\tilde\lambda$ and η_{Λ^c}.

We conclude that in $\mathscr{H}$

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \left(\Gamma_\Lambda g - \mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g | \eta_{\Lambda^c}) \right) = 0. \tag{4.12}$$

If $n(\Lambda)$ satisfies (4.5), this follows from Proposition 3 and Lemma 3. Otherwise, by large deviations results [Ellis (1984)], the probability is exponentially small in $|\Lambda|$.

Note that if $\lambda(\rho_\Lambda, \eta_{\Lambda^c})$ satisfies (3.2) then by (3.3a) we may replace in (4.12) $\mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g | \eta_{\Lambda^c})$ by $\mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g)$ since their difference is exponentially small. □

We next expand $\mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g)$ around $\mu_{\lambda(\rho_0, \eta_{\Lambda^c})}(g)$ where, to simplify notation, we denote by λ_0 the chemical potential of our (fixed) Gibbs state $\mu(= \mu_{\lambda_0})$, and by ρ_0 its density.

We denote the zero order term by:

$$\psi_\Lambda := \mu_{\lambda(\rho_0, \eta_{\Lambda^c})}(g) \tag{4.13}$$

and first show that ψ_Λ does to zero in $\mathscr{H}$. Since $\mu_{\lambda_0}(g) = 0$ we have

$$\psi_\Lambda = \mu'_{\tilde{\lambda}}(g) \left[\lambda(\rho_0, \eta_{\Lambda^c}) - \lambda_0 \right], \tag{4.14}$$

where $'$ refers to the derivative with respect to λ and $\tilde{\lambda}$ is suitably chosen between λ_0 and $\lambda(\rho_0, \eta_{\Lambda^c})$. According to Lemma 3 we have to center ψ_Λ and then to estimate its L^2-norm. Since by centering the norm decreases and since $\mu'_{\tilde{\lambda}}(g)$ is bounded, it suffices to estimate the L^2-norm of $\lambda(\rho_0, \eta_{\Lambda^c}) - \lambda_0$.

LEMMA 4. *For a hypercube* Λ *let* $\partial\Lambda$ *be its boundary and*

$$\delta\Lambda = \left\{ x \in \Lambda : d(x, \partial\Lambda) \le (\log|\Lambda|)^2 \right\}. \tag{4.15}$$

Then there exists a constant a, *independent of* Λ, *such that*

$$\left\langle \left(\lambda(\rho_0, \eta_{\Lambda^c}) - \lambda_0 \right)^2 \right\rangle \le a \left(|\delta\Lambda| / \Lambda| \right)^2 \tag{4.16}$$

and therefore

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \psi_\Lambda | \psi_\Lambda \rangle = 0. \tag{4.17}$$

PROOF. Let $F(\cdot, \eta_{\Lambda^c})$ be the inverse function of $\rho \to \lambda(\rho, \eta_{\Lambda^c})$ i.e.,

$$F(\lambda, \eta_{\Lambda^c}) := \mu_\lambda\left(n(\Lambda)/|\Lambda| | \eta_{\Lambda^c} \right). \tag{4.18}$$

We already know that when $\Lambda \uparrow \mathbb{Z}^d$, $\lambda(\rho_0, \eta_{\Lambda^c})$ goes to λ_0 uniformly in η_{Λ^c}. We expand

$$F(\lambda, \eta_{\Lambda^c}) = F(\lambda_0, \eta_{\Lambda^c}) + F'(\tilde{\lambda}, \eta_{\Lambda^c})(\lambda - \lambda_0), \tag{4.19}$$

where $\tilde{\lambda}$ is between λ and λ_0. Since

$$F'(\lambda, \eta_{\Lambda^c}) = \sum_{x, y \in \Lambda} |\Lambda|^{-1} \mu_\lambda\left(\tilde{\eta}(x) \tilde{\eta}(y) \right) | \eta_{\Lambda^c}) \tag{4.20}$$

with $\tilde{\eta}(x) = \eta(x) - \mu_\lambda(\eta(x) | \eta_{\Lambda^c})$, and since

$$\sum_x \mu_{\lambda_0}\left((\eta(0) - \rho_0)(\eta(x) - \rho_0 \right) > 0, \tag{4.21}$$

$F'(\tilde{\lambda}, \eta_{\Lambda^c})$ is bounded away from zero uniformly in η_{Λ^c} for large Λ, provided in (4.19) λ is in a suitably small neighborhood of λ_0. From (4.19) and (4.18) for Λ large enough and all η_{Λ^c} we have

$$\lambda(\rho_0, \eta_{\Lambda^c}) - \lambda_0 = K(\lambda(\rho_0, \eta_{\Lambda^c}), \lambda_0, \eta_{\Lambda^c})(\rho_0 - F(\lambda_0, \eta_{\Lambda^c})) \tag{4.22}$$

with $K(\lambda, \lambda_0, \eta_{\Lambda^c}) = F'(\tilde{\lambda}, \eta_{\Lambda^c})^{-1}$. K is uniformly bounded in η_{Λ^c}.

We have to show then that there exists a constant a_1, such that

$$\left\langle (F(\lambda_0, \eta_{\Lambda^c}) - \rho_0)^2 \right\rangle \le a_1(|\delta\Lambda|/|\Lambda|)^2. \tag{4.23}$$

We have

$$\begin{aligned} &\left\langle (F(\lambda_0, \eta_{\Lambda^c}) - \rho_0)^2 \right\rangle \\ &\quad = |\Lambda|^{-2} \sum_{x, y \in \Lambda} \left\langle \mu_{\lambda_0}(\eta(x) - \rho_0 | \eta_{\Lambda^c}) \mu_{\lambda_0}(\eta(y) - \rho_0 | \eta_{\Lambda^c}) \right\rangle. \end{aligned} \tag{4.24}$$

Because of the exponential decay of the correlations there exists a_2 such that

$$\sum_{x \in \Lambda \setminus \delta\Lambda} \left| \mu_{\lambda_0}(\eta(x) - \rho_0 | \eta_{\Lambda^c}) \right| \le a_2 |\Lambda \setminus \delta\Lambda| / |\Lambda|^2, \tag{4.25}$$

Therefore

$$\begin{aligned} \left\langle (F(\lambda_0, \eta_{\Lambda^c}) - \rho_0)^2 \right\rangle &\le 2a_2/|\Lambda|^2 + |\Lambda|^{-2} \left\langle \mu_{\lambda_0}\left(\sum_{x \in \delta\Lambda} (n(x) - \rho_0) \middle| \eta_{\Lambda^c} \right)^2 \right\rangle \\ &\le 2a_2/|\Lambda|^2 + |\Lambda|^{-2} \left\langle \mu_{\lambda_0}\left(\left[\sum_{x \in \delta\Lambda} (\eta(x) - \rho_0) \right]^2 \middle| \eta_{\Lambda^c} \right) \right\rangle \\ &\le 2a_2/|\Lambda|^2 + |\Lambda|^{-2} \mu_{\lambda_0}\left(\left[\sum_{x \in \delta\Lambda} (\eta(x) - \rho_0 \right]^2 \right) \\ &\le 2a_2/|\Lambda|^2 + a_3 |\Lambda|^{-2} |\delta\Lambda|^2. \quad \square \end{aligned} \tag{4.26}$$

PROOF OF PROPOSITION 2. Using (4.12), Lemma 4 and Lemma 3 we only have to show that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda| \left\langle \left[\mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g) - \mu_{\lambda(\rho_0, \eta_{\Lambda^c})}(g) - a(g)(\rho_\Lambda - \rho_0) \right]^2 \right\rangle = 0. \tag{4.27}$$

We denote by $\chi(\rho_\Lambda, \eta_{\Lambda^c})$ the characteristic function of the event "$\lambda(\rho_\Lambda, \eta_{\Lambda^c})$ satisfies (3.2)." In the set $\{\chi(\rho_\Lambda, \eta_{\Lambda^c}) = 1\}$ we expand $\mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g) - \mu_{\lambda(\rho_0, \eta_{\Lambda^c})}(g)$ around ρ_0 and we obtain

$$\chi(\rho_\Lambda, \eta_{\Lambda^c}) \left[\mu'_{\lambda(\rho_0, \eta_{\Lambda^c})}(g)(\rho_\Lambda - \rho_0) + \tfrac{1}{2} \mu''_{\lambda(\tilde{\rho}_0, \eta_{\Lambda^c})}(g)(\rho_\Lambda - \rho_0)^2 \right], \tag{4.28}$$

where μ', μ'' denote the first and second derivatives of $\mu_{\lambda(\rho, \eta_{\Lambda^c})}(g)$ with respect to ρ and $\tilde{\rho}$ is suitably chosen between ρ_0 and $\rho_\Lambda = n(\Lambda)/|\Lambda|$. A large deviation result, Ellis (1984), shows that we can neglect

$$(1 - \chi(\rho_\Lambda, \eta_{\Lambda^c}))\Big[\mu'_{\lambda(\rho_0, \eta_{\Lambda^c})}(g)(\rho_\Lambda - \rho_0) + \big(\mu_{\lambda(\rho_\Lambda, \eta_{\Lambda^c})}(g) - \mu_{\lambda(\rho_0, \eta_{\Lambda^c})}(g)\big)\Big],$$

since its L^2-norm is exponentially small in $|\Lambda|$. (4.27) follows then from the fact that there exists a constant $a > 0$ so that $[\mu'_{\lambda_0}(g) = a(g) = d/d\rho\, \mu_{\lambda(\rho)}(g)|_{\rho=\rho_0}$ below]

$$(4.29)\qquad \begin{aligned} \left|\mu'_{\lambda(\rho_0, \eta_{\Lambda^c})}(g) - \mu'_{\lambda_0}(g)\right| &\le a|\partial\Lambda|/|\Lambda|, \\ \chi(\rho_\Lambda, \eta_{\Lambda^c})\left|\mu''_{\lambda(\tilde{\rho}_0, \eta_{\Lambda^c})}(g)\right| &\le a \\ \left\langle(\rho_\Lambda - \rho_0)^2\right\rangle &\le a|\Lambda|^{-1} \\ \left\langle(\rho_\Lambda - \rho_0)^4\right\rangle &\le a|\Lambda|^{-2}. \end{aligned}$$

For $d = 1$ we follow the same proof and use (i) the analyticity of the pressure in a complex neighborhood of the chemical potential λ uniformly in the region Λ and (ii) the exponential decay of the correlations [Cassandro and Olivieri (1981) and Dobrushin (1973)]. □

5. Equilibrium fluctuations for exclusion processes of gradient type.

We want to prove that for exclusion processes with speed change

$$(5.1)\qquad \lim_{\varepsilon\to 0} Y_t^\varepsilon(\phi)$$

exists as a process with values in $\mathscr{S}'(\mathbb{R}^d)$ and that the limit is an infinite-dimensional Ornstein–Uhlenbeck process. In general even in one dimension or with the smallness condition (3.2) this problem is open. However with a further condition, called gradient type, (5.1) can be established. All zero-range processes and Brownian particles interacting by a pair potential are of gradient type [Brox and Rost (1983) and Spohn (1986)]. The meaning of this condition is more fully explored in De Masi et al. (1984).

To motivate our definition let us first consider the simple symmetric exclusion process defined by

$$(5.2)\qquad c(x, y, \eta) = p(x - y)(\eta(x) - \eta(y))^2$$

with $p(x) = p(-x)$. Then

$$(5.3)\qquad L\eta(0) = \sum p(x)(\eta(x) - \eta(0)).$$

A more general class of systems is obtained by requiring (5.3) but with $\eta(x)$ replaced by $\tau_x h$ for some $h \in \mathscr{D}_0$.

Definition 3. An exclusion process with speed change is said to be of gradient type if there exist $h_m \in \mathscr{D}_0$ and functions p_m on $\mathbb{Z}^d$ of bounded support with $\sum_x p_m(x) = 0$, $\sum_x x p_m(x) = 0$, $m = 1, \dots, M$ such that

$$(5.4)\qquad L\eta(0) = \sum_{m=1}^{M} \sum_x p_m(x)\tau_x h_m(\eta).$$

In one dimension with nearest-neighbor jumps depending on the nearest neighbors of the bond only, for a given temperature the jump rates satisfying detailed balance depend on four free parameters. The gradient condition then singles out a two-dimensional hypersurface in this parameter space [Katz et al. (1983) and Spohn (1982)].

For a gradient system the bulk diffusion matrix is given by:

$$D_{ij} = (4\chi)^{-1}\sum_{x} x_i x_j \left\langle c(0, x, \eta)(\eta(x) - \eta(0))^2 \right\rangle, \tag{5.5}$$

$i, j = 1, \ldots, d$ with $x = (x_1, \ldots, x_d)$ [Spohn (1982)]. $D > 0$, as a matrix because of (1.1). Given D we define the Gaussian kernel

$$C_t(q) = \chi(2\pi|t|)^{-d/2}(\det D)^{-1/2}\exp\left[-\sum_{i,j=1}^{d} q_i D_{ij}^{-1} q_j/2|t|\right]. \tag{5.6}$$

THEOREM 2. *Let the jump rates of the exclusion process η_t be of gradient type. Let either $d = 1$ or $d \geq 2$ with the potential ϕ and ρ satisfying the smallness condition* (3.2). *Then*:

(i)

$$\lim_{\varepsilon \to 0} E(Y_t^\varepsilon(\phi_1)Y_s^\varepsilon(\phi_2)) = \int_{\mathbb{R}^d} dq \int_{\mathbb{R}^d} dq'\, \phi_1(q) C_{|t-s|}(q - q')\phi_2(q') \tag{5.7}$$

with $C_t(q)$ given by (5.6) *and D by* (5.5).

(ii) *Let P^ε be the path measure of $Y_t^\varepsilon(\cdot)$ considered as a stochastic process on $D(\mathbb{R}, \mathscr{S}'(\mathbb{R}^d))$ and let P be the path measure of the Gaussian process on $D(\mathbb{R}, \mathscr{S}'(\mathbb{R}^d))$ with mean zero and covariance* (5.7). *Then weakly*

$$\lim_{\varepsilon \to 0} P^\varepsilon = P.$$

PROOF. Given (2.1) and the gradient type assumption part (i) follows by the same argument as in Brox and Rost (1983), cf. also De Masi et al. (1984), Section 5, and Holley and Stroock (1978). Assuming Theorem 1 and the gradient type assumption, Part (ii) is proved in De Masi et al., Section 6. □

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