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Abstract We consider the symmetric simple exclusion process in the interval $[-N, N]$ with additional birth and death processes respectively on $(N-K, N]$, $K>0$, and $[-N, -N+K)$. The exclusion is speeded up by a factor N^2, births and deaths by a factor N. Assuming propagation of chaos (a property proved in a companion paper, De Masi et al., http://arxiv.org/abs/1104.3447) we prove convergence in the limit $N \to \infty$ to the linear heat equation with Dirichlet condition on the boundaries; the boundary conditions however are not known a priori, they are obtained by solving a non-linear equation. The model simulates mass transport with current reservoirs at the boundaries and the Fourier law is proved to hold.

Keywords Hydrodynamic limits · Fourier law · Non-linear boundary processes

1 Introduction

A basic question in statistical mechanics is to understand the structure of the stationary non-equilibrium states characterized by the presence of steady currents flowing through the system. Currents are produced by external driving forces: driving forces and currents are then

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"conjugate variables", as the chemical potential and the number of particles in equilibrium statistical mechanics. We may then expect the existence of a "current ensemble" where the current is fixed and a "driving force ensemble" where instead the driving force is assigned and only as a consequence a current is obtained: in the equilibrium statistical mechanics analogue they would be the canonical and grand-canonical ensembles where respectively the number of particles and the chemical potential are fixed. Purpose of the present paper is to investigate these questions in the simplest case of the one dimensional symmetric simple exclusion process (SSEP).

In general the driving forces (which produce the desired currents) are of two types, they either act in the bulk of the system or only at the boundaries. In the stochastic interacting particle systems framework, to which we restrict hereafter, bulk forces are usually introduced by making the jumps of the particles asymmetric. This is truly non-equilibrium dynamics, as the stationary states, which may even be spatially homogeneous, are in general non-Gibbsian and unknown (with few exceptions that include the noticeable case of ASEP, the asymmetric simple exclusion). Much simpler is the case where the forces act only at the boundaries, since in the bulk hydrodynamic behavior is then expected (and in many cases proved). Namely for large systems and in a "macroscopic limit" a locally Gibbsian equilibrium profile establishes throughout the system and the current flows proportionally to the order parameter gradient (that is according to Fourier's law, if there is an energy current, or to the Fick's law if it is a mass current). Thus to a first order the stationary state is determined by hydrodynamics and its characteristic non-equilibrium features appear only at higher orders. Non-equilibrium thermodynamics for these states is related to the large deviations of the stationary measures and a clear picture is emerging, see for instance [1, 2, 5, 6] and references therein.

The driving forces acting at the boundaries are physically interpreted in terms of reservoirs. As claimed in the beginning we may think of two types of reservoirs, density and current reservoirs. Those used in all the previous references are density reservoirs, the local density at the boundaries being determined by the reservoirs attached there. Thinking of a one dimensional system in an interval the reservoirs add and subtract at unit rate particles on the right trying to keep fixed a given density ρ_+ close to the right boundary; same is done on the left and if the two densities $\rho_\pm$ are different, say $\rho_+ > \rho_-$, then we have a positive density gradient which by the Fourier law induces a negative current (inversely proportional to the size of the region).

In a current reservoir instead we fix the current so that we send in particles from the right at the desired rate (i.e. inversely proportional to the size of the region in agreement with the Fourier or rather Fick law) and take out at same rate particles from the left. Notice that the density reservoir fixes the extremal densities and consequently via hydrodynamics the current j; in the current reservoir (which produces the same current j) hydrodynamics does not fix the density profile, as many profiles are compatible with the same j. We thus do not expect equivalence of ensembles and one of our goals was indeed to investigate this issue, but as we shall argue below we are still very far from this, the same implementation of a current reservoir being non-trivial.

To our knowledge there is no mention in the literature about current reservoirs, even though they look as the most natural to produce a current. Our purpose here is to start their analysis in the simplest possible context. For this reason we consider the $d = 1$ SSEP in an interval $\Lambda_N = [-N, N]$, N a positive integer (we are interested in the behavior as $N \to \infty$). The process takes place in $\{0, 1\}^{\Lambda_N}$ (at most one particle per site) and time is speeded up by a factor N^2 (to match the length of the interval Λ_N). Thus, independently each particle tries to jump at rate $N^2/2$ to each one of its nearest neighbor sites, the jump then takes place if and only if the chosen site is empty, see the next section for a formal definition; jumps outside Λ_N are suppressed. To induce a current we modify the process by sending in from the right and taking out from the left particles at rate $Nj/2$, $j>0$ a fixed parameter independent of N (to compare with the previous statements where the rates were said to be proportional to N^{-1}, we should recall that here times are speeded up by a factor N^2). As we want the boundary processes localized at the boundaries we fix two intervals $I_\pm$ of length K at the boundaries and we send in particles only in I_+ and take out particles only from I_-. This is not unfortunately the action of a proper current reservoir because it may happen that I_+ is already full or I_- empty, and then the proposed mechanisms abort, so that the current really flowing in the system will not be exactly what desired (but hopefully close if K is large). This seems unavoidable if we insist to localize at the boundaries the birth-death processes or to consider lattice gases. We shall come back on this issue at the end of the next section in the paragraph "Discussions, conjectures and open problems".

In this paper we derive the hydrodynamic equations in the limit as $N\to\infty$ under the hypothesis of "propagation of chaos", a property proved in a companion paper, [4]. The hydrodynamic equation is the linear heat equation in the "macroscopic" interval $(-1,1)$ with Dirichlet boundary conditions at time 0 and at ±1: the values at ±1 are however unknown and can be obtained by solving a coupled system of two non-linear integral equations. We also prove the validity of the Fourier law; in particular the currents which enter and exit from the system are at all times equal to the local density gradient at ±1.

2 Model and Main Results

2.1 Notation and Definitions

$\Lambda_N := [-N,N]$ is the interval in $\mathbb{Z}$ with endpoints $\pm N$, denoted by $\Lambda_N := [-N,N]$. We write $\epsilon \equiv 1/N$, fix an integer $K\ge 1$, write $I_+\equiv[N-K+1,N]$ and $I_-\equiv[-N,-N+K-1]$. Particle configurations are elements η of $\{0,1\}^{\Lambda_N}$, $\eta(x)=0,1$ being the occupation number at $x\in\Lambda_N$.

We shall study the Markov process on $\{0,1\}^{\Lambda_N}$ with generator $L_\epsilon := \epsilon^{-2}(L_0+\epsilon L_b)$, where $L_b = L_{b,+}+L_{b,-}$ and

$$\begin{aligned} L_0 f(\eta) &:= \frac{1}{2}\sum_{x=-N}^{N-1}[f(\eta^{(x,x+1)})-f(\eta)],\\ L_{b,\pm} f(\eta) &:= \frac{j}{2}\sum_{x\in I_\pm} D_\pm\eta(x)[f(\eta^{(x)})-f(\eta)], \end{aligned} \tag{1}$$

$\eta^{(x)}$ being the configuration obtained from η by changing the occupation number at x, $\eta^{(x,x+1)}$ by exchanging the occupation numbers at $x,x+1$; for any $u:\Lambda_N\to[0,1]$

$$\begin{aligned} D_+u(x) &= [1-u(x)]u(x+1)u(x+2)\dots u(N), \quad x\in I_+,\\ D_-u(x) &= u(x)[1-u(x-1)][1-u(x-2)]\dots[1-u(-N)], \quad x\in I_-. \end{aligned} \tag{2}$$

L_0 is the generator of the SSEP (and of the stirring process as well). $L_{b,+}$ and $L_{b,-}$ are generators of birth respectively death processes, the former is active in I_+ the latter in I_-. The parabolic nature of the stirring process suggests to scale time as the square of space, hence the factor ϵ^{-2} in the definition of L_ϵ. It readily follows from the structure of the generators that the expectations $\mathbb{E}_\epsilon[\eta(x,t)]$ satisfy the relations

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_\epsilon[\eta(x,t)] &= \frac{1}{2}\Delta_\epsilon\mathbb{E}_\epsilon[\eta(x,t)]\\ &\quad+\epsilon^{-1}\frac{j}{2}\Big(\mathbf{1}_{x\in I_+}\mathbb{E}_\epsilon\big[D_+\eta(x,t)\big]-\mathbf{1}_{x\in I_-}\mathbb{E}_\epsilon\big[D_-\eta(x,t)\big]\Big)\end{aligned} \tag{3}$$

where $\Delta_\epsilon = \epsilon^{-2}\Delta$, Δ the discrete Laplacian in Λ_N with reflecting boundary conditions:

$$\begin{aligned}&\Delta u(x) = u(x+1)+u(x-1)-2u(x), \quad |x|<\epsilon^{-1},\\ &\Delta u(\pm N) = u(\pm(N-1),t)-u(\pm N,t).\end{aligned} \tag{4}$$

2.2 Propagation of Chaos

Due to the last term, (3) is not a closed equation in $\mathbb{E}_\epsilon[\eta(x,t)]$, but since the stirring generator is the leading term in L_ϵ and the invariant measures for the stirring process on the line are product Bernoulli measures, it looks natural to conjecture "propagation of chaos", i.e. that the measures at time $t>0$ are approximately product (as $\epsilon\to 0$). If the law at time $t>0$ were a true product measure, then, instead of (3) the expectations $\mathbb{E}_\epsilon[\eta(x,t)]$ would satisfy the closed equation:

$$\frac{d}{dt}\rho_\epsilon(x,t) = \frac{1}{2}\Delta_\epsilon\rho_\epsilon(x,t)+\epsilon^{-1}\frac{j}{2}\Big(\mathbf{1}_{x\in I_+}D_+\rho_\epsilon(x,t)-\mathbf{1}_{x\in I_-}D_-\rho_\epsilon(x,t)\Big), \tag{5}$$

which will be referred to as "the discretized time evolution".

The Cauchy problem for (5) with [0, 1]-valued initial datum $\rho_\epsilon(\cdot,0)$ has a unique global solution which also takes values in [0, 1]. Indeed (5) is a first order system of ordinary differential equations with polynomial non-linearity hence local existence and uniqueness. Global existence follows because the solution has values in [0, 1], which in turns is a consequence of the fact that $D_\pm u(x)$ vanishes when $u(x)=1$, respectively $u(x)=0$. A formal proof is given in Proposition 3.1.

2.3 Hydrodynamic Limit

The first result in this paper (proved in Sect. 5) shows that ρ_ϵ converges as $\epsilon\to 0$ to a limit function which then identifies the hydrodynamics of the system.

Theorem 1 *Suppose that the initial datum $\rho_\epsilon(\cdot,0)$ defined on Λ_N, with values in* [0, 1], *converges weakly as $\epsilon\to 0$ to $u_0(\cdot)\in L^\infty([-1,1],[0,1])$ in the sense that*

$$\lim_{\epsilon\to 0}\epsilon\sum_{x\in\Lambda_N}\rho_\epsilon(x,0)\phi(\epsilon x) = \int_{[-1,1]}u_0(r)\phi(r)dr, \quad \textit{for any } \phi\in L^\infty([-1,1],\mathbb{R}). \tag{6}$$

Then there is $\rho(r,t)$, $r\in[-1,1]$, $t>0$ so that for any $t_1>t_0>0$:

$$\lim_{\epsilon\to 0}\sup_{x\in\Lambda_N}\sup_{t_0\le t\le t_1}|\rho_\epsilon(x,t)-\rho(\epsilon x,t)| = 0. \tag{7}$$

The function $\rho(r,t)$ solves and is the unique solution of the integral equation

$$\rho(r,t) = \int_{[-1,1]} P_t(r,r')u_0(r')dr' + \frac{j}{2}\int_0^t \Big\{P_s(r,1)(1-\rho(1,t-s)^K) \\ - P_s(r,-1)(1-(1-\rho(-1,t-s))^K)\Big\}ds, \tag{8}$$

where $P_t(r,r')$ is the density kernel of the semigroup (also denoted as P_t) with generator $\Delta/2$, Δ the Laplacian in $[-1,1]$ with reflecting, Neumann, boundary conditions (see the Remarks below).

2.4 Remarks

- The density kernel $P_t(r,r')$ can be expressed in terms of the Gaussian kernel

$$G_t(r,r') = \frac{e^{-(r-r')^2/(2t)}}{\sqrt{2\pi t}}, \qquad r,r' \in \mathbb{R}, \tag{9}$$

as follows: if $\psi : \mathbb{R} \to [-1,1]$ denotes the usual reflection map, i.e. $\psi(x) = x$ for $x \in [-1,1]$, $\psi(x) = 2-x$ for $x \in [1,3]$, with ψ extended to the whole line as periodic of period 4, then

$$\begin{aligned} P_t(r,r') &= \sum_{r'':\psi(r'')=r'} G_t(r,r'') \quad \text{for } r' \neq \pm 1, \\ P_t(r,\pm 1) &= \sum_{r'':\psi(r'')=\pm 1} 2G_t(r,r''). \end{aligned} \tag{10}$$

- From the expressions above and (8) it follows that $\rho(\cdot,t)$ is "smooth" for any $t>0$: we are calling "smooth" a function $f(r)$, $r \in [-1,1]$, if it is C^∞ in $(-1,1)$, continuous in $[-1,1]$, and if for each n exist the limits $\frac{d^n f(r)}{dr^n}$ as $r \to \pm 1$.
- Since ρ is smooth we can write (8) in differential form: it then becomes the heat equation with Dirichlet boundary conditions:

$$\begin{aligned} &\frac{\partial}{\partial t}\rho(r,t) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\rho(r,t), \qquad r \in (-1,1), t>0, \\ &\rho(r,0) = u_0(r), \qquad \rho(\pm 1,t) = u_\pm(t). \end{aligned} \tag{11}$$

However the boundary conditions $u_\pm(t)$ are not a priori known, they must be obtained by solving a non-linear system of two integral equations:

$$\begin{aligned} &u_\pm(t) = \int_0^t \{p(s)f_\pm(u_\pm(t-s)) - q(s)f_\mp(u_\mp(t-s))\}ds + w_{\pm,t}, \\ &f_+(u) = \frac{j}{2}\big(1-u^K\big), \qquad f_-(u) = \frac{j}{2}\big(1-(1-u)^K\big), \end{aligned} \tag{12}$$

where, writing $G_t(r) = G_t(0,r)$, the latter as in (9),

$$p(t) = 2\sum_{k\in\mathbb{Z}} G_t(4k), \qquad q(t) = 2\sum_{k\in\mathbb{Z}} G_t(4k+2),$$

$$w_{+,t} = \sum_{k\in\mathbb{Z}} \int_{-1}^{1} u_0(r')2G_t(1-r'+4k)dr', \tag{13}$$

$$w_{-,t} = \sum_{k\in\mathbb{Z}} \int_{-1}^{1} u_0(r')2G_t(r'+1+4k)dr'.$$

- By a simple computation one can check that

$$\left.\frac{\partial\rho(r,t)}{\partial r}\right|_{r=1} = j(1-\rho(1,t)^K), \qquad \left.\frac{\partial\rho(r,t)}{\partial r}\right|_{r=-1} = j(1-(1-\rho(-1,t))^K). \tag{14}$$

  This remark will be important in the analysis of the Fourier law.
- To characterize the asymptotic behavior of the invariant measure as $N\to\infty$ it will be important to study the evolution starting from arbitrary initial configurations $\eta^{(N)}$. Since the functions $f^{(N)}$ defined through $f^{(N)}(r) = \eta^{(N)}([Nr])$ are in a ball of $L^2([-1,1],\mathbb{R})$ they converge weakly by subsequences in $L^2([-1,1],\mathbb{R})$, and we can then apply Theorem 1 to any convergent subsequence.
- The identification of (8) and (11) as the hydrodynamic equation of the system is based on the assumption that ρ_ϵ gives an accurate description of the process. This is indeed correct because the "empirical averages" are close to the functions ρ_ϵ in the following sense. There is $\tau>0$ so that calling $J_M(x) = [x-M, x+M]\cap\Lambda_N$, M the integer part of N^a, $a\in(0,1)$, then for any $t_0>0$

$$\lim_{\delta\to0}\lim_{\epsilon\to0}\sup_{t_0\le t\le\tau\log\epsilon^{-1}}\sup_{\eta}\mathbb{P}_\epsilon\left(\sup_{x\in\Lambda_N}\left|\frac{1}{|J_M(x)|}\sum_{y\in J_M(x)}\{\eta(y,t)-\rho_\epsilon(y,t)\}\right|\ge\delta\right)=0. \tag{15}$$

  Equation (15) follows from Theorem 4 using the Chebyshev inequality, Theorem 4 is proved in a companion paper, [4].

2.5 Fourier Law

In the "hydrodynamic limit literature" the limit function $\rho(r,t)$ is usually interpreted as the "density profile" at time t: this comes from attributing to each particle a mass ϵ so that $\mathbb{E}_\epsilon[\epsilon\eta(x,t)]$ is the average mass in the interval $[x-\frac{1}{2}, x+\frac{1}{2}]$ which in macroscopic units has length ϵ (as $[-N,N]$ in the macroscopic limit shrinks to $[-1,1]$). Thus $\mathbb{E}_\epsilon[\eta(x,t)]$ is the mass density, which in the limit converges to $\rho(r,t)$ (when $\epsilon x\to r$). Analogously, the expected current through a point is the *signed* average mass crossing that point per unit time. Let x be away from the boundaries in the sense that $|x|\le N-K$. Then it follows from (1) that the expected current through $x+\frac{1}{2}$ is

$$j^{(\epsilon)}(x,t) = \frac{\epsilon^{-2}}{2}\,\mathbb{E}_\epsilon\big[\epsilon\{\eta(x,t)-\eta(x+1,t)\}\big] = -\frac{1}{2}\mathbb{E}_\epsilon\left[\frac{\eta(x+1,t)-\eta(x,t)}{\epsilon}\right]. \tag{16}$$

By a similar argument the expected currents through N and $-N$ are:

$$j_\pm^{(\epsilon)}(t) = -\frac{\epsilon^{-1}j}{2}\sum_{y\in I_\pm}\mathbb{E}_\epsilon\big[\epsilon D_\pm(\eta(y,t))\big] = -\frac{j}{2}\sum_{y\in I_\pm}\mathbb{E}_\epsilon\big[D_\pm(\eta(y,t))\big]. \tag{17}$$

By (16) $j^{(\epsilon)}(x,t)$, $|x|\le N-K$, is equal to $-\frac{1}{2}$ times the discrete gradient of the density in agreement with the Fourier's law, which is then satisfied before the macroscopic limit

$\epsilon \to 0$, (but not necessarily in the limit, as this requires that the limit of the derivative is the derivative of the limit). One would expect that also $j_{\pm}^{(\epsilon)}(t)$ are equal to $-\frac{1}{2}$ times the discrete gradient of the density, at least in the limit as $\epsilon \to 0$. This is settled in the next theorem where using the factorization properties proved in [4] we show that the limit of the current is both in the bulk and at the boundaries equal to $-\frac{1}{2}$ times the gradient of the density.

Theorem 2 (Validity of the Fourier law) *Suppose that the process starts with a product measure μ^ϵ such that $\mu^\epsilon[\eta(x) = 1] = u_0(\epsilon x)$, where $u_0 \in C([-1, 1], [0, 1])$ and has bounded derivative in $(-1, 1)$. Let ρ be the solution of (8). Then, for any $t \ge 0$ and $r \in (-1, 1)$, denoting by $[u]$ the integer part of u, we have*

$$\lim_{\epsilon\to 0} j^{(\epsilon)}([\epsilon^{-1}r], t) = -\frac{1}{2}\,\frac{\partial \rho(r, t)}{\partial r}. \tag{18}$$

Moreover for any $t > 0$

$$\begin{aligned} \lim_{\epsilon\to 0} j_+^{(\epsilon)}(t) &= -\frac{1}{2}\,\frac{\partial \rho(r, t)}{\partial r}\Big|_{r=1} = \frac{j}{2}(1 - \rho(1, t)^K), \\ \lim_{\epsilon\to 0} j_-^{(\epsilon)}(t) &= -\frac{1}{2}\,\frac{\partial \rho(r, t)}{\partial r}\Big|_{r=-1} = \frac{j}{2}(1 - (1 - \rho(-1, t))^K). \end{aligned} \tag{19}$$

Theorem 2 is proved in Sect. 6.

2.6 Discussions, Conjectures and Open Problems

In a forthcoming paper we prove that the unique stationary measure for finite N is supported, in the limit as $N \to \infty$, by a linear profile ρ on $[-1, 1]$. ρ is a stationary solution of the limit equation being determined by imposing that the entering current at 1 equals the current flowing in the bulk: $\frac{j}{2}(1 - \rho(1)^K) = \frac{\rho(1)-\rho(-1)}{2}$ and that this is also equal to the current exiting from -1: $\frac{j}{2}(1 - (1 - \rho(-1))^K) = \frac{\rho(1)-\rho(-1)}{2}$. These two relations determine $\rho(\pm 1)$ and hence the linear profile $\rho(r)$. We also prove that this is the unique stationary solution of the hydrodynamic equation and establish uniform in N convergence of the process to the stationary measure using stochastic domination and maximum principle for the limit evolution.

The characteristic feature of the system studied in this paper is that the limit hydrodynamic equation is complemented by equations specifying its boundary conditions. This is due to the fact that the boundary processes are "weak", the birth-death densities are proportional to N and not to N^2. When $K > 1$ the boundary processes create "correlations" as well, the hardest technical part being the proof that they remain small (as proved in [4]). The proof is "robust" and extends to more general boundary processes where both births and deaths are allowed at both boundaries (provided they are weak in the above sense). Much harder is the problem when the births and deaths occur at rate N^2. Our analysis does not extend to such cases. A result in this direction is due to Bodineau, Derrida and Lebowitz, [3], with their analysis of the so called "battery problem".

Our methods in this paper do not allow to study the large deviations of the stationary measure. It would be interesting to try to apply Derrida's techniques as in [5].

Maybe Varadhan techniques could be applied to study other interacting models where propagation of chaos is replaced by a local Gibbs property. It would also be interesting to see what happens with the ASEP.

3 The Discretized Evolution

We begin the analysis of (5) by proving:

Proposition 3.1 *The Cauchy problem for* (5) *with initial datum* $\rho_\epsilon(\cdot,0)\in[0,1]$ *has a unique global solution* ρ_ϵ. *Moreover* $\rho_\epsilon(x,t)\in[0,1]$ *for all* $x\in\Lambda_N, t>0$.

Proof Write

$$D_+^* u(x) = (1-u(x))\big|u(x+1)u(x+2)\dots u(N)\big|, \quad x\in I_+,$$
$$D_-^* u(x) = u(x)\big|(1-u(x-1))(1-u(x-2))\dots(1-u(-N))\big|, \quad x\in I_-.$$

If $0\le u(x)\le 1$ then $D_\pm^*\equiv D_\pm$. A local existence and uniqueness theorem holds for the Cauchy problem (5) as well as for the problem with $D_\pm$ replaced by $D_\pm^*$ (as these are Lipschitz functions of u in the sup-norm topology). Denote the solution of the latter by $\rho_\epsilon^*(x,t)$, $t\le\tau$, $\tau>0$, recalling that the initial datum $\rho_\epsilon(x,0)$ verifies $0\le\rho_\epsilon(x,0)\le 1$ for any $x\in\Lambda_N$. We shall next prove that $0\le\rho_\epsilon^*(x,t)\le 1$ for all x and $t\le\tau$. Define $u(s)=\max_{x\in\Lambda_N}\rho_\epsilon^*(x,s)$ and suppose by contradiction that there is $T\le\tau$ such that $u(T)>1$. Then there is $t\le T$ so that (i) $u(t)>1$ and (ii) $du(t)/dt>0$ (because $u(0)\le 1$ and it cannot be that $du(s)/ds\le 0$ for almost all $s\le T$ such that $u(s)>1$). Moreover there exists x such that (a) $\rho_\epsilon^*(x,t)=u(t)$ and (b) $du(t)/dt=d\rho_\epsilon^*(x,t)/dt$. All that leads to a contradiction because $d\rho_\epsilon^*(x,t)/dt=\frac{1}{2}\Delta_\epsilon\rho_\epsilon^*(x,t)+\frac{j}{2}(D_+^*-D_-^*)\rho_\epsilon^*(x,t)\le 0$. Indeed $\Delta_\epsilon\rho_\epsilon^*(x,t)\le 0$, because (x,t) maximizes $\rho_\epsilon^*(\cdot,t)$. $D_+^*\rho_\epsilon^*(x,t)=0$ if $x\notin I_+$ and ≤ 0 in I_+, because $\rho_\epsilon^*(x,t)>1$. $D_-^*\rho_\epsilon^*=0$ if $x\notin I_-$ and ≥ 0 in I_-, because $\rho_\epsilon^*(x,t)\ge 0$. Thus $(D_+^*-D_-^*)\rho_\epsilon^*(x,t)\le 0$.

Analogous arguments show that the solution cannot exit $[0,1]$ through 0, hence $\rho_\epsilon^*(x,t)\in[0,1]$. As a consequence $D_\pm^*\rho_\epsilon^*=D_\pm\rho_\epsilon^*$ and therefore ρ_ϵ^* solves (5) as well. By iteration, the previous argument extends to all times. □

We shall study (5) in its integral form:

$$\rho_\epsilon(x,t)=\sum_{y\in\Lambda_N}P_t^{(\epsilon)}(x,y)\rho_\epsilon(y,0)+\epsilon^{-1}\frac{j}{2}\int_0^t ds\Bigg(\sum_{y\in I_+}P_s^{(\epsilon)}(x,y)D_+\rho_\epsilon(y,t-s)$$
$$-\sum_{y\in I_-}P_s^{(\epsilon)}(x,y)D_-\rho_\epsilon(y,t-s)\Bigg), \tag{20}$$

where $P_t^{(\epsilon)}$ is the semigroup with generator $\frac{1}{2}\Delta_\epsilon$ and $P_t^{(\epsilon)}(x,y)$ its kernel:

$$P_t^{(\epsilon)}:=e^{\frac{1}{2}\Delta_\epsilon t},\qquad P_t^{(\epsilon)}(x,y)=P_t^{(\epsilon)}(y,x). \tag{21}$$

The analysis of (20) will exploit the nice regularity properties of $P_t^{(\epsilon)}(x,y)$ which are established in the next section.

4 Probability Estimates for a Random Walk with Reflections

In this section we shall consider a simple random walk on Λ_N which jumps with intensity $\epsilon^{-2}/2$ to each of its nearest neighbor sites, the jumps outside Λ_N being suppressed. We denote by $P_t^{(\epsilon)}$ its law and call $Q_t^{(\epsilon)}$ the law of the corresponding unrestricted random walk on the whole $\mathbb{Z}$. In the sequel we shall prove (in many cases just recall) bounds and estimates on $P_t^{(\epsilon)}$ which will be used in the next sections to prove Theorems 1 and 2. We start by relating $P_t^{(\epsilon)}$ and $Q_t^{(\epsilon)}$, through a "reflection map" from $\mathbb{Z}$ to Λ_N which is a discrete analogue of the map ψ defined in the first remark after Theorem 1. Since the jump rate from $\pm N$ is $\epsilon^{-2}/2$ for $P_t^{(\epsilon)}$ and ϵ^{-2} for $Q_t^{(\epsilon)}$ to relate the two it just suffices to identify $N+1$ with N (as well as $-N-1$ with $-N$), then the jumps of $Q_t^{(\epsilon)}$ from N to $N+1$ and $-N$ to $-N-1$ are like suppressed. We thus define:

Definition The "reflection map" $\psi_N : \mathbb{Z} \to \Lambda_N$ is:

- $|x| \le N$: $\psi_N(x) = x$
- $x < -N$: $\psi_N(x) = -\psi_N(-x)$
- $x > N$: $\psi_N(N + j(2N+1) + k) = \psi_N(N + j(2N+1) - (k-1))$, $k = 1, \dots, 2N+1$, $j = 0, 1, \dots$.

Proposition 4.1 *With the above notation,*

$$P_t^{(\epsilon)}(x,z) = \sum_{y:\psi_N(y)=z} Q_t^{(\epsilon)}(x,y). \tag{22}$$

Proof Let f be a function on $\mathbb{Z}$ which is ψ_N-measurable, i.e. $f(x) = f(y)$ whenever $\psi_N(x) = \psi_N(y)$, $x, y \in \mathbb{Z}$, and let g be its restriction to Λ_N. Calling L_Q and L_P the generators of $Q_t^{(\epsilon)}$ and $P_t^{(\epsilon)}$ we have

$$L_Q f(x) = L_P g(\psi_N(x))$$

so that $e^{L_Q t} f = e^{L_P t} g$, hence (22). □

By the local central limit theorem (see for instance [7]):

Theorem 3 *There exist positive finite constants $c_1, \dots, c_5$ so that*

$$\begin{aligned} &|Q_t^{(\epsilon)}(x,y) - G_{\epsilon^{-2}t}(x,y)| \le \frac{c_1}{\sqrt{\epsilon^{-2}t}} G_{\epsilon^{-2}t}(x,y), \quad |x-y| \le (\epsilon^{-2}t)^{5/8}, \\ &Q_t^{(\epsilon)}(x,y) \le \min\left\{c_2 e^{-c_3|x-y|^2/(\epsilon^{-2}t)}, c_4 e^{-|y-x|(\log|y-x|-c_5)}\right\}, \quad |x-y| > (\epsilon^{-2}t)^{5/8}, \end{aligned} \tag{23}$$

G_t being the Gaussian kernel defined in (9).

The next corollary follows directly from Theorem 3 and Proposition 4.1.

Corollary 4.2 *For any $T > 0$ there exists finite positive c so that the following holds.*

– *For all ϵ, all $t \in (0,T]$ and all x, y in Λ_N,*

$$P_t^{(\epsilon)}(x,y) \le c\, G_{\epsilon^{-2}t}(x,y). \tag{24}$$

– *For all ϵ, all $t \in (0,T]$ and all $-N \le x \le N-1$,*

$$\left|P_t^{(\epsilon)}(x,y) - P_t^{(\epsilon)}(x+1,y)\right| \le \frac{c}{\sqrt{\epsilon^{-2}t}} G_{\epsilon^{-2}t}(x,y). \tag{25}$$

– *For all* ϵ, *all* $t \in (0, T]$, $s > 0$ *and all* $x \in \Lambda_N$,

$$\sum_{y \in \Lambda_N} \left| P^{(\epsilon)}_{t+s}(x, y) - P^{(\epsilon)}_t(x, y) \right| \le c\sqrt{\frac{s}{t}}, \tag{26}$$

$$\left| P^{(\epsilon)}_{t+s}(x, y) - P^{(\epsilon)}_t(x, y) \right| \le c\frac{\sqrt{\epsilon^{-2}s}}{\epsilon^{-2}t}. \tag{27}$$

Proof Inequalities (24) and (25) follow directly from (23). By (24) and (25) we can bound the left hand side of (26) by

$$\sum_{y \in \Lambda_N} \sum_{z \in \Lambda_N} P^{(\epsilon)}_s(x, z) \left| P^{(\epsilon)}_t(z, y) - P^{(\epsilon)}_t(x, y) \right| \le c \sum_{z \in \Lambda_N} G_{\epsilon^{-2}s}(x, z) \frac{|z - x|}{\sqrt{\epsilon^{-2}t}}$$

hence (26). The estimate (27) is obtained similarly, recalling that $G_t(x, y) \le ct^{-1/2}$. □

In the proof of Theorem 1 we shall use the following convergence results:

Lemma 4.3 *As in Theorem* 1, *suppose that* $\rho_\epsilon(\cdot, 0)$ *converges weakly to* $u_0(\cdot)$ *in the sense of* (6). *Then, for any* $t > 0$

$$\lim_{\epsilon \to 0} \sup_{x \in \Lambda_N} \left| \sum_{y \in \Lambda_N} P^{(\epsilon)}_t(x, y)\rho_\epsilon(y, 0) - \int_{[-1,1]} P_t(\epsilon x, r)u_0(r)dr \right| = 0. \tag{28}$$

Proof By (25) the family of functions f_ϵ defined by

$$f_\epsilon(r) := \sum_{y \in \Lambda_N} P^{(\epsilon)}_t([\epsilon^{-1}r], y)\rho_\epsilon(y, 0), \quad r \in [-1, 1] \tag{29}$$

is uniformly Lipschitz, so that it will suffice to prove pointwise convergence. We thus fix $r^* \in [-1, 1]$ and take $x = [\epsilon^{-1}r^*]$. By (22) and (23)

$$\begin{aligned}\sum_{y \in \Lambda_N} P^{(\epsilon)}_t(x, y)\rho_\epsilon(y, 0) &= \sum_{y \in \Lambda_N} \rho_\epsilon(y, 0) \sum_{z:\psi_N(z)=y} Q^{(\epsilon)}_t(x, z) \\ &= \sum_{y \in \Lambda_N} \rho_\epsilon(y, 0) \sum_{z:\psi_N(z)=y} G_{\epsilon^{-2}t}(x, z) + O\big((\epsilon^{-2}t)^{-1/2}\big) + O\big(e^{-\epsilon^{-1}}\big).\end{aligned}$$

Call Ψ_N the discrete analogue of the reflection map ψ of the Remarks after Theorem 1, i.e. $\Psi_N(x) = N\psi(x/N)$ for $x \in \Lambda_N$. It differs from ψ_N by shifts and we have:

$$\left| \sum_{z:\psi_N(z)=y} G_{\epsilon^{-2}t}(x, z) - \sum_{z:\Psi_N(z)=y} G_{\epsilon^{-2}t}(x, z) \right| \le \frac{c}{\sqrt{\epsilon^{-2}t}}.$$

But (see (10))

$$\sum_{z:\Psi_N(z)=y} G_{\epsilon^{-2}t}(x, z) = \sum_{r':\psi(r')=\epsilon y} \epsilon G_t(\epsilon x, r') = \epsilon P_t(\epsilon x, \epsilon y)$$

and we conclude that

$$\left| \sum_{y\in\Lambda_N} P_t^{(\epsilon)}(x,y)\rho_\epsilon(y,0) - \epsilon \sum_{y\in\Lambda_N} P_t(\epsilon x,\epsilon y)\rho_\epsilon(y,0) \right| \le \frac{c}{\sqrt{\epsilon^{-2}t}}.$$

Letting $x=[\epsilon^{-1}r^*]$, then by (6) with $\phi(r):=P_t(r^*,r)$ we have

$$\lim_{\epsilon\to 0} \sum_{y\in\Lambda_N} P_t^{(\epsilon)}([\epsilon^{-1}r^*],y)\rho_\epsilon(y,0) = \int_{[-1,1]} P_t(r^*,r)u_0(r)dr \tag{30}$$

which proves pointwise convergence and hence the lemma, as argued at the beginning of the proof. □

Lemma 4.4 *Let $h_\epsilon(t)$ be a continuous function with values in $[0,1]$ which converges pointwise to $h(t)$. Then for any $t>0$, $r\in[-1,1]$ and $y\in I_+$*

$$\lim_{\epsilon\to 0}\int_0^t \epsilon^{-1} P_s^{(\epsilon)}([\epsilon^{-1}r],\pm y)h_\epsilon(t-s)ds = \int_0^t P_s(r,\pm 1)h(t-s)ds. \tag{31}$$

Proof Again, this follows easily from (22) and (23), after recalling also (10). Details are omitted. □

5 Proof of the Hydrodynamic Limit

In this section we shall prove Theorem 1. We start by proving equicontinuity, which is a direct consequence of the estimates of the previous section:

Proposition 5.1 *For any $T>0$ there exists a finite constant c so that for any solution ρ_ϵ of (5) with $\rho_\epsilon(\cdot,0)\in[0,1]$ the following holds. For any $x\in[-N,N-1]$, any $t\in(0,T]$ and any $\epsilon>0$*

$$|\rho_\epsilon(x,t)-\rho_\epsilon(x+1,t)| \le \min\left\{1, c\left(\epsilon\log_+(\epsilon^{-2}t)+\frac{1}{\sqrt{\epsilon^{-2}t}}\right)\right\}, \tag{32}$$

where $\log_+ u=\max\{\log u,1\}$. For any $0<s<t$, $x\in\Lambda_N$ and $\epsilon>0$:

$$|\rho_\epsilon(x,t)-\rho_\epsilon(x,t+s)| \le \min\left\{1, c\left(\sqrt{\frac{s}{t}}+\sqrt{s}\log\left(\frac{t}{s}\right)\right)\right\}. \tag{33}$$

Proof By (25)

$$\left|\sum_y \left(P_t^{(\epsilon)}(x,y)-P_t^{(\epsilon)}(x+1,y)\right)\rho_\epsilon(y,0)\right| \le \frac{c}{\sqrt{\epsilon^{-2}t}}$$

and for any $y\in I_+\cup I_-$,

$$\int_0^t \epsilon^{-1}\left|P_s^{(\epsilon)}(x,y)-P_s^{(\epsilon)}(x+1,y)\right|ds \le \epsilon + \int_{\epsilon^2}^t \frac{c\epsilon^{-1}}{\epsilon^{-2}s}ds,$$

hence (32). By (26)

$$\Big|\sum_{y}\big(P^{(\epsilon)}_{t+s}(x,y)-P^{(\epsilon)}_{t}(x,y)\big)\rho_\epsilon(y,0)\Big|\le c\sqrt{\frac{s}{t}}.$$

By (27) for any $y\in I_+\cup I_-$ and denoting by $f(t):=D_\pm\rho_\epsilon(y,t)$, $y\in I_\pm$,

$$\begin{aligned}
&\epsilon^{-1}\Big|\int_0^{t+s}P^{(\epsilon)}_{s'}(x,y)f(t+s-s')ds'-\int_0^{t}P^{(\epsilon)}_{s'}(x,y)f(t-s')ds'\Big|\\
&\quad\le\epsilon^{-1}\Big(\int_0^{2s}P^{(\epsilon)}_{s'}(x,y)ds'+\int_0^{s}P^{(\epsilon)}_{s'}(x,y)ds'+\int_s^{t}|P^{(\epsilon)}_{s'+s}(x,y)-P^{(\epsilon)}_{s'}(x,y)|ds'\Big)\\
&\quad\le c\Big(\sqrt{s}+\sqrt{s}\log\Big(\frac{t}{s}\Big)\Big).
\end{aligned}\tag{34}$$

□

Turning now to the proof of Theorem 1, we fix $T>0$ and study the evolution in the finite time interval $[0,T]$. Since we only have that $\rho_\epsilon(\cdot,0)\to u_0(\cdot)=\rho(\cdot,0)$ weakly, it is convenient to introduce a regularized equation. We denote by $\rho_\epsilon(x,t|u,s)$, $t\ge s\ge 0$, the solution of (5) for $t\ge s$ with u the initial datum at time s, $u=u(x)$, $x\in\Lambda_N$, $u(x)\in[0,1]$. With such notation we then set for any $\delta\in(0,T)$

$$\rho^{(\delta)}_\epsilon(x,t)=\begin{cases}\sum_{y\in\Lambda_N}P^{(\epsilon)}_t(x,y)\rho_\epsilon(y,0) & 0\le t\le\delta,\\ \rho_\epsilon(x,t|\rho^{(\delta)}_\epsilon(\cdot,\delta),\delta) & t\in(\delta,T].\end{cases}\tag{35}$$

By Proposition 5.1 the family of functions $(r,t)\mapsto\rho^{(\delta)}_\epsilon([\epsilon^{-1}r],t)$, $r\in[-1,1]$, $t\in[\delta,T]$ is equicontinuous and bounded, hence it converges in sup norm by subsequences to a limit function $u^{(\delta)}(r,t)$. By Lemma 4.3

$$u^{(\delta)}(r,\delta)=\int_{[-1,1]}P_\delta(r,r')\rho(r',0)dr'.\tag{36}$$

Moreover for any integer $0\le m\le K$

$$\begin{aligned}
&\lim_{\epsilon\to0}\sup_{\delta\le t\le T}\Big|\big\{D_+\rho^{(\delta)}_\epsilon(N-m,t)-\big(1-u^{(\delta)}(1,t)\big)u^{(\delta)}(1,t)^m\big\}\Big|=0,\\
&\lim_{\epsilon\to0}\sup_{\delta\le t\le T}\Big|\big\{D_-\rho^{(\delta)}_\epsilon(-N+m,t)-u^{(\delta)}(-1,t)\big(1-u^{(\delta)}(-1,t)\big)^m\big\}\Big|=0.
\end{aligned}\tag{37}$$

By Lemma 4.3 and Lemma 4.4 it then follows that

$$u^{(\delta)}(r,t)=u(r,t|u^{(\delta)}(\cdot,\delta),\delta),\tag{38}$$

where the latter is the solution of the limit equation in the time interval $[\delta,T]$ with initial datum at time δ equal to $u^{(\delta)}(\cdot,\delta)$. Uniqueness can be easily proved, but it also follows from (40) below where we prove that the solution depends continuously on the initial datum. From the uniqueness one has that $\rho^{(\delta)}_\epsilon$ indeed converges in sup norm to $u^{(\delta)}$ as $\epsilon\to0$, not only by subsequences.

We shall next examine the dependence on δ, and define for $t \in [\delta, T]$

$$h_\epsilon^{(\delta)}(t) = \sup_{x\in\Lambda_N} |\rho_\epsilon^{(\delta)}(x,t) - \rho_\epsilon(x,t)|, \qquad h^{(\delta)}(t) := \sup_{|r|\le 1} |u^{(\delta)}(r,t) - u(r,t)|. \tag{39}$$

We are going to prove that there exists finite c_T so that for all ϵ and δ positive

$$h_\epsilon^{(\delta)}(t) + h^{(\delta)}(t) \le c_T\sqrt{\delta} \tag{40}$$

(which in particular implies uniqueness of the solution of (8)). It follows from (40) that

$$\limsup_{\epsilon\to 0}\ \sup_{t\in[\delta,T]}\ \sup_{x\in\Lambda_N} \left|\rho_\epsilon(x,t) - u(\epsilon x, t)\right| \le c\sqrt{\delta} \tag{41}$$

which then proves Theorem 1.

Proof of (40) From (24) it follows that $h_\epsilon^{(\delta)}(\delta) \le c\sqrt{\delta}$, then using again (24),

$$h_\epsilon^{(\delta)}(t) \le c\sqrt{\delta} + C\int_\delta^t \frac{1}{\sqrt{s}} h_\epsilon^{(\delta)}(t-s)ds. \tag{42}$$

By iteration,

$$h_\epsilon^{(\delta)}(t) \le c\sqrt{\delta}\left(1 + \sum_{n=1}^\infty C^n a_n(t-\delta)\right)$$
$$a_n(t) := \int_0^t \frac{1}{\sqrt{s_1}} ds_1 \int_0^{t-s_1} \frac{1}{\sqrt{s_2}} ds_2 \dots \int_0^{t-s_1\cdots-s_{n-1}} \frac{1}{\sqrt{s_n}} ds_n.$$

By (43) below we then have $h_\epsilon^{(\delta)}(t) \le c'\sqrt{\delta}(1 + e^{\pi C^2 T})$. The same argument applies to $h^{(\delta)}(t)$, hence proving (40). □

Lemma 5.2 *With $a_n(t)$ as above,*

$$a_n(t) \le (\pi t)^{\frac{n}{2}} e^{-\frac{n}{2}[\log(\frac{n}{2})-1]} \tag{43}$$

Proof We have

$$a_n(t) = \int_{[0,t]^n} \mathbf{1}_{s_1+\cdots+s_n\le t} \prod_{i=1}^n \frac{1}{\sqrt{s_i}}\, ds_1 \dots ds_n. \tag{44}$$

We change variables by setting $s_i = t_i\, t$ and get

$$a_n(t) = (\sqrt{t})^n \int_{[0,1]^n} \mathbf{1}_{t_1+\cdots+t_n\le 1} \prod_{i=1}^n \frac{1}{\sqrt{t_i}}\, dt_1 \dots dt_n. \tag{45}$$

Multiplying and dividing by $\exp\{-\alpha(t_1 + \cdots + t_n)\}$ we have

$$a_n(t) \le (\sqrt{t})^n e^\alpha \int_{[0,1]^n} \prod_{i=1}^n \frac{e^{-\alpha t_i}}{\sqrt{t_i}}\, dt_1 \dots dt_n \le (\sqrt{t})^n e^\alpha \left[\int_0^1 \frac{e^{-\alpha s}}{\sqrt{s}}\, ds\right]^n$$

$$\le (\sqrt{t})^n e^{\alpha} \left(\frac{\sqrt{\pi}}{\sqrt{\alpha}} \right)^n . \tag{46}$$

By choosing $\alpha = \frac{n}{2}$ we get (43). □

6 Proof of the Fourier Law

In this section we shall prove Theorem 2. The proof relies on Theorem 4 below which is proved in [4]. Writing $\Lambda_N^{n,\neq}$, $n \ge 1$, for the set of all sequences $\underline{x} = (x_1, \dots, x_n)$ in Λ_N^n with distinct entries, we first define the v-functions

$$v^{\epsilon}(\underline{x}, t | \mu^{\epsilon}) := \mathbb{E}_{\epsilon} \left[\prod_{i=1}^{n} \{\eta(x_i, t) - \rho_{\epsilon}(x_i, t)\} \right], \quad \underline{x} \in \Lambda_N^{n,\neq}, \; n \ge 1, \tag{47}$$

where the process starts with a product measure μ^{ϵ}, in particular a single configuration, and $\rho_{\epsilon}(x,t)$ is the solution of (5) with initial datum $\rho_{\epsilon}(x, 0) = \mu^{\epsilon}[\eta(x, 0) = 1]$.

Theorem 4 *There exist $\tau > 0$ and $c^* > 0$ so that the following holds. For any $\beta^* > 0$ and for any positive integer n there is a constant $c_n < \infty$ so that for any $\epsilon > 0$, any initial product measure μ^{ϵ}*

$$\sup_{\underline{x} \in \Lambda_N^{n,\neq}} |v^{\epsilon}(\underline{x}, t | \mu^{\epsilon})| \le \begin{cases} c_n(\epsilon^{-2} t)^{-c^* n}, & t \le \epsilon^{\beta^*}, \\ c_n \epsilon^{(2-\beta^*)c^* n}, & \epsilon^{\beta^*} \le t \le \tau \log \epsilon^{-1}. \end{cases} \tag{48}$$

Proof of (19) Recalling (17) and applying Theorem 4 we have:

$$\lim_{\epsilon \to 0} j_+^{(\epsilon)}(t) = -\frac{j}{2}(1 - \rho(1,t)^K), \qquad \lim_{\epsilon \to 0} j_-^{(\epsilon)}(t) = -\frac{j}{2}(1 - (1 - \rho(-1,t))^K) \tag{49}$$

and (19) follows from (14). □

Proof of (18) We can express $\rho(\cdot, t)$ using the Green function for (11) and get

$$\rho(r,t) = \tilde{P}_t \rho(r, 0) + \int_0^t \{\rho(1, t-s)\mathbb{P}_{r,1}(ds) + \rho(-1, t-s)\mathbb{P}_{r,-1}(ds)\}, \tag{50}$$

where $\tilde{P}_t \rho(r, 0) = \mathbb{E}_r(\rho(B(t), 0)\mathbf{1}_{\tau > t})$ with $B(t)$ the standard Brownian motion starting from r, and τ the hitting time of $\{-1, 1\}$; $\mathbb{P}_{r,\pm 1}(ds) = \mathbb{P}_r(\tau \in ds, B(\tau) = \pm 1)$ are the corresponding hitting time distributions. Since $\rho'(r,t) := \partial \rho(r,t)/\partial r$ satisfies the heat equation we can write similarly to (50)

$$\rho'(r,t) = \tilde{P}_t \rho'(r, 0) + \int_0^t \{\rho'(1, t-s)\mathbb{P}_{r,1}(ds) + \rho'(-1, t-s)\mathbb{P}_{r,-1}(ds)\} \tag{51}$$

with $\rho'(\pm 1, t-s)$ explicitly given in (14). The idea then is to write $j^{(\epsilon)}(x,t)$ (which is defined in (16)) in a similar way. We are going to prove that for $-N + K < x < N - K - 1$,

$$j^{(\epsilon)}(x,t) = -\frac{\bar{\phi}_{\epsilon}(x+1, t)}{2} - \frac{1}{2} \int_0^t \left\{ \Theta_+^{(\epsilon)}(t-s)\mathbb{P}_{x,\epsilon;+}(ds) + \Theta_-^{(\epsilon)}(t-s)\mathbb{P}_{x,\epsilon;-}(ds) \right\} \tag{52}$$

with

$$\begin{aligned} \Theta_+^{(\epsilon)}(t) &= \epsilon^{-1}\Big(E_\epsilon[\eta(N-K,t)] - E_\epsilon[\eta(N-K-1,t)]\Big), \\ \Theta_-^{(\epsilon)}(t) &= \epsilon^{-1}\Big(E_\epsilon[\eta(-N+K+1,t)] - E_\epsilon[\eta(-N+K,t)]\Big) \end{aligned} \tag{53}$$

and

$$\bar{\phi}_\epsilon(x+1,t) = \epsilon^{-1}\sum_y \Big(P_{x+1}(y(t)=y,\tau'>t) - P_x(y(t)=y,\tau>t)\Big)\rho_\epsilon(y,0), \tag{54}$$

where we have used the following notation: $y(t)$ is a random walk on Λ_N with transition kernel $P_t^{(\epsilon)}$, P_x its law when $y(0)=x$, τ' its first hitting time of $\{-N+K+1, N-K\}$ and τ the first hitting time of $\{-N+K, N-K-1\}$, and $\mathbb{P}_{x;\epsilon,\pm}(ds)$ refers to the hitting time distribution of the boundary. More precisely, $\mathbb{P}_{x;\epsilon,+}(ds) = P_x(\tau\in ds, y(\tau)=N-K-1)$ and $\mathbb{P}_{x;\epsilon,-}(ds) = P_x(\tau\in ds, y(\tau)=-N+K)$.

To prove (52) we use (3), observing that when $x\notin I_-\cup I_+$, the second term on the r.h.s. of (3) vanishes, so that

$$\frac{d}{dt}\mathbb{E}_\epsilon[\eta(x,t)] = \frac{1}{2}\Delta_\epsilon\mathbb{E}_\epsilon[\eta(x,t)], \quad |x|\le N-K.$$

This allows to express $g_\epsilon(x,t) := \mathbb{E}_\epsilon[\eta(x,t)] = P_x[g_\epsilon(y(\bar\tau\wedge t), t-\bar\tau\wedge t)]$ where P_x refers to the expectation with respect to a random walk $y(\cdot)$ that starts at x, with $\bar\tau$ the first hitting time of $\{-N+K, N-K\}$, which corresponds to the (time variable) boundary condition. Doing the same for each of the terms in $E_\epsilon[\eta(x,t)] - E_\epsilon[\eta(x+1,t)]$ we arrive to (52).

We need to compare (51) and (52) recalling (14). By the weak convergence of the random walk to the Brownian motion, $\tilde\phi(x+1,t)$, $x=[\epsilon^{-1}r]$, converges to $\tilde P_t\rho_0'(r)$ and $\mathbb{P}_{x,\epsilon;\pm}(ds)$ converges weakly to $\mathbb{P}_{r,\pm}(ds)$ $(x=[\epsilon^{-1}r])$. Therefore, recalling (14), (51) and (52), the proof of (18) will follow from: for any $t>0$

$$\lim_{\epsilon\to 0}\Theta_+^{(\epsilon)}(t) = j(1-\rho(1,t)^K), \qquad \lim_{\epsilon\to 0}\Theta_-^{(\epsilon)}(t) = j(1-(1-\rho(-1,t))^K) \tag{55}$$

which will be proved in the remaining part of this section. As the analysis of $\Theta_\pm^{(\epsilon)}(t)$ are similar we shall only prove (55) for $\Theta_+^{(\epsilon)}(t)$.

Recalling (3) we can write:

$$\Theta_+^{(\epsilon)}(t) = \phi_\epsilon(N-K,t) + \sum_{y\in I_+}\Gamma_{\epsilon,t,y} - \sum_{y\in I_-}\Gamma_{\epsilon,t,y} \tag{56}$$

where

$$\phi_\epsilon(x,t) := \epsilon^{-1}\sum_{y\in\Lambda_N}\Big(P_t^{(\epsilon)}(x,y) - P_t^{(\epsilon)}(x-1,y)\Big)\rho_\epsilon(y,0), \quad x\in\Lambda_N, t>0 \tag{57}$$

and for $y\in I_\pm$, respectively,

$$\begin{aligned} \Gamma_{\epsilon,t,y} &:= \epsilon^{-2}\int_0^t ds\,\big(P_s^{(\epsilon)}(N-K,y) - P_s^{(\epsilon)}(N-K-1,y)\big) \\ &\quad\times \mathbb{E}_\epsilon\left(\frac{j}{2}(D_\pm\eta(\cdot,t-s))(y)\right). \end{aligned} \tag{58}$$

As the analysis of (56) will involve several steps, we give first an outline.

- We shall first prove that $\phi_\epsilon(N-K,t)$ vanishes as $\epsilon\to 0$ (this will be simple).
- We will then show that also $\Gamma_{\epsilon,t,y}$ with $y\in I_-$ vanishes as $\epsilon\to 0$. This is less simple and involves couplings of random walks.
- The analysis in the previous step is then used to prove that

$$\begin{aligned}&\lim_{\epsilon\to 0}|\Gamma_{\epsilon,t,y}-\Gamma^*_{\epsilon,t,y}|=0,\quad \text{for all } y\in I_+,\ \text{where:}\\&\Gamma^*_{\epsilon,t,y}:=\epsilon^{-2}\int_0^t ds\,\big(P_s^{(\epsilon)}(N-K,y)-P_s^{(\epsilon)}(N-K-1,y)\big)\,\frac{j}{2}(D_+\rho_\epsilon(\cdot,t-s))(y).\end{aligned}\tag{59}$$

- It is then proved that there exist numbers $a(h)$, $h=0,\dots,K-1$, so that

$$\lim_{\epsilon\to 0}\sum_{y\in I_+}\Gamma^*_{\epsilon,t,y}=\frac{j}{2}\sum_{h=0}^{K-1}a(h)\big(1-\rho(1,t)\big)\rho(1,t)^h.\tag{60}$$

- The final step consists in recognizing that the right hand side of (60) is indeed equal to $j(1-\rho(1,t)^K)$.

By (22)

$$\phi_\epsilon(x,t)=\epsilon^{-1}\sum_{z\in\mathbb{Z}}Q_t^{(\epsilon)}(x,z)\,(\rho_\epsilon(\psi_N(z),0)-\rho_\epsilon(\psi_N(z-1),0))\,.\tag{61}$$

Recalling that $\rho_\epsilon(y,0)=\rho_0(\epsilon y)$ and that ρ_0', the derivative of ρ_0, is by assumption bounded, we then have for any $r\in[-1,1]$

$$\lim_{\epsilon\to 0}\phi_\epsilon([\epsilon^{-1}r],t)=\int_{\mathbb{R}}G_t(r,r')(-1)^{S(r')}\rho_0'(\psi(r'))dr'=:\phi(r,t),\tag{62}$$

where $S(r')=1$ if r' in $[-1,1]$, $\pm[3,5],\dots$ and $=-1$ in the complement. By symmetry $\phi(\pm 1,t)=0$ so that

$$\lim_{\epsilon\to 0}\phi_\epsilon(x,t)=0,\quad x=N-K.\tag{63}$$

By rescaling the time we rewrite $\Gamma_{\epsilon,t,y}$ as

$$\begin{aligned}\Gamma_{\epsilon,t,y}:=\int_0^{\epsilon^{-2}t}ds\,\Big(P_{\epsilon^2 s}^{(\epsilon)}(N-K,y)-P_{\epsilon^2 s}^{(\epsilon)}(N-K-1,y)\Big)\\\times\,\mathbb{E}_\epsilon\Big[\frac{j}{2}(D_\pm\eta(\cdot,t-\epsilon^2 s))(y)\Big]\end{aligned}\tag{64}$$

and recall that $P_{\epsilon^2 s}^{(\epsilon)}(x,y)$, which in this proof we denote by $p_s^{(N)}(x,y)$, is the transition probability of a reflected random walk in Λ_N with jump intensity $1/2$ for each pair of nearest neighbor sites in Λ_N. In the sequel we shall also consider the transition probabilities $p_t(x,y)$ of the random walk on $\mathbb{Z}_+=\{0,1,2,\dots\}$ with jump intensity $1/2$ among nearest neighbors. □

Lemma 6.1 *There exists a constant c so that for any $h = 0, \dots, K-1$ and any t*

$$\left| p_t(K,h) - p_t(K+1,h) \right| \le \frac{c}{1+t^{3/2}}. \tag{65}$$

The integrals below are well defined:

$$\int_0^\infty \{p_t(K,h) - p_t(K+1,h)\} dt =: a(h). \tag{66}$$

Proof The second statement follows at once from the first, which we now prove with a coupling argument. We write

$$p_t(K,h) - p_t(K+1,h) = \mathcal{E}\left[\mathbf{1}_{y_1(t)=h} - \mathbf{1}_{y_2(t)=h}\right],$$

where $\mathcal{E}$ is the expectation in a process which couples two simple random walks on $\mathbb{Z}_+$, denoted by $y_1(s)$ and $y_2(s)$, $s \in [0,t]$, with $y_1(0) = K$, $y_2(0) = K+1$. The coupling (whose law will be denoted by $\mathcal{P}$) is defined as follows: $y_2(s)$ moves as the random walk on $\mathbb{Z}_+$ (i.e. with transition probability $p_s(x,y)$) for all $s \in [0,t]$. Let $t_1 = t/3$: in the time interval $[0,t_1]$, $y_1(s)$ copies exactly the jumps of $y_2(s)$ for all $s < \min\{\tau, t_1\}$, where τ is the first time when y_2 jumps to 0. If $\tau \le t_1$ then we set $y_1(s) = y_2(s)$ for all $s \in [\tau, t]$. When $\tau > t_1$, we let y_1 move independently of y_2 in $[t_1, \tau^*]$, where τ^* is the first time when $y_1 = y_2$, and for $s > \tau^*$ we set $y_1(s) = y_2(s)$. $\mathcal{P}$ is evidently a coupling and we have:

$$p_t(K,h) - p_t(K+1,h) = \mathcal{E}\left[(\mathbf{1}_{y_1(t)=h} - \mathbf{1}_{y_2(t)=h})\mathbf{1}_{\tau^*>t}\right].$$

Letting $t_2 = 2t/3$ and

$$\begin{aligned} g(z_1,z_2) &:= \mathcal{E}\left[\mathbf{1}_{y_1(t)=h} + \mathbf{1}_{y_2(t)=h} \,\middle|\, y_1(t_2) = z_1, y_2(t_2) = z_2\right], \\ h(z_1,z_2) &:= \mathcal{P}\left[\tau^* > t_2; \,\middle|\, y_1(t_1) = z_1, y_2(t_1) = z_2\right], \end{aligned}$$

the l.h.s. of (65) is bounded by:

$$\mathcal{E}\Big[g(y_1(t_2), y_2(t_2)) h(y_1(t_1), y_2(t_1)) \mathbf{1}_{\tau > t_1}\Big]$$

and (65) follows after recalling that $y_1(t_1) - y_2(t_1) = 1$ if $\tau > t_1$. □

Lemma 6.2 *There is a constant c so that for any $h = 0, \dots, K-1$, any N and any t*

$$\left| p_t^{(N)}(N-K, N-h) - p_t^{(N)}(N-K-1, N-h) \right| \le \frac{c}{1+t^{3/2}}. \tag{67}$$

Moreover for any $t \le N$ and $\bar{c}$ suitable positive constant,

$$|p_t^{(N)}(x, N-h) - p_t(N-x, h)| \le c e^{-\bar{c}N}, \quad x = N-K, N-K-1. \tag{68}$$

Proof The same argument used in the proof of Lemma 6.1 proves (67). Details are omitted. As for (68), just notice that $p_t(N-x,h)$ is the probability for the random walk on $\{y \in \mathbb{Z} \colon y \le N\}$ reflected at N and starting at x at time 0 to be at $N-h$ at time t, while $p_t^{(N)}(x, N-h)$ refers to the walk that is also reflected at $-N$. Letting the two walks move together before reaching $-N$, the difference on the l.h.s. is bounded from above by the probability of reaching $-N$ by time N, and the estimate follows at once by very simple exponential bound on the Poisson clock process (or still using (23)). □

Lemma 6.3 *For any $y \in I_-$*

$$\lim_{\epsilon \to 0} \Gamma^-_{\epsilon,t,y} = 0, \quad y \in I_-. \tag{69}$$

Proof By (64)

$$|\Gamma_{\epsilon,t,y}| \le c \int_0^{\epsilon^{-2}t} |p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)| ds. \tag{70}$$

We bound the probability difference by coupling the two random walks as in the beginning of the proof of Lemma 6.1, namely the random walk $y_1(s)$ starting at $N-K$ copies the jumps of $y_2(s)$, the one starting at $N-K-1$. Calling τ_N the first hitting time of N by y_2, the two random walks become identical after τ_N. Let τ_- be the first hitting time of $-N+K$ by y_2. Thus the contribution to (70) comes from the event $\tau_- < \tau_N$. Calling $\mathcal{P}$ the law of the above coupling, $\mathcal{E}$ its expectation and $\mathcal{F}_{\tau_-}$ the canonical σ-algebra, we have

$$|\Gamma_{\epsilon,t,y}| \le c \int_0^{\epsilon^{-2}t} \mathcal{E}\left[\mathbf{1}_{\tau_- < s}\mathbf{1}_{\tau_N > \tau_-}\left|\mathcal{E}\left[\mathbf{1}_{y_1(s)=y} - \mathbf{1}_{y_2(s)=y}|\mathcal{F}_{\tau_-}\right]\right|\right] ds. \tag{71}$$

Since $\tau_N > \tau_-$, $y_1(\tau_-) = -N+K+1$ and $y_2(\tau_-) = -N+K$, the above conditional expectation can be bounded using (67) (changing x to $-x$). Thus

$$|\Gamma_{\epsilon,t,y}| \le c \int_0^{\epsilon^{-2}t} \mathcal{E}\left[\mathbf{1}_{\tau_- < s}\mathbf{1}_{\tau_N > \tau_-}\frac{c}{1+(s-\tau_-)^{3/2}}\right] ds. \tag{72}$$

The r.h.s. of (72) involves only the random walk y_2, and due to the initial conditions we are considering (K is fixed), $\mathcal{P}(\tau_- < \tau_N) \le \tilde{c}\epsilon$ for a positive constant $\tilde{c}$. Calling $m(dt)$ the law of τ_- conditioned to $\tau_- < \tau_N$, we may write for $y \in I_-$

$$\begin{aligned} |\Gamma_{\epsilon,t,y}| &\le c'\epsilon \int_0^{\epsilon^{-2}t} ds \int_{(0,s]} m(du) \frac{1}{1+(s-v)^{3/2}} \\ &\le c'\epsilon \int_{(0,\epsilon^{-2}t]} m(du) \int_u^{\epsilon^{-2}t} \frac{1}{1+(s-u)^{3/2}} ds \le c''\epsilon, \end{aligned}$$

proving the lemma. □

Proof of (59) We split the integral in (64) into $s \le \epsilon^{-c^*}$ and $s > \epsilon^{-c^*}$, where c^* is as in Theorem 4 (assuming without any loss that $c^* < 2$). For the second one we use (67) to see that

$$\left|\int_{\epsilon^{-c^*}}^{\epsilon^{-2}t} ds \left(p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)\right) \mathbb{E}_\epsilon\left((D_\pm \eta(\cdot, t-\epsilon^2 s))(y)\right)\right| \le C\epsilon^{c^*/2}.$$

The same estimates hold for $\Gamma^*_{\epsilon,t,y}$ so that using Theorem 4 we get

$$|\Gamma_{\epsilon,t,y} - \Gamma^*_{\epsilon,t,y}| \le c\epsilon^{(2-\beta^*)c^*} \int_0^{\epsilon^{-c^*}} |p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)| ds + 2C\epsilon^{c^*/2},$$

hence (59). □

Proof of (60) Again by Lemma 6.2 we have

$$\Big|\Gamma^*_{\epsilon,t,y} - \int_0^{\epsilon^{-1}} \big(p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)\big)\frac{j}{2}(D_+\rho_\epsilon(\cdot,t-\epsilon^2 s))(y)ds\Big|$$
$$\le c\sqrt{\epsilon}.$$

We replace $(D_+\rho_\epsilon(\cdot,t-\epsilon^2 s))(y)$ by $(D_+\rho_\epsilon(\cdot,t))(y)$, the error being bounded by $c\sqrt{\frac{\epsilon}{t}}$, by (33). Using again Lemma 6.2 we obtain

$$\Big|\Gamma^*_{\epsilon,t,y} - \frac{j}{2}(D_+\rho_\epsilon(\cdot,t))(y)\int_0^{\infty} \big(p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)\big)ds\Big|$$
$$\le c'\left(\sqrt{\frac{\epsilon}{t}} + \sqrt{\epsilon}\right).$$

By (7),

$$\lim_{\epsilon\to 0}(D_+\rho_\epsilon(\cdot,t))(y) = (1-\rho(1,t))\rho(1,t)^{N-y}, \quad y\in I_+$$

which, by (66), proves (60). □

We are left with the final step, namely to recognize that the right hand side of (60) is equal to $1-\rho(1,t)^K$. We use conservation of mass, namely from (5) it follows that

$$\begin{aligned} &2\epsilon\left(\sum_{x=N-K}^{N}\rho_\epsilon(x,t+\tau) - \sum_{x=N-K}^{N}\rho_\epsilon(x,t)\right) \\ &\quad = \int_t^{t+\tau}\left(-\frac{1}{2}J_+^{(\epsilon)}(s) + \sum_{y\in I_+}\frac{j}{2}D_+\rho_\epsilon(y,s)\right)ds \end{aligned} \tag{73}$$

with $J_+^{(\epsilon)}(s)$ the analogue of $\Theta_+^{(\epsilon)}(s)$, namely

$$J_+^{(\epsilon)}(t) := \epsilon^{-1}\left(\rho_\epsilon(N-K,t) - \rho_\epsilon(N-K-1,t)\right).$$

Then, analogously to (56)

$$J_+^{(\epsilon)}(t) = \phi_\epsilon(N-K,t) + \sum_{y\in I_+}\Gamma^*_{\epsilon,t,y} - \sum_{y\in I_-}\Gamma^*_{\epsilon,t,y}.$$

The same arguments used for $\Gamma_{\epsilon,t,y}$, for $y\in I_-$ show that $\lim_{\epsilon\to 0}\sum_{y\in I_-}\Gamma^*_{\epsilon,t,y} = 0$, so that using (60) we get from (73) in the limit $\epsilon\to 0$

$$0 = \int_t^{t+\tau}\frac{j}{2}\sum_{h=0}^{K-1}\left(-\frac{1}{2}a(h)(1-\rho(1,s))\rho(1,s)^h + (1-\rho(1,s))\rho(1,s)^h ds\right)ds \tag{74}$$

which by the continuity in t gives for any $t>0$

$$\frac{1}{2}\sum_{h=0}^{K-1}a(h)(1-\rho(1,t))\rho(1,t)^h = \sum_{h=0}^{K-1}(1-\rho(1,t))\rho(1,t)^h = 1-\rho(1,t)^K. \tag{75}$$

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