

Truncated correlations in the stirring process with births and deaths

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Abstract

We consider the stirring process in the interval $\Lambda_N := [-N, N]$ of \mathbb{Z} with births and deaths taking place in the intervals $I_+ := (N - K, N]$, and respectively $I_- := [-N, -N + K)$, $1 \leq K < N$. We prove bounds on the truncated moments uniform in N which yield strong factorization properties.

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1 Introduction

This paper is part of a wide and long standing effort for the understanding of basic questions in non-equilibrium statistical mechanics, with a rigorous derivation of typical macroscopic behavior and thermodynamical properties. A common feature to many such initiatives consists in the use of markovian interacting particle systems to mimic the time evolution of physical systems at the microscopic level (see e.g. [6, 14, 15]). With the model studied here we address the problem of mass transport in a diffusive system under the action of external driving forces that act at the boundaries of the system. This has been the object of many articles, and substantial progress has been made in the description of non-equilibrium thermodynamics in terms of the macroscopic behavior of the particle system, as examined in [2, 3, 4, 9, 11], and references therein. These external forces at the boundaries might be interpreted and modeled in terms of reservoirs, and all the above quoted references consider systems where this is achieved as “particle reservoirs”, that is, one tries to fix different densities on two “sides” of the boundary (say right/left side, in a one-dimensional case).

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We want to understand a different (and very natural) type of reservoirs: instead of adding and subtracting mass at the boundaries in such a way to keep the above mentioned densities, we would like to consider what might be called "current reservoirs", i.e. we try to send in particles from one side of our volume at suitable rate (this ought to be inversely proportional to the volume, to match Fick's law) and to take them out on the other side at the same rate. This should produce a current field throughout the system, and the description of the steady-state is one of our main goals. A first step towards this program consists in the derivation of the macroscopic equation for the evolution of the density field. The approach brings in different challenges in comparison with the "particle reservoir" case, and we start from the simplest case: bulk dynamics given by a symmetric simple exclusion, one dimensional case. In this paper we establish the propagation of chaos of the process subject to "current reservoirs". This is the fundamental tool for our derivation of its hydrodynamical equation in the companion paper [8], where the reader finds a more detailed discussion of the physical motivation and related problems.

The model. Let $K, N \in \mathbb{N}$ with $K \leq N$. (K is fixed and N will be sent to infinity.) We consider a system of particles in the interval $\Lambda_N := [-N, N]$ of \mathbb{Z} . Particle configurations are elements η of $\{0, 1\}^{\Lambda_N}$, $\eta(x) = 0, 1$ being the occupation number at $x \in \Lambda_N$. The evolution is a Markov process defined by the generator

$$L_\epsilon := \epsilon^{-2}(L_0 + \epsilon L_b), \quad \epsilon \equiv \frac{1}{N}, \quad (1.1)$$

where L_0 is the generator of the stirring process in Λ_N , namely

$$L_0 f(\eta) := \frac{1}{2} \sum_{x=-N}^{N-1} [f(\eta^{(x,x+1)}) - f(\eta)] \quad (1.2)$$

with $\eta^{(x,x+1)}$ the configuration obtained from η by interchanging its values in x and $x+1$, namely $\eta^{(x,x+1)}(z) = \eta(z)$ if $z \notin \{x, x+1\}$ and $\eta^{(x,x+1)}(x) = \eta(x+1)$, $\eta^{(x,x+1)}(x+1) = \eta(x)$. Moreover, calling $\eta^{(x)}$ the configuration obtained from η by changing its value in x , namely $\eta^{(x)}(z) = \eta(z)$ if $z \neq x$ and $\eta^{(x)}(x) = 0, 1$ if $\eta(x) = 1, 0$, then

$$L_b = L_{b,+} + L_{b,-}, \quad L_{b,\pm} f(\eta) := \frac{j}{2} \sum_{x \in I_\pm} D_\pm \eta(x) [f(\eta^{(x)}) - f(\eta)], \quad (1.3)$$

with $j > 0$, $I_+ = [N - K + 1, N]$, $I_- = [-N, -N + K - 1]$,

$$D_+ u(x) = [1 - u(x)]u(x+1)u(x+2) \cdots u(N), \quad x \in I_+,$$

$$D_- u(x) = u(x)[1 - u(x-1)][1 - u(x-2)] \cdots [1 - u(-N)], \quad x \in I_-.$$

The generator $N^2 L_0$ describes a process where nearest neighbor sites exchange their content at rate $N^2/2$, the factor N^2 ensuring that information propagates to the whole Λ_N in time of order one, with positive probability uniformly in N . $N L_{b,+}$ describes a birth process: at rate $jN/2$ a particle is created in the first (starting from N) empty site in I_+ , if no site is empty the birth is aborted. Symmetrically $N L_{b,-}$ describes deaths: at rate $jN/2$ the first particle (starting from $-N$) in I_- is removed, if I_- is empty there is no death. Our results will show that the factor N is the correct one to match the stirring rate.

As explained before, the basic question to address concerns the actual particle flux as a function of the external parameter j which rules the birth-death mechanisms in

I_{\pm} . Moreover, how does the system relax from some initial state? Being the system a Döblin chain, there is a unique invariant measure and convergence is exponentially fast, but we look for bounds and estimates uniform in N as $N \rightarrow \infty$.

In this paper we shall study the evolution till times which grow like $\tau \log N$, $\tau > 0$ suitably small and shall prove strong factorization properties for the evolution starting from any single configuration; we prove rather sharp bounds on the truncated correlation functions. Together with the results in [8], this yields the convergence of empirical averages to a limit macroscopic equation; see the next section for details. Our project for the future is to prove (i) that the time-flow defined by the limit macroscopic equation leads to a unique stationary profile as $t \rightarrow \infty$; (ii) that the unique stationary measure of the process when N is finite is in the limit $N \rightarrow \infty$ supported by the macroscopic stationary profile. The strong factorization properties proved here seem to be the main ingredients for this program to be fulfilled, but the full analysis of the limit equation has not yet been done.

Besides the natural physical interpretation described above, one might view the system as a model in queuing theory. “Files” are sent in to the first available “server” in I_+ , they are elaborated and then finally sent out from a server in I_- . While the “public” only sees how many files enter and come out, what really happens to the files is just that they go from one server to the other back and forth (as described by the stirring process) in a random fashion. This pessimistic view of the action of the servers is unfortunately not too unrealistic as experienced by some of the authors. In both cases the relevant questions are: how efficient is the system?

A few comments. With the techniques developed in this paper, one might expect to treat also higher dimensional situations, though the technical complications for the analysis pursued here increase considerably. On the other hand, these techniques are restricted to systems with propagation of chaos (and the duality of the bulk dynamics plays a crucial role). It is very interesting trying to use instead Varadhan’s techniques (see [12] and [16]) to handle situations where this is not the case.

In a forthcoming paper we prove that in the limit $N \rightarrow \infty$ the unique stationary measure is supported by a linear macroscopic profile and that this is the unique stationary solution of the hydrodynamic equation. It would also be important to address the large deviations of the stationary measure, and to see whether Derrida’s techniques (see e.g. [9]) can be applied here.

2 Main result

We shall study the process described in the introduction starting from an arbitrary initial configuration η , denoting by \mathbb{P}_{ϵ} its law and \mathbb{E}_{ϵ} the corresponding expectation. We shall not make explicit the dependence on η unless ambiguities may arise. We are interested in the expectations $\mathbb{E}_{\epsilon}[\eta(x, t)]$ and in the truncated correlation functions $\mathbb{E}_{\epsilon}\left[\prod_{i=1}^n \tilde{\eta}(x_i, t)\right]$, $\tilde{\eta}(x, t) := \eta(x, t) - \mathbb{E}_{\epsilon}[\eta(x, t)]$ (it is actually more convenient to study a slightly different expression as defined in (2.3) below). Define first $\rho_{\epsilon}(x, t)$ as the

solution of

$$\begin{aligned} \frac{d}{dt} \rho_\epsilon(x, t) &= \frac{1}{2} \Delta_\epsilon \rho_\epsilon(x, t) + \epsilon^{-1} \frac{j}{2} \left(\mathbf{1}_{x \in I_+} D_+ \rho_\epsilon(x, t) - \mathbf{1}_{x \in I_-} D_- \rho_\epsilon(x, t) \right) \\ \rho_\epsilon(x, 0) &= \mu^\epsilon[\eta(x, 0) = 1] \end{aligned} \quad (2.1)$$

where $\Delta_\epsilon = \epsilon^{-2} \Delta$, Δ the discrete Laplacian in Λ_N with reflecting boundary conditions:

$$\begin{aligned} \Delta u(x) &= u(x+1) + u(x-1) - 2u(x), \quad |x| < \epsilon^{-1} \\ \Delta u(\pm N) &= u(\pm(N-1), t) - u(\pm N, t) \end{aligned} \quad (2.2)$$

and μ^ϵ is a product measure [in particular it may be supported by a single configuration]. Global existence and uniqueness for (2.1) are proved in [8] where it is also shown that the solution has values in $[0, 1]$. Writing $\Lambda_N^{n, \neq}$, $n \geq 1$, for the set of all sequences $\underline{x} = (x_1, \dots, x_n)$ in Λ_N^n such that $x_i \neq x_j$, we then define the v -functions

$$v^\epsilon(\underline{x}, t | \mu^\epsilon) := \mathbb{E}_\epsilon \left[\prod_{i=1}^n \{ \eta(x_i, t) - \rho_\epsilon(x_i, t) \} \right], \quad \underline{x} \in \Lambda_N^{n, \neq}, \quad n \geq 1 \quad (2.3)$$

where the process starts with a product measure μ^ϵ and $\rho_\epsilon(x, t)$ is the solution of (2.1). In the particular case when μ^ϵ is the point mass at a configuration η , we write $v^\epsilon(\underline{x}, t | \eta)$ instead.

The main result in this paper is a bound on the v -functions:

Theorem 2.1. *There exist $\tau > 0$ and $c^* > 0$ so that the following holds. For any $\beta^* > 0$ and for any positive integer n there is a constant $c_n < \infty$ so that for any $\epsilon > 0$, any initial product measure μ^ϵ*

$$\sup_{\underline{x} \in \Lambda_N^{n, \neq}} |v^\epsilon(\underline{x}, t | \mu^\epsilon)| \leq \begin{cases} c_n (\epsilon^{-2t})^{-c^* n}, & t \leq \epsilon^{\beta^*} \\ c_n \epsilon^{(2-\beta^*)c^* n} & \epsilon^{\beta^*} \leq t \leq \tau \log \epsilon^{-1}. \end{cases} \quad (2.4)$$

Theorem 2.1 proves that $\sup_x |\mathbb{E}_\epsilon[\eta(x, t)] - \rho_\epsilon(x, t)| \rightarrow 0$ as $N \rightarrow \infty$ and that the “empirical averages” behave as ρ_ϵ in the following sense. Call $J_N(x) = [x - N^a, x + N^a] \cap \Lambda_N$, for $a \in (0, 1)$, and let $|A|$ denote the cardinality of $A \subset \mathbb{Z}$. Then by the Chebyshev inequality,

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \sup_{\epsilon^{\beta^*} \leq t \leq \tau \log \epsilon^{-1}} \sup_{\eta} \mathbb{P}_\epsilon \left[\sup_{x \in \Lambda_N} \frac{1}{|J_N(x)|} \sum_{y \in J_N(x)} \{ \eta(y, t) - \rho_\epsilon(y, t) \} \geq \delta \right] = 0. \quad (2.5)$$

Thus, under suitable assumptions on the initial configurations, it follows that the above empirical averages converge to the solution of the hydrodynamic equation for the system, as we are going to explain. Suppose that the initial configurations $\rho_\epsilon(x, 0)$ (see the second line in (2.1)) are such that for some smooth function $u_0(r)$, $r \in [-1, 1]$,

$$\lim_{\epsilon \rightarrow 0} \sup_{x \in \Lambda_N} \left| \frac{1}{|J_N(x)|} \sum_{y \in J_N(x)} \{ \rho_\epsilon(y, 0) - u_0(\epsilon y) \} \right| = 0. \quad (2.6)$$

In [8] it has been proved that under such an assumption for any $t > 0$,

$$\lim_{\epsilon \rightarrow 0} \sup_{x \in \Lambda_N} |\rho_\epsilon(x, t) - \rho(\epsilon x, t)| = 0 \quad (2.7)$$

with $\rho(r, t)$ the unique solution of the limit hydrodynamic equation, namely the heat equation with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial}{\partial t} \rho(r, t) &= \frac{1}{2} \frac{\partial^2}{\partial r^2} \rho(r, t), & r \in (-1, 1), t > 0 \\ \rho(r, 0) &= u_0(r), & \rho(\pm 1, t) = u_{\pm}(t). \end{aligned} \quad (2.8)$$

However the boundary conditions $u_{\pm}(t)$ are not a-priori known, they must be obtained by solving a nonlinear system of two integral equations:

$$\begin{aligned} u_{\pm}(t) &= \int_0^t \{p(s)f_{\pm}(u_{\pm}(t-s)) - q(s)f_{\mp}(u_{\mp}(t-s))\} ds + w_{\pm,t} \\ f_{+}(u) &= \frac{j}{2} (1 - u^K), & f_{-}(u) = \frac{j}{2} (1 - (1 - u)^K), \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} p(t) &= 2 \sum_{k \in \mathbb{Z}} G_t(4k), & q(t) &= 2 \sum_{k \in \mathbb{Z}} G_t(4k + 2), & G_t(r) &= \frac{e^{-r^2/2t}}{\sqrt{2\pi t}} \\ w_{+,t} &= \sum_{k \in \mathbb{Z}} \int_{-1}^1 u_0(r') 2G_t(1 - r' + 4k) dr', & w_{-,t} &= \sum_{k \in \mathbb{Z}} \int_{-1}^1 u_0(r') 2G_t(r' + 1 + 4k) dr'. \end{aligned} \quad (2.10)$$

Theorem 2.1 then shows that (2.5) holds with $\rho_{\epsilon}(y, t)$ replaced by $\rho(\epsilon y, t)$ and for t in a compact interval.

Scheme of proofs. In Section 3 we shall state some mostly elementary properties of simple random walks on Λ_N which have been proved or recalled in [8]. In Section 4 we shall prove sharp probability estimates on the stirring process, which extend analogous estimates proved in [6] for the process on the whole \mathbb{Z} . The reflections at $\pm N$ make the extension not trivial at all. In the remaining sections we write an integral equation for some truncated correlation functions called “the v -functions” that we study by iteration. For further results and applications of v -functions see e.g. [5, 7, 10]. The terms which arise are interpreted as a branching process with stirring evolution between the branching events. The main point will be to prove that typically branching events are well separated in time so that the stirring has time to mix up things in the proper way; here we use extensively the estimates in Section 4.

3 A single random walk with reflections

In this section we state some properties of a single random walk in Λ_N with reflections at $\pm N$, referring to [8] for the proofs.

Let $Q_t^{(\epsilon)}(x, y)$ denote the transition probability of a simple random walk in \mathbb{Z} with jumps of intensity $\epsilon^{-2}/2$ between nearest neighbor (*n.n.*) sites, and let $P_t^{(\epsilon)}(x, y)$ denote the transition probability of a corresponding random walk on Λ_N , with the jumps outside Λ_N being suppressed (and the same jump rates within Λ_N). $P_t^{(\epsilon)}$ and $Q_t^{(\epsilon)}$ are thus related via the “reflection map” $\psi_N : \mathbb{Z} \rightarrow \Lambda_N$ defined as follows.

- $|x| \leq N$: $\psi_N(x) = x$.

- $x < -N$: $\psi_N(x) = -\psi_N(-x)$;
- $x > N$: $\psi_N(N + j(2N + 1) + k) = \psi_N(N + j(2N + 1) - (k - 1))$, $k = 1, \dots, 2N + 1$, $j = 0, 1, \dots$

Then

$$P_t^{(\epsilon)}(x, z) = \sum_{y: \psi_N(y)=z} Q_t^{(\epsilon)}(x, y). \quad (3.1)$$

Let

$$G_t(x, y) = \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}. \quad (3.2)$$

By the local central limit theorem, [13], there exist positive constants c_1, \dots, c_5 so that

$$|Q_t^{(\epsilon)}(x, y) - G_{\epsilon^{-2}t}(x, y)| \leq \frac{c_1}{\sqrt{\epsilon^{-2}t}} G_{\epsilon^{-2}t}(x, y), \quad |x - y| \leq (\epsilon^{-2}t)^{5/8}, \quad (3.3)$$

while for $|x - y| > (\epsilon^{-2}t)^{5/8}$

$$Q_t^{(\epsilon)}(x, y) \leq \min \left\{ c_2 e^{-c_3|x-y|^2/(\epsilon^{-2}t)}, c_4 e^{-|y-x|(\log|y-x|-c_5)} \right\}. \quad (3.4)$$

As a corollary, for any $T > 0$ there exists c so that the following holds:

- For all ϵ , all $t \in (0, T]$ and all x, y in Λ_N

$$P_t^{(\epsilon)}(x, y) \leq c G_{\epsilon^{-2}t}(x, y). \quad (3.5)$$

- For all ϵ , all $t \in (0, T]$ and all $-N \leq x \leq N - 1$,

$$\left| P_t^{(\epsilon)}(x, y) - P_t^{(\epsilon)}(x + 1, y) \right| \leq \frac{c}{\sqrt{\epsilon^{-2}t}} G_{\epsilon^{-2}t}(x, y). \quad (3.6)$$

In Proposition 5.1 of [8] it is proved that for any $S > 0$ there is a constant c so that for any solution $\rho_\epsilon(x, t)$ of (2.1) with $\rho_\epsilon(\cdot, 0) \in [0, 1]$ the following holds. For any $x \in [-N, N - 1]$, any $t \in (0, S]$ and any $\epsilon > 0$

$$|\rho_\epsilon(x, t) - \rho_\epsilon(x + 1, t)| \leq \min \left\{ 1, c \left(\epsilon \log_+(\epsilon^{-2}t) + \frac{1}{\sqrt{\epsilon^{-2}t}} \right) \right\} \quad (3.7)$$

where $\log_+ u = \max\{\log u, 1\}$. By the arbitrariness of the initial datum we can iterate the bound in (3.7), and from the previous result with $S = 1$ we get that for all $x \in [-N, N - 1]$ and all $t > 0$:

$$|\rho_\epsilon(x, t) - \rho_\epsilon(x + 1, t)| \leq \min \left\{ 1, c \left(\epsilon \log_+(\epsilon^{-2}[t]_1) + \frac{1}{\sqrt{\epsilon^{-2}[t]_1}} \right) \right\} \quad (3.8)$$

where $[t]_1 = t$ if $t \leq 1$ and $= 1$ otherwise. We shall use a weaker version of (3.8), namely that for any $\zeta > 0$ and $\tau > 0$ there is c so that

$$\sup_{x, y \in \Lambda_N: |x-y| \leq 1} |\rho_\epsilon(x, t) - \rho_\epsilon(y, t)| \leq \frac{c}{(\epsilon^{-2}t)^{1/2-\zeta} + 1} \quad \text{for any } t \leq \tau \log \epsilon^{-1}. \quad (3.9)$$

4 Probability estimates for the stirring process.

In this section we shall study the stirring process. The generator L_0 of the stirring process defined in (1.2) can be interpreted by saying that after independent exponential times of rate $1/2$ sites x and $x + 1$ exchange their content. This leads to the following realization of the process which will be extensively used in the sequel.

Definition 1. [*The active/passive marks process*]

- The active/passive marks process is realized in a probability space denoted by $(\Omega, \mathbf{P}_\epsilon)$. It is a product of Poisson processes indexed by $\{x, x + 1\}$, $x \in \mathbb{Z}$: for each pair $\{x, x + 1\}$ we have a Poisson point process of intensity ϵ^{-2} , its events are called “marks” and each mark is independently given the attribute “active” or “passive” with probability $1/2$. The processes relative to different pairs are mutually independent and their common law is \mathbf{P}_ϵ , its expectation being also denoted by \mathbf{E}_ϵ .

- For any $\omega \in \Omega$ we define the following particle evolution in Λ_N : a particle at $x \in \Lambda_N$ moves as soon as an active mark appears at a pair x, y with $y \in \Lambda_N$. The particle then moves to y (note that if another particle was at y then it would simultaneously move to x). Passive marks do not play any role so far as well as all marks $\{x, x + 1\}$ with either x or $x + 1$ not in Λ_N ; they will be used later to construct couplings. We denote by $X(t) \subset \Lambda_N$ the set of all sites occupied by the particles at time t , so that $\eta(x, t) = 1$ iff $x \in X(t)$.

Proposition 4.1. $(X(t))_{t \geq 0}$ has the law of the stirring process with generator $\epsilon^{-2}L_0$.

The proof of the proposition is elementary and omitted. The evolution defined in the active/passive marks process can clearly be inverted: given ω and $X(t)$ by following backwards the marks we uniquely determine the initial position $X(0)$. This remark leads to the proof of the well known proposition, see for instance [14]:

Proposition 4.2 (Duality). For any $X \subset \Lambda_N$, $\eta_0 \in \{0, 1\}^{\Lambda_N}$ and $t > 0$,

$$\mathbf{E}_\epsilon \left[\prod_{x \in X} \eta(x, t) \mid \eta(\cdot, 0) = \eta_0 \right] = \mathbf{E}_\epsilon \left[\prod_{x \in X(t)} \eta_0(x) \mid X(0) = X \right]. \quad (4.1)$$

The importance of Proposition 4.2 is that it allows to compute the probability that the sites in X are all occupied at time t by studying the stirring process with “only” $|X|$ particles (no matter what the initial configuration η_0 is). The realization of the process in terms of active/passive marks allows us to identify particles and hence to label them:

Definition 2. [*The labeled process*]

Given $\omega \in \Omega$, we can follow unambiguously the motion of each individual particle so that we can give them labels at time 0 which then remain attached to the particles during their motion. We shall denote by $\underline{x} = (x_{i_1}, \dots, x_{i_n})$ a labeled configuration of n particles, (i_1, \dots, i_n) the labels, x_{i_j} the positions; configurations obtained under permutations of the labels are now considered distinct. We write $\underline{x}(t)$ for the labeled process induced by the active/passive marks and denote by $X(t)$ the unlabeled configuration obtained from $\underline{x}(t)$, (i.e. only the positions in $\underline{x}(t)$ are recorded by $X(t)$). By an abuse of notation we shall write \mathbf{P}_ϵ and \mathbf{E}_ϵ both for law and expectation in the active/passive marks process and for the marginal over $\underline{x}(t)$, the labeled process realized in this space. By adding a subscript \underline{x} we mean that the initial distribution of particles has support on the single labeled configuration \underline{x} .

The advantage of having defined the process with also passive marks is exemplified in Lemma 4.3 below, where we use the following:

Definition 3. [The variables \mathcal{T}_{x_1, x_2} , τ_{x_1, x_2} and $N_{x_1, x_2, t}$]

Given the initial position x_1 and x_2 of particles 1 and 2 define in Ω the random multi-interval $\mathcal{T}_{x_1, x_2} = \{s \geq 0 : |x_1(s) - x_2(s)| = 1\}$, calling $\mathcal{T}_{x_1, x_2, t} := \mathcal{T}_{x_1, x_2} \cap [0, t]$ and $I_{x_1, x_2, t} = \{(s, y_1, y_2) : s \in \mathcal{T}_{x_1, x_2, t}, y_i = x_i(s) \text{ and a mark appears at } s \text{ between } y_1 \text{ and } y_2\}$, we define the stopping time τ_{x_1, x_2} as the smallest s in $I_{x_1, x_2, \infty}$ and $N_{x_1, x_2, t}$ the total number of elements in $I_{x_1, x_2, t}$. Notice that the presence of other particles does not affect the values of \mathcal{T}_{x_1, x_2} , τ_{x_1, x_2} and $N_{x_1, x_2, t}$.

Lemma 4.3. Let $\underline{x} = (x_1, \dots, x_n)$, $t > s > 0$ and $f(y_1, \dots, y_n)$ a function antisymmetric under the exchange of y_1 and y_2 . Then

$$\mathbf{E}_{\epsilon, \underline{x}} \left[\mathbf{1}_{\tau_{x_1, x_2} \leq s} f(\underline{x}(t)) \right] = 0, \quad (4.2)$$

where the suffix \underline{x} indicates the initial condition i.e. $\mathbf{E}_{\epsilon, \underline{x}}(\cdot) = \mathbf{E}_{\epsilon}[\cdot | \underline{x}(0) = \underline{x}]$.

Proof. Given x_1 , x_2 and $s > 0$ call two elements ω and ω' in Ω "similar" if all the marks at all pairs $x, x+1$ occur at the same time in ω and ω' ; their active/passive attribute must also be the same in both except for the marks in $I_{x_1, x_2, s}$ (see Definition 3): a mark at the times and between the pairs indicated by $I_{x_1, x_2, s}$ may be active in one sample and passive in the other, or the same in both. If ω and ω' are similar then the configurations evolved from the same \underline{x} with ω and ω' are at all times the same except at most for an exchange of the particles with label 1 and 2. The above similarity relation is an equivalence. $N_{x_1, x_2, s}$ is constant in all ω in an equivalence class, so that if $N_{x_1, x_2, s} = p$ there are 2^p elements in the equivalence class. Each element is characterized by the active/passive attribute of the p marks in $I_{x_1, x_2, s}$ and their distribution, conditioned on a given equivalence class, is that of independent symmetric Bernoulli random variables. The set $\{\tau_{x_1, x_2} \leq s\}$ is the union of all such equivalence classes with $p \geq 1$ so that the law of $\underline{x}(t)$ conditioned on $\{\tau_{x_1, x_2} \leq s\}$ is symmetric under exchanging $x_1(t)$ with $x_2(t)$ and (4.2) follows. \square

We shall complement Lemma 4.3 by proving in (4.3) below that the probability that $\tau_{x_1, x_2} \leq s$ goes to 1 as $s \rightarrow \infty$ if $|x_1 - x_2| = 1$, thus establishing exchangeability properties of the stirring process. Since the stirring particles move like independent random walks when at distance larger than 1, the crucial ingredients to construct good couplings between the two processes (as in [10, 6]) are a-priori estimates on the tails of the variables τ_{x_1, x_2} and $N_{x_1, x_2, t}$. The results of [6] cannot be directly applied here because they heavily used that the particles were moving on the whole line, and our process is in the bounded interval Λ_N . Some estimates are helped by being in a bounded domain but in others the inequality goes in the wrong direction and we have extra work to do.

Theorem 4.4. There is c so that for all $\epsilon > 0$

$$\sup_{|x_1 - x_2| = 1} \mathbf{P}_{\epsilon} [\tau_{x_1, x_2} \geq t] \leq \frac{c}{(\epsilon^{-2}t)^{1/2} + 1}. \quad (4.3)$$

Moreover, given any $T > 0$, for any $\zeta > 0$ and $k \geq 1$ there is c so that for all $t \leq T$ and for all $\epsilon > 0$

$$\sup_{x_1, x_2} \mathbf{P}_\epsilon [N_{x_1, x_2, t} \geq (\epsilon^{-2}t)^{1/2+\zeta}] \leq c(\epsilon^{-2}t)^{-k}. \quad (4.4)$$

Proof. The proof of (4.3) will follow by bounding $\mathbf{P}_\epsilon [\tau_{x_1, x_2} \geq t]$ in terms of the probability of the return time to the origin of a single random walk on \mathbb{Z} . Given a realization ω of the active/passive marks process, call $x_i(t)$, $i = 1, 2$, the positions of the particles evolved from x_1 and x_2 by considering only the marks in Λ_N (as in Definition 3 above); denote instead by $x_i^\pm(t)$, $i = 1, 2$, the positions obtained by using all the marks in \mathbb{Z} (thus $x_i^\pm(t)$ are two stirring walks on the whole \mathbb{Z} starting from x_1 and x_2). Suppose without loss of generality that $x_1 < x_2$ then for all $t \leq \tau_{x_1, x_2}$ we have $x_1^\pm(t) \leq x_1(t) < x_2(t) \leq x_2^\pm(t)$ so that $\mathbf{P}_\epsilon [\tau_{x_1, x_2} \geq t]$ is bounded from above by the probability that t is smaller than the first time s when two independent random walks on \mathbb{Z} starting from x_1 and x_2 are on a same site. Thus (4.3) follows from classical estimates on return times of random walks.

The l.h.s. of (4.4) is bounded by

$$\mathbf{E}_\epsilon \left[\mathbf{P}_\epsilon \left[N_{x_1, x_2, t} \geq (\epsilon^{-2}t)^{1/2+\zeta} \mid |\mathcal{T}_{x_1, x_2, t}| < \epsilon^2(\epsilon^{-2}t)^{1/2+\zeta'} \right] \right] + \mathbf{P}_\epsilon \left[|\mathcal{T}_{x_1, x_2, t}| \geq \epsilon^2(\epsilon^{-2}t)^{1/2+\zeta'} \right].$$

Let $p_L(n) = e^{-L} \frac{L^n}{n!}$ the Poisson distribution with parameter L , then

$$\mathbf{P}_\epsilon \left[N_{x_1, x_2, t} \geq (\epsilon^{-2}t)^{1/2+\zeta} \mid |\mathcal{T}_{x_1, x_2, t}| = L \right] = \sum_{n \geq (\epsilon^{-2}t)^{1/2+\zeta}} p_{\epsilon^{-2}L}(n)$$

Since the r.h.s. is an increasing function of L ,

$$\begin{aligned} \mathbf{E}_\epsilon \left[\mathbf{P}_\epsilon \left[N_{x_1, x_2, t} \geq (\epsilon^{-2}t)^{1/2+\zeta} \mid |\mathcal{T}_{x_1, x_2, t}| < \epsilon^2(\epsilon^{-2}t)^{1/2+\zeta'} \right] \right] \\ \leq \sum_{n \geq (\epsilon^{-2}t)^{1/2+\zeta}} p_{(\epsilon^{-2}t)^{1/2+\zeta'}}(n) \end{aligned}$$

which, for any k , is at most $c(\epsilon^{-2}t)^{-k}$ if $\zeta' = \zeta/2$ and c is a suitable constant (dependent on k and ζ). It will therefore suffice to prove that for any $\zeta > 0$ and any k there is c so that

$$\sup_{x_1, x_2} \mathbf{P}_{\epsilon, x_1, x_2} \left[\int_0^t \mathbf{1}_{|x_1(s) - x_2(s)|=1} ds \geq \epsilon^2(\epsilon^{-2}t)^{1/2+\zeta} \right] \leq c(\epsilon^{-2}t)^{-k}, \quad (4.5)$$

where as before the suffix x_1, x_2 in $\mathbf{P}_{\epsilon, x_1, x_2}$ recalls that $x_i(0) = x_i$, $i = 1, 2$. By the Chebyshev inequality:

$$\begin{aligned} \text{l.h.s. of (4.5)} &\leq \sup_{x_1, x_2} (\epsilon^{-2}t)^{-(\frac{1}{2}+\zeta)p} p! \\ &\times \int_{\{0 \leq t_1 \leq t_2 \leq \dots \leq t_p \leq t\}} \epsilon^{-2} dt_1 \dots \epsilon^{-2} dt_p \mathbf{E}_{\epsilon, x_1, x_2} \left[\prod_{i=1}^p \mathbf{1}_{|x_1(t_i) - x_2(t_i)|=1} \right]. \end{aligned} \quad (4.6)$$

By choosing p so that $\zeta p > k$ we are left with the proof that there is a constant c so that

$$\sup_{x_1, x_2} \int_{\{0 \leq t_1 \leq t_2 \leq \dots \leq t_p \leq t\}} \epsilon^{-2} dt_1 \dots \epsilon^{-2} dt_p \mathbf{E}_{\epsilon, x_1, x_2} \left[\prod_{i=1}^p \mathbf{1}_{|x_1(t_i) - x_2(t_i)|=1} \right] \leq c(\epsilon^{-2}t)^{p/2}. \quad (4.7)$$

By conditioning successively on the state at the times $t_{p-1}, t_{p-2}, \dots, 0$ we are reduced to prove the bound

$$\sup_{y_1, y_2} \mathbf{P}_{\epsilon, y_1, y_2} \left[|x_1(s) - x_2(s)| = 1 \right] \leq \frac{c}{(\epsilon^{-2}s)^{1/2}} \quad (4.8)$$

where s stands for $t_{i+1} - t_i$. (4.7) follows from (4.8) after some simple computations (details are omitted) and we are left with the proof of (4.8). We use the Liggett's inequality to bound

$$\mathbf{P}_{\epsilon, y_1, y_2} \left[|x_1(s) - x_2(s)| = 1 \right] \leq \sum_x \mathbf{P}_{\epsilon, y_1} \left[x_1(s) \in \{x, x+1\} \right] \mathbf{P}_{\epsilon, y_2} \left[x_2(s) \in \{x, x+1\} \right], \quad (4.9)$$

and (4.8) then follows using (3.3) and (3.4). \square

The above estimates will be used to prove that n stirring particles move like n independent random walks. We construct a coupling between the two processes by realizing both of them in the active/passive mark processes. The definition adapts to the present case the one considered in [6].

Definition 4. [Couplings]

Denoting by \underline{x} and \underline{x}^0 stirring and independent labeled particles, respectively, without loss of generality we suppose that the labels are $1, \dots, n$ and write $\underline{x} = (x_1, \dots, x_n)$, $\underline{x}^0 = (x_1^0, \dots, x_n^0)$. We also assign (arbitrarily) a priority list σ , σ a permutation of $\{1, \dots, n\}$, and say that particle i has priority over particle j if $\sigma(i) < \sigma(j)$. We consider the active/passive marks process realized on the whole \mathbb{Z} and according to Definition 1 in this Section we define $\underline{x}(t)$ by looking only at the marks $\{x, x+1\}$ with both x and $x+1$ in Λ_N . In the same space we define $\underline{x}^0(t) = (x_1^0(t), \dots, x_n^0(t)) \in \Lambda_N^n$ given $\underline{x}(0)$, $\underline{x}^0(0)$ and σ by the following rules (we shall later prove that this is a process of independent random walks in Λ_N). $\underline{x}^0(t)$ is defined by giving for each $i \in \{1, \dots, n\}$ the times $t_{i,r}$, $t_{i,l}$ when $x_i^0(\cdot)$ "tries" to move to the right, respectively to the left, as those jumps which lead out of Λ_N are suppressed. Thus the sequence of all such times determines $\underline{x}^0(\cdot)$ (while the viceversa is not true as we cannot recover from $\underline{x}^0(\cdot)$ the attempts to jump out of Λ_N). While the times $t_{i,r}$, $t_{i,l}$ cannot be recovered from \underline{x}^0 they can be read out of an auxiliary process $\underline{y}(t)$ that we define next.

Definition of $\underline{y}(t)$. Let $t > 0$ be the first time when a mark appears at a pair $(x, x+1)$ such that $\underline{x}(0) \cap \{x, x+1\} \neq \emptyset$. We set $\underline{y}(s) = \underline{x}^0(0)$ for $s \in (0, t)$ and define $\underline{y}(t)$ (which will then define $\underline{x}^0(s)$, $s \leq t$, as well) in the following way.

CASE 1, $\underline{x}(0) \cap \{N, -N\} = \emptyset$.

- Subcase (1.i): the intersection $\underline{x}(0) \cap \{x, x+1\}$ is a singleton, for instance $\underline{x}(0) \cap \{x, x+1\} = x_i(0)$.

(1.i.a): the mark is passive, then $\underline{y}(t) = \underline{x}^0(0)$.

(1.i.b): the mark is active, then $y_k(t) = x_k^0(0)$, $k \neq i$, and $y_i(t) = x_i^0(0) \pm 1$ if $x_i(0) = x$, respectively if $x_i(0) = x+1$.

- Subcase (1.ii): the intersection is a doubleton, for instance $\underline{x}(0) \cap \{x, x+1\} = \{x_i(0), x_j(0)\}$ and $\sigma(i) < \sigma(j)$; we then say that "particles i and j collide at time t ".

(1.ii.a): the mark is passive, then $y_k(t) = x_k^0(0)$, $k \neq j$ and $y_j(t) = x_j^0(0) - [x_j(0) - x_i(0)]$

(1.ii.b): the mark is active. Then $y_k(t) = x_k^0(0)$, $k \neq i$ and $y_i(t) = x_i^0(0) + [x_j(0) - x_i(0)]$

CASE 2, $\underline{x}(0) \cap \{N, -N\} \neq \emptyset$.

If both x and $x+1$ are in Λ_N the same rules given before apply. It remains to consider the subcases where $x = -N-1$ and $x = N$, the definitions are analogous and we only consider the former case. Let $x_i(0) = -N$, so that $x = -N-1$ and $x+1 = x_i(0)$.

- Subcase (2.i): The mark is passive, then $\underline{y}(t) = \underline{x}^0(0)$.

- Subcase (2.ii): The mark is active, then $y_k(t) = x_k^0(0)$, $k \neq i$, $y_i(t) = x_i^0(0) - 1$.

Having defined $\underline{y}(s)$ for $s \leq t$ we then set $\underline{x}^0(s) = \underline{y}(s) = \underline{x}^0(0)$, $s < t$; and $\underline{x}^0(t) = \underline{y}(t)$ if $\underline{y}(t) \in \Lambda_N^n$, otherwise $\underline{x}^0(t) = \underline{x}^0(0)$. Having $\underline{x}(t)$ and $\underline{x}^0(t)$ we can then define $\underline{y}(\cdot)$ from time t till the time of the next mark by using the same rules used starting from time 0. In particular they imply that $\underline{y}(t^+) = \underline{x}^0(t)$ (so that it may happen that the process $\underline{y}(\cdot)$ jumps twice at time t ; if the first jump takes \underline{y} out of Λ_N , then “instantaneously” \underline{y} comes back from where it jumped and have $\underline{y}(t^-) = \underline{y}(t^+) \neq \underline{y}(t)$). By applying repeatedly this procedure we define (with probability 1) $\underline{y}(\cdot)$ and $\underline{x}^0(\cdot)$ at all times. The collections of times when y_i jumps to its right/left are called respectively $\{t_{i,r}\}$ and $\{t_{i,l}\}$ (right/left refers to the first jump if there are two jumps at the same time). These are the “attempted jumps” of x_i^0 because when y_i jumps twice (the first time out of Λ_N and the second time back to the initial position) then x_i^0 does not change, i.e. the jump is suppressed.

It readily follows from the definition (see [6] for details)

- The jump times $\{t_{i,r}, t_{i,l}, i = 1, \dots, n\}$ are mutually independent Poisson processes with intensity $\epsilon^{-2}/2$ and $\underline{x}^0(\cdot)$ has the same law as n independent random walks in Λ_N , with jump rate $\epsilon^{-2}/2$ to each n.n. site in Λ_N .
- The times when x_i and x_i^0 have different jumps can only occur when one of them is at $\pm N$ or when “ x_i collides with x_j ” and $\sigma(j) < \sigma(i)$ (see subcase (1.ii)).
- For any i and $t > 0$, $x_i(t)$ is completely determined by $y_j(s)$, $s \in [0, t]$ with $j : \sigma(j) \leq \sigma(i)$.
- If $\sigma(\ell) = 1$ ($\sigma(\cdot)$ the “priority list”) and $x_\ell(0) = x_\ell^0(0)$ then, with probability 1, $x_\ell(t) = x_\ell^0(t)$ for all $t \geq 0$.

The following theorem proves bounds on $|x_i(t) - x_i^0(t)|$ as those established in [6] for processes on \mathbb{Z} , but the proof is more involved due to the reflections at $\pm N$.

Theorem 4.5. *Let $T > 0$ and $\underline{x}(0) = \underline{x}^0(0)$. Then for any $\zeta > 0$ and k there is c so that for all $t \leq T$ and for all $\epsilon > 0$*

$$\mathbf{P}_\epsilon[|x_\ell(t) - x_\ell^0(t)| \geq (\epsilon^{-2}t)^{1/4+\zeta}] \leq c(\epsilon^{-2}t)^{-k}. \quad (4.10)$$

Proof. Without loss of generality we may and will suppose that σ is the identity permutation (just rename the particles following the priorities). Since $x_1(t) = x_1^0(t)$ for all $t \geq 0$ we only need to prove (4.10) with $\ell > 1$, the label ℓ being hereafter fixed.

The idea of the proof is the following. Define the vectors

$$D(t) = (x_1(t) - x_1^0(t), \dots, x_\ell(t) - x_\ell^0(t)), \quad \xi(t) = (x_1(t), \dots, x_\ell(t)).$$

The first point will be to check that $D^2(t) - \frac{\epsilon^{-2}}{2}H(\xi(t))$ is a \mathbf{P}_ϵ super-martingale, where $H(\xi)$ is the number of pairs x_i, x_j in ξ such that $|x_i - x_j| = 1$. We shall then need to extend the analysis to the higher moments $D^{2n}(t)$ and in this way we shall relate the moments of $x_\ell(t) - x_\ell^0(t)$ to bounds on the probability of the time length when pairs of

stirring particles are close-by. All that however is more easily accomplished by studying the skeleton of the process, i.e. by looking at the times when particles jump and since the process is realized in the active/passive marks process, at the times when the marks appear.

Thus, following the proof of Lemma 4.3, we call two elements ω and ω' of Ω (the active/passive marks space) “similar” if all the marks in the two realizations occur at same time and their attribute, active/passive, is the same unless a mark occurs at a time t at a pair $(x, x + 1)$ such that $x = x_i(t^-)$, $x + 1 = x_j(t^-)$ and both $i, j \leq \ell$: in such a case the mark attributes in ω and ω' may either be equal or opposite. It is readily seen that this is an equivalence relation. Common to all ω in a same equivalence class are all the times $s_1 < \dots < s_{M'}$ in $[0, t]$ where a mark appears involving sites with at least one stirring particle with label $\leq \ell$. We call $t_1 < \dots < t_M$ its subset when both sites indicated by the mark are occupied by particles with label $\leq \ell$. We define $\delta(t_i) = \pm 1$ if the mark at time t_i is active, respectively passive. Then, conditioned on the equivalence class, the variables $\delta(t_i)$ are independent, taking the values $+1$ or -1 with probability $\frac{1}{2}$ each.

We define

$$D(s) = (d_1(s), \dots, d_\ell(s)), \quad d_i(s) := x_i(s) - x_i^0(s), \quad s \leq t$$

and we observe that for a mark $\{x, x + 1\}$ at time $s_i \notin \{t_1, \dots, t_M\}$ we have $D(s_i^+) = D(s_i^-)$ if $\{x, x + 1\} \cap \{-N, N\} = \emptyset$; if instead $\{x, x + 1\} \cap \{-N, N\} \neq \emptyset$, then $|d_k(s_i^+)| \leq |d_k(s_i^-)|$ for all k .

We call $L(s_i)$ the label of the particle involved by the mark at time s_i if $s_i \notin \{t_1, \dots, t_M\}$, otherwise $L(s_i)$ is the largest of the two labels involved. $L(s_i)$ is specified by the values of all the $\delta(t_k)$ with $t_k < s_i$. Let $\alpha(t_i) \in \{-1, 1\}$ (which depends on all the previous history) be such that $d_{L(t_i)}(t_i^+) - d_{L(t_i)}(t_i^-) = \alpha(t_i)\delta(t_i)$, namely if $\{x, x + 1\}$ is the mark at time t_i , then $\alpha(t_i) = 1$ ($= -1$) if $x_{L(t_i)}(t_i^-) = x$ ($= x + 1$ resp.).

We now define an auxiliary process $D^* = (d_1^*(s), \dots, d_\ell^*(s))$, $s \leq t$, that jumps only at the times t_i when only the $L(t_i)$ component varies (by ± 1). Suppose we have specified the values $\delta(t_j)$, $j < i$, so that we know $D(s)$, $s < t_i$, and suppose inductively that we know $D^*(s)$ as well. Let e_k denote the vector $\mathbf{1}_{i=k}$, $i = 1, \dots, \ell$. If $k = L(t_i)$ we define

$$d_k^*(t_i^+) - d_k^*(t_i^-) = \beta(t_i)\delta(t_i), \quad D^*(t_i^+) - D^*(t_i^-) = e_{L(t_i)}\beta(t_i)\delta(t_i) \quad (4.11)$$

where $\beta(t_i) = \alpha(t_i)$ if $d_k^*(t_i^-)$ and $d_k(t_i^-)$ have same sign otherwise $\beta(t_i) = -\alpha(t_i)$ (with the convention that two numbers have same sign also when one of them is 0). The above defines inductively $D^*(s)$ at all times having supposed $D^*(0) = 0$.

From the definition we can inductively check that $|d_i(s)| \leq |d_i^*(s)| + 1$ for all $s \leq t$ and all $i \in \{1, \dots, \ell\}$. Indeed it is enough to prove that if this holds up to t_j^- then it is also true at t_j^+ . If $d_i(t_j^-)$ and $d_i^*(t_j^-)$ have the same sign (including the case when one or both are 0) then $d_i(t_j^+) - d_i(t_j^-) = d_i^*(t_j^+) - d_i^*(t_j^-)$ and so the inequality holds by the induction hypothesis. If instead they have opposite sign $d_i(t_j^+) - d_i(t_j^-) = -(d_i^*(t_j^+) - d_i^*(t_j^-))$ and again the inequality holds by the induction hypothesis. We have

$$D(t)^2 \leq 2D^*(t)^2 + 2\ell. \quad (4.12)$$

By (4.11)

$$D^*(t_M^+)^2 = D^*(t_M^-)^2 + 2\beta(t_M)\delta(t_M)\langle D^*(t_M^-), e_{L(t_M)} \rangle + 1. \quad (4.13)$$

Denoting by \mathcal{P} and \mathcal{E} the law and expectation conditioned to an equivalence class and letting $\mathcal{F}_{t_M^-}$ denote the σ -algebra generated by all $\delta(t_i)$ with $i < M$, we have, using

(4.13),

$$\mathcal{E} \left[|D^*(t_M^+)|^{2n} | \mathcal{F}_{t_M^-} \right] = |D^*(t_M^-)|^{2n} + \sum_{m_1+m_2 \leq n, m_2 \neq 0 \text{ even}} c_{m_1, m_2} |D^*(t_M^-)|^{2m_1} (\langle D^*(t_M^-), e_{L(t_M)} \rangle)^{m_2}, \tag{4.14}$$

the sum being over even m_2 because $\beta(t_M)$ and $D^*(t_M^-)$ are $\mathcal{F}_{t_M^-}$ -measurable and

$$\mathcal{E}[\delta(t_M) | \mathcal{F}_{t_M^-}] = \mathcal{E}[\delta(t_M)] = 0$$

because the $\delta(t_i)$ are independent. Bounding $|\langle D^*(t_M^-), e_{L(t_M)} \rangle|^{m_2} \leq |D^*(t_M^-)|^{m_2}$ and recalling that m_2 is even, we get (for suitable coefficients c_k)

$$\mathcal{E} \left[|D^*(t_M^+)|^{2n} | \mathcal{F}_{t_M^-} \right] \leq |D^*(t_M^-)|^{2n} + \sum_{k=1}^{n-1} c_k |D^*(t_M^-)|^{2k}, \tag{4.15}$$

and by iteration,

$$\mathcal{E} \left[|D^*(t)|^{2n} \right] \leq \sum_{m=1}^M \sum_{k=1}^{n-1} c_k \mathcal{E} \left[|D^*(t_m^-)|^{2k} \right]. \tag{4.16}$$

Thus, by induction in n , there are new coefficients c_n so that

$$\mathcal{E} \left[|D^*(t)|^{2n} \right] \leq c_n M^n,$$

and (4.10) follows by the Chebyshev inequality and (4.4). Details are omitted. □

5 Integral inequalities for the v -functions

The difference between the expectation of $\eta(x, t)$ and $\rho_\epsilon(x, t)$ will be controlled by the v -functions:

Definition. [v -functions] We fix arbitrarily $\eta \in \{0, 1\}^{\Lambda_N}$, shorthand by \mathbb{E}_ϵ the expectation for the process with generator $L_\epsilon = \epsilon^{-2}L_0 + \epsilon^{-1}L_b$ which starts from η and write $\rho_\epsilon(x, t)$ for the solution of (2.1) with initial condition η . Recall that the v -functions are defined in (2.3) and, for brevity, we shall write $v(\underline{x}, t) \equiv v^\epsilon(\underline{x}, t | \eta)$.

The notation $v(\underline{x}, t)$ with $\underline{x} \in \Lambda_N^{n, \neq}$ has been chosen just for convenience when using duality the labeled process (Definition 2 in Section 4). Of course, $v(\underline{x}, t)$ is symmetric under any permutation of the \underline{x} coordinates, and we may write it as $v(X, t)$, with X a non-empty subset of Λ_N . Set $v(\emptyset, t) = 1$.

Lemma 5.1. For any $X \subset \Lambda_N$ and any $t \geq 0$,

$$\frac{d}{dt} v(X, t) = \epsilon^{-2}(L_0 v)(X, t) + (C_\epsilon v)(X, t), \tag{5.1}$$

where $L_0 v$ denotes the action of L_0 on $v(\cdot, t)$ as a function of X with X regarded as a particle configuration, $(C_\epsilon v)$ is the result of a linear operator acting on v , given as

$$(C_\epsilon v)(X, t) := \epsilon^{-2}(A v)(X, t) + \epsilon^{-1}(B v)(X, t), \tag{5.2}$$

with $(Av)(X, t) = 0$ if $|X| < 2$, while for $|X| \geq 2$

$$(Av)(X, t) := \sum_{x, y \in X} \mathbf{1}_{|x-y|=1} \{ [\rho_\epsilon(x, t) - \rho_\epsilon(y, t)] [v(X \setminus x, t) - v(X \setminus y, t)] - \frac{1}{2} [\rho_\epsilon(x, t) - \rho_\epsilon(y, t)]^2 v(X \setminus (x \cup y), t) \}, \tag{5.3}$$

and $Bv := B_+v + B_-v$, with $(B_\pm v)(X, t)$ that can be written as

$$(B_\pm v)(X, t) = \sum_{Z' \subset I_\pm} b([X \cap I_\pm], Z', t) v(X \setminus [X \cap I_\pm] \cup Z', t) \tag{5.4}$$

with coefficients $b(Z, Z', t)$ satisfying the following properties:

$$b(\emptyset, Z', t) = 0, \quad b(Z, \emptyset, t) = 0 \text{ if } |Z| = 1 \tag{5.5}$$

for any integer M : $\sup_{t, |Z| \leq M, |Z'| \leq M} |b(Z, Z', t)| < \infty$.

Proof. We obviously have

$$\frac{d}{dt} v(X, t) = \mathbb{E}_\epsilon \left[L_\epsilon \prod_{x \in X} \{ \eta(x, t) - \rho_\epsilon(x, t) \} \right] + \mathbb{E}_\epsilon \left[\frac{\partial}{\partial t} \prod_{x \in X} \{ \eta(x, t) - \rho_\epsilon(x, t) \} \right]$$

where the partial derivative acts only on $\rho_\epsilon(\cdot, t)$. Recalling (2.1), when the time derivative acts on the factor $\rho_\epsilon(x, t)$ it gives rise to the sum $\epsilon^{-2} \Delta \rho_\epsilon(x, t) + \epsilon^{-1} D \rho_\epsilon(x, t)$ with $D \rho_\epsilon = \mathbf{1}_{x \in I_+} D_+ \rho_\epsilon(x, t) - \mathbf{1}_{x \in I_-} D_- \rho_\epsilon(x, t)$. All terms with $\epsilon^{-2} \Delta \rho_\epsilon$ combined with those arising from $\epsilon^{-2} L_0$ are the same as when L_b is absent and it is proved in Lemma 10.1.2 of [6] that their sum is equal to $\epsilon^{-2} [L_0 v + Av]$. Thus we need only to prove that the remaining terms (arising from the action of L_b and from the terms with $\epsilon^{-1} D \rho_\epsilon$) can be written as described in (5.4) and (5.5). Considering the terms arising from the boundary generator in I_+ (the one in I_- is similar and the analysis omitted) we get, modulo a pre-factor ϵ^{-1} ,

$$\mathbb{E}_\epsilon \left[L_{b,+} \prod_{x \in X} [\eta(x, t) - \rho_\epsilon(x, t)] \right] - \sum_{x \in X} \mathbb{E}_\epsilon \left[D_+ \rho_\epsilon(x, t) \prod_{y \in X \setminus x} [\eta(y, t) - \rho_\epsilon(y, t)] \right]. \tag{5.6}$$

We write, for $x \in I_+$,

$$L_{b,+} \eta(x, t) = \frac{j}{2} \left([1 - \rho_\epsilon(x, t)] - [\eta(x, t) - \rho_\epsilon(x, t)] \right) \prod_{x+1 \leq y \leq N} \left([\eta(y, t) - \rho_\epsilon(y, t)] + \rho_\epsilon(y, t) \right)$$

After expanding the products and denoting below by Z' a subset of $\{x + 1, \dots, N\}$,

$$L_{b,+} \eta(x, t) = D_+ \rho_\epsilon(x, t) + \frac{j}{2} (1 - \rho_\epsilon(x, t)) \sum_{Z' \neq \emptyset} \left\{ \prod_{z \notin Z'} \rho_\epsilon(z, t) \right\} \left\{ \prod_{y \in Z'} [\eta(y, t) - \rho_\epsilon(y, t)] \right\} - \frac{j}{2} [\eta(x, t) - \rho_\epsilon(x, t)] \sum_{Z'} \left\{ \prod_{z \notin Z'} \rho_\epsilon(z, t) \right\} \left\{ \prod_{y \in Z'} [\eta(y, t) - \rho_\epsilon(y, t)] \right\}$$

(in the first sum $Z' = \emptyset$ is absent because it has been included in $D_+ \rho_\epsilon(x, t)$). The term with $D_+ \rho_\epsilon(x, t)$ cancels with the second term in (5.6) if $X \cap I_+ = \{x\}$. Hence all remaining terms have at least one factor $\eta(y, t) - \rho_\epsilon(y, t)$ with $y \in I_+$. The other properties of the coefficients b stated in (5.5) easily follow. \square

For the stirring process defined in Section 4, we let $\mathbf{P}_\epsilon(X \xrightarrow{s} Y) = \mathbf{P}_\epsilon(X(s) = Y | X(0) = X)$, for $X \subset \Lambda_N$ to $Y \subset \Lambda_N$, $|X| = |Y|$. (In particular $\mathbf{P}_\epsilon(X \xrightarrow{s} Y) = P_s^{(\epsilon)}(x, y)$ as in (3.1), if $X = \{x\}$, $Y = \{y\}$.) Since $v(X, 0) = 0$, the integral form of (5.1) is

$$v(X, t) = \int_0^t ds \sum_Y \mathbf{P}_\epsilon(X \xrightarrow{s} Y) (C_\epsilon v)(Y, t - s). \tag{5.7}$$

We start by bounding the contribution of $B_\pm v$ to $C_\epsilon v$ in the right hand side of (5.7):

Lemma 5.2. *For any n and any $\zeta > 0$ there is a constant c so that for any $X \subset \Lambda_N$, $|X| = n$, and any $s < t \leq \log \epsilon^{-1}$*

$$\begin{aligned} \sum_{Y \subset \Lambda_N} \mathbf{P}_\epsilon(X \xrightarrow{s} Y) \epsilon^{-1} |(B_\pm v)(Y, t - s)| &\leq \sum_{Z' \subset I_\pm} \sum_{\emptyset \neq X'' \subset X} (1 - \mathbf{1}_{|Z'|=0, |X''|=1}) \\ &\times \frac{c\epsilon^{-1}}{(\epsilon^{-2}s)^{|X''|/2} + 1} \sum_{W \subset I_\pm^c} \mathbf{P}_\epsilon(X \setminus X'' \xrightarrow{s} W) |v(W \cup Z', t - s)|. \end{aligned} \tag{5.8}$$

Proof. We consider explicitly only the case with B_+ . The left hand side of (5.8) without absolute values is equal to:

$$\Psi := \epsilon^{-1} \sum_{Y, Z' \subset I_+} \mathbf{P}_\epsilon(X \xrightarrow{s} Y) \mathbf{1}_{Y \cap I_+ \neq \emptyset} b(Y \cap I_+, Z') v([Y \setminus (Y \cap I_+)] \cup Z', t - s). \tag{5.9}$$

We decompose $Y = W \cup Z$, $W \subset I_+^c$ and $Z \subset I_+$, $|Z| > 0$. Then the sum over Y becomes a sum over W and Z with the condition that $|W \cup Z| = |X|$. For each fixed W and Z we apply Andjel’s inequality, see [1], and get

$$\mathbf{P}_\epsilon(X \xrightarrow{s} W \cup Z) \leq \left\{ \sum_{Y \supset W} \mathbf{P}_\epsilon(X \xrightarrow{s} Y) \right\} \left\{ \sum_{Y' \supset Z} \mathbf{P}_\epsilon(X \xrightarrow{s} Y') \right\}.$$

By Liggett’s inequality, see [14],

$$\mathbf{P}_\epsilon(X \xrightarrow{s} Y') \leq \prod_{x \in X} \sum_{y \in Y'} \mathbf{P}_\epsilon(x \xrightarrow{s} y)$$

and by (3.5) there is $c = c_{|X|}$ so that

$$\sum_{Y' \supset Z} \mathbf{P}_\epsilon(X \xrightarrow{s} Y') \leq \frac{c}{(\epsilon^{-2}s)^{|Z|/2} + 1}.$$

Observe that

$$\sum_{Y \supset W} \mathbf{P}_\epsilon(X \xrightarrow{s} Y) = \sum_{X' \subset X} \mathbf{P}_\epsilon(X' \xrightarrow{s} W),$$

and collecting the estimates we have

$$\begin{aligned} |\Psi| &\leq \sum_{Z \subset I_+, |Z| \geq 1} \sum_{W \subset I_+^c, |W \cup Z| = |X|} \frac{c\epsilon^{-1}}{(\epsilon^{-2}s)^{(|Z|)/2} + 1} \sum_{Z' \subset I_+} |b(Z, Z')| \\ &\times \sum_{X' \subset X, |X'| = |W|} \mathbf{P}_\epsilon(X' \xrightarrow{s} W) |v(W \cup Z', t - s)|. \end{aligned}$$

From (5.5) we get that $|b(Z, Z')| \leq c(1 - \mathbf{1}_{|Z|=1, |Z'|=0})$. Thus, denoting by $p(K)$ the number of subsets of I_+ ,

$$|\Psi| \leq \sum_{\emptyset \neq X'' \subset X} \sum_{Z' \subset I_+} (1 - \mathbf{1}_{|X''|=1, |Z'|=0}) \frac{c p(K) \epsilon^{-1}}{(\epsilon^{-2}s)^{(|X''|)/2} + 1} \times \sum_{W \subset I_+^c: |W|=|X \setminus X''|} \mathbf{P}_\epsilon(X \setminus X'' \xrightarrow{s} W) |v(W \cup Z', t - s)|.$$

□

We are left with the bound of the contribution in (5.7) due to $\epsilon^{-2}A$, see (5.3). It is crucial here to exploit the smallness of the gradients, namely the differences $\rho_\epsilon(x, t) - \rho_\epsilon(y, t)$ and $v(X \setminus x, t) - v(X \setminus y, t)$ (recall that $|x - y| = 1$). Both bounds use the parabolic nature of the evolution, but the latter requires a more delicate analysis which, following [6], is based on the realization of the stirring process given in Definition 1 of Section 4.

Recalling Definition 2 in section 4, we may order arbitrarily the sites of X as $\underline{x} = (x_1, \dots, x_n)$ and set $v(\underline{x}, t) := v(X, t)$. Let $\mathbf{E}_{\epsilon, \underline{x}}$ denote the expectation with respect to the stirring process defined in Section 4 starting at time 0 from \underline{x} and when the starting point will be clear from the context we use instead $\mathbf{E}_\epsilon \equiv \mathbf{E}_{\epsilon, \underline{x}}$. We then rewrite (5.8) as

$$\sum_Y \mathbf{P}_\epsilon(X \xrightarrow{s} Y) \epsilon^{-1} |(B_\pm v)(Y, t - s)| \leq \sum_{\emptyset \neq J \subset \{1, \dots, n\}} \sum_{\underline{z}' \subset I_\pm} [1 - \mathbf{1}_{|J|=1, |\underline{z}'|=0}] \times \frac{c\epsilon^{-1}}{(\epsilon^{-2}s)^{|J|/2} + 1} \mathbf{E}_{\epsilon, \underline{x}} \left[\mathbf{1}_{\underline{x}^{(J)}(s) \subset I_\pm^c} |v(\underline{x}^{(J)}(s) \cup \underline{z}', t - s)| \right] \quad (5.10)$$

where $\underline{x}^{(J)}$ is the configuration obtained by erasing from \underline{x} all x_j with $j \in J$ and we then say that all the particles $x_j, j \in J$ have died (and their labels will not be used again) and that the particles \underline{z}' are born at time s . Our general rule to label a new particle is to use the smallest integer never used earlier in the labeling (the order in which the particles in \underline{z}' are born is chosen arbitrarily). In conclusion (5.10) describes a labeled stirring evolution with a death/birth process at time s (notice that it is the same to erase the particles $x_j, j \in J$ either at time 0 or at time s). In an analogous way we write the labeled version of (5.3) as

$$(Av)(\underline{x}, t) := \sum_{x_i, x_j \in \underline{x}} \mathbf{1}_{|x_i - x_j|=1} \left\{ [\rho_\epsilon(x_i, t) - \rho_\epsilon(x_j, t)] [v(\underline{x}^{(i)}, t) - v(\underline{x}^{(j)}, t)] - \frac{1}{2} [\rho_\epsilon(x_i, t) - \rho_\epsilon(x_j, t)]^2 v(\underline{x}^{(i,j)}, t) \right\} \quad (5.11)$$

so that we have:

$$|v(\underline{x}, t)| \leq \int_0^t ds \left(\sum_{u=\pm} \sum_{\emptyset \neq J \subset \{1, \dots, n\}} \sum_{\underline{z}' \subset I_u} \frac{c\epsilon^{-1}}{(\epsilon^{-2}s)^{|J|/2} + 1} [1 - \mathbf{1}_{|J|=1, |\underline{z}'|=0}] \times \mathbf{E}_\epsilon \left[\mathbf{1}_{\underline{x}^{(J)}(s) \subset I_u^c} |v(\underline{x}^{(J)}(s) \cup \underline{z}', t - s)| \right] + \epsilon^{-2} \mathbf{E}_\epsilon \left[|(Av)(\underline{x}(s), t - s)| \right] \right), \quad (5.12)$$

with Av as in (5.11).

The difference of the ρ_ϵ 's in (5.11) is bounded by using (3.8). Indeed by using a weaker form as in (3.9) we bound the last term in (5.12) by

$$\mathbf{E}_\epsilon \left[\left| \epsilon^{-2} (Av)(\underline{x}(s), t-s) \right| \right] \leq \sum_{i,j} \mathbf{E}_\epsilon \left[\mathbf{1}_{|x_i(s)-x_j(s)|=1} \times c\epsilon^{-2} \left(\frac{|v(\underline{x}^{(i)}(s), t-s) - v(\underline{x}^{(j)}(s), t-s)|}{(\epsilon^{-2}(t-s))^{1/2-\zeta} + 1} + \frac{|v(\underline{x}^{(i,j)}(s), t-s)|}{[\epsilon^{-2}(t-s)]^{1-\zeta} + 1} \right) \right]. \quad (5.13)$$

To bound the v -gradients appearing in (5.13) we shall use the following lemma where we take advantage for the first time of the features of the active/passive marks process (analogous estimates are given in Sect.10.1 of [6]).

Definition. [The stopping time τ_{i,j,t_0}]

Let $\{\underline{x}(t)\}_{t \geq 0}$ be the labeled process realized in the active/passive marks process, let i and j be the labels of two of its particles and $t_0 \geq 0$. We then define τ_{i,j,t_0} as the first time $\tau > t_0$ when (i) $|x_i(\tau) - x_j(\tau)| = 1$ and (ii) at τ there is a mark (either active or passive) between $x_i(\tau)$ and $x_j(\tau)$; otherwise we set $\tau = \infty$. When $t_0 = 0$ we just write $\tau_{i,j}$.

Lemma 5.3. We have

$$\begin{aligned} |v(\underline{x}^{(i)}, t) - v(\underline{x}^{(j)}, t)| &\leq \int_0^t ds \mathbf{E}_\epsilon \left[\mathbf{1}_{\tau_{i,j} \geq \frac{s}{2}} \sum_{\underline{y}} \{ \mathbf{P}_\epsilon(\underline{x}^{(j)}(\frac{s}{2}) \xrightarrow{s/2} \underline{y}) \right. \\ &\quad \left. + \mathbf{P}_\epsilon(\underline{x}^{(i)}(\frac{s}{2}) \xrightarrow{s/2} \underline{y}) \} |(C_\epsilon v)(\underline{y}, t-s)| \right]. \end{aligned} \quad (5.14)$$

Proof. Denoting by $\underline{x}(s)$ the process starting from \underline{x} with both particles i and j ,

$$v(\underline{x}^{(i)}, t) - v(\underline{x}^{(j)}, t) = \int_0^t ds \mathbf{E}_\epsilon \left[(C_\epsilon v)(\underline{x}(s) \setminus x_i(s), t-s) - (C_\epsilon v)(\underline{x}(s) \setminus x_j(s), t-s) \right]$$

which is the expectation at time s of the function $f(\underline{y}) = (C_\epsilon v)(\underline{y}^{(i)}, t-s) - (C_\epsilon v)(\underline{y}^{(j)}, t-s)$. Since f is antisymmetric under the exchange of particles i and j , (5.14) follows from Lemma 4.3. \square

The reason for the time interval $s/2$ in the lemma is to be able to exploit (5.10). We have in fact from (5.14) and (5.10)

$$\begin{aligned} |v(\underline{x}^{(i)}, t) - v(\underline{x}^{(j)}, t)| &\leq \int_0^t ds \left(\sum_{u=\pm} \sum_{\emptyset \neq J \subset \{1, \dots, n\}} \frac{c\epsilon^{-1}}{(\epsilon^{-2}\frac{s}{2})^{|J|/2} + 1} \sum_{\underline{z}' \subset I_u} [1 - \mathbf{1}_{|J|=1, |z'|=0}] \right. \\ &\quad \left. \times \mathbf{E}_\epsilon \left[\mathbf{1}_{\tau_{i,j} > \frac{s}{2}} \mathbf{1}_{\underline{x}^{(j)}(s) \in I_u^c} |v(\underline{x}^{(j)}(s) \cup \underline{z}', t-s)| \right] \right. \\ &\quad \left. + \mathbf{E}_\epsilon \left[\mathbf{1}_{\tau_{i,j} > \frac{s}{2}} \epsilon^{-2} \{ |(Av)(\underline{x}^{(i)}(s), t-s)| + |(Av)(\underline{x}^{(j)}(s), t-s)| \} \right] \right), \end{aligned} \quad (5.15)$$

while the v -functions alone are bounded by:

$$\begin{aligned}
|v(\underline{x}, t)| \leq & c \int_0^t ds \left(\sum_{u=\pm} \sum_{\emptyset \neq J \subset \{1, \dots, n\}} \frac{\epsilon^{-1}}{(\epsilon^{-2}s)^{|J|/2} + 1} \sum_{z' \subset I_u} [1 - \mathbf{1}_{|J|=1, |z'|=0}] \right. \\
& \times \mathbf{E}_\epsilon \left[\mathbf{1}_{\underline{x}^{(J)}(s) \subset I_u^c} |v(\underline{x}^{(J)}(s) \cup z', t-s)| \right] \\
& + \sum_{i, j \subset \{1, \dots, n\}} \mathbf{E}_\epsilon \left[\mathbf{1}_{|x_i(s) - x_j(s)|=1} \epsilon^{-2} \left(\frac{|v(\underline{x}^{(i)}(s), t-s) - v(\underline{x}^{(j)}(s), t-s)|}{(\epsilon^{-2}(t-s))^{1/2-\zeta} + 1} \right. \right. \\
& \left. \left. + \frac{|v(\underline{x}^{(i,j)}(s), t-s)|}{[\epsilon^{-2}(t-s)]^{1-\zeta} + 1} \right) \right]. \tag{5.16}
\end{aligned}$$

We shall derive the desired bound for $|v(\underline{x}, t)|$ by iterating (5.16) and using (5.15) (complemented by (5.11) to write the terms Av whenever a v -gradient appears. The series obtained in this way is described in the next section and studied in Section 7 and Section 8.

6 The truncated hierarchy

By iterating (5.15)–(5.16) we can write the solution as a formal series, but we do not know whether it converges. We shall therefore truncate the expansion proving that at least for small times the remainder is small. Thus, in a first step we only prove short time estimates:

Theorem 6.1. *For any $c^* < \frac{1}{4(K+2)}$ the following holds. For any $\beta^* > 0$, any initial configuration η_0 and any positive integer n there is c so that*

$$|v(\underline{x}, t | \eta_0)| \leq c(\epsilon^{-2}t)^{-c^*n}, \quad t \leq \epsilon^{\beta^*}, \quad |\underline{x}| = n. \tag{6.1}$$

For brevity in the sequel we shall simply write $v(\underline{x}, t)$ for $v(\underline{x}, t | \eta_0)$. The theorem will be proved in Section 9 using the results of Sections 7 and 8. Of course we only need to prove (6.1) when $t \geq \epsilon^2$ because for all values of its arguments $|v(\underline{x}, t)| \leq 1$.

The setup. As mentioned above, Theorem 6.1 will be proved by finitely many iterations of the integral inequalities (5.15)–(5.16). The series obtained in this way will be referred to as “the truncated hierarchy”. The number of iterations will depend on n and β^* , and will be denoted by M ; its actual value will be specified later in (9.6) and (9.12). At each iteration the number of “particles”, i.e. elements in the argument of the v -function, increases at most by $K - 1$ so that the total number of particles is not larger than $n + (K - 1)M$. Hence all constants that appear in the previous section which depend on the cardinality of the configuration in the v -functions are bounded by a constant (once n and β^* are fixed). The various terms which appear in the expansion will be classified in terms of sequences called skeletons. We shall first define the skeletons and then establish the correspondence with the terms appearing in the expansion. The positions of the particles are not recorded in the skeleton; it says which particles are alive at each step of the process, as well as those which die and are born, specifying also the positions of the new-born particles at their birth.

Definition. [*The skeleton*]

Skeletons are denoted by π . Each π consists of a sequence $\pi = (\pi_i)_{i=1, \dots, m(\pi)}$, $m(\pi) \leq M$ (see the “setup” paragraph). “ i ” is a “branching time” and π_i describes the nature of the branching (which particles die and which are born). As we shall explain the particles alive at step i , denoted by A_i , are determined by the values π_j with $j \leq i$ while the particles initially alive are $A_0 = \{1, \dots, n\}$.

- for each i , $\pi_i = (\delta_i, J_i, u_i, \underline{z}_i)$, $\delta_i \in \{0, 1, 2\}$, J_i is a finite increasing sequence of distinct positive integers, $u_i \in \{0, +, -\}$, \underline{z}_i is a labeled configuration, its labels will be denoted by J_i^+ . There are several constraints relating the elements π_i of π which we state inductively. We suppose that we have already chosen the elements π_j with $j < i$ and thus know the sequence A_j , $j < i$, of particles alive at the branching times j . We then want to specify the possible values of π_i , and for each choice of π_i we shall define A_i . (When $i = 1$ we only need A_0 which is $\{1, \dots, n\}$, hence the induction is complete).

With a small abuse of notation we sometimes identify J_i with the set of its constituents.

- Before entering into all the details we just say that the particles which die are those with labels in J_i except when $\delta_i = 0$ in which case J_i consists of two particles but only one of them dies; in all cases $J_i \subset A_{i-1}$. If $u_i = 0$ then no particle is born and $\underline{z}_i = \emptyset$. If $u_i \neq 0$ there may be new particles. The configuration of the new particles is \underline{z}_i which is contained in I_{u_i} . The labels in \underline{z}_i are J_i^+ , J_i^+ is a sequence of consecutive integers, the first one is $h + 1$ if h is the max over all integers in the union of A_j over $j < i$. The positions of the particles in \underline{z}_i are increasing functions of the labels.

- If $\delta_i = 0, 1$ then J_i is an ordered pair, $J_i = (k_i, \ell_i)$, $k_i < \ell_i$, $u_i = 0$ and $\underline{z}_i = \emptyset$. If $\delta_i = 1$, then both particles die, so that $A_i = A_{i-1} \setminus \{k_i, \ell_i\}$. If $\delta_i = 0$, then only particle ℓ_i dies so that $A_i = A_{i-1} \setminus \ell_i$.

- If $\delta_i = 2$, J_i is non empty ($|J_i| \geq 1$), $u_i \neq 0$ and $A_i = (A_{i-1} \setminus J_i) \cup J_i^+$ if J_i^+ are the labels of the particles in \underline{z}_i . The configuration \underline{z}_i may also be empty but not when $|J_i| = 1$, in which case $|J_i^+| > 0$.

- Finally if $m(\pi) = m < M$ then $\delta_m > 0$, $\underline{z}_m = \emptyset$ and $|J_m| = |A_{m-1}|$, i.e. $A_m = \emptyset$. If on the other hand $m(\pi) = M$ then there is no restriction on A_M , which could be either empty or non empty.

Definition. [*The branching process*]

Given an element $\omega \in \Omega$, the active/passive marks space, \underline{x} (the initial configuration), a skeleton π and $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = t$, $m = m(\pi)$, we define $\underline{x}(t)$ by following in the time intervals (t_i, t_{i+1}) the active/passive marks. At time t_i all particles $x_j(t_i^-)$, $j \in J_i$, disappear from $\underline{x}(t_i^-)$ except when $\delta_i = 0$: in that case the particle with label k_i remains alive, the one with label ℓ_i survives but it will die at time $t_i + (t_{i+1} - t_i)/2$. We also require that if $\delta_i = 2$ then $x_j(t_i^-) \in I_{u_i}^c$ for all $j \in A_{i-1} \setminus J_i$, we shall shorthand this event by R_i . We complete the definition of $\underline{x}(t)$ by saying that at time t_i^+ we add the labeled particles \underline{z}_i .

In order to write “the truncated hierarchy” we introduce the factors γ_i which depend on the realization of the active/passive marks process, \underline{x} (the initial configuration), the skeleton π and the sequence of times $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = t$, $m = m(\pi)$.

- If $\delta_i = 0$ then

$$\gamma_i = \mathbf{1}_{|x_{k_i}(t_i) - x_{\ell_i}(t_i)|=1} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{\frac{1}{2} - \zeta} + 1} \mathbf{1}_{\tau_{k_i, \ell_i, t_i} > t_i + \frac{t_{i+1} - t_i}{2}}. \tag{6.2}$$

It means that we are considering the first term in the second expectation on the

right hand side of (5.16) with i, j equal to (k_i, l_i) and then, when writing the v -gradient via (5.15), we take the term in the second expectation where the label l_i is missing (i.e. particle k_i survives, particle l_i dies).

- If $\delta_i = 1$ then

$$\gamma_i = \mathbf{1}_{|x_{k_i}(t_i) - x_{l_i}(t_i)|=1} \frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{1-\zeta} + 1}. \tag{6.3}$$

It means that we are considering the second term in the second expectation on the right hand side of (5.16) with (i, j) equal to (k_i, l_i) .

- If $\delta_i = 2$ then

$$\gamma_i = \mathbf{1}_{R_i} \frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i - t_{i-1})]^{p_i/2} + 1}, \quad p_i \equiv |J_i|, \tag{6.4}$$

recalling that $R_i = \{x_j(t_i^-) \in I_{u_i}^c, j \notin J_i\}$. In (6.4) we are considering the first term on the right hand side of (5.16) with $u = u_i, J = J_i$ and $\underline{z}' = \underline{z}_i$.

In this way we have classified all possible terms of the truncated hierarchy and

$$|v(\underline{x}, t)| \leq c \sum_{\pi} w_{\pi}(\underline{x}, t) \tag{6.5}$$

with $c = c(n, \beta)$ is a constant (as discussed in the setup definition) and $w_{\pi}(\underline{x}, t)$ is obtained by integrating the product of all the γ_i defined above:

$$w_{\pi}(\underline{x}, t) = \int_0^t dt_1 \dots \int_{t_{m-1}}^t dt_m \prod_{\delta_i=1} \left[\frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{1-\zeta} + 1} \right] \prod_{\delta_i=0} \left[\frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{1/2-\zeta} + 1} \right] \prod_{\delta_i=2} \left[\frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i - t_{i-1})]^{p_i/2} + 1} \right] \mathbf{E}_{\epsilon} \left[\prod_{\delta_i=0,1} \mathbf{1}_{|x_{k_i}(t_i) - x_{l_i}(t_i)|=1} \prod_{\delta_i=0} \mathbf{1}_{T_i} \prod_{\delta_i=2} \mathbf{1}_{R_i} \right] \tag{6.6}$$

where $m = m(\pi)$, p_i is defined in (6.4), the product over $\{\delta_i = k\}$, $k = 0, 1, 2$, means the product over $\{i \in \{1, \dots, m\} : \delta_i = k\}$ and

$$T_i := \{\tau_{k_i, l_i, t_i} > t_i + \frac{t_{i+1} - t_i}{2}\}, \quad R_i := \{x_j(t_i^-) \in I_{u_i}^c, j \notin J_i\} \tag{6.7}$$

The expectation \mathbf{E}_{ϵ} is with respect to the active/passive marks process and $\underline{x}(t)$ is the branching process defined above (in terms of π and of the realization of the active/passive marks process). If $m(\pi) = M$ there could be surviving particles at t_M , i.e. a v -function $|v(\underline{x}(t_M), t - t_M)|$ that in (6.6) has been bounded by 1.

7 Bounds when times do not cluster

The proof of Theorem 6.1 is based on bounds for $w_{\pi}(\underline{x}, t)$ (cf. (6.6)) which will be proved in this and in the next sections. As in the statement of Theorem 6.1 n , the cardinality of \underline{x} , is fixed and $t \leq \epsilon^{\beta^*} \beta^* > 0$. As already mentioned in Section 6 we then introduce a parameter M which depends on n and β^* and we only consider skeletons π such that $m(\pi) \leq M$. The choice of M will be specified in (9.6) and (9.12). Hereafter

n, M and β^* are to be considered fixed and any parameter which depends only on n, M and β^* will be called constant. Setting

$$a = \frac{K}{K + 1} \tag{7.1}$$

we introduce the quantity $\Delta = \Delta(a, t)$ as

$$\Delta = \begin{cases} \epsilon^a & \text{if } t > (M + 1)\epsilon^a \\ \frac{t}{M+1} & \text{if } t \leq (M + 1)\epsilon^a. \end{cases} \tag{7.2}$$

The choice of a will be explained later in the course of the proofs. The parameter Δ is used to distinguish cases when the times t_1, \dots, t_m “cluster to t ” or not, i.e. if $t_m \geq t - \Delta$ or $t_m < t - \Delta$. We accordingly split (6.6) writing

$$w_\pi(\underline{x}, t) = w'_\pi(\underline{x}, t) + w''_\pi(\underline{x}, t) \tag{7.3}$$

where $w'_\pi(\underline{x}, t)$ is defined by the right hand side of (6.6) with the integral over $t_m, m = m(\pi)$, restricted to $\{t_m < t - \Delta\}$; $w''_\pi(\underline{x}, t)$ is instead the integral over $\{t_m \geq t - \Delta\}$. The analysis of both $w'_\pi(\underline{x}, t)$ and $w''_\pi(\underline{x}, t)$ consists of two steps: we first apply the results of Section 4 to bound the expectation in (6.6); after this we are reduced to a rather explicit integral over $t_1 \dots t_m$ which is bounded in the second step. Convergence problems in the latter motivate the type of inequalities used in the first step. The case $\{t_m < t - \Delta\}$ is much simpler and examined first in this section where we shall prove:

Proposition 7.1. *For any $\zeta > 0$ there is c so that for all $\pi : m = m(\pi) \leq M$, for all $\underline{x} : |\underline{x}| = n$, for all $\epsilon > 0$ and all $t \leq \epsilon^{\beta^*}$:*

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} \Delta^{-S_1(m)} \epsilon^{S_2(m)} t^{S_3(m)} \tag{7.4}$$

where, recalling that $p_i = |J_i|$ as specified by π ,

$$\begin{aligned} S_1(m) &= |\{i \leq m : \delta_i = 1\}| + \frac{1}{2} |\{i \leq m : \delta_i = 0\}| \\ S_2(m) &= |\{i \leq m : \delta_i = 0, 1\}| + |\{i \leq m : \delta_i = 2, p_i \geq 2, \delta_{i-1} \neq 0\}| \\ S_3(m) &= \frac{1}{2} \left(|\{i \leq m : \delta_i = 1\}| + |\{i \leq m : \delta_i = 2, p_i = 1\}| + |\{i \leq m : \delta_i = 2, p_i \geq 2, \delta_{i-1} = 0\}| \right). \end{aligned} \tag{7.5}$$

The proof to Proposition 7.1 is given after stating and proving Lemma 7.2 below. We fix π , write $m = m(\pi)$, $t_0 := 0$, $t_{m+1} := t$ and, with the sets T_i and R_i defined in (6.7), we set

$$\psi_h := \prod_{i \leq h: \delta_i=0,1} \mathbf{1}_{|x_{k_i}(t_i) - x_{l_i}(t_i)|=1} \prod_{i \leq h: \delta_i=0} \mathbf{1}_{T_i} \prod_{i \leq h: \delta_i=2} \mathbf{1}_{R_i}, \quad 1 \leq h \leq m \tag{7.6}$$

$$\phi_h := \prod_{i > h: \delta_i=0,1} \frac{1}{[\epsilon^{-2}(t_i - t_{i-1})]^{1/2} + 1} \prod_{i > h: \delta_i=0} \frac{1}{[\epsilon^{-2}(t_{i+1} - t_i)]^{1/2} + 1}, \quad 0 \leq h < m,$$

letting $\phi_m := 1$ and $\psi_0 = 1$. Then:

Lemma 7.2. *There is c so that for any $h \leq m$*

$$\phi_h \mathbf{E}_\epsilon [\psi_h] \leq c \phi_{h-1} \mathbf{E}_\epsilon [\psi_{h-1}]. \tag{7.7}$$

Proof. Let

$$\mathcal{F}(t) := \sigma\text{-algebra generated by the active/passive marks in the interval } [0, t] \tag{7.8}$$

Suppose first $\delta_h = 0$. We then condition on $\mathcal{F}(t_h)$ getting

$$\mathbf{E}_\epsilon [\psi_h] = \mathbf{E}_\epsilon \left[\psi_{h-1} \mathbf{1}_{|x_{k_h}(t_h^-) - x_{l_h}(t_h^-)|=1} \left(\mathbf{P}_\epsilon [T_h | \mathcal{F}(t_h)] \right) \right]. \tag{7.9}$$

By (4.3)

$$\mathbf{P}_\epsilon [T_h | \mathcal{F}(t_h)] \leq \frac{c}{[\epsilon^{-2}(t_{h+1} - t_h)]^{1/2} + 1}. \tag{7.10}$$

We shorthand $t_{h,+} := t_h + \frac{t_{h+1} - t_h}{2}$ and have

$$\mathbf{E}_\epsilon \left[\psi_{h-1} \mathbf{1}_{|x_{k_h}(t_h^-) - x_{l_h}(t_h^-)|=1} \right] = \mathbf{E}_\epsilon \left[\psi_{h-1} \left(\mathbf{P}_\epsilon [|x_{k_h}(t_h^-) - x_{l_h}(t_h^-)| = 1 | \mathcal{F}(t_{h-1,+})] \right) \right]. \tag{7.11}$$

By (4.8)

$$\mathbf{P}_\epsilon [|x_{k_h}(t_h^-) - x_{l_h}(t_h^-)| = 1 | \mathcal{F}(t_{h-1,+})] \leq \frac{c}{[\epsilon^{-2}(t_h - t_{h-1})]^{1/2} + 1} \tag{7.12}$$

thus completing the proof of (7.7) when $\delta_h = 0$. When $\delta_h = 1$, (7.7) follows from (7.11)–(7.12), while when $\delta_h = 2$ we simply bound $\mathbf{1}_{R_h} \leq 1$. \square

Proof of Proposition 7.1.

Recalling the definitions (7.3) and (7.6), we apply repeatedly Lemma 7.2 to get

$$w'_\pi(\underline{x}, t) \leq \int_0^t dt_1 \dots \int_{t_{m-1}}^{t-\Delta} dt_m f_{1,\dots,m}(t_1, \dots, t_m) \tag{7.13}$$

where

$$f_{1,\dots,m} := \prod_{\delta_i=1} \left[\frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{1-\zeta} + 1} \frac{1}{[\epsilon^{-2}(t_i - t_{i-1})]^{1/2} + 1} \right] \prod_{\delta_i=2} \frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i - t_{i-1})]^{p_i/2} + 1} \prod_{\delta_i=0} \left[\frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{1/2-\zeta} + 1} \frac{1}{[\epsilon^{-2}(t_{i+1} - t_i)]^{1/2} + 1} \frac{1}{[\epsilon^{-2}(t_i - t_{i-1})]^{1/2} + 1} \right] \tag{7.14}$$

We bound the factors on the right hand side of (7.14) as follows. Since $t - t_i > \Delta$ for all i we have

$$\frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{1-\zeta} + 1} \leq \epsilon^{-2\zeta} \Delta^{-1} \quad \text{and} \quad \frac{\epsilon^{-2}}{[\epsilon^{-2}(t - t_i)]^{1/2-\zeta} + 1} \leq \epsilon^{-2\zeta} \epsilon^{-1} \Delta^{-1/2}, \tag{7.15}$$

where we have used that $\Delta^\zeta \leq 1$ since $\Delta \leq \epsilon^a$, $a > 0$. Moreover we obviously have

$$\frac{1}{[\epsilon^{-2}(t_{i+1} - t_i)]^{1/2} + 1} \leq \frac{\epsilon}{(t_{i+1} - t_i)^{1/2}}. \tag{7.16}$$

When $\delta_i = 2$, $p_i \geq 2$ and $\delta_{i-1} \neq 0$, for any $\zeta > 0$ we bound

$$\frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i - t_{i-1})]^{p_i/2} + 1} \leq \frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i - t_{i-1})]^{1-\zeta}} = \epsilon^{-2\zeta} \frac{\epsilon}{(t_i - t_{i-1})^{1-\zeta}}. \tag{7.17}$$

The first inequality is proved as follows: if $\epsilon^{-2}(t_i - t_{i-1}) \geq 1$ then we drop the +1 in the denominator and replace $p_i/2$ by $1 - \zeta$ (since $p_i/2 \geq 1 - \zeta$); if $\epsilon^{-2}(t_i - t_{i-1}) \leq 1$, then the denominator is at least one and so not smaller than $[\epsilon^{-2}(t_i - t_{i-1})]^{1-\zeta}$, hence (7.17).

By the same argument, for $p_i \geq 2$ and $\delta_{i-1} = 0$ as well as when $p_i = 1$

$$\frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i - t_{i-1})]^{p_i/2} + 1} \leq \frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i - t_{i-1})]^{1/2-\zeta}} \leq \epsilon^{-2\zeta} \frac{1}{(t_i - t_{i-1})^{1/2-\zeta}} \tag{7.18}$$

The product of all terms with powers of $\epsilon^{-2\zeta}$ is bounded by $\epsilon^{-2\zeta M}$ uniformly in π . The product of the ϵ -factors gives:

$$C(\epsilon, \Delta) := \epsilon^{|\{\delta_i=0,1\}| \Delta - |\{\delta_i=1\}| - \frac{1}{2}|\{\delta_i=0\}| \epsilon^{|\{\delta_i=2, p_i \geq 2, \delta_{i-1} \neq 0\}|}. \tag{7.19}$$

Hence, using arguments analogous the the ones used to get (7.17), we have

$$\int_0^t dt_1 \dots \int_{t_{m-1}}^{t-\Delta} dt_m f_{1,\dots,m} \leq c \epsilon^{-2\zeta M} C(\epsilon, \Delta) \int_0^t dt_1 \dots \int_{t_{m-1}}^{t-\Delta} dt_m \tilde{f}_{1,\dots,m} \tag{7.20}$$

where the new integrand $\tilde{f}_{1,\dots,m}$ is independent of ϵ and given by

$$\tilde{f}_{1,\dots,m} := \prod_{\delta_i=1} \left[\frac{1}{(t_i - t_{i-1})^{1/2-\zeta}} \right] \prod_{\delta_i=2} \left[\frac{1}{(t_i - t_{i-1})^{q_i-\zeta}} \right] \prod_{\delta_i=0} \left[\frac{1}{(t_{i+1} - t_i)^{1/2}} \frac{1}{(t_i - t_{i-1})^{1/2-\zeta}} \right] \tag{7.21}$$

q_i being defined for $i : \delta_i = 2$ as follows: $q_i = \frac{1}{2}$ when either $p_i = 1$ or $p_i \geq 2$ and $\delta_{i-1} = 0$; in all other cases $q_i = 1$, i.e. when $p_i \geq 2$ and $\delta_{i-1} > 0$. To prove that the integral is finite we observe that for any $u < v$, $\alpha < 1$, $\beta < 1$,

$$\int_u^v \frac{1}{(s-u)^\alpha (v-s)^\beta} ds = c_{\alpha,\beta} (v-u)^{1-\alpha-\beta}$$

with $c_{\alpha,\beta} = \int_0^1 \frac{1}{s^\alpha (1-s)^\beta} ds < \infty$. We use the above formula when integrating successively t_m, t_{m-1}, \dots observing that the sum $\alpha + \beta$ at each step is < 1 so that the resulting expression is bounded by a constant. Once established that the integral is finite a *scaling* argument yields

$$\int_0^t dt_1 \dots \int_{t_{m-1}}^t dt_m \mathbf{1}_{t_m < t-\Delta} \tilde{f}_{1,\dots,m} \leq c t^S \tag{7.22}$$

where

$$S \geq \frac{1}{2} \left(|\{\delta_i = 1\}| + |\{\delta_i = 2, p_i = 1\}| + |\{\delta_i = 2, p_i \geq 2, \delta_{i-1} = 0\}| \right) - \zeta M. \tag{7.23}$$

Proposition 7.1 is proved. □

8 Bounds when times cluster

The expectation which appears in $w''_\pi(\underline{x}, t)$ could be bounded exactly as in $w'_\pi(\underline{x}, t)$, the problem is that the power S in (7.22) could then be negative and spoil the final bound. For $w'_\pi(\underline{x}, t)$ in fact the powers $(t - t_i)^{-1/2}$ and $(t - t_i)^{-1}$ could be bounded by $\Delta^{-1/2}$ and Δ^{-1} respectively, now $t - t_i$ might be smaller than Δ . The factors $(t - t_i)^{-1/2}$

and $(t - t_i)^{-1}$ may produce a negative S in the integral in (7.22). The argument used in the proof of Lemma 7.2 to bound the expectation was based on an iterative argument where at each step we conditioned on the “previous time” bounding the conditional expectation uniformly on the positions of the particles at the conditioning time. We should here do better taking into account the fact that the conditioning configuration may be “favorable”.

Using throughout the sequel the notation $t_0 \equiv 0$ and $t_{m+1} \equiv t$, we let, for $1 \leq H \leq m$

$$\mathcal{T}_H := \{t_1, \dots, t_m : t_{i+1} - t_i < \Delta, i = H, \dots, m; t_H - t_{H-1} \geq \Delta\} \tag{8.1}$$

and call t_H, \dots, t_m, t the “last cluster”. Since we are supposing $t > (M + 1)\Delta$, we readily see that the domain of integration $\{t_m \geq t - \Delta\}$ in $w''_\pi(\underline{x}, t)$ can be decomposed as the union of \mathcal{T}_H for $H = 1, \dots, m$. In any \mathcal{T}_H there is a time t_{H-1} not belonging to the last cluster (this is $t_0 \equiv 0$ if $H = 1$). Recalling that A_i denotes the set of labels at time t_i^+ (the particles alive), given H and $t_1, \dots, t_m \in \mathcal{T}_H$, we give the following definition.

Definition. We denote by $w''_{\pi,H}(\underline{x}, t)$ the integral in (6.6) extended to $t_1, \dots, t_m \in \mathcal{T}_H$. We then write G_H for the set of all indices $i \geq H$ such that $\delta_i < 2$ and there is $\ell \in A_{H-1} \cap \{k_i, l_i\}$ such that: $\ell \notin \{k_j, l_j\}$ for any $j \in [H, i)$ with $\delta_j = 0$.

Proposition 8.1. For any $\zeta > 0$ there is c so that for all $\pi : m = m(\pi) \leq M$, for all $\underline{x} : |\underline{x}| = n$, for all $\epsilon > 0$ and all $t \leq \epsilon^{\beta^*}$, recalling (7.5), we have

$$w''_{\pi,H}(\underline{x}, t) \leq \{c(\epsilon^2 t)^{-\zeta M} \Delta^{-S_1(H-1)} \epsilon^{S_2(H-1)} t^{S_3(H-1)}\} \{c(\epsilon^2 \Delta)^{-\zeta M} (\epsilon^{-2} \Delta)^{-\frac{1}{4}|G_H|} \Delta^{\frac{1}{2}|\{i \in [H, m] : \delta_i = 2\}|\}\} \tag{8.2}$$

where the first curly bracket is equal to the right hand side of (7.4) with m replaced by $H - 1$.

Recalling the definition of ϕ_h in (7.6) we define for $h \geq H$

$$\phi_h^* := \prod_{i>h: i \in G_H} \frac{1}{[\epsilon^{-2} \Delta]^{1/4-\zeta} + 1} \prod_{i>h: \delta_i=0} \frac{1}{[\epsilon^{-2}(t_{i+1} - t_i)]^{1/2} + 1} \tag{8.3}$$

and set $\phi_h^* = \phi_h$ for $h < H$. The analogue of Lemma 7.2 then holds:

Lemma 8.2. There is c and, for any k, c' so that for all H , all $t_1, \dots, t_m \in \mathcal{T}_H$ and for any h such that $H \leq h \leq m$

$$\phi_h^* \mathbf{E}_\epsilon [\psi_h] \leq c \phi_{h-1}^* \mathbf{E}_\epsilon [\psi_{h-1}] + c' (\epsilon^{-2} \Delta)^{-k}. \tag{8.4}$$

Proof. Let $h \geq H$. If $h \notin G_H$ and $\delta_h < 2$ we bound $\mathbf{1}_{|x_{k_h}(t_h) - x_{l_h}(t_h)|=1} \leq 1$. Analogously if $\delta_h = 2$ so that $h \notin G_H$, we bound $\mathbf{1}_{R_h} \leq 1$ and in both cases we proceed as in Lemma 7.2. It thus remains to consider the case $h \in G_H$ where the analysis is much more complex and relies heavily on the results in Section 4.

Recalling (7.6) and calling ℓ the particle-label entering in the definition of G_H , we factorize $\psi_h = \psi_{H-1} \psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell,+)}$ where $\psi_{H,h}^{(\neq \ell)}$ does not depend on $x_\ell(\cdot)$ and it is given by

$$\psi_{H,h}^{(\neq \ell)} := \prod_{H \leq i < h : \delta_i = 0, 1} \mathbf{1}_{|x_{k_i}(t_i) - x_{l_i}(t_i)|=1} \prod_{H \leq i < h : \delta_i = 0} \mathbf{1}_{T_i} \prod_{H \leq i < h : \delta_i = 2} \mathbf{1}_{(x_j(t_i^-), j \neq \ell) \in I_{u_i}^c}. \tag{8.5}$$

Then $\psi_{H,h}^{(\ell,+)}$ is equal to

$$\psi_{H,h}^{(\ell,+)} := \psi_{H,h}^{(\ell)} \mathbf{1}_{T_h}, \quad \psi_{H,h}^{(\ell)} := \mathbf{1}_{|x_{k_h}(t_h) - x_{l_h}(t_h)|=1} \prod_{i \in \mathcal{I}} \mathbf{1}_{x_\ell(t_i^-) \in I_{u_i}^c} \tag{8.6}$$

$$\mathcal{I} = \{i \in [H, h] : \delta_i = 2\}.$$

By (7.9)–(7.10) we get

$$\mathbf{E}_\epsilon [\psi_h] \leq \frac{c}{[\epsilon^{-2}(t_{h+1} - t_h)]^{1/2} + 1} \mathbf{E}_\epsilon \left[\psi_{H-1} \psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell)} \right]. \tag{8.7}$$

Since $h \in G_H$, ℓ is one of the two labels, k_h, l_h , for the sake of definiteness suppose $l_h = \ell$. We compute the expectation on the right hand side of (8.7) by conditioning on $\mathcal{F}(t_{H-1})$, then ψ_{H-1} drops from the conditional expectation and we are left with

$$\mathbf{E}_\epsilon^* (\psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell)}) := \mathbf{E}_\epsilon \left[\psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell)} \mid \mathcal{F}(t_{H-1}) \right]. \tag{8.8}$$

Denote by \underline{x}^* the configuration at time t_{H-1}^+ , A_{H-1} being their labels; one of the particles is ℓ with position x_ℓ^* . We shall estimate (8.8) using the coupling with independent particles defined in Section 4. More precisely in each interval (t_i, t_{i+1}) , $i \geq H - 1$, we realize (couple) the stirring process in terms of independent random walks $\underline{x}^0(s)$ adding and removing particles at the end of each interval as specified by π , independent particles are added on the same sites as the stirring ones, the sites are specified by the skeleton π ; the independent particles start, as the stirring ones, at t_{H-1} from \underline{x}^* . The coupling is defined in terms of a priority list which is arbitrary except for the requirement that particle ℓ has the lowest priority. The reason for this is that any stirring particles position $x_m(s)$, $m \neq \ell$, $t_{H-1} \leq s \leq t$, is then a function of only the independent processes $x_k^0(s)$, $t_{H-1} \leq s \leq t$, and $k \neq \ell$. Recall that the stirring particles $x_j(s)$ are functions of the $\underline{x}^0(\cdot)$, as described above, so that in the end the expression (8.8) becomes an expectation for independent particles.

We decompose the identity by writing $1 = \chi + (1 - \chi)$ where

$$\chi = \mathbf{1}_{|x_\ell(t_h) - x_\ell^0(t_h)| \leq (\epsilon^{-2}\Delta)^{1/4+\zeta}} \prod_{i \in \mathcal{I}} \mathbf{1}_{|x_\ell(t_i) - x_\ell^0(t_i)| \leq (\epsilon^{-2}\Delta)^{1/4+\zeta}}. \tag{8.9}$$

By (4.10) for any n there is c so that

$$\mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell)} (1 - \chi) \right] \leq (h - H + 1)c(\epsilon^{-2}\Delta)^{-n}. \tag{8.10}$$

Writing F_i for all sites $x \in \Lambda_N$ which have distance $> (\epsilon^{-2}\Delta)^{1/4+\zeta}$ from I_{u_i} we introduce the variable ω with values in \mathcal{I} , see (8.6), which has value $i < h$ if i is the smallest integer in \mathcal{I} such that $x_\ell^0(t_i) \notin F_i$. We set $\omega = h$ if such i does not exist. We then have $1 = \sum_{i \in \mathcal{I}} \mathbf{1}_{\omega=i}$. We bound for $i < h$

$$\begin{aligned} \mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell)} \chi \mathbf{1}_{\omega=i} \right] &\leq \mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \mathbf{1}_{x_\ell^0(t_i) \notin F_i} \right] = \mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \right] \mathcal{P} \left[x_\ell^0(t_i) \notin F_i \right] \\ &\leq \frac{c}{(\epsilon^{-2}\Delta)^{1/4-\zeta} + 1} \mathbf{E}_\epsilon \left[\psi_{H,h}^{(\neq \ell)} \mid \mathcal{F}(t_{H-1}) \right]. \end{aligned} \tag{8.11}$$

When $\omega = h$ we have for all $i \in \mathcal{I}, i < h$

$$\mathbf{1}_{x_\ell(t_i) \in I_{u_i}^c} \chi \mathbf{1}_{\omega=h} = \chi \mathbf{1}_{\omega=h} \tag{8.12}$$

and

$$\mathbf{1}_{|x_\ell(t_h) - x_{k_h}(t_h)|=1} \chi \mathbf{1}_{\omega=h} \leq \chi \mathbf{1}_{\omega=h} \mathbf{1}_{|x_\ell^0(t_h) - x_{k_h}(t_h)| \leq 1 + (\epsilon^{-2}\Delta)^{1/4+\zeta}} \tag{8.13}$$

so that $\mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell)} \chi \mathbf{1}_{\omega=h} \right] \leq \mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \mathbf{1}_{|x_\ell^0(t_h) - x_{k_h}(t_h)| \leq 1 + (\epsilon^{-2}\Delta)^{1/4+\zeta}} \right]$. We condition on $\mathcal{F}^{(\ell)}$, the σ -algebra generated by the variables x_j^0 , $j \neq \ell$, (including their “attempted jumps”, see Section 4) observing that under $\mathbf{P}_\epsilon[\cdot | \mathcal{F}^{(\ell)}]$ the variable x_ℓ^0 is a simple random walk. Then

$$\begin{aligned} \mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \psi_{H,h}^{(\ell)} \chi \mathbf{1}_{\omega=h} \right] &\leq \mathbf{E}_\epsilon^* \left[\psi_{H,h}^{(\neq \ell)} \left(\mathbf{P}_\epsilon \left[|x_\ell^0(t_h) - x_{k_h}(t_h)| \leq 1 + (\epsilon^{-2}\Delta)^{1/4+\zeta} | \mathcal{F}^{(\ell)} \right] \right) \right] \\ &\leq \frac{c}{(\epsilon^{-2}\Delta)^{1/4-\zeta} + 1} \mathbf{E}_\epsilon \left[\psi_{H,h}^{(\neq \ell)} \mid \mathcal{F}(t_{H-1}) \right]. \end{aligned} \tag{8.14}$$

From (8.7), (8.8), (8.11), (8.14) and observing that $\mathbf{E}_\epsilon(\psi_{H-1} \psi_{H,h}^{(\neq \ell)}) = \mathbf{E}_\epsilon(\psi_{h-1})$, we get (8.4). \square

Proof of Proposition 8.1. Using Lemma 8.2 we get

$$w''_{\pi,H}(\underline{x}, t) \leq \int_0^t dt_1 \dots \int_{t_{m-1}}^t dt_m \mathbf{1}_{\mathcal{T}_H} f_{1,\dots,H-1} g_{H,\dots,m} \tag{8.15}$$

where \mathcal{T}_H is defined in (8.1), $f_{1,\dots,H-1}$ in (7.14) and

$$\begin{aligned} g_{H,\dots,m} &:= (\epsilon^{-2}\Delta)^{(-\frac{1}{4}+\zeta)|G_H|} \prod_{\delta_i=1} \left[\frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1-\zeta} + 1} \right] \prod_{\delta_i=2} \left[\frac{\epsilon^{-1}}{[\epsilon^{-2}(t_i-t_{i-1})]^{p_i/2} + 1} \right] \\ &\prod_{\delta_i=0} \left[\frac{\epsilon^{-2}}{[\epsilon^{-2}(t-t_i)]^{1/2-\zeta} + 1} \frac{1}{[\epsilon^{-2}(t_{i+1}-t_i)]^{1/2} + 1} \right]. \end{aligned} \tag{8.16}$$

We bound all factors in $g_{h,\dots,m}$ with $\delta_i = 2$ as in (7.18), in all the others we drop the +1 addendum in the denominator so that we have a product of pure powers. The same argument used in the proof of Lemma 7.2 shows that the integral is finite. After observing that $t_H \geq t - (m - H + 1)\Delta$, a scaling argument yields:

$$\text{l.h.s. of (8.16)} \leq c(\epsilon^2\Delta)^{-\zeta M} (\epsilon^{-2}\Delta)^{-\frac{1}{4}|G_H|} \Delta^{\frac{1}{2}|\{\delta_i=2\}|} \tag{8.17}$$

which gives the second curly bracket in (8.2). We are left with

$$\int_0^t dt_1 \dots \int_{t_{H-2}}^t dt_{H-1} \mathbf{1}_{t_{H-1} < t - \Delta} f_{1,\dots,H-1}$$

which is the same as in Proposition 7.1, thus giving the first curly bracket in (8.2). \square

9 Proof of Theorem 6.1

Since M is fixed the number of skeletons π is finite. Hence recalling (6.5), (7.3) and the definition of $w''_{\pi,H}$ before Proposition 8.1, it will suffice to prove

$$\max_{\pi} w'_{\pi}(\underline{x}, t) \leq c(\epsilon^{-2}t)^{-c^*n}, \quad \max_{\pi,H} w''_{\pi,H}(\underline{x}, t) \leq c(\epsilon^{-2}t)^{-c^*n}. \tag{9.1}$$

We will need to distinguish various cases: firstly if $m(\pi) < M$ or $m(\pi) = M$; then if $t > (M + 1)\epsilon^\alpha$ (see (7.1)) or the opposite and in all these cases we will have different arguments for w'_π and $w''_{\pi,H}$.

- $m(\pi) = M$. **Estimate of $w'_\pi(\underline{x}, t)$ for $t > (M + 1)\epsilon^\alpha$.**

From (7.4)–(7.5) we get

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} [\epsilon \Delta^{-1}]^{|\delta_i=1|} \Delta^{-\frac{1}{2}(|\delta_i=0| + |\delta_i=2, p_i \geq 2, \delta_{i-1} \neq 0|)} t^{\frac{1}{2}|\delta_i=2, p_i=1|} \quad (9.2)$$

having used $t^{[S_3(m)]} \leq t^{\frac{1}{2}|\delta_i=2, p_i=1|}$. Observing that

$$|\delta_i = 0| + |\delta_i = 2, p_i \geq 2, \delta_{i-1} > 0| \geq |\delta_i = 2, p_i \geq 2| \quad (9.3)$$

we bound

$$\begin{aligned} \Delta^{-\frac{1}{2}(|\delta_i=0| + |\delta_i=2, p_i \geq 2, \delta_{i-1} \neq 0|)} &\leq \Delta^{-\frac{1}{2}(|\delta_i=0| + \frac{1}{2}|\delta_i=2, p_i \geq 2, \delta_{i-1} \neq 0|)} \epsilon^{\frac{1}{2}|\delta_i=2, p_i \geq 2, \delta_{i-1} \neq 0|} \\ &\leq [\epsilon \Delta^{-\frac{1}{2}}]^{\frac{1}{2}(|\delta_i=0| + |\delta_i=2, p_i \geq 2|)}. \end{aligned}$$

We thus get

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} [\epsilon \Delta^{-1}]^{|\delta_i=1|} [\epsilon \Delta^{-\frac{1}{2}}]^{\frac{1}{2}(|\delta_i=0| + |\delta_i=2, p_i \geq 2|)} t^{\frac{1}{2}|\delta_i=2, p_i=1|}. \quad (9.4)$$

Let $\epsilon^b := \max\{\epsilon \Delta^{-1}; [\epsilon \Delta^{-\frac{1}{2}}]^{\frac{1}{2}}; \epsilon^{\frac{\beta^*}{2}}\}$ then

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} [\epsilon^b]^{|\delta_i=1| + |\delta_i=0| + |\delta_i=2, p_i \geq 2| + |\delta_i=2, p_i=1|} \leq (\epsilon^{2+\beta^*})^{-\zeta M} \epsilon^{bM}. \quad (9.5)$$

We choose M so that

$$bM \geq 2n \quad (9.6)$$

which implies

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^{2+\beta^*})^{-\zeta M} \epsilon^{2n} \leq c' \epsilon^n \quad (9.7)$$

by choosing ζ small enough. Hence this term is compatible with (9.1) provided $c^* < \frac{1}{2}$.

- $m(\pi) = M$. **Estimate of $w'_\pi(\underline{x}, t)$ for $t \leq (M + 1)\epsilon^\alpha$.**

In this case t is proportional to Δ hence by (7.4)–(7.5)

$$w'_\pi(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} [\epsilon t^{-\frac{1}{2}}]^{|\delta_i=0| + |\delta_i=1|} \epsilon^{|\delta_i=2, p_i \geq 2, \delta_{i-1} > 0|} t^{\frac{1}{2}\{|\delta_i=2, p_i=1| + |\delta_i=2, p_i \geq 2, \delta_{i-1}=0|\}}.$$

We may suppose without loss of generality that $\epsilon^{-2}t \geq 1$, in such a case

$$w'_\pi(\underline{x}, t) \leq c\epsilon^{-4\zeta M} \epsilon^{|\delta_i=2, p_i \geq 2, \delta_{i-1} > 0|} t^{\frac{1}{2}\{|\delta_i=2, p_i=1| + |\delta_i=2, p_i \geq 2, \delta_{i-1}=0|\}}$$

hence

$$\begin{aligned} w'_\pi(\underline{x}, t) &\leq c\epsilon^{-4\zeta M} \epsilon^{\beta^* \{|\delta_i=2, p_i \geq 2, \delta_{i-1} > 0| + \frac{1}{2}(|\delta_i=2, p_i=1| + |\delta_i=2, p_i \geq 2, \delta_{i-1}=0|)\}} \\ w'_\pi(\underline{x}, t) &\leq c\epsilon^{-4\zeta M} \epsilon^{\frac{\beta^*}{2}|\delta_i=2|}. \end{aligned} \quad (9.8)$$

Observe that

$$\begin{aligned} |\delta_i = 1| + |\delta_i = 0| + |\delta_i = 2, p_i \geq 2| &\leq M \\ |\delta_i = 1| + |\delta_i = 0| &\leq n + (K - 1)|\delta_i = 2, p_i \geq 2| \end{aligned} \quad (9.9)$$

so that

$$|\delta_i = 1| + |\delta_i = 0| \leq n + \frac{K-1}{K}M \tag{9.10}$$

Since $|\delta_i = 2| + |\delta_i = 1| + |\delta_i = 0| = M$, for $M > 2Kn$

$$|\delta_i = 2| \geq M - \{n + \frac{K-1}{K}M\} = \frac{M}{K} - n \geq \frac{M}{2K}. \tag{9.11}$$

We choose M so that

$$\frac{\beta^* M}{2 \cdot 2K} \geq 2n. \tag{9.12}$$

Then

$$w'_\pi(\underline{x}, t) \leq c\epsilon^{-4\zeta M} \epsilon^{2n} \tag{9.13}$$

which for ζ small enough proves compatibility with (9.1) provided $c^* < \frac{1}{2}$.

• $m(\pi) = M$. **Estimate of $w''_{\pi,H}(\underline{x}, t)$ for $t > (M + 1)\epsilon^a$.**

Suppose first $H \geq \frac{M}{2}$. We bound by 1 the second curly bracket in (8.2) and use the same arguments as those used to get (9.4). In this case, writing $|\delta_i = k|_H$ for $|\{i < H : \delta_i = k\}|$:

$$w''_{\pi,H}(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} [\epsilon \Delta^{-1}]^{|\delta_i=1|_H} [\epsilon \Delta^{-\frac{1}{2}}]^{\frac{1}{2}(|\delta_i=0|_H + |\delta_i=2, p_i \geq 2|_H)} t^{\frac{1}{2}|\delta_i=2, p_i=1|_H}$$

so that for b as before (9.4), we have

$$w''_{\pi,H}(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} \epsilon^{b(H-1)} \leq (\epsilon^2 t)^{-\zeta M} \epsilon^{b(M/2-1)}$$

Taking M such that $b(M/2 - 1) > 2n$ we get, as in (9.6), $w''_{\pi,H}(\underline{x}, t) \leq c\epsilon^n$ compatible with (9.1) provided $c^* < \frac{1}{2}$.

Suppose next that $H < \frac{M}{2}$. We bound by 1 the first curly bracket in (8.2) to write

$$w''_{\pi,H}(\underline{x}, t) \leq c(\epsilon^2 \Delta)^{-\zeta M} (\epsilon^{-2} \Delta)^{-\frac{1}{4}|G_H|} \Delta^{\frac{1}{2}|\{i \in [H, m] : \delta_i=2\}|} \tag{9.14}$$

We have that

$$\begin{aligned} |G_H| &\geq \frac{1}{2} |\{i \geq H : \delta_i = 1, 0 \text{ and } \{k_i, \ell_i\} \cap A_{H-1} \neq \emptyset\}| \\ K|\{i \in [H, m] : \delta_i = 2\}| &\geq |\{i \in [H, m] : \delta_i = 1, 0 \text{ and } \{k_i, \ell_i\} \cap A_{H-1} = \emptyset\}| \end{aligned} \tag{9.15}$$

Letting $\epsilon^{b'} := \max\{(\epsilon^{-2} \Delta)^{-\frac{1}{8}}; \Delta^{\frac{1}{2}}\}$, one has

$$w''_{\pi,H}(\underline{x}, t) \leq c(\epsilon^2 \Delta)^{-\zeta M} \epsilon^{b'(|i \in [H, m] : \delta_i=1, 0 \text{ and } \{k_i, \ell_i\} \cap A_{H-1} \neq \emptyset| + |i \in [H, m] : \delta_i=2|)} \tag{9.16}$$

First assume that (also simplifying a bit the notation)

$$|i \geq H : \delta_i = 1, 0 \text{ and } \{k_i, \ell_i\} \cap A_{H-1} = \emptyset| \geq \frac{1}{2} |i \geq H : \delta_i = 1, 0|, \tag{9.17}$$

and the second relation in (9.15) yields

$$|i \geq H : \delta_i = 2| \geq \frac{1}{2K} |i \geq H : \delta_i = 1, 0|$$

so that (since $4K > 2$)

$$\begin{aligned} |i \geq H : \delta_i = 1, 0 \text{ and } \{k_i, \ell_i\} \cap A_{H-1} \neq \emptyset| + |i \geq H : \delta_i = 2| &\geq |i \geq H : \delta_i = 2| \\ &\geq \frac{1}{2} |i \geq H : \delta_i = 2| + \frac{1}{4K} |i \geq H : \delta_i = 1, 0| \geq \frac{M-H}{4K} \geq \frac{M}{8K} \end{aligned} \tag{9.18}$$

From (9.16) and if $\frac{M}{8K} \geq 2n$, we get $|w''_{\pi,H}(\underline{x}, t)| \leq (\epsilon^2 \Delta)^{-\zeta M} \epsilon^{b'2n}$, compatible with (9.1).

For the case of complementary inequality in (9.17), we have from (9.16):

$$w''_{\pi,H}(\underline{x}, t) \leq c(\epsilon^2 \Delta)^{-\zeta M} \epsilon^{b'[\frac{1}{2}|i \geq H:\delta_i=1,0|+|i \geq H:\delta_i=2|]} \leq c(\epsilon^2 \Delta)^{-\zeta M} \epsilon^{b'[\frac{1}{2}(M-H)]}$$

with $M - H \geq \frac{M}{2}$.

- $m(\pi) = M$. **Estimate of $w''_{\pi,H}(\underline{x}, t)$ for $t \leq (M + 1)\epsilon^a$.**

In (8.2) we estimate the first curly bracket as in (9.8) and we bound the second one with $\Delta^{|i \geq H:\delta_i=2|}$, getting

$$w''_{\pi,H}(\underline{x}, t) \leq c\epsilon^{-4\zeta M} \epsilon^{\frac{\beta^*}{2}|i < H:\delta_i=2|} \Delta^{|i \geq H:\delta_i=2|} \tag{9.19}$$

and the same argument used for $m(\pi) = M$, $w'_{\pi}(\underline{x}, t)$, $t \leq (M + 1)\epsilon^a$ applies. We thus have compatibility with (9.1) if $c^* < \frac{1}{2}$.

- $m(\pi) < M$. **Estimate of $w'_{\pi}(\underline{x}, t)$ for $t \leq (M + 1)\epsilon^a$.**

In this case t is proportional to Δ hence by (7.4)–(7.5) we get after multiplying and dividing by $t^{-\frac{1}{2}|\delta_i=2,p_i \geq 2,\delta_{i-1} > 0|}$

$$w'_{\pi}(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} [\epsilon t^{-\frac{1}{2}}]^{|\delta_i=0|+|\delta_i=1|} [\epsilon t^{-\frac{1}{2}}]^{|\delta_i=2,p_i \geq 2,\delta_{i-1} > 0|} t^{\frac{1}{2}\{|\delta_i=2,p_i=1|+|\delta_i=2,p_i \geq 2|\}}$$

By using the inequality

$$n \leq |\delta_i = 0| + 2|\delta_i = 1| + K|\delta_i = 2, p_i \geq 2|, \tag{9.20}$$

and bounding $|\delta_i = 2, p_i \geq 2|$ by (9.3), we get

$$n \leq (K + 1)|\delta_i = 0| + 2|\delta_i = 1| + K|\delta_i = 2, p_i \geq 2, \delta_{i-1} > 0|. \tag{9.21}$$

Notice that since $K \geq 1$ the largest factor is $K + 1$. Hence

$$w'_{\pi}(\underline{x}, t) \leq c(\epsilon^2 t)^{-\zeta M} [\epsilon t^{-\frac{1}{2}}]^{\frac{n}{1+K}} \leq c(\epsilon^{-2} t)^{-\frac{n}{2(1+K)}}, \tag{9.22}$$

and for ζ small enough we have compatibility with (9.1) if $c^* < \frac{1}{2(1+K)}$.

- $m(\pi) < M$. **Estimate of $w'_{\pi}(\underline{x}, t)$ for $t > (M + 1)\epsilon^a$.**

By (7.4)–(7.5), the inequality (9.21) and by dropping the factor $t^{S_3(m)}$ we get

$$\begin{aligned} w'_{\pi}(\underline{x}, t) &\leq c(\epsilon^2 t)^{-\zeta M} (\epsilon \Delta^{-1})^{|\delta_i=1|} (\epsilon \Delta^{-\frac{1}{2}})^{|\delta_i=0|} \epsilon^{|\delta_i=2, p_i \geq 2, \delta_{i-1} > 0|} \\ &\leq c(\epsilon^2 t)^{-\zeta M} \max\{(\epsilon \Delta^{-1})^{\frac{1}{2}}, (\epsilon \Delta^{-\frac{1}{2}})^{\frac{1}{K+1}}, \epsilon^{\frac{1}{K}}\}^n \leq c(\epsilon^2 t)^{-\zeta M} \epsilon^{\frac{1}{2(K+1)}n}, \end{aligned}$$

where the last inequality is true for our choice of Δ with $a = \frac{K}{K+1} \geq \frac{K-1}{K}$. (Note that for $a \geq \frac{K-1}{K}$ the dominant term is $(\epsilon \Delta^{-1})^{\frac{1}{2}}$.)

Thus, for ζ small enough we have compatibility with (9.1) if $c^* < \frac{1}{4(K+1)}$.

- $m(\pi) < M$. **Estimate of $w''_{\pi,H}(\underline{x}, t)$ for $t > (M + 1)\epsilon^a$.**

First observe that since for all $\ell \in A_{H-1}$ there is $i \geq H$ such that $\delta_i = 0, 1$ and $\ell \in \{k_i, \ell_i\}$ then

$$|A_{H-1}| \leq 2|G_H| + K|\delta_i = 2| \tag{9.23}$$

Suppose first that $|A_{H-1}| < n$. From (9.23) we get that the second factor in (8.2) is bounded by

$$c(\epsilon^2 \Delta)^{-\zeta M} \left(\max\{(\epsilon^{-2} \Delta)^{-\frac{1}{8}}, \Delta^{\frac{1}{2K}}\} \right)^{|A_{H-1}|} \leq c(\epsilon^2 \Delta)^{-\zeta M} \epsilon^{\frac{1}{2(K+1)}|A_{H-1}|} \tag{9.24}$$

by our choice of Δ , since for $a \leq \frac{2K}{K+4}$ the dominant term is $\Delta^{\frac{1}{2K}}$.

On the other hand, for the first factor of (8.2) we apply the result as for the case $m(\pi) < M$, $w'_\pi(\underline{x}, t)$, $t > (M + 1)\epsilon^a$ which gives the bound $c(\epsilon^2 t)^{-\zeta M} \epsilon^{\frac{1}{2(K+1)}(n-|A_{H-1}|)}$.

Overall we have:

$$w''_{\pi,H}(\underline{x}, t) \leq c(\epsilon^2 \Delta)^{-2\zeta M} \epsilon^{\frac{1}{2(K+1)}(n-|A_{H-1}|)} \epsilon^{\frac{1}{2(K+1)}|A_{H-1}|}$$

which for ζ small enough is compatible with (9.1) if $c^* < \frac{1}{4(K+1)}$. Same conclusions hold for $|A_{H-1}| \geq n$ as it suffices to use only the bound (9.24).

- $m(\pi) < M$. **Estimate of $w''_{\pi,H}(\underline{x}, t)$ for $t \leq (M + 1)\epsilon^a$.**

As in the previous case it suffices to investigate the case $|A_{H-1}| < n$. The second factor in (8.2) is bounded from above by

$$c(\epsilon^2 \Delta)^{-\zeta M} \left(\max\{(\epsilon^{-2} t)^{-\frac{1}{8}}, t^{\frac{1}{2K}}\} \right)^{|A_{H-1}|}$$

We have for $t \geq \epsilon^{\frac{2K}{K+4}}$ (which is $\geq \epsilon^2$),

$$\max\{(\epsilon^{-2} t)^{-\frac{1}{8}}, t^{\frac{1}{2K}}\} \leq t^{\frac{1}{2K}}.$$

For $\alpha := \frac{1}{2(K+2)}$

$$t^{\frac{1}{2K}} \leq (\epsilon^2 t^{-1})^\alpha, \quad \text{for all } \epsilon^{\frac{2K}{K+4}} \leq t \leq \epsilon^a.$$

For $\epsilon^{\frac{2K}{K+4}} \geq t \geq \epsilon^2$

$$\max\{(\epsilon^{-2} t)^{-\frac{1}{8}}, t^{\frac{1}{2K}}\} \leq (\epsilon^{-2} t)^{-\frac{1}{8}}.$$

By (9.22) the first factor in (8.2) is bounded from above by

$$c(\epsilon^2 t)^{-\zeta M} [\epsilon t^{-\frac{1}{2}}]_{1+K}^{\frac{n}{2}} \leq c(\epsilon^{-2} t)^{-\frac{n-|A_{H-1}|}{2(1+K)}} \tag{9.25}$$

Overall we have compatibility with (9.1) if $c^* < \frac{1}{2(2+K)}$. □

10 Long times

In this section we extend the estimate on the v -functions to times of order $\log \epsilon^{-1}$. In Theorem 6.1 we have proved bounds for times $t \leq \epsilon^{\beta^*}$, where β^* is any given positive number. Here we shall study times $t \geq \epsilon^{\beta^*}$:

Theorem 10.1. For suitable $\tau > 0$ (which depends on c^* in Theorem 6.1) and for any n there is c_n so that for all $\epsilon > 0$

$$\sup_{\eta_0} \sup_{\underline{x}: |\underline{x}|=n} \sup_{\epsilon^{\beta^*} \leq t \leq \tau \log \epsilon^{-1}} |v^\epsilon(\underline{x}, t | \eta_0)| \leq c_n \epsilon^{(2-\beta^*)c^* n}, \quad (10.1)$$

where we have made explicit the dependence of the v -function on ϵ and on the initial configuration η_0 .

The theorem will be proved later in this section. We start with some definitions:

Definition

- For any $b \in (0, 1)$ and $f : \Lambda_N \rightarrow \mathbb{R}$ we define

$$\|f\| := \sup_{x \in \Lambda_N} |f|_x, \quad |f|_x := \left| \sum_{y \in \Lambda_N} P_{\epsilon^{1+b}}^{(\epsilon)}(x, y) f(y) \right| \quad (10.2)$$

- $\rho_\epsilon(x, t | f, s)$, $t \geq s$, denotes the solution of (2.1) with initial datum f at time s .

The following holds:

Lemma 10.2. For any $\gamma > 0$ such that $\gamma \leq \inf \left\{ (2 - \beta^*)c^*, \frac{1-b}{4} \right\}$ the following holds. Given any n there is c_n so that for all ϵ and for all configurations η ,

$$\mathbb{P}_\epsilon^\eta (\| \rho_\epsilon(\cdot, \epsilon^{\beta^*} | \eta, 0) - \eta(\cdot, \epsilon^{\beta^*}) \| \leq \epsilon^\gamma) \geq 1 - c_n \epsilon^n \quad (10.3)$$

where $\eta(\cdot, \epsilon^{\beta^*})$ is the random configuration at time ϵ^{β^*} starting from η at time 0.

Proof. We first write

$$\mathbb{P}_\epsilon^\eta (\| \rho_\epsilon(\cdot, \epsilon^{\beta^*} | \eta, 0) - \eta(\cdot, \epsilon^{\beta^*}) \| \geq \epsilon^\gamma) \leq (2N + 1) \sup_{x \in \Lambda_N} \mathbb{P}_\epsilon^\eta (| \rho_\epsilon(\cdot, \epsilon^{\beta^*} | \eta, 0) - \eta(\cdot, \epsilon^{\beta^*}) |_x \geq \epsilon^\gamma).$$

By the Chebyshev inequality with power $2m$

$$\begin{aligned} \mathbb{P}_\epsilon^\eta (| \rho_\epsilon(\cdot, \epsilon^{\beta^*} | \eta, 0) - \eta(\cdot, \epsilon^{\beta^*}) |_x \geq \epsilon^\gamma) &\leq \epsilon^{-2m\gamma} \sum_{x_1, \dots, x_{2m}} \prod_{i=1}^{2m} P_{\epsilon^{1+b}}^{(\epsilon)}(x, x_i) \\ &\quad \times \mathbb{E}_\epsilon^\eta \left(\prod_{i=1}^{2m} [\eta(x_i, \epsilon^{\beta^*}) - \rho(x_i, \epsilon^{\beta^*} | \eta, 0)] \right). \end{aligned}$$

There are constants c and c' (dependent on m) so that

$$\left| \mathbb{E}_\epsilon^\eta \left(\prod_{i=1}^{2m} [\eta(x_i, \epsilon^{\beta^*}) - \rho(x_i, \epsilon^{\beta^*} | \eta, 0)] \right) \right| \leq c |v^\epsilon(Y, \epsilon^{\beta^*} | \eta)| \leq c' \epsilon^{-(2+\beta^*)c^* |Y|}$$

where $Y \subset \{x_1, \dots, x_{2m}\}$ is the set of singletons, i.e. all x_i such that $x_j \neq x_i$ for $j \neq i$; the last inequality follows from Theorem 6.1. On the other hand

$$P_{\epsilon^{1+b}}^{(\epsilon)}(x, x_i) \leq c(\epsilon^{-2}\epsilon^{1+b})^{-1/2} = c\epsilon^{(1-b)/2}$$

and (10.3) follows by the arbitrariness of m . □

We shall use the iterative approach used in Chapter 5 of [6], the presence of the factors ϵ^{-1} at the boundaries makes however the analysis different. Thus, given $\tau > 0$ (which will be specified later) we fix a time $\epsilon^{\beta^*} < T \leq \tau \log \epsilon^{-1}$. Let $r \in [\frac{1}{2}, 1]$ be such that $T = (m + 1)r\epsilon^{-\beta^*}$, $m \in \mathbb{N}$. We then partition the interval $[0, T] = \bigcup_{k=1}^{m+1} [t_{k-1}, t_k)$, $t_0 = 0$ where

$$[0, T] = \bigcup_{k=1}^{m+1} [t_{k-1}, t_k), \quad t_k = kr\epsilon^{\beta^*}, \quad m + 1 = r^{-1}\epsilon^{\beta^*}T. \tag{10.4}$$

Given any initial configuration η_0 , we denote by $\eta_k \equiv \eta(t_k)$ the random configurations at times t_k .

We study the evolution by successively conditioning the process at the times t_k . The conditioning at time t_{k-1} fixes the configuration η_{k-1} and by Lemma 10.2 the evolution in the next time step $[t_{k-1}, t_k)$ is well approximated by $\rho_\epsilon(x, t|\eta_{k-1}, t_{k-1})$. To iterate this estimate we need to bound the difference $\rho_\epsilon(x, t|\eta_k, t_k) - \rho_\epsilon(x, t|\eta_{k-1}, t_{k-1})$:

Proposition 10.3. *Let γ and b be positive numbers as in Lemma 10.2. Given any η_{k-1} , let η_k be such that $|||\rho_{k-1}(\cdot, t_k|\eta_{k-1}) - \eta_k||| \leq \epsilon^\gamma$ then there is c so that*

$$|\rho_\epsilon(x, t|\eta_k, t_k) - \rho_\epsilon(x, t|\eta_{k-1}, t_{k-1})| \leq c[\epsilon^\gamma + \epsilon^b]\epsilon^{-\tau\pi}\tau \log \epsilon^{-1}, \quad t \in (t_k, T]. \tag{10.5}$$

Proof. We denote by

$$h_k(x, t) = |\rho_\epsilon(x, t|\eta_k, t_k) - \rho_\epsilon(x, t|\eta_{k-1}, t_{k-1})|, \quad \in \Lambda_N; \quad h_k(t) = \sup_{x \in \Lambda_N} h_k(x, t). \tag{10.6}$$

Since $\rho_\epsilon(\cdot) \leq 1$, for any $t \in [t_k + \epsilon^{1+b}, T]$ (using the weak form of equation (2.1))

$$\begin{aligned} h_k(x, t) &\leq \sum_y P_{t-t_k-\epsilon^{1+b}}^{(\epsilon)}(x, y)|\rho_\epsilon(\cdot, t_k|\eta_{k-1}, t_{k-1}) - \eta_k|_y \\ &\quad + c_1\epsilon^{-1} \int_{t_k}^t \sum_{y \in I_+ \cup I_-} P_s^{(\epsilon)}(x, y)h_k(y, t-s)ds. \end{aligned} \tag{10.7}$$

Since (recalling (3.1))

$$\epsilon^{-1} \sum_{y \in I_+ \cup I_-} P_s^{(\epsilon)}(x, y) \leq \frac{c_2\tau \log \epsilon^{-1}}{\sqrt{s}}.$$

We then bound

$$\epsilon^{-1} \int_{t_k}^t \sum_{y \in I_+ \cup I_-} P_s^{(\epsilon)}(x, y)h_k(x, t-s) \leq 2K\epsilon^b + c_2\tau \log \epsilon^{-1} \int_{t_k+\epsilon^{1+b}}^t \frac{1}{\sqrt{s}}h_k(t-s)ds.$$

We thus get, using the hypothesis on η_k ,

$$h_k(x, t) \leq [\epsilon^\gamma + 2K\epsilon^b] + c_3\tau \log \epsilon^{-1} \int_{t_k+\epsilon^{1+b}}^t \frac{1}{\sqrt{s}}h_k(t-s)ds \quad t \in [t_k + \epsilon^{1+b}, T]. \tag{10.8}$$

Iterating (10.8) and letting $a_n(t)$ be as in (10.10), we get that for all $t \in [t_k + \epsilon^{1+b}, T]$

$$h_k(x, t) \leq c_4[\epsilon^\gamma + \epsilon^b](1 + \tau \log \epsilon^{-1} \sum_{n=1}^{\infty} a_n(t-t_k)). \tag{10.9}$$

Indeed the series converges as follows from Lemma 10.4 below. Since $t - t_k \leq T \leq \tau \log \epsilon^{-1}$ from (10.12) we get

$$\sum_{n=1}^{\infty} a_n(t - t_k) \leq c e^{\pi \tau \log \epsilon^{-1}} = c \epsilon^{-\pi \tau}.$$

Thus from (10.9) and (10.15) we finally get (10.5) for suitable τ . □

Lemma 10.4. *Let*

$$a_n(t) := \int_0^t \frac{1}{\sqrt{s_1}} ds_1 \int_0^{t-s_1} \frac{1}{\sqrt{s_2}} ds_2 \dots \int_0^{t-s_1-\dots-s_{n-1}} \frac{1}{\sqrt{s_n}} ds_n. \tag{10.10}$$

Then

$$a_n(t) \leq (\pi t)^{\frac{n}{2}} e^{-\frac{n}{2} [\log(\frac{n}{2}) - 1]} \tag{10.11}$$

and there is c so that

$$\sum_{n=1}^{\infty} a_n(t) \leq c e^{\pi t}. \tag{10.12}$$

Proof. We have

$$a_n(t) = \int_{[0,t]^n} \mathbf{1}_{s_1+\dots+s_n \leq t} \prod_{i=1}^n \frac{1}{\sqrt{s_i}} ds_1 \dots ds_n. \tag{10.13}$$

We change variable by setting $t_i = s_i t$ and get

$$a_n(t) = (\sqrt{t})^n \int_{[0,1]^n} \mathbf{1}_{t_1+\dots+t_n \leq 1} \prod_{i=1}^n \frac{1}{\sqrt{t_i}} dt_1 \dots dt_n. \tag{10.14}$$

Multiplying and dividing by $\exp\{-\alpha(t_1 + \dots + t_n)\}$ we have

$$a_n(t) \leq (\sqrt{t})^n e^{\alpha} \int_{[0,1]^n} \prod_{i=1}^n \frac{e^{-\alpha t_i}}{\sqrt{t_i}} dt_1 \dots dt_n \leq (\sqrt{t})^n e^{\alpha} \left[\int_0^1 \frac{e^{-\alpha s}}{\sqrt{s}} ds \right]^n \leq (\sqrt{t})^n e^{\alpha} \left(\frac{\sqrt{\pi}}{\sqrt{\alpha}} \right)^n. \tag{10.15}$$

By choosing $\alpha = \frac{n}{2}$ we get (10.11) and (10.12) easily follows. □

Proof of Theorem 10.1. Recalling the definitions given in (10.4) and below and letting $\rho_0 = \eta_0$, we write

$$-\rho_{\epsilon}(x, T | \eta_0, 0) = \sum_{k=1}^m [\rho_{\epsilon}(x, T | \eta_k, t_k) - \rho_{\epsilon}(x, T | \eta_{k-1}, t_{k-1})] - \rho_{\epsilon}(x, T | \eta_m, t_m). \tag{10.16}$$

Thus

$$\begin{aligned} v^{\epsilon}(X, T | \eta_0) &= \mathbb{E}_{\epsilon}^{\eta_0} \left(\prod_{x \in X} \left[(\eta(x, T) - \rho_{\epsilon}(x, T | \eta_m, t_m)) \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^m [\rho_{\epsilon}(x, T | \eta_k, t_k) - \rho_{\epsilon}(x, T | \eta_{k-1}, t_{k-1})] \right] \right) \\ &= \sum_{Y \subset X} \mathbb{E}_{\epsilon}^{\eta_0} \left(\prod_{y \in Y} (\eta(y, T) - \rho_{\epsilon}(y, T | \eta_m, t_m)) \right. \\ &\quad \left. \times \prod_{x \in X \setminus Y} \sum_{k=1}^m [\rho_{\epsilon}(x, T | \eta_k, t_k) - \rho_{\epsilon}(x, T | \eta_{k-1}, t_{k-1})] \right). \end{aligned}$$

We take conditional expectation with respect to $\mathcal{F}(\{\eta_t, t \leq t_m\})$ and since the bound on the v function in Theorem 6.1 is uniform in the initial configuration (η_m in this case) we get

$$|v(X, t)| \leq \sum_{Y \subset X} c_Y \epsilon^{c^* |Y|} \mathbb{E}_\epsilon^{\eta_0} \left(\prod_{x \in X \setminus Y} \sum_{k=1}^m |\rho_\epsilon(x, T | \eta_k, t_k) - \rho_\epsilon(x, T | \eta_{k-1}, t_{k-1})| \right).$$

Let

$$\mathcal{A}_\gamma := \{(\eta_1, \dots, \eta_m) : \sup_k \|\rho_\epsilon(\cdot, t_k | \eta_{k-1}, t_{k-1}) - \eta_k\| \leq \epsilon^\gamma\}.$$

Since

$$\mathbb{P}_\epsilon^{\eta_0} \left(\sup_k \|\rho_\epsilon(\cdot, t_k | \eta_{k-1}, t_{k-1}) - \eta_k\| \geq \epsilon^\gamma \right) \leq m \sup_\eta \mathbb{P}_\epsilon^\eta \left(\|\rho_\epsilon(\cdot, \epsilon^{\beta^*} | \eta, 0) - \eta_{\epsilon^{\beta^*}}\| \geq \epsilon^\gamma \right)$$

and by Lemma 10.2 for any integer q there is c_q so that

$$\mathbb{P}_\epsilon^\eta(\mathcal{A}_\gamma) \geq 1 - c_q \epsilon^q,$$

so that, using (10.5) we get

$$|v(X, t)| \leq \sum_{Y \subset X} c_Y \epsilon^{c^* |Y|} \left\{ \bar{c}_q \epsilon^q + [m \epsilon^{-d} (\epsilon^\gamma + \epsilon^b)]^{X \setminus Y} \right\}. \quad (10.17)$$

We choose the parameters in such a way that

$$\epsilon^{-\beta^*} d \log \epsilon^{-1} \epsilon^{-d} (\epsilon^\gamma + \epsilon^b) \leq \epsilon^{c^*}. \quad (10.18)$$

Theorem 10.1 then follows from (10.17) and (10.18). \square

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