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Fourier Law, Phase Transitions and the Stationary Stefan Problem

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Abstract

We study the one-dimensional stationary solutions of the integro-differential equation which, as proved in Giacomin and Lebowitz (J Stat Phys 87:37–61, 1997; SIAM J Appl Math 58:1707–1729, 1998), describes the limit behavior of the Kawasaki dynamics in Ising systems with Kac potentials. We construct stationary solutions with non-zero current and prove the validity of the Fourier law in the thermodynamic limit showing that below the critical temperature the limit equilibrium profile has a discontinuity (which defines the position of the interface) and satisfies a stationary free boundary Stefan problem. Under-cooling and over-heating effects are also studied: we show that if metastable values are imposed at the boundaries then the mesoscopic stationary profile is no longer monotone and therefore the Fourier law is not satisfied. It regains its validity however in the thermodynamic limit where the limit profile is again monotone away from the interface.

1. Introduction

When hydrodynamic or thermodynamic limits are performed in systems which are in the phase transitions regime, we may observe perfectly smooth profiles develop singularities with the appearance of sharp interfaces. We shall study the phenomenon in stationary non-equilibrium states which carry non-zero steady currents; the general context is the one where the Fourier law applies, but here it is complemented by a free boundary problem due to the presence of interfaces. We work at the mesoscopic level considering a model which has been derived in [5,6] from Ising systems with Kac potentials and Kawasaki dynamics, and derive in the hydrodynamic limit macroscopic profiles with interfaces which satisfy a stationary Stefan problem and obey the Fourier law.

The mesoscopic model is defined in terms of a free energy functional, the Lebowitz and Penrose (L–P) functional (see (2.1) in the next section) which is a non-local version of the scalar Ginzburg–Landau (or Allen–Cahn or Cahn–Hilliard) functional. Its thermodynamic free energy density is obtained by minimizing the L–P functional over profiles with fixed total magnetization density, and then taking the thermodynamic limit where the spatial size of the system diverges. It is found that the phase diagram (of free energy density versus magnetization density) obtained in this way has a non-trivial flat interval $[-m_\beta, m_\beta]$ (indicative of a phase transition) when the inverse temperature β is above the critical value (equal to 1 here). This is in qualitative and quantitative agreement with the thermodynamics of the underlying Ising model with Kac potentials; see Chapter 9 in [10] and references therein. The axiomatic theory for such phase diagrams predicts that the values inside $(-m_\beta, m_\beta)$ do not appear in any stationary local equilibrium state, so that a macroscopic magnetization density profile will have a discontinuity if it assumes values both smaller than $-m_\beta$ and larger than m_β.

This is just what we see. We fix $\beta > 1$ and study the stationary solutions of the equations of motion $\frac{\mathrm{d}m}{\mathrm{d}t} = -\mathrm{div}\ I$, that is div $I = 0$, I the local current (of the conserved order parameter, the magnetization density m here). By a gradient flow assumption on its constitutive law, I is supposed proportional to the gradient of the functional derivative of the L–P functional: due to the non-local structure of the latter, $\frac{\mathrm{d}m}{\mathrm{d}t} = -\mathrm{div}\ I$ is an integro-differential equation (see (2.19) in the next section), which is the same as the one derived by Giacomin and Lebowitz from the Ising system [5,6], and which has been much studied in the past years [1,7,8]. We look for solutions of div $I = 0$ with a planar symmetry, thus reducing to a one dimensional problem, and prove existence and smoothness of solutions with a steady non-zero current. However in the hydrodynamic limit where the size L of the system diverges, the stationary profile, once expressed in macroscopic space units (that is, proportional to L), is proved to converge to a discontinuous limit profile, solution of a stationary free boundary problem, the stationary Stefan problem, in agreement with the axiomatic macroscopic theory. The mesoscopic theory is in this respect in complete agreement with the macroscopic one; the mesoscopic profiles are smooth versions of the macroscopic ones. They are monotone as well and the current is proportional to [minus] the magnetization density gradient in agreement with the Fourier law, which we may then say to be valid at the mesoscopic level as well.

The mesoscopic theory has, however, a richer and more complex structure even in the macroscopic limit. This is seen for instance if we impose boundary conditions which force metastable values at the boundaries, the metastable region being made of two separate intervals called the plus and the minus metastable phases (according to the sign of the magnetization) which (together with the spinodal region) are contained in the "forbidden region" $(-m_\beta, m_\beta)$. With boundary conditions, one in the minus, the other in the plus metastable phases, the mesoscopic stationary magnetization density profiles are not monotone anymore. We have the "paradoxical" result of a positive [magnetization] current when the total magnetization gradient is also positive, having fixed at the left and right, respectively, a negative and a positive metastable value of the magnetization. The mesoscopic stationary profile is then first decreasing, then increasing and then again decreasing. The Fourier's law is therefore not satisfied but, in the thermodynamic limit, the region where the profile increases shrinks to a point, which is where the limit profile has a discontinuity (a sharp interface). Elsewhere the profile is always decreasing in agreement with the Fourier's law (as the current is positive). The stationary profile, therefore, has values all in the metastable region (except at the interface, which macroscopically is only a point). All the issues presented in this introduction are discussed in more detail in the next section; proofs are given in the remaining ones.

2. Model, Backgrounds and Main Results

The free energy functional to which we have been referring so far is defined on functions $m \in L^\infty(\Lambda, [-1, 1])$, Λ a bounded measurable subset of $\mathbb{R}^d$, as

$$\begin{aligned} F_{\beta,\Lambda}(m|m_{\Lambda^c}) &= F_{\beta,\Lambda}(m) + \frac{1}{2}\int_\Lambda \int_{\Lambda^c} J(x, y)[m(x) - m_{\Lambda^c}(y)]^2 \mathrm{d}x\, \mathrm{d}y \\ F_{\beta,\Lambda}(m) &= \int_\Lambda \phi_\beta(m)\mathrm{d}x + \frac{1}{4}\int_\Lambda \int_\Lambda J(x, y)[m(x) - m(y)]^2 \mathrm{d}x\, \mathrm{d}y, \end{aligned} \tag{2.1}$$

where $J(x, y) = J(|x - y|)$ is a smooth, translational invariant, probability kernel of range 1; $m_{\Lambda^c} \in L^\infty(\Lambda^c, [-1, 1])$ is a fixed external profile and

$$\phi_\beta(m) = -\frac{1}{2}m^2 - \frac{1}{\beta}S(m), \ -S(m) = \frac{1+m}{2}\log\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\log\left(\frac{1-m}{2}\right). \tag{2.2}$$

To simplify the analysis we suppose Λ a cube and consider Neumann boundary conditions, namely the functional

$$F^{\text{neum}}_{\beta,\Lambda}(m) = \int_\Lambda \phi_\beta(m)\mathrm{d}x + \frac{1}{4}\int_\Lambda \int_\Lambda J^{\text{neum}}(x, y)[m(x) - m(y)]^2 \mathrm{d}x\, \mathrm{d}y, \tag{2.3}$$

where $J^{\text{neum}}(x, y) = \sum_{z \in R_\Lambda(y)} J(x, z)$ with $R_\Lambda(y)$ the set image of y under reflections of the cube Λ around its faces. In $d = 1$, if $\Lambda = \epsilon^{-1}[-1, 1]$, $J^{\text{neum}}(x, y) = J(x, y) + J(x, 2\epsilon^{-1} - y) + J(x, -2\epsilon^{-1} - y)$ ($\epsilon > 0$ is a scaling parameter which will vanish in the thermodynamic limit). With minor modification what follows in the next item, "Equilibrium thermodynamics", holds as well for general boundary conditions as those considered in (2.1).

2.1. *Equilibrium Thermodynamics of the Mesoscopic Model*

(The statements in this paragraph are proved in Section 6.1 of [10]). The thermodynamic free energy density $a_\beta(s)$, $s \in [-1, 1]$, is defined as

$$a_\beta(s) := \lim_{\Lambda \to \mathbb{R}^d} \inf \left\{ F^{\text{neum}}_{\beta,\Lambda}(m) \;\middle|\; \fint_\Lambda m \, \mathrm{d}x = s \right\} \tag{2.4}$$

The limit on the right-hand side exists and it is equal to:

$$a_\beta = \phi^*_\beta = \text{ convex envelope of } \phi_\beta(\cdot) \tag{2.5}$$

$\phi^*_\beta \equiv \phi_\beta$ when $\beta \leqq 1$ and $\phi^*_\beta \neq \phi_\beta$ when $\beta > 1$. More precisely let m_β be the positive solution of

$$m_\beta = \tanh\{\beta m_\beta\}, \quad \beta > 1, \tag{2.6}$$

then $\phi^*_\beta(s)$, $s \in (-m_\beta, m_\beta)$, is constant and strictly smaller than $\phi_\beta(s)$, while $\phi^*_\beta(s) = \phi_\beta(s)$ elsewhere. The values of the magnetization in the interval $(-m_\beta, m_\beta)$ are "forbidden". This is best seen working in the grand canonical ensemble (in other words, using Lagrange multipliers). To this end we add a constant magnetic field h so that the free energy functional becomes

$$F^{\text{neum}}_{\beta,h,\Lambda}(m) = F^{\text{neum}}_{\beta,\Lambda}(m) - h \int_\Lambda m \, \mathrm{d}x. \tag{2.7}$$

The grand canonical thermodynamic pressure $p_\beta(h)$ is defined by a minimization problem without constraints:

$$p_\beta(h) = \lim_{\Lambda \to \mathbb{R}^d} \sup \left\{ -F^{\text{neum}}_{\beta,h,\Lambda}(m) \;\middle|\; m \in L^\infty(\Lambda, [-1,1]) \right\}. \tag{2.8}$$

Existence of the limit is, again, a fact, and the thermodynamics defined by the free energy a_β and by the pressure p_β are equivalent, a property called in statistical mechanics "equivalence of ensembles". Namely, p_β and a_β are interrelated as in thermodynamics, one being the Legendre transform of the other:

$$p_\beta(h) = \sup\left\{hs - a_\beta(s) \mid s \in [-1,1]\right\}, \quad a_\beta(s) = \sup\left\{hs - p_\beta(h) \mid h \in \mathbb{R}\right\}. \tag{2.9}$$

For any $\beta > 1$ and any $h \in \mathbb{R}$, any maximizer of (2.8), at least for Λ large enough, is a constant function equal to $m_{\beta,h}$ where $m_{\beta,h}$ is the solution of the mean field equation

$$m_{\beta,h} = \tanh\{\beta(m_{\beta,h} + h)\}, \tag{2.10}$$

which minimizes $\phi_\beta(s) - hs$ and therefore it is not in $(-m_\beta, m_\beta)$, the values in $(-m_\beta, m_\beta)$ "are therefore forbidden".

2.2. *Gibbsian Equilibrium Thermodynamics*

The thermodynamics obtained above are in qualitative and quantitative agreement with the thermodynamics of the underlying microscopic model, that is, the Ising system with Kac potential. The Gibbs canonical equilibrium free energy $f_{\beta,\gamma}(m)$ is defined as

$$f_{\beta,\gamma}(m) := \lim_{\delta \to 0} \lim_{\Lambda_n \to \mathbb{Z}^d} \frac{-1}{\beta|\Lambda_n|} \log Z_{\Lambda_n,\beta,\gamma} \tag{2.11}$$

$$Z_{\Lambda_n,\beta,\gamma} = \sum_{\sigma_{\Lambda_n} \in \{-1,1\}^{\Lambda_n}} \mathbf{1}\left(\Big| \sum_{x \in \Lambda_n} (\sigma_{\Lambda_n}(x) - m) \Big| \leqq \delta |\Lambda_n| \right) \mathrm{e}^{-\beta H_{\gamma,\Lambda_n}(\sigma_{\Lambda_n})},$$

where Λ_n is a sequence of increasing cubes and

$$H_{\gamma,\Lambda}(\sigma_\Lambda) = -\frac{1}{2}\sum_{x\neq y\in\Lambda} J_\gamma(x,y)\sigma_\Lambda(x)\sigma_\Lambda(y), \quad J_\gamma(x,y) = \gamma^d J_\gamma(\gamma|x-y|). \tag{2.12}$$

(The same free energy is obtained for more general regions and boundary conditions). As discussed in Chapter 9 of [10], in $d \geqq 2$ for any $\beta > 1$ and $\gamma > 0$ small enough, $f_{\beta,\gamma}(m)$ is flat in an interval $[-m_{\beta,\gamma}, m_{\beta,\gamma}]$ and $m_{\beta,\gamma} \to m_\beta$ as $\gamma \to 0$ (in $d = 1$ $f_{\beta,\gamma}$ is instead strictly convex for any $\gamma > 0$). The original result has been proved in [3] and [2] while the fact that in any $d \geqq 1$, $\lim_{\gamma\to 0} f_{\beta,\gamma}(m) = a_\beta(m)$ is much older and proved by Lebowitz and Penrose [9].

2.3. Axiomatic Non-equilibrium Macroscopic Theory

The basic postulates are (i)–(iv).

(i) *local equilibrium and barometric formula*. The free energy of a macroscopic profile m in the macroscopic (bounded) region $\Omega \subset \mathbb{R}^d$ is given by the local functional:

$$F^{\text{macro}}_{\beta,\Omega}(m) := \int_\Omega a_\beta(m)\,\mathrm{d}x, \quad m \in L^\infty(\Omega, [-1,1]). \tag{2.13}$$

(ii) *gradient dynamics*. The evolution equation in the interior of Ω is the conservation law (D below denoting functional derivative)

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\nabla\cdot j, \quad j = -\chi\nabla D F^{\text{macro}}_{\beta,\Omega} = -\chi\nabla a'_\beta,\ a'_\beta(s) := \frac{\mathrm{d}a_\beta(s)}{\mathrm{d}s}. \tag{2.14}$$

(iii) *mobility coefficient*. χ is a mobility coefficient which depends on the dynamical characteristics of the system. we take

$$\chi(s) = \beta\left(1 - s^2\right), \tag{2.15}$$

as this is what is found when deriving (2.14) from the Ising spins [5,6,8].

In the usual setup for Fourier law Ω is a parallelepiped and different values of the order parameter are imposed on its right and left faces, while Neumann (or periodic) conditions are imposed on the other faces. By assuming a planar symmetry the problem becomes one dimensional, and from now on we are restricted to $d = 1$ taking $\Omega = [-\ell, \ell]$. A stationary profile m is then an element of $C^1((-\ell,\ell), [-1,1])$ such that

$$D_\beta\frac{\mathrm{d}m}{\mathrm{d}x} = -j = \text{constant},\ D_\beta(m) = \chi(m)a''_\beta(m), \tag{2.16}$$

where j is the constant current flowing in the system. (2.16) is then supplemented by Dirichlet boundary conditions at $\pm\ell$, namely $m(x) \to m_\pm$ as $x \to \pm\ell$. When $\beta < 1$, $D_\beta > 0$ and the problem has a unique solution, while when $\beta > 1$,

$D_\beta(m) = 0$ when $m \in (-m_\beta, m_\beta)$ and in the present formulation the problem has no solution if $m_- < -m_\beta$ and $m_+ > m_\beta$ (or vice-versa). The theory then needs a further postulate:

(iv) *The stationary Stefan problem.* Suppose $\beta > 1$ and boundary conditions $m_- < -m_\beta$ and $m_+ > m_\beta$ (the opposite case, $m_- > -m_\beta$ and $m_+ < -m_\beta$ being recovered by symmetry). We then say that m is stationary if there is $x_0 \in (-\ell, \ell)$ so that m is stationary in the sense of (2.16) both in $(-\ell, x_0)$ and (x_0, ℓ) with boundary conditions $m_-, -m_\beta$ and, respectively, m_β, m_+, with the additional condition that there is a same current j in both regions. The profile $m(x)$ is therefore discontinuous at x_0 and we say that at x_0 there is a sharp interface which separates the positive and negative phases.

A different formulation of the problem is, however, more convenient for our purposes. We start by a change of variables, going from the magnetization m to the magnetic field h. There is a one-to-one correspondence between the two when $\{m \geqq m_\beta\}$ and $\{h \geqq 0\}$, as well as when $\{m \leqq -m_\beta\}$ and $\{h \leqq 0\}$. We set $h = a'_\beta(m)$; its inverse gives m as a function of h which is obtained by solving $m = \tanh\{\beta h + \beta m\}$. Expressed in terms of the magnetic field, (2.16) becomes

$$h(x) = \int_{x_0}^{x} \frac{-j}{\chi(m)} \mathrm{d}x', \quad m = (a'_\beta)^{-1}(h). \tag{2.17}$$

In (2.17), $\chi(m) = \chi(m(h))$ is regarded as a function of h and (2.17) becomes an integral equation in $h(\cdot)$ where, however, x_0 and j are also unknown; they must be determined by imposing the boundary conditions $h(\pm\ell) = h_\pm := a'_\beta(m_\pm)$. All this suggests a new formulation alternative to the Dirichlet problem where we assign x_0 and j instead of $m_\pm$. In this way the Stefan problem is written in a compact way as in (2.17) above, which is now a "pure" integral equation for $h(\cdot)$ with x_0 and j known data. Clearly any solution of (2.16) defines a solution of (2.17) and vice-versa. In the sequel we shall mostly use the formulation (2.17) when proving that the Stefan problem with assigned x_0 and j can be derived from the mesoscopic theory.

As a difference with the Dirichlet problem, in the "x_0, j problem" there is no "global existence theorem", in the sense that given x_0 and j there are no solutions if ℓ is too large. Take the antisymmetric solution of (2.16) with $x_0 = 0$ and $j < 0$. It can be easily seen that $D_\beta(m)$ is strictly positive for all $m \in [m_\beta, 1]$ (actually it can be seen that $D_\beta(m) \to 1$ as $m \to 1$) so that $m(x)$ reaches the value 1 at some $\ell_j < \infty$, and therefore the problem with $(x_0 = 0, \ell > \ell_j)$ has no solution. In conclusion (2.17) with $x_0 = 0$ and $j < 0$ has a "maximal solution" $(m_j(x), h_j(x))$. Namely, there is a bounded interval $(-\ell_j, \ell_j)$ such that

$$\lim_{x \to \pm\ell_j} m_j(x) = \pm 1, \quad \lim_{x \to \pm\ell_j} h_j(x) = \pm\infty. \tag{2.18}$$

Equation (2.17) has no solution if $\ell > \ell_j$, while any other solution of (2.17) with the same j is obtained, modulo translations, by restricting the maximal solution to a suitable interval contained in $(-\ell_j, \ell_j)$. The collection of all the maximal solutions $(m_j(x), h_j(x))$ when $j \in \mathbb{R} \setminus \{0\}$ determines, in the sense explained above, all the possible solutions of (2.17). Since $\ell_j \to 0$ as $j \to \infty$ and $\ell_j \to \infty$ as $j \to 0$, it then follows that for any ℓ the Dirichlet problem with data $m_\pm$ at $\pm\ell$ ($m_+ \neq m_-$, $m_\pm$ in the complement of $[-m_\beta, m_\beta]$) can be obtained as described above from the collection of all the maximal solutions. By taking limits we can also include m_β and $-m_\beta$.

By restricting to intervals strictly contained in the maximal interval $[-\ell_j, \ell_j]$, the solution (m, h) of (2.17) is smooth, $\|m\| < 1$, $\chi(m)$ bounded away from 0 and $\|h\| < \infty$. These are the properties of the macroscopic solution which will be repeatedly used in the sequel.

2.4. Mesoscopic Theory and Stationary Profiles

Dynamics are defined using the same postulate of the macroscopic theory, that is, they are the gradient flow of the free energy functional which, in the mesoscopic theory is (2.3) (supposing again Neumann conditions). The gradient flow is (D below denoting functional derivative)

$$\begin{aligned} &\frac{\mathrm{d}m}{\mathrm{d}t} = -\nabla \cdot I, \quad I = -\chi \nabla \left(D F_{\beta,\Lambda} \right) \\ &I = -\chi \nabla \left(\frac{1}{2\beta} \log \frac{1+m}{1-m} - \int J^{\mathrm{neum}}(x,y) m(y)\, \mathrm{d}y \right). \end{aligned} \tag{2.19}$$

With the choice $\chi = \beta(1-m^2)$ (that we adopt hereafter) (2.19) becomes the one found in [5,6] from the Ising spins. We suppose, again, a planar symmetry to reduce to one dimension, take $\Lambda = \epsilon^{-1}[-\ell, \ell]$ interpreting ϵ^{-1} as the ratio of macroscopic and mesoscopic lengths so that (2.19) becomes

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}x} \left(-\frac{\mathrm{d}m}{\mathrm{d}x} + \beta(1-m^2) \frac{\mathrm{d}}{\mathrm{d}x} J^{\mathrm{neum}} * m \right). \tag{2.20}$$

As in the macroscopic theory, it is now convenient to change variables. Define $h(x)$ as

$$h := \frac{1}{2\beta} \log \frac{1+m}{1-m} - J^{\mathrm{neum}} * m. \tag{2.21}$$

Then the current I in (2.19) has the expression

$$I = -\chi(m) \frac{\mathrm{d}h}{\mathrm{d}x}, \quad m = \tanh\{\beta J^{\mathrm{neum}} * m + \beta h\}. \tag{2.22}$$

The stationary mesoscopic problem in the x_0, j formulation. Given any $x_0 \in (-\ell, \ell)$ and $j < 0$, find

$$(m, h) \in \mathcal{X}_{\epsilon,\ell} := L^\infty(\epsilon^{-1}(-\ell,\ell); [-1,1]) \times L^\infty(\epsilon^{-1}(-\ell,\ell); \mathbb{R})$$

so that

$$m = \tanh\{\beta J^{\mathrm{neum}} * m + \beta h\}, \quad h(x) = \int_{\epsilon^{-1}x_0}^{x} \frac{-\epsilon j}{\chi(m)} \mathrm{d}x'. \tag{2.23}$$

Denoting by (m_ϵ, h_ϵ) a solution of (2.23) we define:

$$(m^{(\epsilon)}, h^{(\epsilon)}) \in \mathcal{X}_{1,\ell}, \quad (m^{(\epsilon)}(x), h^{(\epsilon)}(x)) = (m_\epsilon(\epsilon^{-1}x), h_\epsilon(\epsilon^{-1}x)). \quad (2.24)$$

We first consider the simpler case where $x_0 = 0$ and look for solutions where m and h are both odd functions.

Theorem 1. *Let $j \neq 0$, $x_0 = 0$, $\ell > 0$ and smaller than ℓ_j (see (2.18)). Then for any $\epsilon > 0$ small enough there is a continuous, antisymmetric pair $(m_\epsilon, h_\epsilon) \in \mathcal{X}_{\epsilon,\ell}$ which solves (2.23); h_ϵ and m_ϵ are both strictly increasing if $j < 0$ and strictly decreasing if $j > 0$. Moreover the pair $(m^{(\epsilon)}, h^{(\epsilon)})$ defined in (2.24) converges as $\epsilon \to 0$ to the pair $(m, h) \in \mathcal{X}_{1,\ell}$ solution of the Stefan problem (2.17) in the following sense: for any $\delta > 0$*

$$\lim_{\epsilon \to 0} \sup_{|x|>\delta} \Big(|m^{(\epsilon)}(x) - m(x)| + |h^{(\epsilon)}(x) - h(x)| \Big) = 0. \quad (2.25)$$

Remarks. (a) Theorem 1 is proved in Section 3 and in Appendices A, B and C. The proof is based on a fixed point argument. We shall prove that in a suitable subset of $\mathcal{X}_{\epsilon,\ell}$ the following map is well defined: given a function h, solve the first one in (2.23) to get m and use the second one to find the new h. Existence of a fixed point is proved by showing convergence of the iterates h_n and of the corresponding m_n. Since $x_0 = 0$, if we start with an antisymmetric function, the whole orbit remains antisymmetric and indeed the limit macroscopic solution is antisymmetric as well. As we shall see, restricting to the space of odd functions greatly simplifies the problem. We start the iteration from a profile m_0 which is almost a fixed point: m_0 is in fact the [scaled by ϵ^{-1}] macroscopic solution away from 0, while it is equal to the "instanton" (the "diffuse interface" defined in Section 3) in a neighborhood of 0 (in sup-norm). We shall prove that all the profiles m_n obtained by iterating (2.23) are contained in a small neighborhood of m_0 and that the iterates converge in sup-norm to a continuous limit profile m. Also, the corresponding magnetic fields h_n converge in sup-norm to a continuous limit h and the pair (m, h) is the desired fixed point which solves (2.23). The crucial point in the analysis is to control the change δm of m in the first equality in (2.23) when we slightly vary h by δh. To linear order, δm and δh are related by $(A_{m,h} - 1)\delta m = -p_{m,h}\delta h$ where $A_{m,h} = p_{m,h} J*$, $J*$ the convolution operator with kernel J, and

$$p_{m,h} = \frac{\beta}{\cosh^2\{\beta J^{\text{neum}} * m + \beta h\}} \quad (2.26)$$

$$p_{m,h} = \chi(m) \quad \text{if } m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}. \quad (2.27)$$

(The equality $p_{m,h} = \chi(m)$ in (2.27) will be often exploited in the sequel). Thus $\delta m = L_{m,h}^{-1}(-p_{m,h}\delta h)$, provided $L_{m,h} := A_{m,h} - 1$ is invertible. In [4] it is shown that the largest eigenvalue of $L_{m,h}$ converges to 0 as $\epsilon \to 0$ and that there is a spectral gap bounded away from 0 uniformly in ϵ. By restricting to odd functions the leading eigenvalue disappears (because the corresponding eigenvector is an even function) and the invertibility problem can then be solved. As is clear from this outline the proof does not give uniqueness, which is left open.

(b) With Neumann conditions the non-local convolution term is completely defined, but since the evolution also involves derivatives other conditions are needed to determine the solution: our choice was to fix j and x_0. Dirichlet conditions would, instead, prescribe the limits $m_\pm$ of $m(x)$ as $x \to \pm\epsilon^{-1}\ell$. There are two types of boundary conditions here, those which fix m outside the domain and are used to define the convolution (in our case replaced by Neumann conditions), and those which prescribe the values of m when going to the boundary from the interior (in our case are replaced by j and x_0). The distinction is not as clear in other models as, for instance, in the Cahn–Hilliard equation. We are indebted to N. Alikakos and G. Fusco for many enlightening discussions on such issues.

We have a slightly weaker result when $x_0 \neq 0$ (as we do not have a full control of the zeros of m and h).

Theorem 2. *Let $j \neq 0$ and $x_0 \neq 0$ in $(-\ell, \ell)$ with* $\mathrm{dist}(x_0, \{\pm\ell\}) < \ell_j$. *Then for any $\epsilon > 0$ small enough there is a pair $(m_\epsilon, h_\epsilon) \in \mathcal{X}_{\epsilon,\ell}$ which solves* (2.23) *in $\epsilon^{-1}(-\ell, \ell)$ with the following properties: (m_ϵ, h_ϵ) is continuous, there is $x_\epsilon \in \epsilon^{-1}(-\ell, \ell)$ such that $h_\epsilon(x_\epsilon) = 0$, $\epsilon x_\epsilon \to x_0$ (see* (G.26) *in Appendix G), the pair $(m^{(\epsilon)}, h^{(\epsilon)})$ defined in* (2.24) *converges as $\epsilon \to 0$ to the pair $(m, h) \in \mathcal{X}_{1,\ell}$ solution of the Stefan problem* (2.17) *in the sense that for any $\delta > 0$*

$$\lim_{\epsilon\to 0} \sup_{|x-\epsilon^{-1}x_0|>\delta} \Big(|m^{(\epsilon)}(x) - m(x)| + |h^{(\epsilon)}(x) - h(x)|\Big) = 0. \tag{2.28}$$

Remarks. (a) Theorem 2 is proved in Section 4 and in Appendices D, E, F and G, where we derive explicit bounds on the speed of convergence. The idea of the proof is as follows. By Theorem 1 we can construct a quasi-solution (m_0, h_0) of (2.23) with an error which around the interface $\epsilon^{-1}x_0$ is exponentially small in ϵ^{-1} (we shall exploit this with the introduction of suitable weighted norms). (m_0, h_0) is then used as the starting point of an iterative scheme similar to the one in the proof of Theorem 1, from which, however, it differs significantly due to the absence of symmetries. The problem is that we can no longer restrict to the space of anti-symmetric functions, and thus need to check that the maximal eigenvalue of the operator L obtained by linearizing the first equation in (2.23) is non-zero. We know however from [4] that it is indeed dangerously close to zero and that it actually vanishes as $\epsilon \to 0$. In our specific case, though, we can be more detailed as we shall prove that it is negative and bounded away from 0 proportionally to ϵ. Thus we can invert L but get a dangerous factor ϵ^{-1} in the component along the direction of the maximal eigenvector, which spoils the iterative scheme as it is; it thus needs to be modified. The idea, roughly speaking, is to slightly shift from $\epsilon^{-1}x_0$ to make the component along the maximal eigenvector smaller, which we enforce by requiring that $\int_{-\epsilon^{-1}\ell}^{\epsilon^{-1}\ell} hu^* \mathrm{d}x = 0$, where u^* (whose dependence on ϵ is not made explicit) is a suitable positive function on $\mathbb{R}$, symmetric around $\epsilon^{-1}x_0$ and which decays exponentially as $|x - \epsilon^{-1}x_0| \to \infty$ uniformly in ϵ. (If u^* was a delta at $\epsilon^{-1}x_0$ we would then recover the condition $h(\epsilon^{-1}x_0) = 0$). In this way the iteration converges but we do not have a sharp control of the zero of the magnetization profile, which may not coincide with the zero of the magnetic field.

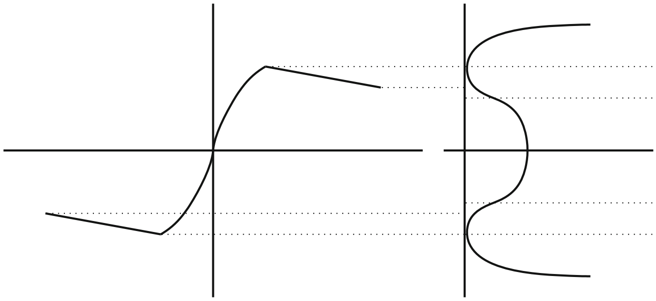

Fig. 1. On the *right* a sketch of the diagram of $\phi_\beta(m)$ at $\beta > 1$ where $-m_\beta$, $-m^*$, m^* and m_β are successively indicated. On the *left* a sketch of the stationary magnetization $m(x)$ when $m_- \in (-m_\beta, -m^*)$ and $m_+ = -m_-$. The profile $m(x)$ decreases to $\approx -m_\beta$, then (in an interval called I_ϵ) increases to $\approx m_\beta$ and decreases again to m_+

(b) By Theorem 1 and 2 it then follows that there are solutions of the stationary mesoscopic equation which converge as $\epsilon \to 0$ to the solution of any Dirichlet problem with $m_- < -m_\beta$ and $m_+ > m_\beta$, or vice-versa. At the mesoscopic level, though, the boundary values may differ from the prescribed ones but the difference is infinitesimal in ϵ. We thus have a complete theory of the derivation of the Stefan problem from (2.23), gaining a deeper insight on the sense in which the values in $(-m_\beta, m_\beta)$ are forbidden. At the mesoscopic level, in fact, such a restriction is absent and in the approximating profiles (m_ϵ, h_ϵ), which at each ϵ solve (2.23), the values in $(-m_\beta, m_\beta)$ are indeed present in m_ϵ. However, the fraction of space where they are attained, which we shall prove to go as $\epsilon \log \epsilon^{-1}$, becomes negligible as $\epsilon \to 0$. They concentrate at the interface, which in macroscopic units becomes a point and in mesoscopic units is described to leading order by the instanton which converges exponentially fast to $\pm m_\beta$.

2.5. Limits which do not Satisfy the Stefan Problem

So far, we have studied cases where the mesoscopic stationary solution converges in the limit $\epsilon \to 0$ to the solution of the macroscopic Stefan problem. We shall see next that it may happen that the limit exists but does not satisfy the Stefan problem. There will be cases where the limit profile is monotone, as in Theorem 3 below, but also cases where it is not, see Theorem 4. The physical origin of the pathology comes from under-cooling and over-heating effects which are related to the structure of the "mesoscopic free energy" $\phi_\beta(m)$ for m in the "forbidden" region $(-m_\beta, m_\beta)$. We distinguish two regions inside $(-m_\beta, m_\beta)$, the first one $[-m^*, m^*]$, $m^* = \sqrt{1 - 1/\beta}$, is called "spinodal", the other one, $\{m_\beta > |m| > m^*\}$, is called metastable, see Fig. 1.

The latter splits according to the sign of m into two disjoint intervals, the plus and minus metastable phases. In the spinodal region ϕ_β is concave, see (2.2), while in $(m^*, 1)$ [as well as in $(-1, -m^*)$] ϕ_β is strictly convex. At the mesoscopic level the metastable phases look as if they were pure thermodynamic phases, as shown in the next theorem where, for the sake of definiteness, we restrict to the negative phase (by symmetry the result extends to the positive one). We state without proof the following theorem:

Theorem 3. *Let $j \neq 0$ and $\ell > 0$ small enough. Then for any $\epsilon > 0$ small enough there is a pair $(m_\epsilon, h_\epsilon) \in \mathcal{X}_{\epsilon,\ell}$ which solves (2.23) with $m_\epsilon < -m^*$ in the whole interval $\epsilon^{-1}(-\ell, \ell)$ and having values in $(-m_\beta, -m^*)$. The pair $(m^{(\epsilon)}, h^{(\epsilon)})$ defined in (2.24) converges in sup-norm as $\epsilon \to 0$ to the pair $(m, h) \in \mathcal{X}_{1,\ell}$ where*

$$D^*_\beta \frac{\mathrm{d}m}{\mathrm{d}x} = -j, \quad D^*_\beta(m) = \chi(m)\phi''_\beta(m) = 1 - \beta(1 - m^2). \tag{2.29}$$

Thus the limit profile m satisfies a macroscopic stationary equation where the "true free energy" a_β is replaced by the "metastable free energy" ϕ_β, from which it differs in the non-empty region where $m \in (-m_\beta, -m^*)$. As its name suggests, metastable behavior is expected only on "short time scales". In Ising spin systems with Kawasaki dynamics and Kac potentials on a suitable scaling limit, it is proved in [8] that if the initial datum is in $(-1, -m^*)$, then the limit is also in $(-1, -m^*)$ and it satisfies

$$\frac{\partial m}{\partial t} = \nabla \cdot \left(D^*_\beta \nabla m\right), \quad D^*_\beta = 1 - \beta\left(1 - m^2\right). \tag{2.30}$$

On much longer times, which scale exponentially in ϵ^{-1}, large deviations and tunnelling effects enter into play with the metastable phase becoming unstable, see [1]. Such effects are lost in the mesoscopic description, where Theorem 3 shows that there is a stationary solution which lives entirely in the negative metastable phase.

The above results refer to cases where the profile is in only a single phase, either the plus or the minus metastable phase. Metastable behavior is not, in general, expected when the two phases coexist. Theorems 1 and 2 go in this direction, as they show that the whole interval $(-m_\beta, m_\beta)$ shrinks in the thermodynamic limit to a point, not distinguishing between metastable and spinodal values (thus in agreement with the macroscopic thermodynamics of the model). Our next theorem proves that there are also stationary solutions of (2.23) where the plus and minus metastable phases coexist; the stationary profile however is not monotonic, see Fig. 1.

Theorem 4. *Let $j > 0$. Then for any ℓ small enough there is an antisymmetric pair $(m_\epsilon, h_\epsilon) \in \mathcal{X}_{\epsilon,\ell}$ which solves (2.23) and such that $(m^{(\epsilon)}, h^{(\epsilon)})$ defined in (2.24) converges as $\epsilon \to 0$ and in the sense of (2.25) to $(m, h) \in \mathcal{X}_{1,\ell}$ solution of the "metastable" Stefan problem:*

$$h^*(x) = \int_0^x \frac{-j}{\chi(m)} \mathrm{d}x', \qquad m = \phi'^{-1}_\beta(h^*) \ \text{ in}(-\ell, \ell) \setminus \{0\}. \tag{2.31}$$

While h_ϵ is strictly decreasing, m_ϵ, instead (to leading orders in ϵ), first decreases then increases (around the origin) and then again decreases. The interval where it increases has length I_ϵ and $\epsilon I_\epsilon \to 0$ as $\epsilon \to 0$.

The proof of Theorem 4 is completely similar to the proof of Theorem 1 and is therefore omitted. We did not check that the result extends to the case $x_0 \neq 0$. Notice finally that while the current is positive the magnetization at the right endpoint is larger than the magnetization at the left endpoint ($m_+ > m_-$) in apparent contrast with the Fourier's law. However, in agreement with the Fourier's law, in most of the space the magnetization decreases and it is only at the interface (which shrinks to a point) that the profile increases.

3. Proof of Theorem 1

In this section we shall prove Theorem 1, which will be a corollary of three theorems stated below and proved later in three successive appendices. For notational simplicity we suppose $j < 0$ and, as discussed in Remark (a) after Theorem 1, we restrict to odd functions, so that by default in this section all functions are antisymmetric. The analysis is based on an iterative scheme which is outlined in the next two paragraphs. We shall define a sequence (m_n, h_n) which for each n satisfies the equality $m_n = \tanh\{\beta J^{\text{neum}} * m_n + \beta h_n\}$ and prove that (m_n, h_n) converges as $n \to \infty$ in sup-norm to a limit (m, h) which is the desired solution of (2.23).

3.1. The Starting Element

We define h_0 using (2.21) with m set equal to m_0, m_0 the odd function defined for $x > 0$ as

$$m_0(x) = \bar{m}(x)\mathbf{1}_{[0,\xi_\epsilon]}(x) + u(\epsilon[x - \xi_\epsilon])\mathbf{1}_{(\xi_\epsilon,\epsilon^{-1}\ell]}(x), \tag{3.1}$$

where $\bar{m}$ is the instanton (see the paragraph *Instanton: notation and properties* in Appendix A); $\xi_\epsilon = x_\epsilon + 2n_0$, x_ϵ : $\bar{m}(x_\epsilon) = m_\beta - \epsilon$, n_0 a large integer independent of ϵ, its value will be specified in the course of the proof of Lemma 9; as shown in Appendix A x_ϵ scales as $\log \epsilon^{-1}$. Finally, $u(r)$, $r \in [0, \ell - \epsilon\xi_\epsilon]$, is the solution of the macroscopic equation (2.17) (which in(2.17) is denoted by m). Since h_0 is obtained from m_0 by (2.21) then

$$m_0 = \tanh\left\{\beta J^{\text{neum}} * m_0 + \beta h_0\right\}, \tag{3.2}$$

a property which will be satisfied by all the elements of the sequence (m_n, h_n). Moreover, denoting by $\|\cdot\|$ the sup-norm,

$$\sup_\epsilon \|m_0\| \leqq c_{(3.3)} < 1, \tag{3.3}$$

because $\|\bar{m}\| \leqq m_\beta$ and $\|u\| < 1$ since $\ell < \ell_j$, see (2.18) and the paragraph *Axiomatic non-equilibrium macroscopic theory* in Section 2.

3.2. The Iterative Scheme

As discussed in Remark (a) after Theorem 1, the idea is to define a transformation $h \to T(h)$ [from antisymmetric into antisymmetric functions] in two steps. We first find an antisymmetric function m such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and then define for $x \geqq 0$

$$T(h)(x) = -\epsilon j \int_0^x \chi(m(y))^{-1}, \quad m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}. \tag{3.4}$$

(Here and in the sequel we drop dy from the integral). The definition of $T(h)$ thus rests on the possibility of finding an "auxiliary function" m which solves the second equality in (3.4) and is such that $\chi(m)^{-1}$ is integrable. By construction we already know that the auxiliary function m_0 associated to h_0 exists and $\|m_0\| \leqq c_{(3.3)} < 1$ uniformly in ϵ. The crucial step will then be to prove that if h is "close" to h_0, then (at least for ϵ small enough) there is a unique m "close" to m_0 so that the second equality in (3.4) is satisfied. $\|m\| < 1$ and $T(h)$ is thus well defined (we do not have general uniqueness as we are in the phase transition regime: we cannot exclude that there are other solutions not close to m_0). We shall then prove recursively that all images $h_n = T^n(h_0)$ are well defined and close to h_0, while the auxiliary functions m_n are close to m_0; moreover $(m_n, h_n) \to (m, h)$ in sup-norm as $n \to \infty$. h will then be a fixed point of T with auxiliary function m and Theorem 1 will be proved.

3.3. Notation

Our basic accuracy parameter will be ϵ^a, $a \in (0, 1)$. ϵ^a defines quantitatively the a-priori closeness to h_0 (the elements h_k in the iteration will actually be much closer to h_0, $\|h_k - h_0\| \leqq c\epsilon \log \epsilon^{-1}$):

$$\|h - h_0\| \leqq \epsilon^a, \quad \|f\| := \sup_{|x| \leqq \epsilon^{-1}\ell} |f(x)|, \tag{3.5}$$

being understood that all functions we deal with in this section are odd. While the basic accuracy parameter clearly depends on ϵ, $a \in (0, 1)$ above as well as all the constants that we shall write in the sequel, denoted by a, b, c and C with or without suffixes, will be independent of ϵ. The existence of the auxiliary function m in (3.4) is established next:

Theorem 5. *There are constants $c_{(3.6)} > 1$, $\alpha_{(3.6)} > 0$, $c'_{(3.6)} := \frac{2c_{(3.6)}}{\alpha_{(3.6)}}$ so that for all ϵ small enough the following holds. For any $h : \|h - h_0\| \leqq \epsilon^a$ there is a unique m_h in the ball $\{m : \|m - m_0\| \leqq c'_{(3.6)}\epsilon^a\}$ such that $m_h = \tanh\{\beta J^{\text{neum}} * m_h + \beta h\}$ and for any $h' : \|h' - h_0\| \leqq \epsilon^a$*

$$|m_h(x) - m_{h'}(x)| \leqq c_{(3.6)} \int_0^{\epsilon^{-1}\ell} e^{-\alpha_{(3.6)}|x-y|} |h(y) - h'(y)|, \quad x \geqq 0. \tag{3.6}$$

We postpone to Appendix A the proof of Theorem 5 and proceed with the proof of Theorem 1, observing that as a consequence of Theorem 5 if $\|h - h_0\| \leqq \epsilon^a$ then $T(h)$ is well defined (for all ϵ small enough) because $\chi(m)$ in the first of (3.4) is bounded away from 0. To prove this it suffices to show that $\|m\| < 1$. By (3.6) with $m_h = m$ and $m_{h'} = m_0$,

$$\|m - m_0\| \leqq c'_{(3.6)}\|h - h_0\| \leqq c'_{(3.6)}\epsilon^a. \tag{3.7}$$

Then by (3.3) $\|m\| \leqq c_{(3.3)} + c'_{(3.6)}\epsilon^a < 1$ for ϵ small enough.

Theorem 6. *There are constants $c_{(3.8)}$ and $c_{(3.9)} > 0$ so that for all ϵ small enough the following holds. Let m' and m'' be both in the ball $\left\{m : \|m - m_0\| \leqq c'_{(3.6)}\epsilon^a\right\}$, then denoting by $h' = \int_0^x \frac{-\epsilon j}{\chi(m')}$, $h'' = \int_0^x \frac{-\epsilon j}{\chi(m'')}$,*

$$|h'(x) - h''(x)| \leqq c_{(3.8)}\epsilon|j| \int_0^x |m'(y) - m''(y)|, \quad x > 0 \tag{3.8}$$

$$\|h_1 - h_0\| \leqq c_{(3.9)}\epsilon \log \epsilon^{-1}, \quad h_1 = T(h_0). \tag{3.9}$$

We postpone to Appendix B the proof of Theorem 6 and observe that since the transformation T is well defined in the ball $\|h - h_0\| \leqq \epsilon^a$ we are in business once we show that any iterate of T is in the ball $\|h - h_0\| \leqq \epsilon^a$. We postpone to Appendix C the proof of:

Theorem 7. *There is a constant $c_{(3.10)} > 0$ such that the following holds. Suppose there is n such that for all $k < n$, $h_k = T^k(h_0)$ is well defined, $\|h_k - h_0\| \leqq \epsilon^a$ and $\|m_k - m_0\| \leqq \epsilon^a$, m_k the auxiliary function in the definition of $T(h_k)$. Then h_n is well defined and*

$$\|h_{k+1} - h_k\| \leqq c_{(3.10)}\left(\frac{1}{2}\right)^k \|h_1 - h_0\|, \quad k < n. \tag{3.10}$$

It is now easy to prove Theorem 1. We restrict to $\epsilon > 0$ so small that

$$2c_{(3.10)}c_{(3.13)}c_{(3.9)}\epsilon \log \epsilon^{-1} < \epsilon^a \tag{3.11}$$

(with $c_{(3.13)}$ defined in (3.13) below) and prove by induction that (m_k, h_k) exists for all k and, moreover, $\|h_k - h_0\| \leqq \epsilon^a$ and $\|m_k - m_0\| \leqq \epsilon^a$. Since the statement is obviously true for $k = 0$ we only need to prove that if it is verified for $k < n$, then it holds for n as well. By (3.10) and (3.9) for all $k < n$,

$$\|h_{k+1} - h_k\| \leqq c_{(3.10)}\left(\frac{1}{2}\right)^k c_{(3.9)}\epsilon \log \epsilon^{-1}, \tag{3.12}$$

which, by (3.11) shows that $\|h_n - h_0\| < \epsilon^a$ (for ϵ small enough). Then by Theorem 5 m_n is well defined and by (3.6) for all $k < n$

$$\|m_{k+1} - m_k\| \leqq c_{(3.13)}\|h_{k+1} - h_k\|, \quad c_{(3.13)} = \max\{1, c'_{(3.6)}\}. \tag{3.13}$$

Then, using (3.12),

$$\|m_{k+1} - m_k\| \leqq c_{(3.10)} c_{(3.13)} \left(\frac{1}{2}\right)^k c_{(3.9)} \epsilon \log \epsilon^{-1}, \tag{3.14}$$

which by (3.11) proves that $\|m_n - m_0\| \leqq \epsilon^a$. Thus the induction is proved and we know that for all k, (m_k, h_k) exists, $\|h_k - h_0\| \leqq \epsilon^a$ and $\|m_k - m_0\| \leqq \epsilon^a$.

As a consequence of (3.14) and (3.12), $(m_n, h_n) \to (m, h)$ in sup-norm with $h = T(h)$, $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$, and

$$\|h - h_0\| \leqq c\epsilon \log \epsilon^{-1}, \quad \|m - m_0\| \leqq c\epsilon \log \epsilon^{-1}. \tag{3.15}$$

Making explicit the dependence on ϵ we write the limit as (m_ϵ, h_ϵ) in agreement with the notation in Theorem 1. Recalling the definition of (m_0, h_0), see (3.1), we then obtain the proof of Theorem 1 except for the statement about the monotonicity of m_ϵ which is proved at the end of Appendix D.

4. Outline of the Proof of Theorem 2

4.1. The Macroscopic Solution

For the sake of definiteness we suppose $j < 0$ and $x_0 > 0$, and for notational simplicity that the interval $(-\ell, \ell)$ is just the interval $(-1, 1)$. By assumption $(-1, 1)$ is then strictly contained in the interval of length $2\ell_j$ and center x_0, that is the maximal interval where the macroscopic problem with parameters (j, x_0) has solution (see the paragraph *Axiomatic non-equilibrium macroscopic theory* in Section 2). We then write $\ell^* = 1 + 2x_0$ so that x_0 is the middle point of the interval $[-1, \ell^*]$ and, for what said above, $\ell^* + 1 < 2\ell_j$ so that the macroscopic problem has a solution $(m_{\text{mac}}(x), h_{\text{mac}}(x))$, $x \in (-1, \ell^*)$ with the following properties: it is a smooth pair of functions antisymmetric around x_0 such that $\|m_{\text{mac}}\| < 1$ and $\|h_{\text{mac}}\| < \infty$ (so that $\inf \chi(m_{\text{mac}}) > 0$).

4.2. The pairs (m^*, h^*) and (m_ϵ, h_ϵ)

By Theorem 1 for any $\epsilon > 0$ small enough there is a pair $(m^*(x), h^*(x))$, $x \in \epsilon^{-1}[-1, \ell^*]$ (dependence on ϵ is not made explicit) which solves (2.23) and is antisymmetric around $\epsilon^{-1}x_0$. Then there is $c_{(4.1)} > 0$ so that

$$\beta \geqq p_{m^*,h^*} \geqq c_{(4.1)} \quad \text{for all } \epsilon > 0 \text{ small enough.} \tag{4.1}$$

$\beta \geqq p_{m,h}$ is true in general, see (2.26); instead $p_{m^*,h^*} \geqq c_{(4.1)}$ because by (2.27) $p_{m^*,h^*} = \chi(m^*) = \beta(1 - (m^*)^2)$ and $\|m^*\| < 1$ uniformly in ϵ. This follows from the inequality $\|m_{\text{mac}}\| < 1$ because, by Theorem 1, $\lim_{\epsilon \to 0} \|m_{\text{mac}}(x) - m^*(\epsilon^{-1}x)\| = 0$. We next define (m_ϵ, h_ϵ):

$$\begin{aligned} m_\epsilon(x) &= m^*(x), \quad h_\epsilon(x) = h^*(x) + R_\epsilon(x), \qquad x \in \epsilon^{-1}[-1, 1] \\ R_\epsilon(x) &= \int_{\epsilon^{-1}}^{\epsilon^{-1}+1} J(x, y)[m^*(y) - m^*(2\epsilon^{-1} - y)]\,dy. \end{aligned} \tag{4.2}$$

We have added the "correction" R_ϵ to have:

$$m_\epsilon = \tanh\{\beta[J^{\text{neum}} * m_\epsilon] + \beta h_\epsilon\}. \tag{4.3}$$

Lemma 8. *There are* $r_{(4.4)} > 0$, $c_{(4.4)} > 0$ *and* $c'_{(4.4)} > 0$ *so that*

$$\left\| \frac{\mathrm{d}m^*}{\mathrm{d}x} \right\| \leqq c'_{(4.4)}, \qquad \sup_{|x-\epsilon^{-1}x_0| \geqq r_{(4.4)} \log \epsilon^{-1}} \left| \frac{\mathrm{d}m^*(x)}{\mathrm{d}x} \right| \leqq c_{(4.4)}\epsilon. \tag{4.4}$$

As a consequence $|R_\epsilon(x)| \leqq c\epsilon \mathbf{1}_{\epsilon^{-1}-1 \leqq x \leqq \epsilon^{-1}}$.

Proof. By differentiating the equality $m^* = \tanh\{\beta J^{\text{neum},*} * m^* + \beta h^*\}$ (valid in the whole interval $\epsilon^{-1}(-1, \ell^*)$, $J^{\text{neum},*}$ the kernel with Neumann conditions at its endpoints) we get

$$\left\| \frac{\mathrm{d}m^*}{\mathrm{d}x} \right\| \leqq \beta \left(\left\| \frac{\mathrm{d}J^{\text{neum},*}}{\mathrm{d}x} \right\| \|m^*\| + \left\| \frac{\mathrm{d}h^*}{\mathrm{d}x} \right\| \right) \leqq c,$$

because $\|\mathrm{d}h^*/\mathrm{d}x\| \leqq c\epsilon$ (as h^* solves (2.23)), hence the first inequality in (4.4). The second one is not as easy; it will be proved at the end of Appendix D. Using this inequality in (4.2) we readily see that $|R_\epsilon(x)| \leqq c\epsilon \mathbf{1}_{\epsilon^{-1}-1 \leqq x \leqq \epsilon^{-1}}$, $c = c_{(4.4)}$. □

Thus R_ϵ is "a small boundary field" and except for the small error R_ϵ, $\chi(m_\epsilon)\frac{\mathrm{d}h_\epsilon}{\mathrm{d}x} = -\epsilon j$ so that the pair (m_ϵ, h_ϵ) is "almost a solution" of the stationary problem (which could be interpreted as a true solution of a problem with suitably redefined boundary conditions).

4.3. An Interpolation Scheme

A natural way to obtain a true solution from a quasi-solution is via the implicit function theorem after writing (2.23) as a single equation $f(m, h) = 0$ on the space of pairs of L^∞ functions. Unfortunately we do not have good control of the derivative of $f(m, h)$ which may, in principle, vanish. The problem simplifies if we try to solve only the first one in (2.23) and then use the second one to redefine h, which opens the way to an iterative scheme as the one used in Section 3. The crucial step is the following: find $\tilde{m}$ such that $\tilde{m} = \tanh\{\beta J^{\text{neum}} * \tilde{m} + \beta\tilde{h}\}$ knowing $\tilde{h}$ and that $\tilde{h}$ is "close" to another field $\hat{h}$, for which there is $\hat{m}$ such that $\hat{m} = \tanh\{\beta J^{\text{neum}} * \hat{m} + \beta\hat{h}\}$. To solve this problem we interpolate writing $h(t) = t\tilde{h} + (1-t)\hat{h}$, $t \in [0, 1]$, and pretending that for all t there is $m(t)$ such that $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$, we differentiate and get an equation for $\mathrm{d}m/\mathrm{d}t$. Its solution will then allow us to obtain $\tilde{m}$ as $\tilde{m} = \hat{m} + \int_0^1 \frac{\mathrm{d}m}{\mathrm{d}s}\,\mathrm{d}s$.

The main point in this procedure is therefore the analysis of the equation for $\mathrm{d}m/\mathrm{d}t$. This is (E.2) in Appendix E; here we just say that it has the form $\psi = (A_{m,h}-1)^{-1}\phi$ (ψ the unknown), where $A_{m,h} = p_{m,h} J^{\text{neum}} *$, $p_{m,h}$ as in (2.26) (and $p_{m,h} = \chi(m) = \beta(1-m^2)$ because $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$), $J^{\text{neum}} *$ is the convolution operator with kernel J^{neum}. The non-linearity of the problem reflects in the fact that (m, h) above is actually $(m(t), h(t))$, which is itself unknown, but the whole problem boils down to an accurate analysis of the operator $A_{m,h}$ in a suitably large set of pairs (m, h) (the set $\mathcal{A}$ in Appendix D). The same problem has appeared in the proof of Theorem 5, where, however, we had the great simplification of restricting to the space of antisymmetric functions. In such a restricted space $\|A^{n_0}_{m,h}\| < 1$ for a suitable integer n_0 uniformly in ϵ (see Appendix A). $(1-A_{m,h})^{-1}$ is then equal to the convergent sum $\sum A^n_{m,h}$ and the bound (A.13) holds. In the case considered in Theorem 2 we do not have symmetries, and the invertibility of $L_{m,h} := A_{m,h} - 1$ becomes a serious issue.

In Appendix D we shall establish fine spectral properties of $A_{m,h}$ for all (m, h) in a set $\mathcal{A}$. We shall prove a Perron–Frobenius theorem for $A_{m,h}$ regarded as an integral operator on $L^\infty(\epsilon^{-1}[-1, 1])$ showing that it has a maximal eigenvalue $\lambda > 0$, that its eigenvector u (called the "maximal eigenvector") has a definite sign (taken positive) and, see Proposition 12, that there are positive constants c, c' and a so that for all ϵ small enough

$$0 < \lambda < 1 - c\epsilon, \quad 0 < u \leqq c' \mathrm{e}^{-a|x-x_0|}. \tag{4.5}$$

Actually, to leading order in ϵ, $\lambda = 1 - C_{(D.9)}\epsilon$, see (D.9). λ is separated from the rest of the spectrum (spectral gap) as stated in Proposition 13.

Our strategy therefore will be to reduce to pairs $(m, h) \in \mathcal{A}$, a task accomplished by showing that we can actually reduce to functions h in the very small neighborhood $\mathcal{G}$ of h_ϵ defined next.

4.4. The Set $\mathcal{G}$

Let $b_{(4.7)}$ and $a_{(4.6)}$ be positive parameters (specified in Appendix F), and for any $f \in L^\infty(\epsilon^{-1}[-1, 1])$

$$N(f) := \sup_{|x| \leqq \epsilon^{-1}} E_\epsilon(x)|f(x)|; \quad E_\epsilon(x) := \begin{cases} \mathrm{e}^{a_{(4.6)}(\epsilon^{-1}-x)} & x \geqq \epsilon^{-1}x_0 \\ \mathrm{e}^{a^-_{(4.6)}(x+\epsilon^{-1})} & x < \epsilon^{-1}x_0 \end{cases} \tag{4.6}$$

with $a^-_{(4.6)}$ such that $a^-_{(4.6)}(x_0 + 1) = a_{(4.6)}(1 - x_0)$. Recalling that h_ϵ is defined in (4.2) and denoting by $u^* \in L^\infty(\epsilon^{-1}[-1, \ell^*], \mathbb{R}^+)$ the " maximal eigenvector" of A_{m^*,h^*} we define $\mathcal{G}$ as

$$\mathcal{G} := \Big\{h : N(h - h_\epsilon) \leqq b_{(4.7)}, \int_{-\epsilon^{-1}}^{\epsilon^{-1}} hu^* = 0$$
$$\left\|\frac{\mathrm{d}(h - h_\epsilon)}{\mathrm{d}x}\right\| \leqq \epsilon \quad \sup_{|x-\epsilon^{-1}x_0| \leqq (\log \epsilon^{-1})^2} \left|\frac{\mathrm{d}(h - h_\epsilon)}{\mathrm{d}x}\right| \leqq \epsilon^2\Big\}. \tag{4.7}$$

4.5. The Iterative Scheme

We shall prove in Proposition 16 that if $h \in \mathcal{G}$, then there is m such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and, moreover, $(m, h) \in \mathcal{A}$, where $\mathcal{A}$ is the nice set with good spectral properties mentioned earlier. Thus $A_{m,h}$ has a maximal eigenvalue λ with maximal eigenvector u, $A_{m,h}u = \lambda u$. In Corollary 19 we shall prove that u is "very close" to the restriction of u^* to $\epsilon^{-1}[-1, 1]$, u^* the maximal eigenvector of A_{m^*,h^*} relative to the problem in $\epsilon^{-1}[-1, \ell^*]$. All this collects the properties needed to define the iterative scheme and to prove its convergence. We define recursively $h_{n+1} := T(h_n)$, $n \geqq -1$, $h_{-1} := h_\epsilon$, as

$$h_{n+1} = \hat{h}_{n+1} - \frac{\displaystyle\int_{-\epsilon^{-1}}^{\epsilon^{-1}} \hat{h}_{n+1} u^*}{\displaystyle\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^*}, \quad \hat{h}_{n+1}(x) := -\epsilon j \int_{\epsilon^{-1}x_0}^{x} \chi(m_n(y))^{-1}, \tag{4.8}$$

(recalling that $\chi(m_n) = p_{m_n,h_n}$ by (2.27)). The definition is well posed once we prove that $h_n \in \mathcal{G}$ for $n \geqq 0$, so that there is a unique m_n such that $(m_n, h_n) \in \mathcal{A}$. We shall indeed prove in Proposition 22 that $N(h_{n+1} - h_n) \leqq c\epsilon N(h_n - h_{n-1})$. Here we use, in an essential way, the subtraction in (4.8) which subtracts [most of] the component along the maximal eigenvector u of $A_{m_{n-1},h_{n-1}}$ of the "forcing term" $p_{m_{n-1},h_{n-1}}(h_n - h_{n-1})$. In this way we shall prove iteratively that $h_n \in \mathcal{G}$ so that there is m_n with $(m_n, h_n) \in \mathcal{A}$; moreover, we shall see in Appendix G that $h_n \to h$ and $m_n \to m$ as $n \to \infty$ with $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$, $h = \hat{h} - \frac{\int \hat{h} u^*}{\int u^*}$, $\hat{h}(x) := -\epsilon j \int_{\epsilon^{-1}x_0}^{x} \chi(m(y))^{-1}$. As a consequence the pair (m, h) satisfies (2.22) with $h(x_\epsilon) = 0$ where x_ϵ is such that:

$$\int_{\epsilon^{-1}x_0}^{x_\epsilon} \chi(m(y))^{-1} = \frac{\int u^*(x) \int_{\epsilon^{-1}x_0}^{x} \chi(m(y))^{-1}}{\int u^*}. \tag{4.9}$$

The proof of Theorem 2 will then be completed by showing at the end of Appendix G that x_ϵ exists and that $\epsilon x_\epsilon \to x_0$ as $\epsilon \to 0$, see (G.26).

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A Proof of Theorem 5

Before proving Theorem 5 we introduce some notation and definitions which will be used throughout the whole sequel.

An Auxiliary Dynamics

To construct and compare solutions of $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ for given h, we introduce some artificial dynamics. Suppose $(m(t), h(t))$, $t \in [0, 1]$, are smooth functions of t and that for all t

$$m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}. \tag{A.1}$$

By differentiating (A.1) with respect to t we get the identity

$$\frac{\mathrm{d}m}{\mathrm{d}t} = A_t \frac{\mathrm{d}m}{\mathrm{d}t} + p_t \frac{\mathrm{d}h}{\mathrm{d}t}, \quad L_t \frac{\mathrm{d}m}{\mathrm{d}t} = -p_t \frac{\mathrm{d}h}{\mathrm{d}t}, \qquad L_t = A_t - 1, \tag{A.2}$$

where $p_t = p_{m(t),h(t)}$, $p_{m,h}$ as in (2.26), and $A_t = p_t J^{\text{neum}}*$, $J^{\text{neum}}*$ the operator on $L^\infty(\epsilon^{-1}[-\ell, \ell])$ with kernel J^{neum}.

By a change of perspective we now regard (A.2) as an equation for the unknown $\frac{\mathrm{d}m}{\mathrm{d}t}$ with p_t and $\frac{\mathrm{d}h}{\mathrm{d}t}$ considered as "known terms". We shall prove in this appendix that, under suitable assumptions on h, a solution exists and it is unique. We then "construct" $m(t) := m(0) + \int_0^t \frac{\mathrm{d}m}{\mathrm{d}s}$ and check that it verifies (A.1). The important point is that the whole procedure works in the same way even if we ask that (A.1) holds only at time $t = 0$, being a by-product of the analysis that it remains valid for all $t \in [0, 1]$. In the actual applications, $m(0) = m_0$ is a given, known function which solves $m_0 = \tanh\{\beta J^{\text{neum}} * m_0 + \beta h_0\}$, $h(t) = h_1 t + (1-t)h_0$ with h_0 and h_1 also known and $m(t)$ the unknown. In particular we are interested in its value m_1 at time $t = 1$ when $h(1) = h_1$. (A.2) then becomes a non-linear evolution equation and it will be crucial to prove first that L_t is invertible, so that the equation can be written in normal form

$$\frac{\mathrm{d}m}{\mathrm{d}t} = L_t^{-1}\left(-p_t \frac{\mathrm{d}h}{\mathrm{d}t}\right), \tag{A.3}$$

and then that $L_t^{-1}\left(-p_t \frac{\mathrm{d}h}{\mathrm{d}t}\right)$ is a Lipschitz function of m.

The Operator $A_{m,h}$

The whole analysis relies on properties of the spectrum of the operator $A_{m,h} = p_{m,h} J^{\text{neum}}*$ (called A_t when $(m, h) = (m(t), h(t))$ as above). We shall study $A_{m,h}$ in a $L^\infty(\epsilon^{-1}[-\ell, \ell])$ setting and, since we want to prove that $A_{m,h} - 1$ is invertible, it is crucial to prove that 1 is not in the spectrum of $A_{m,h}$. Regarded as an operator on $L^2\left(\epsilon^{-1}[-\ell, \ell], p_{m,h}^{-1}(x)\mathrm{d}x\right)$, $A_{m,h}$ is self-adjoint; it has a maximal eigenvalue $\lambda_{m,h}$ which is positive and the corresponding eigenvector $u_{m,h}$, called the maximal eigenvector, can and will be chosen as strictly positive, see [4]. Further assumptions on h and m will allow us to prove that $\lambda_{m,h} \leqq 1 - c\epsilon$, $c > 0$, and that the rest of the spectrum is strictly below 1 uniformly in ϵ. The bound on $\lambda_{m,h}$ will not be used in this appendix, see the proof of (A.8) below.

Instanton: Notation and Properties

The instanton $\bar m$ is a solution of the local mean field equation $\bar m(x) = \tanh\{\beta J * \bar m(x)\}$, $x \in \mathbb{R}$, with the following properties (see Section 8.1 and 8.2 of [10]). $\bar m(x)$ is a strictly increasing, antisymmetric function which converges to $\pm m_\beta$ as $x \to \pm\infty$, more precisely there are $c_{(A.4)}$ and $a_{(A.4)}$ both positive so that for all $x \geqq 0$

$$0 < m_\beta - \bar m(x) \leqq c_{(A.4)} e^{-a_{(A.4)}x}, \quad \frac{d\bar m(x)}{dx} \leqq c_{(A.4)} e^{-a_{(A.4)}x}. \tag{A.4}$$

We write

$$\bar p = \beta(1 - \bar m^2), \quad \bar A = \bar p J *, \quad \bar m' = \frac{d\bar m}{dx}, \quad \tilde m' = \frac{\bar m'}{(\langle(\bar m')^2\rangle_\infty)^{1/2}}, \tag{A.5}$$

where $\langle f\rangle_\infty = \int_{\mathbb{R}} f \bar p^{-1}$. In [4] and Section 8.3 in [10] it is proved that there are $a_{(A.6)} > 0$ and $c_{(A.6)}$, so that for any bounded function f

$$\left| \int \bar A^n(x,y) \tilde f(y) dy \right| \leqq \|\tilde f\|\, c_{(A.6)} e^{-a_{(A.6)}n}, \quad \tilde f = f - \langle f \tilde m'\rangle_\infty \tilde m'. \tag{A.6}$$

We can now turn to the proof of Theorem 5 and restrict hereafter in this appendix to the space of antisymmetric functions. After observing that by (A.4)

$$x_\epsilon \leqq c \log \epsilon^{-1}, \tag{A.7}$$

we complete the definition (3.1) of m_0 by fixing the integer n_0, chosen so that

$$\|\bar A^{n_0}\psi\| \leqq e^{-a_{(A.8)}}\|\psi\|, \quad a_{(A.8)} > 0, \tag{A.8}$$

where ψ above is any bounded antisymmetric function, recall that $\|f\|$ denotes the sup-norm of f. Existence of n_0 follows from (A.6) because $\bar m'$ and $\bar p_{x_0}$ are symmetric and ψ antisymmetric so that $\langle \psi \tilde m'\rangle_\infty = 0$.

Lemma 9. *There is $a_{(A.9)} > 0$ so that for any c, $a > 0$ and all ϵ small enough*

$$\|A^{n_0}_{m,h}\psi\| \leqq e^{-a_{(A.9)}}\|\psi\|, \quad \text{if } \|m - m_0\| \leqq c\epsilon^a, \|h - h_0\| \leqq c\epsilon^a, \tag{A.9}$$

for any bounded odd function ψ.

Proof. As we shall see (A.9) is a straight consequence of (A.8) and of

$$\left\| \frac{p_{m,h}}{p_{m_0,h_0}} - 1 \right\| \leqq c'\epsilon^a, \tag{A.10}$$

which follows directly from (3.3) and the assumptions on h and m. We distinguish "small" and "large" values of x_0 in $A^{n_0}_{m,h}\psi(x_0)$.

(i). $x_0 \in [0, x_\epsilon + n_0]$. We write

$$A^{n_0}_{m,h}\psi(x_0) = \int \psi(x_{n_0}) \prod_{k=1}^{n_0} \{p_{m_0,h_0}(x_{k-1}) J^{\text{neum}}(x_{k-1}, x_k) \frac{p_{m,h}(x_{k-1})}{p_{m_0,h_0}(x_{k-1})}\} \mathrm{d}x_1 \cdots \mathrm{d}x_{n_0}. \tag{A.11}$$

Since J^{neum} has range 1, $|x_i| \leqq x_\epsilon + 2n_0$ for all $i = 1, \dots n_0$. Then by (A.7) for ϵ small enough, $J^{\text{neum}}(x_i, x_{i+1}) = J(x_i, x_{i+1})$. Moreover $p_{m_0,h_0}(x_i) = \bar{p}(x_i)$ (because $m_0(x) = \bar{m}(x)$, $h_0(x) = 0$ for $|x| \leqq x_\epsilon + 2n_0$). Thus by (A.10)

$$\left| A^{n_0}_{m,h}\psi(x_0) - \bar{A}^{n_0}\psi(x_0) \right| \leqq c' n_0 \epsilon^a \|\psi\|,$$

and using (A.8), for all ϵ small enough

$$\left| A^{n_0}_{m,h}\psi(x_0) \right| \leqq \mathrm{e}^{-a_{(A.8)}} \|\psi\| + c' n_0 \epsilon^a \|\psi\| \leqq \mathrm{e}^{-a_{(A.9)}} \|\psi\|.$$

(ii). $x_0 \in [x_\epsilon + n_0, \epsilon^{-1}\ell]$. We then write

$$A^{n_0}_{m,h}\psi(x_0) = \int \psi(x_{n_0}) \prod_{k=1}^{n_0} \{p_{m,h}(x_{k-1}) J^{\text{neum}}(x_{k-1}, x_k) \mathrm{d}x_1 \cdots \mathrm{d}x_{n_0}, \tag{A.12}$$

and since J^{neum} has range 1, $x_i \geqq x_\epsilon$ in (A.12) for all $i = 1, \dots n_0$. When $x_i \in [x_\epsilon, \xi_\epsilon]$, $p_{m_0,h_0}(x_i) = \bar{p}(x_i) \leqq \beta(1-\bar{m}(x_\epsilon)^2)$ and by the definition of x_ϵ, $\bar{m}(x_\epsilon) = m_\beta - \epsilon$. Hence if b' is such that $\beta(1 - m_\beta^2) < b' < 1$, then for all ϵ small enough, $p_{m_0,h_0}(x_i) \leqq b' < 1$. When $x_i > \xi_\epsilon$, $p_{m_0,h_0} = \beta(1 - u^2)$ and since $u \geqq m_\beta$, $p_{m_0,h_0}(x_i) \leqq \beta(1 - m_\beta^2) < b' < 1$. Thus by (A.10) $p_{m,h}(x_i) \leqq b < 1$ $|A^{n_0}_{m,h}\psi(x_0)| \leqq b^{n_0}\|\psi\|$. □

By (A.9), $L_{m,h} = A_{m,h} - 1$ is invertible and

$$L^{-1}_{m,h} = -\sum_{n=0}^{\infty} A^n_{m,h}, \quad \|L^{-1}_{m,h}\| \leqq \frac{c_{(A.13)}}{1 - a_{(A.9)}}, \tag{A.13}$$

where $c_{(A.13)}$ bounds $\sum_{n=1}^{n_0} \|A^n_{m,h}\|$. Moreover:

Lemma 10. *There exist* $\alpha_{(3.6)} > 0$, *(which defines the parameter introduced in* (3.6)*),* $c_{(A.14)}$ *and* $c_{(A.15)}$*, both larger than* $\max\left\{1, \dfrac{c_{(A.13)}}{1 - a_{(A.9)}}\right\}$*, so that for any* c *and all* ϵ *small enough*

$$|L^{-1}_{m,h}\psi(x)| \leqq c_{(A.14)} \int_0^{\epsilon^{-1}} \mathrm{e}^{-\alpha_{(3.6)}|x-y|} |\psi(y)|, \quad \|m - m_0\| \leqq c\epsilon^a, \ \|h - h_0\| \leqq c\epsilon^a, \tag{A.14}$$

for any $x \geqq 0$*. Moreover, if also* $m' : \|m' - m_0\| \leqq c\epsilon^a$*, then*

$$\|L^{-1}_{m,h} - L^{-1}_{m',h}| \leqq c_{(A.15)} \|m - m'\|. \tag{A.15}$$

Proof. To prove (A.14) we write $A^n_{m,h}(x,y)$ as the kernel of $A^n_{m,h}$ and have by (A.13)

$$L^{-1}_{m,h}\psi(x) = -\sum_{n=0}^{\infty}\int A^n_{m,h}(x,y)\psi(y) = -\int \sum_{n=n(x,y)}^{\infty} A^n_{m,h}(x,y)\psi(y),$$

where $n(x,y) \geqq |y-x|$ because $A_{m,h}(x,y) = p_{m,h}(x)J^{\text{neum}}(x,y)$ is supported by $|x-y| \leqq 1$. (A.14) then follows from (A.9). To prove (A.15) we write

$$L^{-1}_{m,h} - L^{-1}_{m',h} = L^{-1}_{m,h}\left(A_{m,h} - A_{m',h}\right)L^{-1}_{m',h},$$

using (A.13) and that $\|A_{m,h} - A_{m',h}\| \leqq c\|m-m'\|$. □

We shall study (A.3) with

$$h(t) = th'' + (1-t)h', \quad h' \text{and } h'' \text{ in the ball } \|h-h_0\| \leqq \epsilon^a \tag{A.16}$$

so that $\|h(t)-h_0\| \leqq \epsilon^a$ and $\|\dfrac{\mathrm{d}h(t)}{\mathrm{d}t}\| \leqq 2\epsilon^a$. The initial datum m' is chosen so that $m' = \tanh\{\beta J^{\text{neum}} * m' + \beta h'\}$ and $\|m'-m_0\| \leqq c'\epsilon^a$ where $c' := \beta c_{(\text{A.14})}$. To prove existence of solutions of (A.3) we need to control the "velocity field"

$$V(m,h,\dot h) = -L^{-1}_{m,h}\left(p_{m,h}\dot h\right) \tag{A.17}$$

where $m, h, \dot h$ are antisymmetric functions. To this end we specify the "free parameter" c, which appears in the previous two lemmas, so that $c > 3c'$, $c' := \beta c_{(\text{A.14})}$.

Lemma 11. *For all ϵ small enough, the Cauchy problem in the interval $t \in [0,1]$*

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = V\left(m(t), h(t), \frac{\mathrm{d}h(t)}{\mathrm{d}t}\right), \quad m(0) = m' \tag{A.18}$$

has a unique solution $m(t)$ such that $\|m(t)-m_0\| \leqq 3c'\epsilon^a$. Moreover, $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$ for all $t \in [0,1]$.

Proof. When $\|m-m_0\| \leqq 3c'\epsilon^a$, $\|h-h_0\| \leqq \epsilon^a$ the velocity field $V(m,h,\dot h)$ is bounded (by (A.13)) and Lipschitz (by (A.15)), recall that by (A.16) $\|\dot h\| \leqq 2\epsilon^a$. We thus have local existence and uniqueness till when $\|m-m_0\| \leqq 3c'\epsilon^a$. Till this time

$$\left\|\frac{\mathrm{d}m(t)}{\mathrm{d}t}\right\| \leqq \beta\left\|L^{-1}_t\frac{\mathrm{d}h(t)}{\mathrm{d}t}\right\| \leqq \beta\frac{c_{(\text{A.13})}}{1-a_{(\text{A.9})}}2\epsilon^a \leqq 2\beta c_{(\text{A.14})}\epsilon^a$$

(recalling from Lemma 10 that $c_{(\text{A.14})} \geqq \max\{1, \dfrac{c_{(\text{A.13})}}{1-a_{(\text{A.9})}}\}$). Hence $\|m(t) - m(0)\| \leqq 2c'\epsilon^a$ which ensures existence till $t=1$. Recalling (A.17) we get from (A.18) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}\right) = 0$$

$m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$ is thus constant and being 0 initially, it is 0 at all times. □

By taking $h' = h_0$ in (A.16) by Lemma 11 we conclude that for any $h : \|h - h_0\| \leqq \epsilon^a$ there is m which satisfies $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and $\|m - m_0\| \leqq 3c'\epsilon^a$, $c' = \beta c_{(A.14)}$.

Finally, to prove (3.6) we write $h(t) = th + (1-t)h'$ so that

$$m_h - m_{h'} = \int_0^1 \frac{\mathrm{d}m(t)}{\mathrm{d}t} = -\int_0^1 L_t^{-1} p_t (h - h')$$

and (3.6) follows from (A.14). The proof of Theorem 5 is complete.

B Proof of Theorem 6

By (3.3) there is $b < 1$ so that for all ϵ small enough $\|m\| \leqq b < 1$ in the ball $\{m : \|m - m_0\| \leqq c_{(3.6)}\epsilon^a\}$; (3.8) then readily follows. To prove (3.9) we observe that $h_0(x) = 0$ for $x \in [0, x_\epsilon + 2n_0]$ because in such interval $m_0 = \bar{m}$ and $\bar{m} = \tanh\{\beta J^{\text{neum}} \star \bar{m}\}$. Thus, if $h_1 = T(h_0)$ by (A.7)

$$|h_1(x) - h_0(x)| \leqq c\epsilon \log \epsilon^{-1}, \quad |x| \leqq \xi_\epsilon := x_\epsilon + 2n_0.$$

Define for $x > \xi_\epsilon$

$$h(x) = \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(u)} = \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(m_0)}, \tag{B.1}$$

then $m_0(x) = u(\epsilon[x - \xi_\epsilon]) = \tanh\{\beta u(\epsilon[x - \xi_\epsilon] + \beta h(\epsilon[x - \xi_\epsilon])\}$, hence

$$m_0(x) = \tanh\{\beta J^{\text{neum}} \star m_0(x) + \beta(u(\epsilon[x - \xi_\epsilon]) - J^{\text{neum}} \star m_0(x) + h(\epsilon[x - \xi_\epsilon]))\}. \tag{B.2}$$

Since $m_0(x) = \tanh\{\beta J^{\text{neum}} \star m_0(x) + \beta h_0(x\}$, by (B.2)

$$|h_0(x) - h(\epsilon[x - \xi_\epsilon])| \leqq c\epsilon, \quad \text{and by (B.1)} \;\; |h_0(x) - \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(m_0)}| \leqq c\epsilon,$$

Since $h_1(x) = \int_0^x \frac{-\epsilon j}{\chi(m_0)}$, $|h_1(x) - h_0(x)| \leqq |h_0(x) - \int_{\xi_\epsilon}^x \frac{-\epsilon j}{\chi(m_0)}| + c\epsilon\xi_\epsilon$ hence (3.9).

C Proof of Theorem 7

By assumption for $x \geqq 0$, $0 \leqq m_k(x) \leqq m_0(x) + \epsilon^a$, $k < n$. By By (3.3) $\|m_0\| < 1$ so that for all ϵ small enough, p_{m_k,h_k} is uniformly bounded away from 0. There is, therefore, $C < \infty$ (recall the current j is a constant) such that

$$|h_{k+1}(x) - h_k(x)| \leqq C\epsilon \int_0^x |m_k(y) - m_{k-1}(y)|. \tag{C.1}$$

By (3.6) for any $y \in [0, \epsilon^{-1}]$,

$$|m_k(y) - m_{k-1}(y)| \leqq c \int_0^{\epsilon^{-1}\ell} \mathrm{e}^{-\alpha|y-z|} |h_k(z) - h_{k-1}(z)|, \tag{C.2}$$

where we have dropped the suffixes from c and α. We define $\psi_{k+1}(x) = |h_{k+1}(\epsilon^{-1}x) - h_k(\epsilon^{-1}x)|$, $x \in [0, \ell]$ and by combining (C.1) and (C.2) we get

$$\psi_{k+1}(x) \leqq c' \int_0^x \mathrm{d}y \int_0^\ell \mathrm{e}^{-\epsilon^{-1}\alpha|y-z|} \psi_k(z)\epsilon^{-1}\mathrm{d}z. \tag{C.3}$$

Define $v_k(x) = \mathrm{e}^{-bx}\psi_k(x)$, $b > 0$ a large constant whose value will be specified later. We have:

$$v_{k+1}(x) \leqq c' \int_0^x \mathrm{e}^{-b(x-y)}\mathrm{d}y \int_0^\ell \mathrm{e}^{-\epsilon^{-1}\alpha|y-z|+b(z-y)} v_k(z)\epsilon^{-1}\mathrm{d}z. \tag{C.4}$$

For ϵ so small that $\epsilon^{-1}\alpha > b$ we have

$$\|v_{k+1}\| \leqq c' \int_0^x \mathrm{e}^{-b(x-y)}\mathrm{d}y \, \frac{2\epsilon^{-1}}{\epsilon^{-1}\alpha - b} \|v_k\| \leqq \frac{c'}{b} \frac{2\epsilon^{-1}}{\epsilon^{-1}\alpha - b} \|v_k\|. \tag{C.5}$$

We choose b so that $\dfrac{4c'}{\alpha b} = \dfrac{1}{2}$. Then for all ϵ so small that $\dfrac{\epsilon^{-1}}{\epsilon^{-1}\alpha - b} \leqq \dfrac{2}{\alpha}$

$$\|v_{k+1}\| \leqq \frac{1}{2}\|v_k\| \quad \text{which yields} \quad \|\psi_{k+1}\| \leqq \mathrm{e}^{b\ell} \left(\frac{1}{2}\right)^k \|\psi_1\|.$$

D Spectral Properties of $A_{m,h}$

In this appendix we shall first define a set $\mathcal{A}$ by weakening properties of the pair (m_ϵ, h_ϵ), and then prove spectral properties of $A_{m,h}$ when (m, h) is in a small neighborhood of $\mathcal{A}$.

Instanton: Additional Notation

Referring to Appendix A for definition and properties of the instanton $\bar{m}$, we denote by $\bar{m}_{x_0}$, $x_0 \in (-1, 1)$, the translate of $\bar{m}$ by $\epsilon^{-1}x_0$:

$$\bar{m}_{x_0}(x) = \bar{m}(x - \epsilon^{-1}x_0), \; \bar{m}'_{x_0} = \frac{\mathrm{d}\bar{m}_{x_0}}{\mathrm{d}x}, \; \bar{p}_{x_0}(x) = \beta(1 - \bar{m}_{x_0}(x)^2), \; \bar{A}_{x_0} := \bar{p}_{x_0} J * . \tag{D.1}$$

Properties of the Pair (m_ϵ, h_ϵ)

- $m_\epsilon = \tanh\{\beta J^{\text{neum}} * m_\epsilon + h_\epsilon\}$, see (4.3).
- There are $r > 0$ and $b > 0$ so that $p_\epsilon(x) \leqq e^{-b}$ for all $|x - \epsilon^{-1}x_0| \geqq r$.
- $\|\frac{dm_\epsilon}{dx}\| < c'_{(4.4)}$ (proved in Lemma 8) and for any $c > 0$ there is $c' > 0$ so that for all ϵ small enough

$$\sup_{|x-\epsilon^{-1}x_0|\leqq c\log\epsilon^{-1}} |m_\epsilon(x) - \bar m_{x_0}(x)| < c_{(D.2)}\epsilon\log\epsilon^{-1}, \tag{D.2}$$

  because by (3.15), $\|m^* - m_0\| \leqq c\epsilon\log\epsilon^{-1}$.
- Since $\frac{dh_\epsilon}{dx} = \frac{-\epsilon j}{p_\epsilon(x)}$ and $\inf p_\epsilon > 0$ then $\|h_\epsilon\| \leqq c_1$, $\|\frac{dh_\epsilon}{dx}\| < c\epsilon$ and, by (D.2),

$$\sup_{|x-\epsilon^{-1}x_0|\leqq c\log\epsilon^{-1}} \left|\frac{dh_\epsilon(x)}{dx} - \frac{-\epsilon j}{\bar p_{x_0}(x)}\right| < c'_1\epsilon^2\log\epsilon^{-1}.$$

The Set $\mathcal{A}$

By default all coefficients a, c, C with or without a suffix are meant to be positive and independent of ϵ; we shall indicate below by item n the nth property of (m_ϵ, h_ϵ) as listed in the previous paragraph and introduce the quantities (with b in (D.3) below the parameter entering in item 2)

$$C_{(D.3)} > 1\ :\ e^{-aC_{(D.3)}(1-x_0)\log\epsilon^{-1}} = \epsilon^2,\quad a := \min\left\{\frac{b}{4}, a_{(A.6)}, a_{(A.4)}\right\} \tag{D.3}$$

$$I = \{x : |x - \epsilon^{-1}x_0| \leqq 2C_{(D.3)}\log\epsilon^{-1}\},\quad I' = \{x : |x - \epsilon^{-1}x_0| \leqq C_{(D.3)}\log\epsilon^{-1}\}. \tag{D.4}$$

(I' will be used later in Proposition 12). With this notation we define $\mathcal{A}$ as the collection of all pairs (m, h) such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$, and the following three inequalities hold:

$$p_{m,h}(x) = \beta\left(1 - m(x)^2\right) \leqq e^{-2a_{(D.5)}},\quad |x - \epsilon^{-1}x_0| \geqq r_{(D.5)} \tag{D.5}$$

$$\left\|\frac{dm}{dx}\right\| \leqq C_{(D.6)},\quad \sup_{x\in I}|m(x) - \bar m_{x_0}(x)| \leqq c'_{(D.6)}\epsilon\log\epsilon^{-1} \tag{D.6}$$

$$\|h\| \leqq C_{(D.7)},\quad \left\|\frac{dh}{dx}\right\| \leqq C_{(D.7)},\quad \sup_{x\in I}\left|\frac{dh(x)}{dx} - \frac{-\epsilon j}{\bar p_{x_0}(x)}\right| \leqq c_{(D.7)}\epsilon^2\log\epsilon^{-1}, \tag{D.7}$$

where $C_{(D.7)} > 2$ and:

- $r_{(D.5)} > r$ and $2a_{(D.5)} = b/2$, b and r are as in item 2,
- $C_{(D.6)} > 2c'_{(4.4)}$ and $c'_{(D.6)} > 2c_{(D.2)}$ (see item 3),
- $C_{(D.7)} > 2\max\{c, c_1\}$ and $c_{(D.7)} > 2c'_1$ (see item 4).

With the above choice of parameters $(m_\epsilon, h_\epsilon) \in \mathcal{A}$.

Spectral Properties in a Neighborhood of $\mathcal{A}$

We continue the analysis of the spectrum of $A_{m,h}$ started in Appendix A, assuming that (m,h) is in the δ ball of $\mathcal{A}$ defined as $\bigcup_{(m,h)\in\mathcal{A}} B_\delta(m,h)$, $B_\delta(m,h) := \{(m',h') : \|h-h'\| \leqq \delta, \|m-m'\| \leqq \delta\}$. Using the notation: $\langle f\rangle_\infty = \int_{\mathbb{R}} f\bar{p}^{-1}$ and $\langle f\rangle_{m,h} = \int_{\mathbb{R}} f p_{m,h}^{-1}$ we have:

Proposition 12. *There are positive constants $C_{(D.8)}$, $c_{(D.9)}$, $c'_{(D.10)}$, $c_{(D.11)}$ so that for any ϵ small enough there is $\delta = \delta(\epsilon) > 0$ such that for any (m,h) in the δ ball of $\mathcal{A}$*

$$p_{m,h} \geqq C_{(D.8)} \tag{D.8}$$

$$|\lambda_{m,h} - [1 - C_{(D.9)}\epsilon]| \leqq c_{(D.9)}(\epsilon\log\epsilon^{-1})^2, \quad C_{(D.9)} = |j|\frac{\langle \bar{m}'\rangle_\infty}{\langle(\bar{m}')^2\rangle_\infty} > 0. \tag{D.9}$$

Moreover, let $u_{m,h} > 0$ be normalized as $\langle u^2_{m,h}\rangle_{m,h} = 1$ and I' is as in (D.4), then

$$\sup_{x\in I'} |u_{m,h}(x) - \tilde{m}'_{x_0}(x)| \leqq c'_{(D.10)}\epsilon(\log\epsilon^{-1})^2 \tag{D.10}$$

$$u_{m,h}(x) \leqq c_{(D.11)} e^{-a_{(D.5)}|x-\epsilon^{-1}x_0|}. \tag{D.11}$$

Proof. We shall first prove, with slightly better coefficients, the inequalities (D.8)–(D.11) when (m,h) is in $\mathcal{A}$ and then use a continuity argument to extend the analysis to a δ ball of $\mathcal{A}$. We thus fix $(m,h)\in\mathcal{A}$ and drop the suffix (m,h) when no ambiguity may arise.

- **Proof of (D.8).** We bound $|m(x)| \leqq \tanh\{\beta J^{\text{neum}} * \|m\| + \beta\|h\|\}$ and $\|h\| \leqq C_{(D.7)}$, hence $p_{m,h} \geqq 2C_{(D.8)}$, with $2C_{(D.8)} = \beta(1-s^2)$, s the positive solution of $s = \tanh\{\beta s + \beta C_{(D.7)}\}$. (D.8) then follows in a δ ball of (m,h) if δ is small enough.

We shall next prove some rough bounds on λ and u, which will then be improved as required in the proposition. We take here (m,h) in a δ-ball of $\mathcal{A}$ with δ small enough. We are going to use repeatedly variants of the obvious equality:

$$\langle fA_{m,h}g\rangle_{m,h} = \langle fA_{m',h'}g\rangle_{m',h'} = \int fJ^{\text{neum}} * g. \tag{D.12}$$

We have the lower bound $\lambda \geqq \dfrac{\langle \bar{m}'_{x_0} A\bar{m}'_{x_0}\rangle_{m,h}}{\langle(\bar{m}'_{x_0})^2\rangle_{m,h}}$, $A \equiv A_{m,h}$ and $\bar{m}'_{x_0}$, here restricted to $\Lambda = \epsilon^{-1}[-1,1]$. Using (D.12) we can rewrite the numerator as

$$\begin{aligned}
\langle \bar{m}'_{x_0} A\bar{m}'_{x_0}\rangle_{m,h} &= \int_{\Lambda\times\Lambda} \bar{m}'_{x_0}(x)J^{\text{neum}}(x,y)\bar{m}'_{x_0}(y) = \int_{\mathbb{R}\times\mathbb{R}} \bar{m}'_{x_0}(x)J(x,y)\bar{m}'_{x_0}(y) + \Delta \\
&= \int_{\mathbb{R}} \bar{m}'_{x_0}(x)^2/\bar{p}_{x_0} + \Delta = \int_\Lambda \bar{m}'_{x_0}(x)^2/\bar{p}_{x_0} + \Delta' \\
&= \langle(\bar{m}'_{x_0})^2\rangle_{m,h} + \int_\Lambda \bar{m}'_{x_0}(x)^2\frac{p-\bar{p}_{x_0}}{p\,\bar{p}_{x_0}} + \Delta', \quad p \equiv p_{m,h},
\end{aligned}$$

where by (A.4) and (D.3), $|\Delta|$ and $|\Delta'|$ are both bounded by $\leqq c e^{-a_{(A.4)}\epsilon^{-1}(1-x_0)} \leqq c\epsilon^2$. The denominator in the last integral is bounded from below because $p \equiv p_{m,h} \geqq C_{(D.8)}$, ((D.8) has already been proved) and $\bar{p}_{x_0} \geqq \beta(1-m_\beta^2)$ (as $\bar{m}(x)$ converges monotonically to m_β as $x \to \infty$). By (D.6) and for δ small enough $|p(x) - \bar{p}_{x_0}(x)| \leqq 2c'_{(D.6)}\epsilon \log \epsilon^{-1}$ when $x \in I$, while in the complement we bound $\bar{m}'_{x_0}$ as in (A.4) (recalling (D.3)) and use that $|p - \bar{p}_{x_0}| \leqq \beta$. In conclusion, we get

$$\lambda \geqq 1 - c_{(D.13)}\epsilon \log \epsilon^{-1}, \tag{D.13}$$

with $c_{(D.13)}$ dependent on $C_{(D.7)}$, $c'_{(D.6)}$, $a_{(D.5)}$.

- **Proof of (D.11).** We use (D.13) and the identity $u(x) = \lambda^{-n}(A^n u)(x)$ to get upper bounds on u. With $n = 1$ we obtain

$$\|u\| \leqq \lambda^{-1}\|J\|\beta\sqrt{2}\left(\int u^2\right)^{1/2} \leqq \lambda^{-1}\|J\|\beta\sqrt{2}\left(\int \frac{\|p\|}{p}u^2\right)^{1/2} \leqq c\langle u^2\rangle_{m,h}^{1/2} \tag{D.14}$$

(having used Cauchy–Schwartz and that $\|p\| \leqq \beta$). By tuning n with the distance from $\epsilon^{-1}x_0$ we get, using (D.5),

$$u(x) \leqq [1 - c_{(D.13)}\epsilon \log \epsilon^{-1}]^{-n} e^{-2a_{(D.5)}n}\|u\|, \quad \text{when } |x - \epsilon^{-1}x_0| \geqq n + r_{(D.5)}, \tag{D.15}$$

which together with (D.14) proves (D.11) for $(m,h) \in \mathcal{A}$.

To prove (D.10) we need an upper bound on $\lambda_{m,h}$ that we shall prove in (D.20) below. Preliminary to the proof of (D.20) is the following estimate

$$\left|\frac{\langle u^2\rangle}{\langle u^2\rangle_\infty} - 1\right| \leqq c\epsilon^2 + c\epsilon \log \epsilon^{-1}, \tag{D.16}$$

which we prove next. Recalling that $p \geqq C_{(D.8)}$, $p \leqq \beta$, $\bar{p}_{x_0} \geqq \beta(1-m_\beta^2)$ and $\bar{p}_{x_0} \leqq \beta$, we have

$$c_{(D.17)}^{-1} \leqq \frac{\langle u^2\rangle}{\langle u^2\rangle_\infty} \leqq c_{(D.17)}. \tag{D.17}$$

Then, by (D.11)

$$\frac{u(y)}{\langle u^2\rangle_\infty^{1/2}} \leqq c e^{-a_{(D.5)}|y-\epsilon^{-1}x_0|}, \quad y \in \Lambda \setminus I'. \tag{D.18}$$

Hence by (D.3) and since $a_{(D.5)} = \dfrac{b}{4}$,

$$\int_{\Lambda\setminus I'} u^2/\bar{p}_{x_0} \leqq c_{(D.19)}\langle u^2\rangle_\infty \epsilon^2, \qquad \langle u^2\rangle_\infty > \int_{I'} u^2/\bar{p}_{x_0} \geqq (1 - c_{(D.19)}\epsilon^2)\langle u^2\rangle_\infty. \tag{D.19}$$

We also have

$$\left|\frac{\langle u^2\rangle}{\langle u^2\rangle_\infty} - \frac{\int_{I'} u^2/p}{\langle u^2\rangle_\infty}\right| \leqq c\epsilon^2, \quad \left|\frac{\langle u^2\rangle}{\langle u^2\rangle_\infty} - \frac{\int_{I'} u^2/\bar{p}_{x_0}}{\langle u^2\rangle_\infty}\right| \leqq c\epsilon^2 + c\epsilon\log\epsilon^{-1}$$

$$\left|\frac{\langle u^2\rangle}{\langle u^2\rangle_\infty} - 1\right| \leqq c\epsilon^2 + c\epsilon\log\epsilon^{-1}.$$

In the first inequality above we have used (D.18), in the second (D.6) and in the third (D.19). (D.16) is proved.

We are now ready for the proof of (D.20). Let $\lambda \equiv \lambda_{m,h}$ and (m, h) in a δ-ball of $\mathcal{A}$, δ suitably small. We start from the operator $\bar{A}_{x_0} = \bar{p}_{x_0} J*$ acting on $L^\infty(\mathbb{R})$ and since 1 is its maximal eigenvalue (with eigenvector $\bar{m}'_{x_0}$), $1 \geqq \dfrac{\langle u\bar{A}_{x_0}u\rangle_\infty}{\langle u^2\rangle_\infty}$ where we choose $u = u_{m,h}$ on $\Lambda = \epsilon^{-1}[-1, 1]$ and $u = 0$ on Λ^c. Denoting $\langle f\rangle_\infty = \displaystyle\int_{\mathbb{R}} \frac{f}{\bar{p}_{x_0}}$, we then have

$$\langle u\bar{A}_{x_0}u\rangle_\infty = \int_{\Lambda\times\Lambda} u(x)J(x, y)u(y) = \int_{\Lambda\times\Lambda} u(x)J^{\text{neum}}(x, y)u(y) + R,$$

with $R = -\displaystyle\int_{\Lambda\times\Lambda^c} u(x)J(x, y)u(y_\Lambda)$, y_Λ the reflection of y into Λ through its endpoints. By (D.11) $|R| \leqq c e^{-a_{(D.5)}|\epsilon^{-1}(1-x_0)}\|u\|^2$, hence writing hereafter $\langle\cdot\rangle = \langle\cdot\rangle_{m,h}$,

$$\begin{aligned}\langle u^2\rangle_\infty &\geqq \langle u\bar{A}_{x_0}u\rangle_\infty \geqq \langle uAu\rangle - c e^{-a_{(D.5)}|\epsilon^{-1}(1-x_0)}\|u\|^2\\ &= \lambda\langle u^2\rangle - c e^{-a_{(D.5)}|\epsilon^{-1}(1-x_0)}\|u\|^2.\end{aligned}$$

Thus, by (D.14), $\lambda \leqq \dfrac{\langle u^2\rangle_\infty}{\langle u^2\rangle} + c e^{-a_{(D.5)}|\epsilon^{-1}(1-x_0)}$. By (D.16) and (D.13)

$$1 - c_{(D.13)}\epsilon\log\epsilon^{-1} \leqq \lambda \leqq 1 + c_{(D.20)}\epsilon\log\epsilon^{-1}, \tag{D.20}$$

with $c_{(D.20)}$ dependent on $C_{(D.7)}$, $c'_{(D.6)}$, $a_{(D.5)}$.

We shall next prove (D.10), which we split into an upper and a lower bound for $u = u_{m,h}$. We take here (m, h) in a δ-ball of $\mathcal{A}$ with δ small enough

- **Proof of (D.10) (the upper bound).** Let $y \in I'$, then, writing below $y_0 \equiv y$,

$$\lambda^n u(y) = \int u(y_n)\prod_{k=1}^n \left\{\bar{A}_{x_0}(y_{k-1}, y_k)\frac{p(y_{k-1})}{\bar{p}_{x_0}(y_{k-1})}\right\} \mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{D.21}$$

We again choose $n = C_{(D.3)}\log\epsilon^{-1}$, observing that since $y_0 \in I'$ all y_k are in I. We bound $\lambda^{-n} \leqq (1 - c_{(D.13)}\epsilon\log\epsilon^{-1})^{-n} \leqq (1 + nc\epsilon\log\epsilon^{-1}) \leqq (1 + c'\epsilon[\log\epsilon^{-1}]^2)$. Since all y_k are in I, by (D.6) and for δ small enough,

$$\prod_{k=1}^n \frac{p(y_{k-1})}{\bar{p}_{x_0}(y_{k-1})} \leqq 1 + c\epsilon[\log\epsilon^{-1}]^2,$$

hence (with a new constant c)

$$u(y) \leqq [1 + c\epsilon(\log \epsilon^{-1})^2] \int u(y_n) \prod_{k=1}^{n} \{\bar{A}_{x_0}(y_{k-1}, y_k)\}\} dy_1 \cdots dy_n. \tag{D.22}$$

We define $\tilde{u}$ so that $u(y_n) = \langle \tilde{m}'_{x_0} u \rangle_\infty \, \tilde{m}'_{x_0}(y_n) + \tilde{u}$. By (A.6)-(D.3) and for all $y \in I'$

$$u(y) \leqq \tilde{m}'_{x_0}(y)[1 + c\epsilon(\log \epsilon^{-1})^2] \langle \tilde{m}'_{x_0} u \rangle_\infty + c\epsilon^2 \|u\|, \tag{D.23}$$

which by (D.14) can be rewritten as

$$\frac{u(y)}{\langle u^2 \rangle^{1/2}} \leqq \left\{ [1 + c\epsilon(\log \epsilon^{-1})^2] \frac{\langle \tilde{m}'_{x_0} u \rangle_\infty}{\langle u^2 \rangle^{1/2}} \right\} \tilde{m}'_{x_0}(y) + c\epsilon^2. \tag{D.24}$$

By Cauchy–Schwartz,

$$\frac{u(y)}{\langle u^2 \rangle^{1/2}} \leqq \left(\frac{\langle u^2 \rangle_\infty}{\langle u^2 \rangle} \right)^{1/2} \left\{ [1 + c\epsilon(\log \epsilon^{-1})^2] \tilde{m}'_{x_0}(y) \right\} + c\epsilon^2, \tag{D.25}$$

which, by (D.16), proves

$$\frac{u(x)}{\langle u^2 \rangle^{1/2}} \leqq \tilde{m}'_{x_0}(x) + \frac{c'_{(D.10)}}{2} \epsilon(\log \epsilon^{-1})^2. \tag{D.26}$$

- **Proof of (D.10) (the lower bound).** Proceeding in a similar way we get the lower bound:

$$\frac{u(y)}{\langle u^2 \rangle^{1/2}} \geqq \left\{ [1 - c\epsilon(\log \epsilon^{-1})^2] \frac{\langle \tilde{m}'_{x_0} u \rangle_\infty}{\langle u^2 \rangle^{1/2}} \right\} \tilde{m}'_{x_0}(y) \; - c\epsilon^2. \tag{D.27}$$

To bound the curly bracket from below, we multiply both sides of (D.23) by $\bar{p}_{x_0}^{-1} u$ and integrate over I'. By (D.19):

$$(1 - c_{(D.19)}\epsilon^2) \langle u^2 \rangle_\infty \leqq \langle \tilde{m}'_{x_0} u \rangle_\infty^2 [1 + c\epsilon(\log \epsilon^{-1})^2] + c\epsilon^2 \log \epsilon^{-1} \|u\|^2.$$

By (D.14), $(1 - c_{(D.19)}\epsilon^2) \leqq \dfrac{\langle \tilde{m}'_{x_0} u \rangle_\infty^2}{\langle u^2 \rangle_\infty} [1 + c\epsilon(\log \epsilon^{-1})^2] + c\epsilon^2 \log \epsilon^{-1}$, hence

$$1 - c\epsilon(\log \epsilon^{-1})^2 \leqq \frac{\langle \tilde{m}'_{x_0} u \rangle_\infty^2}{\langle u^2 \rangle_\infty} \leqq 1, \tag{D.28}$$

which by (D.27) yields $\dfrac{u(y)}{\langle u^2 \rangle^{1/2}} \geqq \left(\dfrac{\langle u^2 \rangle_\infty}{\langle u^2 \rangle} \right)^{1/2} [1 - c\epsilon(\log \epsilon^{-1})^2] \tilde{m}'_{x_0}(y) - c\epsilon^2$. Using (D.16) we then get

$$u_{m,h}(x) \geqq \tilde{m}'_{x_0}(x) - \frac{c'_{(D.10)}}{2} \epsilon(\log \epsilon^{-1})^2. \tag{D.29}$$

- **Proof of (D.9).** We first suppose $(m, h) \in \mathcal{A}$ and use, for the first time, the conditions on $\mathrm{d}m/\mathrm{d}x$ and $\mathrm{d}h/\mathrm{d}x$ contained in the definition of $\mathcal{A}$. Writing f' for the derivative of f with respect to x, we differentiate $m(x) = \tanh\{\beta J^{\mathrm{neum}} * m(x) + \beta h(x)\}$ and get $m' = pJ^{\mathrm{neum}} * m' + ph'$, hence $Lm' = -ph'$, $L = A - 1$. We multiply both sides by $p^{-1}u$ and integrate over x. Recalling that L is self-adjoint in the scalar product with weight p^{-1}, we then have

$$(\lambda - 1)\langle um'\rangle = -\langle uph'\rangle. \tag{D.30}$$

By (D.11), $|\langle um'\rangle - \int_{I'} p^{-1}um'| \leqq c\epsilon^2$, having used that $|m'|$ is bounded, and the first inequality in (D.6). Since $m' = pJ^{\mathrm{neum}} * m' + ph'$, using the second inequality in (D.7),

$$|m'(x) - p(J^{\mathrm{neum}})' * m(x)| \leqq \sup_{y \in I'} |ph'| \leqq c\epsilon, \quad x \in I'.$$

Then, by the second inequality in (D.6),

$$|m'(x) - pJ^{\mathrm{neum}} * \bar{m}'(x)| = |m'(x) - p\left(J^{\mathrm{neum}}\right)' * \bar{m}(x)| \leqq c\epsilon \log \epsilon^{-1}, \quad x \in I'$$

and by (D.10) and (D.11),

$$|\langle um'\rangle - \langle \tilde{m}'\bar{m}'\rangle_\infty| \leqq c\epsilon \log \epsilon^{-1}. \tag{D.31}$$

Analogous estimates hold for $\langle uph'\rangle$ and we get

$$|\lambda - [1 - C_{(\mathrm{D.9})}\epsilon]| \leqq \frac{c_{(\mathrm{D.9})}}{2}(\epsilon \log \epsilon^{-1})^2. \tag{D.32}$$

To conclude the proof of the Proposition we need to extend the previous bounds to $(\hat{h}, \hat{m})$ in a δ-ball around (m, h). By (D.12)

$$\frac{\hat{\lambda}}{\lambda} \geqq \frac{\langle u^2\rangle}{\langle u^2\rangle_{\hat{h},\hat{m}}} \geqq c\delta. \tag{D.33}$$

The analogous bound can be proved for $\lambda/\hat{\lambda}$ and (D.9) follows if δ is small enough. The proof of Proposition 12 is complete. □

The rest of the spectrum is separated from $\lambda_{m,h}$ by a spectral gap, see [4].

Proposition 13. *There are $c_{(\mathrm{D.34})}$, $a_{(\mathrm{D.34})} > 0$, $c_{(\mathrm{D.35})}$ and $a_{(\mathrm{D.35})} > 0$ so that for all ϵ small enough the following holds. For any $(m', h') \in \mathcal{A}$ there is $\delta = \delta(\epsilon)$ so that for all (m, h) in a δ-ball around (m', h'), for all bounded ψ*

$$\|A^n_{m,h}\tilde{\psi}\| \leqq c_{(\mathrm{D.34})}\mathrm{e}^{-a_{(\mathrm{D.34})}n}\|\psi\|, \quad \tilde{\psi} = \psi - \frac{\langle \psi u_{m,h}\rangle_{m,h}}{\langle u^2_{m,h}\rangle_{m,h}} u_{m,h} \tag{D.34}$$

$$|L^{-1}_{m,h}\tilde{\psi}(x)| \leqq c_{(\mathrm{D.35})} \int \mathrm{e}^{-a_{(\mathrm{D.35})}|x-y|}|\tilde{\psi}(y)|\,\mathrm{d}y. \tag{D.35}$$

The Operator A^* and its Spectral Properties

We conclude this appendix with a simple extension of the previous results which will allow us to complete the proof of Theorem 1 and of Lemma 8. Let (m^*, h^*) be the solution of the antisymmetric problem in $\epsilon^{-1}[-1, \ell^*]$, with x_0 the middle point in $[-1, \ell^*]$. We denote by A^* the operator $p^* J^{\text{neum},*} *$ acting on $L^\infty(\epsilon^{-1}[-1, \ell^*])$ with $p^* = p_{m^*,h^*}$ and kernel $J^{\text{neum},*}(x, y)$ (defined with Neumann conditions on $\epsilon^{-1}[-1, \ell^*]$). We denote by $\langle\cdot\rangle_*$ the integral over $\epsilon^{-1}[-1, \ell^*]$ with respect to the measure $(p^*)^{-1}\mathrm{d}x$. We first observe that the pair (m^*, h^*) satisfies the same properties (with the same parameters) as the pair (m_ϵ, h_ϵ) (recall that m_ϵ is the restriction of m^* to $\epsilon^{-1}[-1, 1]$ and that h_ϵ is the restriction of h^* except for the additive term R_ϵ). It then follows that λ^* and u^* satisfy the same properties as $\lambda_{m,h}$ and $u_{m,h}$ stated in Proposition 12. (Without loss of generality we may suppose with the same coefficients). Also, Proposition 13 remains valid; indeed, its validity is quite general as discussed in Section 8.3 of [10].

Conclusion of the proof of Theorem 1 In order to keep the notation used so far, we replace the original interval $\epsilon^{-1}[-\ell, \ell]$ in Theorem 1 by the interval $\epsilon^{-1}[-1, \ell^*]$ and denote the solution (m_ϵ, h_ϵ) of Theorem 1 by (m^*, h^*). Recalling that it only remains to prove that $m^*(x)$ is an increasing function of x (we are supposing $j < 0$), we shorthand $\psi = \frac{\mathrm{d}m^*}{\mathrm{d}x}$ and shall prove that $\psi(x)$ is strictly positive at all x. We have

$$\psi = L^{-1}\left(-p^*\frac{\mathrm{d}h^*}{\mathrm{d}x}\right) = L^{-1}(\epsilon j), \tag{D.36}$$

where $L = A^* - 1$. The positivity of ψ then follows from

$$L^{-1}(\epsilon j) = \sum_{n=0}^{\infty}(A^*)^n(-\epsilon j), \tag{D.37}$$

once we prove that the series converges (as all its elements are positive). Convergence follows because there are $a = a(\epsilon)$ and $c = c(\epsilon)$ positive such that for all n,

$$\|(A^*)^n\| \leqq c\mathrm{e}^{-an}, \tag{D.38}$$

which would be easy if this were the L^2 norm, as we know that λ^* is the maximal eigenvalue and $\lambda^* < 1 - c\epsilon$.

- **Proof of** (D.38). With λ^* and u^* the maximal eigenvalue and eigenvector of A^*, u^* normalized, $\langle(u^*)^2\rangle_* = 1$, we have

$$(A^*)^n\psi = (\lambda^*)^n\langle u^*\psi\rangle_* u^* + (A^*)^n\tilde{\psi}, \quad \tilde{\psi} = \psi - \langle u^*\psi\rangle_* u^*. \tag{D.39}$$

We have $\lambda^* < 1 - C\epsilon$, $C > 0$, (by (D.9)), we bound u^* using (D.11), then by (D.34)

$$\|(A^*)^n\psi\| \leqq c(\lambda^*)^n\|\psi\| + c_{(\mathrm{D}.34)}\mathrm{e}^{-a_{(\mathrm{D}.34)}n}\|\psi\|, \tag{D.40}$$

hence (D.38). □

Conclusion of the proof of Lemma 8 It only remains to prove the second inequality in (4.4). With $\psi = \frac{\mathrm{d}m^*}{\mathrm{d}x}$, by (D.36), and using the previous notation,

$$\psi = \left([\lambda^* - 1]^{-1} \epsilon j \int_{-\epsilon^{-1}}^{\epsilon^{-1}\ell^*} u^* \right) u^* + L^{-1}\phi, \quad \phi = (\epsilon j) - (\epsilon j \int u^*) u^*, \tag{D.41}$$

$|[\lambda^* - 1]^{-1}\epsilon j| \leqq c$ by (D.9) and by (D.11):

$$\int u^* \leqq c, \qquad \sup_{|x-\epsilon^{-1}x_0| \geqq r_{(4.4)} \log \epsilon^{-1}} u^*(x) \leqq c_{(D.11)} \mathrm{e}^{-a_{(D.5)} r_{(4.4)} \log \epsilon^{-1}}.$$

By choosing $a_{(D.5)} r_{(4.4)} > 1$, the first term on the right-hand side of (D.41) is bounded by $c\epsilon$ when $|x - \epsilon^{-1}x_0| \geqq r_{(4.4)} \log \epsilon^{-1}$. The last term is bounded using (D.34) by

$$\leqq c_{(D.34)} \sum_{n=0}^{\infty} \mathrm{e}^{-a_{(D.34)} n} \|\phi\| \leqq c' \epsilon$$

because $\|\phi\| \leqq c\epsilon$. Lemma 8 is proved. □

E Auxiliary Dynamics

We return in this appendix to the analysis of the auxiliary dynamics introduced in Appendix A. We shall study the case where initially $(m_0, h_0) \in \mathcal{A}$ and prove a local existence and uniqueness theorem under suitable assumptions on $h(t)$. We would like to work in $\mathcal{A}$, but $\mathcal{A}$ itself is not nice in the L^∞ topology we are using, as it involves derivatives. For this reason we introduced the δ-balls of $\mathcal{A}$ in the previous appendix, which will play an important role here as well. Our first result is a straight consequence of Proposition 12 and Proposition 13, and its proof is omitted:

Proposition 14. *There is $c > 0$ and for any $\epsilon > 0$ small enough there is $\delta = \delta(\epsilon) > 0$ not larger than the parameter δ in Proposition 12 such that for any $(m_0, h_0) \in \mathcal{A}$ and any (m', h') and (m'', h'') in the δ-ball of (m_0, h_0)*

$$\|L^{-1}_{m',h'}\| \leqq c\epsilon^{-1}, \quad \|L^{-1}_{m',h'} - L^{-1}_{m'',h''}\| \leqq c\epsilon^{-2} \left(\|h' - h''\| + \|m' - m''\| \right). \tag{E.1}$$

Proposition 15. *Let δ and c be as in Proposition 14 and let C be any positive number. Then for any $\epsilon > 0$ small enough there is $T \in (0, \frac{\delta}{2C})$ such that the following holds. For any $(m_0, h_0) \in \mathcal{A}$, and any $h(t)$, $t \in [0, T]$, such that $h(0) = h_0$ and $\|\frac{\mathrm{d}h(t)}{\mathrm{d}t}\| \leqq C$ there is $m(t)$, $t \in [0, T]$, such that:*

$$\frac{\mathrm{d}m}{\mathrm{d}t} = L_t^{-1} \left(-p_t \frac{\mathrm{d}h}{\mathrm{d}t} \right), \quad m(0) = m_0, \qquad L_t = L_{m(t),h(t)}, \ p_t = p_{m(t),h(t)}, \tag{E.2}$$

($L_t = L_{m(t),h(t)}$, $p_t = p_{m(t),h(t)}$), $\|m(\cdot) - m_0\| \leqq \frac{\delta}{2}$ and $m(\cdot) = \tanh\{\beta J^{\text{neum}} * m(\cdot) + \beta h(\cdot)\}$. Finally $m(\cdot)$ is the unique solution of (E.2) in $\|m(\cdot) - m_0\| \leqq \frac{\delta}{2}$.

Proof. T is determined by the following three conditions:

$$T < \frac{\delta}{2C}, \quad c\epsilon^{-1}\beta C T < \frac{\delta}{2}, \quad \left(c\epsilon^{-1}2\beta^2 + \beta c\epsilon^{-2}\right) T < 1. \tag{E.3}$$

The first one ensures that $\|h(\cdot) - h_0\| < \frac{\delta}{2}$ (because $\|\frac{\mathrm{d}h(t)}{\mathrm{d}t}\| \leqq C$); the second one (obtained by bounding $L_t^{-1}\left(-p_t \frac{\mathrm{d}h}{\mathrm{d}t}\right)$ via Proposition 14) will imply that $\|m(\cdot) - m_0\| < \frac{\delta}{2}$, so that $(m(t), h(t))$ is always in the δ-ball of (m_0, h_0) and Proposition 14 can be applied. The third condition will imply that the integral version of (E.2) gives rise to a contraction.

Let $\mathcal{X} := \left\{m \in C\Big([0, T], L^\infty(\epsilon^{-1}[-1, 1]; [-1, 1])\Big) : m(0) = m_0, \|m(\cdot) - m_0\| \leqq \frac{\delta}{2}\right\}$ and for $m \in \mathcal{X}$ let

$$\psi(m)(t) = m_0 + \int_0^t L_s^{-1}\left(-p_s \frac{\mathrm{d}h(s)}{\mathrm{d}s}\right). \tag{E.4}$$

By (E.1) and the second inequality in (E.3), $\|\psi(m)(t) - m_0\| \leqq c\epsilon^{-1}\beta C t < \dfrac{\delta}{2}$. Thus ψ maps $\mathcal{X}$ into itself. By (E.1) and the third inequality in (E.3) ψ is a contraction with sup-norm in x and t. Therefore there is a fixed point $m \in \mathcal{X}$: $m = \psi(m)$, and since ψ maps $\mathcal{X}$ into functions which are differentiable in t with bounded derivative, m is a solution of (E.2). By (E.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}\right) = 0,$$

so that $m(t) - \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\} = m_0 - \tanh\{\beta J^{\text{neum}} * m_0 + \beta h_0\} = 0$. Finally, if m solves (E.2) and $\|m(\cdot) - m_0\| \leqq \frac{\delta}{2}$, then $m \in \mathcal{X}$ and $\psi(m) = m$. Since ψ is a contraction m is unique. □

Properties of the Sets $\mathcal{A}$ and $\mathcal{G}$

The intervals I and I' which appear frequently in the sequel have been defined in (D.4).

Fixing the Parameters in the Set $\mathcal{G}$

The coefficient $a_{(4.6)}$ is a positive number strictly smaller than all the parameters in Appendix D involved with exponential decay; in particular we require $a_{(4.6)} < \min\{a_{(D.5)}(1 - x_0), a_{(D.35)}\frac{1+x_0}{1-x_0}\}$. The other parameter $b_{(4.7)}$ is fixed so that:

$$b_{(4.7)} < 1 \quad \text{and such that} \quad \left\{\frac{3\beta c_{(D.35)}}{a_{(D.35)} - a_{(4.6)}\frac{1-x_0}{1+x_0}}\right\} b_{(4.7)} < \frac{e^{-2a_{(D.5)}} - e^{-4a_{(D.5)}}}{2\beta}. \tag{F.1}$$

Proposition 16. *For all $\epsilon > 0$ small enough, if $h \in \mathcal{G}$ there is m such that $m = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$ and $(m, h) \in \mathcal{A}$.*

Proof. Given $h \in \mathcal{G}$ and $t \in [0, 1]$ we define $h(t) := th + (1-t)h_\epsilon$ observing that $(m(0), h(0)) := (m_\epsilon, h_\epsilon) \in \mathcal{A}$ by the definition of $\mathcal{A}$. Let S be the sup of all $s \leqq 1$ such that there exists $m(t)$, $t \in [0, s]$, which solves (E.2) in $[0, s]$ starting from $m(0) = m_\epsilon$, and such that for all such t, $(m(t), h(t))$ is in the δ-ball of $\mathcal{A}$ with δ as Proposition 12. We shall prove that $S = 1$ and that for all $t \leqq 1$ $(m(t), h(t)) \in \mathcal{A}$, thus proving the Proposition.

Since $\|\frac{\mathrm{d}h(t)}{\mathrm{d}t}\| = \|h - h_\epsilon\| \leqq b_{(4.7)}$ (because $h \in \mathcal{G}$) we can apply Proposition 15 with $C = b_{(4.7)}$ and $(m_0, h_0) = (m_\epsilon, h_\epsilon) \in \mathcal{A}$. As a consequence there is $T = T(\epsilon) > 0$ so that $m(t) = \tanh\{\beta J^{\text{neum}} * m(t) + \beta h(t)\}$, $t \in [0, T]$, and $\delta = \delta(\epsilon)$ so that $\|h(t) - h_\epsilon\| \leqq \delta/2$, $\|m(t) - m_\epsilon\| \leqq \delta/2$, $t \in [0, T]$. Since δ is not larger than the parameter δ of Proposition 12 (see Proposition 14), we then conclude that $S \geqq T$. By the definition of S, the bounds in Proposition 12 hold for $(m(t), h(t))$ at any $t \in [0, S]$ and it is now just a matter of computations to check that $(m(t), h(t)) \in \mathcal{A}$ for all such t. We start by proving that $h(t)$ satisfies the conditions in (D.7).

$$\|h(t)\| \leqq \|h_\epsilon\| + \|h - h_\epsilon\| \leqq \frac{C_{(D.7)}}{2} + b_{(4.7)} \leqq C_{(D.7)}, \left(\text{as } b_{(4.7)} < 1 < \frac{C_{(D.7)}}{2}\right)$$

$$\left\|\frac{\mathrm{d}h(t)}{\mathrm{d}x}\right\| \leqq \left\|\frac{\mathrm{d}h_\epsilon}{\mathrm{d}x}\right\| + \left\|\frac{\mathrm{d}(h - h_\epsilon)}{\mathrm{d}x}\right\| \leqq \frac{C_{(D.7)}}{2} + \epsilon \leqq C_{(D.7)},$$

for ϵ small enough. Finally in I (defined in (D.4))

$$\left|\frac{\mathrm{d}h(t)}{\mathrm{d}x} - \frac{-\epsilon j}{\bar{p}_{x_0}}\right| \leqq \left|\frac{\mathrm{d}h_\epsilon}{\mathrm{d}x} - \frac{-\epsilon j}{\bar{p}_{x_0}}\right| + t\left|\frac{\mathrm{d}(h - h_\epsilon)}{\mathrm{d}x}\right| \leqq \frac{c_{(D.7)}}{2}\epsilon^2 \log \epsilon^{-1} + \epsilon^2$$
$$\leqq c_{(D.7)}\epsilon^2 \log \epsilon^{-1}$$

(for ϵ small enough) so that also the last condition in (D.7) is satisfied.

We shall next prove that $m(t)$ satisfies the conditions required in $\mathcal{A}$. We write $f(t) := -p_t[h - h_\epsilon]$; λ_t, $u(t)$ for the maximal eigenvalue and eigenvector of A_t; $\langle\cdot\rangle_t$ for the integral of the measure $p_t^{-1}\mathrm{d}x$ on $\epsilon^{-1}[-1, 1]$; $\tilde{f}(t) := f(t) - \langle u(t) f(t)\rangle_t u(t)$. By (E.2)

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = L_t^{-1} f(t) = \lambda_t^{-1}\langle u(t) f(t)\rangle_t u(t) + L_t^{-1}\tilde{f}(t), \quad \langle u(t)^2\rangle_t = 1. \tag{F.2}$$

We bound $|f(t)| \leqq \beta N(h - h_\epsilon)E_\epsilon(x)^{-1}$ (using that $h \in \mathcal{G}$ and $p_t \leqq \beta$), $u(t) \leqq c_{(D.11)}\mathrm{e}^{-a_{(D.5)}|x-\epsilon^{-1}x_0|}$ and get

$$|\langle u(t) f(t)\rangle_t| \leqq cN(h - h_\epsilon)\mathrm{e}^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}$$
$$\left|\tilde{f}(t)\right| \leqq N(h - h_\epsilon)\left(c\mathrm{e}^{-a_{(4.6)}(1-x_0)\epsilon^{-1}} c_{(D.11)}\mathrm{e}^{-a_{(D.5)}|x-\epsilon^{-1}x_0|} + \beta E_\epsilon^{-1}\right). \tag{F.3}$$

By (D.35) $|L_t^{-1}\tilde{f}(t)|(x) \leqq c_{(D.35)} \int \mathrm{e}^{-a_{(D.35)}|x-y|}|\tilde{f}(t)|\,\mathrm{d}y$ and

$$\int \mathrm{e}^{-a_{(D.35)}|x-y|}E_\epsilon(y)^{-1} \leqq \frac{2E_\epsilon(x)^{-1}}{a_{(D.35)} - a_{(4.6)}},$$

so that, by (F.2) and (F.3) and since $\lambda_t \leqq c\epsilon^{-1}$

$$|m(t) - m_\epsilon| \leqq N(h - h_\epsilon)\Big(c'\epsilon^{-1}\mathrm{e}^{-a_{(4.6)}(1-x_0)\epsilon^{-1}} c_{(D.11)}\mathrm{e}^{-a_{(D.5)}|x-\epsilon^{-1}x_0|} + \frac{2\beta c_{(D.35)}E_\epsilon(x)^{-1}}{a_{(D.35)} - \bar{a}_{(4.6)}}\Big), \tag{F.4}$$

which (recalling that $N(h - h_\epsilon) \leqq b_{(4.7)}$) proves that for ϵ small enough,

$$\|m(t) - m_\epsilon\| \leqq \kappa := \frac{3\beta c_{(D.35)}b_{(4.7)}}{a_{(D.35)} - \bar{a}_{(4.6)}} \tag{F.5}$$

$$N(m(t) - m_\epsilon) \leqq \Big(c'\epsilon^{-1}c_{(D.11)} + \frac{2\beta c_{(D.35)}}{a_{(D.35)} - \bar{a}_{(4.6)}}\Big)N(h - h_\epsilon). \tag{F.6}$$

By (F.5), $p_t \leqq \beta\left(1 - (|m_\epsilon| - \kappa)^2\right) < \mathrm{e}^{-4a_{(D.5)}} + 2\beta\kappa$ in $\{|x - \epsilon^{-1}x_0| \geqq r_{(D.5)}\}$ and therefore, by (F.1), $m(t)$ satisfies (D.5) for $t \in [0, T]$. To prove the second condition in (D.6) we write for $x \in I$,

$$\begin{aligned}|m(x,t) - \bar{m}_{x_0}(x)| &\leqq |m(x,t) - m_\epsilon(x)| + |m_\epsilon(x) - \bar{m}_{x_0}(x)|\\ &\leqq N(m(t) - m_\epsilon)E_\epsilon(x)^{-1} + c_{(D.2)}\epsilon\log\epsilon^{-1}\\ &\leqq c\epsilon^{-1}E_\epsilon(x)^{-1} + c_{(D.2)}\epsilon\log\epsilon^{-1} \leqq 2c_{(D.2)}\epsilon\log\epsilon^{-1},\end{aligned}$$

for ϵ small enough (because $E_\epsilon(x)^{-1} \leqq \mathrm{e}^{-\bar{a}_{(4.6)}[\epsilon^{-1}(1-x_0)-2C_{(D.3)}\log\epsilon^{-1}]}$). The second inequality in (D.6) then follows, recalling that $c'_{(D.6)} > 2c_{(D.2)}$. To prove the first inequality in (D.6) we take the x-derivative of the equality $m(t) = \tanh\{\beta J^{\mathrm{neum}} * m(t) + \beta h(t)\}$:

$$\begin{aligned}&\frac{\mathrm{d}m(t)}{\mathrm{d}x} = \lambda_t^{-1}\langle u(t)g(t)\rangle_t\, u(t) + L_t^{-1}\tilde{g}(t)\\ &g(t) := \Big\{-p_t\frac{d[h - h_\epsilon]}{\mathrm{d}x}\Big\},\ \tilde{g}(t) = g(t) - \langle u(t)g(t)\rangle_t\, u(t).\end{aligned} \tag{F.7}$$

By (4.7), $\|g(t)\| \leqq \beta\epsilon$ and an argument similar to the previous one shows that $\|\frac{\mathrm{d}m(t)}{\mathrm{d}x}\| \leqq c\epsilon$, so that the first condition in (D.6) is also satisfied.

In conclusion, we have proved so far that for all ϵ small enough, $(m(t), h(t)) \in \mathcal{A}$ for all $t \in [0, S]$. Suppose by contradiction that $S < 1$; write $S' = \min\{1, S + T\}$, then since $(m(S), h(S)) \in \mathcal{A}$ by Proposition 15 there is $m(t)$, $t \in [S, S']$, which solves (E.2) in $[S, S']$ starting from $m(S)$ and such that for all such t, $(m(t), h(t))$ is in the δ-ball of $(m(S), h(S))$. Hence, a fortiori, in the δ-ball of $\mathcal{A}$ with δ as in Proposition 12. This contradicts the maximality of S hence $S = 1$. □

Proposition 17. *There are $a_{(F.8)} > 0$, $r_{(F.8)} > 0$, c, c' and $a_{(F.9)} > 0$ such that for all ϵ small enough the following holds. Let $h \in \mathcal{G}$ and $(m, h) \in \mathcal{A}$ (existence of m follows from Proposition 16), then*

$$\sup_{|x-\epsilon^{-1}x_0| \leqq 2r_{(F.8)}\epsilon^{-1}} |m(x) - m_\epsilon(x)| \leqq \mathrm{e}^{-a_{(F.8)}\epsilon^{-1}} \tag{F.8}$$

$$|\lambda_{m,h} - \lambda_\epsilon| \leqq c\mathrm{e}^{-a_{(F.8)}\epsilon^{-1}}, \quad \|u_{m,h} - u_\epsilon\| \leqq c'\mathrm{e}^{-a_{(F.9)}\epsilon^{-1}}. \tag{F.9}$$

Proof. (F.8) follows from (F.6); in the sequel it is convenient to have $a_{(F.8)}$ small, in particular $a_{(F.8)} < a_{(D.11)}$. Let λ and u be the maximal eigenvalue and eigenvector of $A := A_{m,h}$, $u > 0$ normalized so that $\langle u^2 \rangle = 1$ ($\langle \cdot \rangle := \langle \cdot \rangle_{m,h}$), and λ_ϵ, u_ϵ the maximal eigenvalue and eigenvector of $A_\epsilon := A_{m_\epsilon,h_\epsilon}$ with $u_\epsilon > 0$ normalized so that $\langle u_\epsilon^2 \rangle_\epsilon = 1$ ($\langle \cdot \rangle_\epsilon := \langle \cdot \rangle_{m_\epsilon,h_\epsilon}$). Since (m_ϵ, h_ϵ) and (m, h) are both in $\mathcal{A}$ we can use the bounds established in Proposition 12 and 13 for A and A_ϵ.

We then have

$$\frac{\lambda}{\lambda_\epsilon} \geqq \frac{\langle u_\epsilon^2 \rangle_\epsilon}{\langle u_\epsilon^2 \rangle} = 1 - \frac{\langle u_\epsilon^2 (1 - \frac{p}{p_\epsilon}) \rangle}{\langle u_\epsilon^2 \rangle} \geqq 1 - c e^{-a_{(F.8)} \epsilon^{-1}}, \tag{F.10}$$

the first inequality following from (D.12). To prove the last one we recall that $p_\epsilon = p_{m^*,h^*} \geqq c_{(4.1)}$, $p \equiv p_{m,h} \geqq C_{(D.8)}$. In $\{x : |x - \epsilon^{-1} x_0| \leqq 2 r_{(F.8)} \epsilon^{-1}\}$ we use (F.8) to get

$$\sup_{|x - \epsilon^{-1} x_0| \leqq 2 r_{(F.8)} \epsilon^{-1}} |1 - \frac{p}{p_\epsilon}| \leqq c e^{-a_{(F.8)} \epsilon^{-1}}. \tag{F.11}$$

In $\{x : |x - \epsilon^{-1} x_0| > 2 r_{(F.8)} \epsilon^{-1}\}$ we bound $|1 - \dfrac{p}{p_\epsilon}| \leqq \dfrac{2\beta}{c_{(4.1)} C_{(D.8)}}$ and u_ϵ using (D.11). The same argument is used to bound from below $\dfrac{\lambda_\epsilon}{\lambda}$ and the first inequality in (F.9) follows because λ and λ_ϵ are both close to 1 by $c\epsilon$.

In order to compute the sup in the second inequality in (F.9), we consider first $|x - \epsilon^{-1} x_0| > r_{(F.8)} \epsilon^{-1}$. In this case both u and u_ϵ are smaller than $c_{(D.11)} e^{-a_{(D.5)} \epsilon^{-1} r_{(F.8)}}$, hence their difference is bounded by $c' e^{-a_{(F.9)} \epsilon^{-1}}$, provided $a_{(F.9)} < a_{(D.5)} r_{(F.8)}$. We next take $|x - \epsilon^{-1} x_0| \leqq r_{(F.8)} \epsilon^{-1}$. Analogously to (D.21) and with $y_0 \equiv x$,

$$\lambda^N u(x) = \int u(y_N) \prod_{k=1}^{N} \left\{ A_\epsilon(y_{k-1}, y_k) \frac{p(y_{k-1})}{p_\epsilon(y_{k-1})} \right\} \mathrm{d}y_1 \cdots \mathrm{d}y_N. \tag{F.12}$$

We choose $N = b\epsilon^{-1}$ with $b > 0$ smaller than $r_{(F.8)}$. Then for all $k \leqq N$, $|y_k - \epsilon^{-1} x_0| \leqq (r_{(F.8)} + b)\epsilon^{-1} \leqq 2 r_{(F.8)} \epsilon^{-1}$ so that by (F.11) for all ϵ small enough, $u(x) \leqq \lambda^{-N} [1 + c e^{-a_{(F.8)} \epsilon^{-1}}]^N A_\epsilon^N u(x)$. We then write $A_\epsilon^N u(x) = \lambda_\epsilon^N \langle u u_\epsilon \rangle_\epsilon u_\epsilon + A_\epsilon^N \tilde{u}(x)$ so that in $\{|x - \epsilon^{-1} x_0| \leqq r_{(F.8)} \epsilon^{-1}\}$

$$u \leqq [1 + c e^{-a_{(F.8)} \epsilon^{-1}}]^N \left((\frac{\lambda_\epsilon}{\lambda})^N \langle u u_\epsilon \rangle_\epsilon u_\epsilon + \lambda^{-N} \|u\| c_{(D.34)} e^{-a_{(D.34)} N} \right).$$

By (D.14)

$$u \leqq [1 + c e^{-a_{(F.8)} \epsilon^{-1}}]^N \left((\frac{\lambda_\epsilon}{\lambda})^N \langle u u_\epsilon \rangle_\epsilon u_\epsilon + \lambda^{-N} c e^{-a_{(D.34)} N} \right). \tag{F.13}$$

We bound $\langle u u_\epsilon \rangle_\epsilon \leqq \langle u^2 \rangle_\epsilon^{1/2}$, $\langle u^2 \rangle_\epsilon = 1 - \langle u^2 |1 - \dfrac{p}{p_\epsilon}| \rangle$ and use the previous bounds for $|1 - \frac{p}{p_\epsilon}|$ so that $\langle u u_\epsilon \rangle_\epsilon \leqq 1 + c e^{-a\epsilon^{-1}}$ with a and c suitable positive constants. By (F.10)

$$\left(\frac{\lambda_\epsilon}{\lambda} \right)^N \leqq e^{-N \log\{1 - c e^{-a_{(F.8)} \epsilon^{-1}}\}} \leqq \exp\{c' \epsilon^{-1} e^{a_{(F.8)} \epsilon^{-1}}\} \leqq 1 + c'' \epsilon^{-1} e^{a_{(F.8)} \epsilon^{-1}}.$$

Collecting all these bounds and recalling that $\lambda < 1 - c\epsilon$, we get from (F.13)

$$u(x) \leqq \left(1 + c\epsilon^{-1} e^{-a_{(F.8)}\epsilon^{-1}}\right) u_\epsilon(x) \\ + c e^{-N\left(\log(1-c\epsilon) - a_{(D.34)}\right)}, \tag{F.14}$$

hence the upper bound for u in (F.9). The lower bound is proved similarly. □

Recall that (m^*, h^*) is the solution of the antisymmetric problem in $\epsilon^{-1}[-1, \ell^*]$, with x_0 the middle point in $[-1, \ell^*]$ and m_ϵ the restriction of m^* to $\epsilon^{-1}[-1, 1]$. We denote by λ^* and u^* the maximal eigenvalue and eigenvector of A_{m^*,h^*} and by λ_ϵ and u_ϵ those of $A_\epsilon = A_{m_\epsilon,h_\epsilon}$, writing u_ϵ also for its extension to $\epsilon^{-1}[-1, \ell^*]$ with $u_\epsilon = 0$ outside $\epsilon^{-1}[-1, 1]$. We suppose $\langle u_\epsilon^2\rangle_\epsilon = \langle (u^*)^2\rangle_* = 1$ with the obvious meaning of the symbols.

Proposition 18. *For all ϵ small enough,*

$$|\lambda^* - \lambda_\epsilon| \leqq c e^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}, \quad \|u^* - u_\epsilon\| \leqq c_{(F.15)} e^{-a_{(F.15)}\epsilon^{-1}(1-x_0)}. \tag{F.15}$$

Proof. Since $p_\epsilon = p^*$ in $\epsilon^{-1}[-1, 1]$, $\langle u_\epsilon^2\rangle_* = \langle u_\epsilon^2\rangle_\epsilon = 1$ so that $\lambda^* \geqq \int u_\epsilon J^{\text{neum},*} * u_\epsilon$ and, by (D.11),

$$\|(J^{\text{neum},*} - J^{\text{neum},\epsilon}) * u_\epsilon\| \leqq c e^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}, \tag{F.16}$$

where $J^{\text{neum},\epsilon}$ and $J^{\text{neum},*}$ are the kernel with Neumann conditions, respectively, in $\epsilon^{-1}[-1, 1]$ and $\epsilon^{-1}[-1, \ell^*]$. Thus

$$\lambda^* \geqq \int u_\epsilon J^{\text{neum},\epsilon} * u_\epsilon - c e^{-a_{(D.5)}\epsilon^{-1}(1-x_0)} \geqq \lambda_\epsilon - c e^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}. \tag{F.17}$$

For the reverse inequality we write $\lambda_\epsilon \geqq \dfrac{\int u^* J^{\text{neum},\epsilon} u^*}{\langle (u^*)^2\rangle_\epsilon}$, the integral being extended to $\epsilon^{-1}[-1, 1]$. Using (F.16) we replace the kernel $J^{\text{neum},\epsilon}$ with $J^{\text{neum},*}$ and then extend the integral to $\epsilon^{-1}[-1, \ell^*]$ bounding u^* via (D.11), which holds as well for u^* in the whole $\epsilon^{-1}[-1, \ell^*]$ (see the paragraph "The operator A^* and its spectral properties" at the end of Appendix D). In this way we derive the first inequality in (F.15).

As in the proof of Proposition 17, we bound $|u^*(x) - u_\epsilon(x)| \leqq c' e^{-a_{(F.15)}\epsilon^{-1}}$ when $|x - \epsilon^{-1}x_0| > r_{(F.8)}\epsilon^{-1}$ using (D.11) (supposing $a_{(F.15)} < a_{(D.5)} r_{(F.8)}$). When $|x - \epsilon^{-1}x_0| \leqq r_{(F.8)}\epsilon^{-1}$ we write

$$u^*(x) = (\lambda^*)^{-N}(A^*)^N u^*(x) = (\lambda^*)^{-N} A_\epsilon^N u^*(x), \tag{F.18}$$

provided $(x_0 + r_{(F.8)})\epsilon^{-1} + N \leqq \epsilon^{-1}$, which is satisfied if $N = a\epsilon^{-1}$ with $a > 0$ small enough. Hence

$$u^*(x) = \left(\frac{\lambda_\epsilon}{\lambda^*}\right)^N \langle u^* u_\epsilon\rangle_\epsilon u_\epsilon(x) + A_\epsilon^N \tilde{u}^* \tag{F.19}$$

$$|u^*(x) - \left(\frac{\lambda_\epsilon}{\lambda^*}\right)^N \langle u^* u_\epsilon\rangle_\epsilon u_\epsilon(x)| \leqq c e^{-a_{(D.34)}N}. \tag{F.20}$$

By (F.20) and since by (D.11) $\langle u^*\rangle_\epsilon \leqq c$ and $|\langle (u^*)^2\rangle_\epsilon - 1| \leqq c\mathrm{e}^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}$

$$|1 - \left(\frac{\lambda_\epsilon}{\lambda^*}\right)^N \langle u^* u_\epsilon\rangle_\epsilon^2| \leqq c\mathrm{e}^{-a_{(D.34)}N} + c\mathrm{e}^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}, \tag{F.21}$$

so that the second inequality in (F.15) follows from the first one. □

As a corollary of Proposition 17 and Proposition 18 we have:

Corollary 19. *In the same context of Proposition 17,*

$$|\lambda^* - \lambda_{m,h}| \leqq c\mathrm{e}^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}, \quad \|u^* - u_{m,h}\| \leqq c_{(F.15)}\mathrm{e}^{-a_{(F.15)}\epsilon^{-1}(1-x_0)}. \tag{F.22}$$

G Convergence of the Iterative Scheme

By (4.8) with $n = -1$ we have for $x \in \epsilon^{-1}[-1, 1]$,

$$\hat{h}_0(x) = -\epsilon j \int_{\epsilon^{-1}x_0}^{x} \chi(m_\epsilon(y))^{-1} = h^*(x), \tag{G.1}$$

because $m_\epsilon = m^*$ on $\epsilon^{-1}[-1, 1]$ and (m^*, h^*) is a solution of (2.23) in $\epsilon^{-1}[-1, \ell^*]$. Thus by (4.8)

$$h_0(x) = h^*(x) - \frac{\int h^* u^*}{\int u^*}, \tag{G.2}$$

where the integrals are extended to $\epsilon^{-1}[-1, 1]$. Then, recalling (4.2),

$$h_0(x) - h_\epsilon(x) = -R_\epsilon(x) - \frac{\int h^* u^*}{\int u^*}. \tag{G.3}$$

Proposition 20. *For all ϵ small enough $h_0 \in \mathcal{G}$ and*

$$N(h_0 - h_\epsilon) \leqq c_{(G.4)}\epsilon. \tag{G.4}$$

Proof. $\int_{\epsilon^{-1}(2x_0-1)}^{\epsilon^{-1}} h^* u^* = 0$ because h^* is antisymmetric and u^* symmetric around the middle point $\epsilon^{-1}x_0$ of the interval $\epsilon^{-1}[-1, \ell^*]$ (u^* is symmetric because the eigenvalue λ^* is simple and A^* symmetric). Since the estimates in Proposition 12 apply to u^* as well (see the paragraph *The operator A^* and its spectral properties* at the end of Appendix D), by (D.11) and since $\|h^*\| \leqq c$ we get

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^* h^* = \int_{-\epsilon^{-1}}^{\epsilon^{-1}(2x_0-1)} h^* u^* \leqq c\mathrm{e}^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}. \tag{G.5}$$

Recalling that $c_{(4.1)}$ in (4.1) is strictly positive uniformly in ϵ, we shall next prove that

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^* \geqq \frac{c_{(4.1)}}{c_{(D.11)}}. \tag{G.6}$$

By (D.11) $u^* \leqq c_{(D.11)}$, (G.6) then follows from (4.1):

$$1 = \langle (u^*)^2 \rangle_* = \int \frac{(u^*)^2}{p^*} \leqq \int \frac{u^* c_{(D.11)}}{c_{(4.1)}} = \{\frac{c_{(D.11)}}{c_{(4.1)}}\} \int u^*.$$

Thus, recalling (4.6) and that $a_{(D.5)}(1 - x_0) > a_{(4.6)}$, see the paragraph *Fixing the parameters in the set* $\mathcal{G}$ in Appendix F,

$$N\left(\frac{\int h^* u^*}{\int u^*}\right) = \sup_{|x| \leqq \epsilon^{-1}} E_\epsilon(x) \frac{\int h^* u^*}{\int u^*} \leqq c\mathrm{e}^{-a_{(G.7)}\epsilon^{-1}}, \tag{G.7}$$

with $0 < a_{(G.7)} < a_{(D.5)}(1 - x_0) - a_{(4.6)}$. By Lemma 8, $N(R_\epsilon) \leqq c\epsilon$ which together with (G.7) proves (G.4). Before proving that $h_0 \in \mathcal{G}$ we notice that by (D.11)

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^* |R_\epsilon| \leqq c\epsilon \mathrm{e}^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}, \tag{G.8}$$

a property which will be used in the sequel. We have already proved with (G.4) the first condition for $h_0 \in \mathcal{G}$. Then, since $\dfrac{\mathrm{d}(h_0 - h_\epsilon)}{\mathrm{d}x} = -\dfrac{\mathrm{d}R_\epsilon}{\mathrm{d}x}$ it will suffice to show that

$$\left|\frac{\mathrm{d}R_\epsilon}{\mathrm{d}x}\right| \leqq c\epsilon^2 \mathbf{1}_{x \geqq \epsilon^{-1}-1}. \tag{G.9}$$

We have

$$\frac{\mathrm{d}R_\epsilon}{\mathrm{d}x} = \int_{\epsilon^{-1}}^{\epsilon^{-1}+1} J(x, y)[\psi(y) - \psi(2\epsilon^{-1} - y)]\,\mathrm{d}y, \quad \psi = \frac{\mathrm{d}m^*}{\mathrm{d}x}. \tag{G.10}$$

To bound the term $|\psi(x) - \psi(x + \xi)|$, $\xi = x' - x$, x and x' in $[\epsilon^{-1} - 1, \epsilon^{-1} + 1]$, in (G.10) we use the expression (D.41) for ψ. By (D.34)

$$\left\| L^{-1}\phi + \sum_{n=0}^{N} (A^*)^n \phi \right\| \leqq c' \|\phi\| \mathrm{e}^{-a_{(D.34)}N} \leqq c''\epsilon^3, \quad \phi = (\epsilon j) - (\epsilon j \int u^*) u^*, \tag{G.11}$$

if $N = C \log \epsilon^{-1}$ with C large enough. We have $(A^*)^n \phi = (A^*)^n(\epsilon j) - (\epsilon j \int u^*)(A^*)^n u^*$ By (D.11) and since $\lambda^* \in (0, 1)$

$$(A^*)^n u^*(x) \leqq u^*(x) \leqq c_{(D.11)} \mathrm{e}^{-a_{(D.5)}|x - \epsilon^{-1} x_0|},$$

so that $|\sum_{n=0}^{N}\{(A^*)^n\phi-(A^*)^n(\epsilon j)\}|\leqq c\epsilon^3$ for $x\geqq\epsilon^{-1}-1$ and ϵ small enough. (G.9) will then follow from

$$\sum_{n=0}^{N}\left|\int (A^*)^n(x,y)(\epsilon j)\mathrm{d}y-\int (A^*)^n(x',y)(\epsilon j)\mathrm{d}y\right|\leqq c\epsilon^2 \tag{G.12}$$

x and x' in $[\epsilon^{-1}-1,\epsilon^{-1}+1]$. To prove (G.12) we write $\xi=x'-x$ and

$$\left|\int (A^*)^n(x,y)-\int (A^*)^n(x',y)\right|\leqq\int A^*(x,x_1)\cdots A^*(x_{n-1},x_n)$$
$$\left|1-\prod\frac{p^*(x_i+\xi)}{p^*(x_i)}\right|\leqq cn\epsilon b^n,\quad b<1,$$

as all points above are in $\{x: x-\epsilon^{-1}x_0>\epsilon^{-1}-(N+1)\}$ (as $n\leqq N$) and in such a region $0<c_{(4.1)}<p^*<b<1$ (as $m^*>m_\beta$) and $|p^*(x_i+\xi)-p^*(x_i)|\leqq c\epsilon$, by Lemma 8. (G.12) is thus proved. □

Proposition 21. *There are $c_{(G.13)}$ and $c_{(G.14)}$ so that for all ϵ small enough the following holds. Suppose that for $n\geqq 1$, both h_n and h_{n-1} are in $\mathcal{G}$, then*

$$N(m_n-m_{n-1})\leqq c_{(G.13)}N(h_n-h_{n-1}), \tag{G.13}$$

where $m_i=\tanh\{\beta J^{\text{neum}}*m_i+\beta h_i\}$, $i=n-1,n$. Moreover

$$N(m_0-m_\epsilon)\leqq c_{(G.14)}\epsilon. \tag{G.14}$$

Proof. We first prove (G.13), where we recall that $n\geqq 1$. Let $t\in[0,1]$ and $h(t)=th_n+(1-t)h_{n-1}$. Since h_n and h_{n-1} are in $\mathcal{G}$ then, by convexity, $h(t)\in\mathcal{G}$ and by Proposition 16 there is $m(t)$ such that $(m(t),h(t))\in\mathcal{A}$, in particular $m(t)=\tanh\{\beta J^{\text{neum}}*m(t)+\beta h(t)\}$ and $m(0)=m_{n-1}$, $m(1)=m_n$ so that $|m_n-m_{n-1}|\leqq\sup_{t\in[0,1]}|\frac{\mathrm{d}m(t)}{\mathrm{d}t}|$. By (E.2), (D.11) and (D.35), recalling that $p_t\leqq\beta$ and writing

$$\psi(x):=\left(\left|\int u(t)[h_n-h_{n-1}]\right|\right)\,\mathrm{e}^{-a_{(D.5)}|x-\epsilon^{-1}x_0|} \tag{G.15}$$

$$\left|\frac{\mathrm{d}m(t)}{\mathrm{d}t}\right|\leqq c\epsilon^{-1}\psi+c\int \mathrm{e}^{-a_{(D.35)}|x-y|}\,(|h_n-h_{n-1}|(y)+\psi(y))\,\mathrm{d}y. \tag{G.16}$$

We are going to prove that

$$\psi\leqq c_{(G.17)}\epsilon^{10}\mathrm{e}^{-a_{(D.5)}|x-\epsilon^{-1}x_0|}N(h_n-h_{n-1}). \tag{G.17}$$

By the definition of $\mathcal{G}$, $\int u^*h_i=0$, $i=n-1,n$, then

$$\int u(t)[h_n-h_{n-1}]=\int[u(t)-u^*][h_n-h_{n-1}] \tag{G.18}$$
$$\left|\int[u(t)-u^*][h_n-h_{n-1}]\right|\leqq N(h_n-h_{n-1})\int|u(t)-u^*|E_\epsilon^{-1}\leqq c\epsilon^{10}N(h_n-h_{n-1})$$

(by Corollary 19). (G.17) is proved. Using (G.17) we have

$$\int e^{-a_{(D.35)}|x-y|}\psi(y)\,dy \leqq c\epsilon^{10}e^{-a|x-\epsilon^{-1}x_0|}N(h_n-h_{n-1}),$$
$$a=\min\{a_{(D.5)},a_{(D.35)}\}.$$

The other integral on the right-hand side of (G.16) is bounded by

$$\int e^{-a_{(D.35)}|x-y|}|h_n-h_{n-1}|(y)\,dy \leqq N(h_n-h_{n-1})\int e^{-a_{(D.35)}|x-y|}E_\epsilon(y)\,dy$$
$$\leqq ce^{-a_{(4.6)}|x-\epsilon^{-1}x_0|},$$

because $a_{(D.35)} > a_{(4.6)}$. Collecting all these bounds we have from (G.16)

$$\left|\frac{dm(t)}{dt}\right|(x) \leqq c\left(\epsilon^9 e^{-a_{(4.6)}|x-\epsilon^{-1}x_0|}+(1+\epsilon^{10})e^{-a_{(4.6)}|x-\epsilon^{-1}x_0|}\right)N(h_n-h_{n-1}),$$

which proves (G.13).

The proof of (G.14) goes in the same way, except for (G.18) which becomes

$$\int u(t)[h_0-h_\epsilon]=\int[u(t)-u^*][h_0-h_\epsilon]+\int u^*h_\epsilon. \tag{G.19}$$

By (G.5) and (G.8) the latter integral is bounded by $\leqq ce^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}$ and the bound (G.14) is not affected. □

Proposition 22. *There is $c_{(G.20)} \geqq c_{(G.4)}$ such that for all ϵ small enough the following holds. Given any $n \geqq 0$ if h_k, $k \leqq n$, is well defined and in $\mathcal{G}$ then also h_{n+1} is well defined and*

$$N(h_{k+1}-h_k) \leqq \begin{cases} c_{(G.20)}\epsilon N(h_k-h_{k-1}), & k=1,\dots,n \\ c_{(G.20)}\epsilon, & k=0 \end{cases} \tag{G.20}$$

Proof. By Proposition 16 there is m_k, $0 \leqq k \leqq n$, such that $(m_k,h_k)\in\mathcal{A}$ and, by (D.8), $p_k \equiv p_{m_k,h_k} \geqq C_{(D.8)}$. As a consequence p_n^{-1} is bounded and h_{n+1} is well defined; moreover, $|p_k^{-1}-p_{k-1}^{-1}| \leqq c|m_k-m_{k-1}|$ and (for $x>\epsilon^{-1}x_0$)

$$|h_{k+1}-h_k| \leqq c\epsilon\left(f+\frac{\int u^*f}{\int u^*}\right),\quad f(x)=\int_{\epsilon^{-1}x_0}^x |m_k-m_{k-1}|\,dy.$$

Let $x>\epsilon^{-1}x_0$; then by (G.13) for $k \geqq 1$

$$e^{a_{(4.6)}(\epsilon^{-1}-x)}f(x)=\{\int_{\epsilon^{-1}x_0}^x e^{-a_{(4.6)}(x-y)}c_{(G.13)}\}N(h_k-h_{k-1}) \leqq cN(h_k-h_{k-1}), \tag{G.21}$$

and by (D.11)

$$\int_{\epsilon^{-1}x_0}^{\epsilon^{-1}} u^*f \leqq cN(h_k-h_{k-1})\int_{\epsilon^{-1}x_0}^{\epsilon^{-1}} e^{-a_{(4.6)}(\epsilon^{-1}-x)}e^{-a_{(D.5)}|x-\epsilon^{-1}x_0|}$$
$$\leqq cN(h_k-h_{k-1})\,e^{-a_{(4.6)}\epsilon^{-1}(1-x_0)}. \tag{G.22}$$

By (G.6), $\int u^*$ is bounded away from 0 hence the bound in (G.20) for $k > 0$ and $x \geqq \epsilon^{-1}x_0$. When $k = 0$ we use (G.14) after bounding $|m_0 - m_\epsilon| \leqq N(m_0 - m_\epsilon)E_\epsilon^{-1}$. Analogous bounds hold for $x < \epsilon^{-1}x_0$ and (G.20) is proved. □

Proposition 23. *In the same context of Proposition 22, for any* $k \leqq n+1$

$$N(m_k - m_\epsilon) \leqq c\epsilon, \qquad N(h_k - h_\epsilon) \leqq c'\epsilon, \tag{G.23}$$

where $c = c_{(G.14)} + \dfrac{c_{(G.13)}c_{(G.20)}}{1-\epsilon c_{(G.20)}}$, $c' = c_{(G.20)}\Big(1 + \dfrac{1}{1-\epsilon c_{(G.20)}}\Big)$. *Moreover*

$$N\left(\frac{\mathrm{d}(h_k - h_\epsilon)}{\mathrm{d}x}\right) \leqq c\epsilon^2, \tag{G.24}$$

and, in particular, $h_{n+1} \in \mathcal{G}$.

Proof. By (G.20) for $i \geqq 0$, $N(h_{i+1} - h_i) \leqq (\epsilon c_{(G.20)})^{i+1}$ and by (G.4), $N(h_0 - h_{-1}) \leqq \epsilon c_{(G.4)} \leqq \epsilon c_{(G.20)}$, $h_{-1} = h_\epsilon$. Then

$$N(h_k - h_\epsilon) \leqq \sum_{i=0}^{k} N(h_i - h_{i-1}) \leqq \epsilon c_{(G.20)}\left(1 + \frac{1}{1-\epsilon c_{(G.20)}}\right),$$

hence the statement in (G.23) about h_k. The one about m_k is proved similarly, using (G.14) and (G.13). To prove (G.24) we write

$$\left|\frac{\mathrm{d}(h_k - h_{k-1})}{\mathrm{d}x}\right| \leqq c\epsilon|m_{k-1} - m_{k-2}| \leqq c'\epsilon|h_{k-1} - h_{k-2}|, \tag{G.25}$$

so that by (G.9), (G.20) and (G.4) and with $h_{-1} := h_\epsilon$, for $x > \epsilon^{-1}x_0$,

$$\begin{aligned} \mathrm{e}^{a_{(4.6)}(\epsilon^{-1}-x)}\left|\frac{\mathrm{d}(h_k - h_\epsilon)}{\mathrm{d}x}\right| &\leqq \mathrm{e}^{a_{(4.6)}(\epsilon^{-1}-x)}\left|\frac{\mathrm{d}(h_0 - h_\epsilon)}{\mathrm{d}x}\right| + c'\epsilon\sum_{i=1}^{k-1} N\left(|h_i - h_{i-1}|\right) \\ &\leqq c''\epsilon^2. \end{aligned}$$

An analogous bound holds for $x < \epsilon^{-1}x_0$ hence (G.24). □

Conclusion of the proof of Theorem 2. We shall first prove by induction that $h_n \in \mathcal{G}$ for all n. Indeed $h_0 \in \mathcal{G}$ by Proposition 20 and by Proposition 22 if $h_k \in \mathcal{G}$ for all $k \leqq n$, then $h_{n+1} \in \mathcal{G}$. Thus $h_n \in \mathcal{G}$ for all n and by Proposition 16 there is m_n so that $(m_n, h_n) \in \mathcal{A}$. We shall next prove that (m_n, h_n) converges in sup-norm to a limit (m, h) and that, writing $h_{-1} = h_\epsilon$ and $m_{-1} = m_\epsilon$,

$$h = h_\epsilon + \sum_{n=0}^{\infty}(h_n - h_{n-1}), \qquad m = m_\epsilon + \sum_{n=0}^{\infty}(m_n - m_{n-1}).$$

The first series, in fact, converges because $N(h_{n+1} - h_n) \leqq (c_{(G.20)}\epsilon)^{n+1}$, as remarked in the proof of Proposition 23. The series for m converges for the same reason because $N(m_n - m_{n-1}) \leqq c_{(G.13)} N(h_n - h_{n-1})$. By (G.23), $N(m - m_\epsilon) \leqq c\epsilon$ and $N(h - h_\epsilon) \leqq c\epsilon$; moreover

$$m = \lim_{n\to\infty} m_n = \lim_{n\to\infty} \tanh\{\beta J^{\text{neum}} * m_n + \beta h_n\} = \tanh\{\beta J^{\text{neum}} * m + \beta h\}$$
$$h = \hat{h} - \frac{\int \hat{h} u^*}{\int u^*}, \quad \hat{h}(x) = \int_{\epsilon^{-1}x_0}^{x} \frac{-\epsilon j}{\chi(m)}$$

because $h = \lim_{n\to\infty}\{\hat{h}_n - \frac{\int \hat{h}_n u^*}{\int u^*}\}$, $\quad \hat{h}_n(x) = \int_{\epsilon^{-1}x_0}^{x} \frac{-\epsilon j}{\chi(m_n)}$. As a consequence, for any $z \in \epsilon^{-1}(-1, 1)$,

$$h(x) = h(z) + \int_z^x \frac{-\epsilon j}{\chi(m)},$$

so that the proof of Theorem 2 will be complete once we show that:

- there is x_ϵ such that $h(x_\epsilon) = 0$
- $\lim_{\epsilon\to 0} \epsilon x_\epsilon = x_0$.

The existence of x_ϵ is proved using the implicit function theorem. We thus want to prove that $h(\epsilon^{-1}x_0)$ is "small". Since $\hat{h}(\epsilon^{-1}x_0) = 0$ we need to control $|\int \hat{h} u^*|$. We write $\int \hat{h} u^* = \int (\hat{h} - \hat{h}_n) u^* + \int (\hat{h}_n - \hat{h}_0) u^* + \int \hat{h}_0 u^*$. The first term vanishes as $n \to \infty$ because $\int u^* < \infty$ and for any $x > \epsilon^{-1}x_0$ (for instance)

$$|\hat{h}(x) - \hat{h}_n(x)| \leqq |\epsilon j| \int_{\epsilon^{-1}x_0}^{x} |\chi(m)^{-1} - \chi(m_{n-1})^{-1}|$$
$$\leqq c\epsilon |x - \epsilon^{-1}x_0| \|m - m_{n-1}\| \leqq c' \|m - m_{n-1}\| \to 0,$$

having used that $\chi(m_n) = p_{m_n,h_n} \geqq C_{(D.8)}$ and therefore $\chi(m) \geqq C_{(D.8)}$ as $m_n \to m$ in sup-norm. Analogously, $|\chi(m_{n-1})^{-1} - \chi(m_\epsilon)^{-1}| \leqq c|m_{n-1} - m_\epsilon| \leqq cN(m_{n-1} - m_\epsilon)E_\epsilon^{-1} \leqq c'\epsilon E_\epsilon^{-1}$, so that for $x > \epsilon^{-1}x_0$

$$|\hat{h}_n(x) - \hat{h}_0(x)| \leqq |\epsilon j| \int_{\epsilon^{-1}x_0}^{x} |\chi(m_{n-1})^{-1} - \chi(m_\epsilon)^{-1}| \leqq c\epsilon^2 \int_{\epsilon^{-1}x_0}^{x} E_\epsilon(y)^{-1}$$
$$\leqq c\epsilon^2 |x - \epsilon^{-1}x_0| \mathrm{e}^{-a_{(4.6)}\epsilon^{-1}(1-x)}.$$

Thus by (D.11) the second term is bounded by $\int |\hat{h}_n - \hat{h}_0| u^* \leqq c\epsilon^2 \mathrm{e}^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}$. Finally, since by (G.1) $\hat{h}_0 = h^*$, by (G.5) $\int_{-\epsilon^{-1}}^{\epsilon^{-1}} \hat{h}_0 u^* \leqq c_0 \mathrm{e}^{-a_{(D.5)}\epsilon^{-1}(1-x_0)}$. In conclusion, letting $n \to \infty$,

$$\left|\int u^* \hat{h}\right| \leqq c\epsilon^2 \mathrm{e}^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}, \quad |h(\epsilon^{-1}x_0)| \leqq c'\epsilon^2 \mathrm{e}^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}.$$

We shall next prove that h is continuous and that it changes sign in a small interval around $\epsilon^{-1}x_0$, thus concluding that there is x_ϵ in this interval where h vanishes. We have $\frac{\mathrm{d}h}{\mathrm{d}x}(x) = \frac{-\epsilon j}{\beta(1-m^2(x))}$ which, by (D.8), is bounded. Moreover $N(m - m_\epsilon) \leqq c\epsilon$ and $|m_\epsilon - \bar{m}_{x_0}| \leqq c\epsilon$ in $[\epsilon^{-1}x_0 - 1, \epsilon^{-1}x_0 + 1]$. In this interval, therefore, $|\frac{\mathrm{d}h}{\mathrm{d}x}(x)| \geqq a\epsilon$, $a > 0$. Hence there is x_ϵ where $h(x_\epsilon) = 0$ and

$$|x_\epsilon - \epsilon^{-1}x_0| \leqq c''\epsilon \mathrm{e}^{-a_{(4.6)}(1-x_0)\epsilon^{-1}}. \tag{G.26}$$

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