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Escape from the Unstable Equilibrium in a Random Process with Infinitely Many Interacting Particles[1]

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We consider a one-dimensional version of the model introduced in Ref. 1. At each site of Z there is a particle with spin ± 1. Particles move according to the Stirring Process and spins change according to the Glauber dynamics. In the hydrodynamical limit, with the stirring process suitably speeded up, the local magnetic density $m_t(r)$ is proven in Ref. 1 to satisfy the reaction-diffusion equation

$$\partial_t m_t(r) = \tfrac{1}{2}\,\partial_r^2 m_t(r) - V'(m_t) \qquad (*)$$

$V(m) = -\frac{1}{2}\alpha m^2 + \frac{1}{4}\beta m^4$, α and $\beta > 0$, α and β being determined by the parameters of the Glauber dynamics. In the present paper we consider an initial state with zero magnetization, $m_0(r) = 0$. We then prove that at long times, before taking the hydrodynamical limit, the evolution departs from that predicted by $(*)$ and that the microscopic state becomes a nontrivial mixture of states with different magnetizations.

KEY WORDS: Interacting particle systems; reaction-diffusion equations; unstable equilibria.

1. INTRODUCTION

There are by now several examples where it has been possible to derive macroscopic equations from underlying microscopic evolutions, mostly in the frame of stochastic processes with infinitely many interacting particles and under appropriate space-time (hydrodynamic) scaling limits.[2–28]

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Their validity is usually proven for finite times, so that, rigorously speaking, they are useless when the long-time behavior of the system is investigated. Notions like stationary states, stable and unstable orbits, complexity of an orbit, and Lyapunov coefficients, require a more detailed analysis, still at a microscopic level.

The purpose of this paper is to discuss a model where some of these questions have an answer. The model is a one-dimensional version of that introduced in Ref. 1, namely a system of spin particles which move on the lattice Z according to the Stirring process, i.e., by simple exchanges. During the motion particles keep their own spin, which, in turn, may change according to a nearest neighbor, ferromagnetic, Glauber interaction.

The stirring evolution is speeded up by ε^{-2} (ε goes to zero) while the typical space length is scaled like ε^{-1}. In this limit the macroscopic local magnetization $m(r, t)$ satisfies,[1] the following reaction-diffusion equation:

$$\partial_t m = \tfrac{1}{2}\,\partial_r^2 m - V'(m) \tag{1.1}$$

where

$$V(m) = -\tfrac{1}{2}\alpha m^2 + \tfrac{1}{4}\beta m^4 \tag{1.2}$$

Clearly, the term $\partial_r^2 m$ in Eq. (1.1) is due to the Stirring process, while the $-V'(m)$ term originate from the Glauber dynamics. For a suitable choice of the latter [cf. Eq. (2.1)], we get Eq. (1.2), with α and β positive. Equation (1.1) describes the behavior of the system in the "pure" hydrodynamical regime, i.e., after the continuum limit has been performed. To take into account the "corrections" due to the actual microscopic structure of the system, one generally adds to the macroscopic equation a stochastic disturbance, hoping to catch, at least qualitatively, the main features of the evolution.

The simplest possibility is to consider a stationary stochastic force uncorrelated in space and time. So one arrives at the following stochastic differential equation:

$$dm(r, t) = (\tfrac{1}{2}\,\partial_r^2 m(r, t) - V'(m))\,dt + \varepsilon^{1/2}\,dW(r, t) \tag{1.3}$$

where W is a white noise (in space and time) and ε is the "small parameter" of the theory.

The finite-dimensional version of Eq. (1.3), i.e.,

$$dx = -U'(x)\,dt + \varepsilon^{1/2}\,dW \tag{1.4}$$

has also been extensively studied. In Eq. (1.4), $x \in R^d$, U is a "double-well" potential, and W is the standard Wiener motion.

The escape from the unstable equilibrium in Eqs. (1.3) and (1.4) [the spatially homogeneous one, $m=0$, in the case of Eq. (1.3)] is of interest in physics and chemistry, and several theoretical and numerical investigations have appeared on this subject.[27–32]

In particular, a perturbative expansion, as proposed by De Pasquale and Tombesi,[30] seems to be in good agreement with the experimental and numerical results. The main point behind their approach plays also an important role in our analysis, as we shall see in the next Sections.

The escape from equilibrium, as described by De Pasquale and Tombesi for the system corresponding to Eq. (1.3), and for times of order $\log \varepsilon^{-1}$, is in qualitative agreement with the behavior we observe in our microscopic system, in the same time scale.

In the present paper, for technical simplicity, we restrict ourselves to macroscopically finite regions. In this case we have an explicit solution at time $T_\varepsilon + t$, when ε vanishes, T_ε being an initial time layer which diverges when ε goes to zero like $\log \varepsilon^{-1}$. In such a limit the microscopic state becomes a nontrivial mixture of Bernoulli states, i.e., measures which make the spins mutually independent and identically distributed.

The weight of the decomposition is proven to obey the law of a random variable which solves an equation analogous to Eq. (1.3) (at time $T_\varepsilon + t$ and in the limit as ε goes to zero), the profile $m(r, t)$ becoming flat. We think, however, that in the infinite volume case the profile should have a nontrivial spatial structure.

The profiles entering in the decomposition evolve according to Eq. (1.1), so that in the limit as t diverges the state becomes a $\frac{1}{2}$–$\frac{1}{2}$ mixture of states with support on $\pm m^*$, the stable solutions of Eq. (1.1). We have here an example where at each r there is convergence in the continuum limit to a measure rather than to a real number (i.e., the value at r of the magnetization m, in our specific example). The role of measure-valued solutions in nonlinear PDE has been underlined in a different context by Di Perna[33] in his analysis of hyperbolic equations.

In our case their appearance is due to the fact that we take the infinite time limit along with the hydrodynamical one. Therefore the small fluctuations, inherited from the discrete nature of the model and intrinsically connected to the stochasticity of the evolution, are enhanced by the hyperbolic structure of the PDE and blow up exponentially, leading to finite effects after logarithmic times.

Another point we want to underline is the microscopic interpretation that we get for the values of $m(r, t)$ as the parameters which specify the "pure phases" entering in the decomposition of the microscopic state, in the limit when ε goes to zero.

In the hydrodynamical regime, $\varepsilon \to 0$, at fixed t, Eq. (1.1) holds and the state around $\varepsilon^{-1}r$ is pure Bernoulli with parameter $m(r, t)$. As time increases and for small but positive ε, the state becomes closer to a mixture of Bernoulli measures. At T_ε the mixture has finite width and its later evolution defines a statistical solution of Eq. (1.1), for finite times, in the limit when ε goes to zero. At such times the values of the magnetization are in the domain of attraction of $\pm m^*$, no instability is present, and Eq. (1.1) describes correctly the evolution also for longer times. At even longer times new phenomena take place: they are due to the tunnelling between the two stable magnetizations and to the nontrivial spatial structure, in the case of infinite systems.

Such effects are lost in our limiting procedure, i.e., if we first fix a time $T_\varepsilon + t$ and then take the limit of ε going to zero. In a much longer time scale large deviation effects enter into play and our techniques become inadequate. Results have been obtained in other models,[34-39] which might be useful also in our case.

In Sec. 2 we briefly recall the model introduced in Ref. 1. We then state precisely our results and give a qualitative idea of their proofs. The proofs are reported in Sec. 3 and Sec. 4. Some more technical estimates are given in the Appendices.

2. RESULTS

We fix $L>0$ and for each $\varepsilon>0$ we consider the thorus Z_ε of length $[\varepsilon^{-1}L]$ ($[a]$ = integer part of a). Z_ε is the set of all integers with the identification $x = x + [\varepsilon^{-1}L]$. We then consider the space $\{-1, 1\}^{Z_\varepsilon}$ of all spins configurations $\sigma = (\sigma_x)_{x\in Z}$ which are periodic with period $[\varepsilon^{-1}L]$. On $\{-1, 1\}^{Z_\varepsilon}$ we define a Markov process whose generator L^ε acts on the cylinder functions[5] f as

$$\begin{aligned} L^\varepsilon f &= L_G f + \varepsilon^{-2} L_e f \\ &= \sum_{x\in Z_\varepsilon} c(x, \sigma)[f(\sigma^x) - f(\sigma)] \\ &\quad + \tfrac{1}{2}\varepsilon^{-2} \sum_x [f(\sigma^{x,x+1}) - f(\sigma)] \end{aligned} \tag{2.1a}$$

$$\begin{aligned} &\sigma^x(y) = \sigma(y) \quad \text{for } y \neq x; \qquad \sigma^x(x) = -\sigma(x) \\ &\sigma^{x,x+1}(y) = \sigma(y) \quad \text{for } y \neq x, x+1; \\ &\sigma^{x,x+1}(x) = \sigma(x+1); \qquad \sigma^{x,x+1}(x+1) = \sigma(x) \end{aligned} \tag{2.1b}$$

[5] That is, functions depending only on finitely many spins.

$$c(x, \sigma) = 1 - \gamma\sigma(x)[\sigma(x+1) + \sigma(x-1)] + \gamma^2\sigma(x+1)\,\sigma(x-1) \quad (2.1c)$$

$$\alpha = 2(2\gamma - 1) > 0; \qquad \beta = 2\gamma^2 \quad (2.1d)$$

As mentioned in the introduction, L_G is the generator of an n.n. ferromagnetic Glauber dynamics, L_e is the generator of the Stirring Process (namely, each pair of n.n. sites waits independently of the others for a Poisson time of mean $\frac{1}{2}$; then, when "the clock rings," the particles, and hence the spins, of the sites involved exchange with each other).

We denote by ν_m, $m \in [-1, 1]$, the Bernoulli measure[6] on $\{-1, 1\}^Z$ with average spin m, i.e.,

$$\nu_m[\sigma(x)] = m \qquad \forall x \in Z$$

where $\mu[\cdot]$ is the expectation of $[\cdot]$. By ν_m^ε we denote the "Bernoulli measure" on Z_ε such that the law of the spins in an interval of length $[\varepsilon^{-1}L] - 1$ is that inherited from ν_m and, given their values, all the other spins are then specified in agreement with the periodic structure of Z_ε.

Let μ_t^ε be the law of the spins at time t when their initial distribution is ν_0^ε and the evolution is determined by L^ε. Our first result is:

2.1. Theorem

Let μ_t^ε be as above. For $a > 0$, let

$$T_\varepsilon(a) = (2\alpha)^{-1}\, a \log \varepsilon^{-1} \quad T_\varepsilon = T_\varepsilon(1) \quad (2.2)$$

Then, for any $t \geqslant 0$, there is a probability $\lambda_t(dm)$ on $[-1, 1]$ such that

$$\lim_{\varepsilon \to 0} \mu_{T_\varepsilon + t}^\varepsilon = \int \lambda_t(dm)\, \nu_m \quad (2.3a)$$

ν_m being the Bernoulli measure with parameter m. Furthermore,

$$\lim_{t \to \infty} \lambda_t = \tfrac{1}{2}\,\delta_{m^*} + \tfrac{1}{2}\,\delta_{-m^*} \quad (2.3b)$$

where $\pm m^*$ are the stable solutions of

$$d_t m = -V'(m), \qquad \text{where} \quad m \in [-1, 1] \quad (2.4)$$

while, on the other hand, if $a < 1$

$$\lim_{\varepsilon \to 0} \mu_{T_\varepsilon(a)}^\varepsilon = \nu_0 \quad (2.5)$$

[6] That is, the measure for which spins at different sites are independent.

We have a rather explicit knowledge of $\lambda_t(dm)$ also at finite times, namely:

2.2. Theorem

Let $\lambda_t(dm)$ be as in Theorem 2.1, and let $S_t(m)$ be the flow which solves Eq. (2.4). For $a<1$, let

$$\Gamma(a)=\tfrac{1}{2}(1-a) \tag{2.6}$$

Then there is a map $\psi\colon R\to[-1,1]$ such that

$$\lim_{\varepsilon\to 0} S_{T_\varepsilon - T_\varepsilon(a)}(\varepsilon^{\Gamma(a)}x)=\psi(x) \tag{2.7}$$

$\lambda_t(dm)$ is then the image under $S_t\circ\psi$ of the measure $G(x)\,dx$ on R, where

$$G(x)=(2\pi V)^{-1/2}\exp(-(2V)^{-1}x^2) \tag{2.8}$$

and

$$V=(2\alpha)^{-1}\,8\gamma \tag{2.9}$$

Remarks. (1) ψ^{-1} is the analogue of the map that De Pasquale and Tombesi[30] (cf. Refs. 29–32) introduce in their perturbative approach.

(2) $\lambda_t(dm)$ is also the limiting law, as ε vanishes, of the distribution of the variable $m^\varepsilon(0,\cdot)$ at time $T_\varepsilon+t$, where $m^\varepsilon(r,0)\equiv 0$ and

$$dm^\varepsilon(r,t)=[\tfrac{1}{2}\,\partial_r^2 m^\varepsilon(r,t)-V'(m(r,t))]\,dt+(\varepsilon 8\gamma)^{1/2}\,dW(r,t) \tag{2.10}$$

The choice of the noise in Eq. (2.10) is consistent with the analysis of Ref. 1. By linearizing Eq. (2.10) around $m=0$, we get, in fact, the fluctuation field $\phi(r,t)$ considered in Ref. 1. As time increases, ϕ grows exponentially,[1] and one needs to look at the magnetization itself. When m becomes finite, the strength $(\varepsilon 8\gamma)^{1/2}$ of the noise is no longer correct. In Ref. 1, in fact, it is proven that it depends effectively on the values of the magnetization [cf. Eqs. (2.17) and (2.18) of Ref. 1]. This is irrelevant, however, for the determination of $\lambda_t(dm)$ which is only sensitive to the initial fluctuations, when m is small.

(3) In the above $m^\varepsilon(r,t)$ labels the Bernoulli measures whose superposition (with weights given by the probability distribution of m) approximates the microscopic state around $\varepsilon^{-1}r$ at time t. There is also a relation between $m^\varepsilon(r,t)$ and the magnetic fluctuation fields mentioned in (2) above.

The fluctuation fields $X_t^\varepsilon(\varphi)$, $t \geqslant 0$, $\varphi \in \mathscr{S}(\mathbb{R})$ are defined as

$$X_t^\varepsilon(\varphi) = \varepsilon^{1/2} \sum_x \varphi(\varepsilon x)\, \sigma(x, t) \tag{2.11}$$

Let P^ε be the law induced by the $X_t^\varepsilon(\varphi)$ on $D(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$.

Then in Ref. 1 it is shown that P^ε converges weakly to P, which is the generalized Ornstein–Uhlenbeck process with mean zero and a covariance whose kernel $C_t(r - r')$ has a delta singularity and a regular part. The latter at $r = r'$ coincides with the limiting covariance of "our" $\varepsilon^{-1/2} m^\varepsilon(r, t)$. In Ref. 1 it is also proven that this diverges when t goes to infinity. We generalize such result as in the following theorem.

2.3. Theorem

Let $T_\varepsilon(a)$, $a < 1$, be as in Theorem 2.1. Let $\Gamma(a)$ be as in Eq. (2.6), and for $\varphi \in \mathscr{S}(\mathbb{R})$ let

$$X_t^{\varepsilon, a}(\varphi) = \varepsilon^{1 - \Gamma(a)} \sum_x \varphi(\varepsilon x)\, \sigma(x, T_\varepsilon(a) + t) \tag{2.12}$$

Let $P^{\varepsilon, a}$ be the law they induce on $D(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$. Then $P^{\varepsilon, a}$ converges weakly to P. P is a Gaussian process. It has support on those distributions which act as multiplication by a constant. Furthermore the paths $m(t)$, $t \geqslant 0$, on which P is supported are such that

$$d_t m = 2\alpha m$$

which is the linearization of Eq. (2.4). The distribution of $m(0)$ is G as defined in Eq. (2.8).

Below we give a brief sketch of the proofs.

Clearly one has to distinguish between two time regimes. In the first one the magnetization is infinitesimally small, as ε goes to zero, while, in the second one, it has become finite. In the latter the evolution is governed by Eq. (1.1), and in the first the nonlinear effects will be negligible.

The problem is to find a suitable intermediate region which connects the previous two. The breakthrough comes from the fact that it is possible to extend the validity of Eq. (1.1) to cases where the initial magnetization m_0^ε is still infinitesimal but "much larger" than $\varepsilon^{1/2}$, which is the typical value of the magnetic fluctuations. Namely, assume $m_0^\varepsilon = \varepsilon^\Gamma$, $0 < \Gamma < \frac{1}{2}$, and let Γ be suitably small. One can then prove that the solution of Eq. (1.1) correctly describes the evolution of the state for times of order $\log \varepsilon^{-1}$, i.e., when the magnetization becomes finite.

The first time regime when the magnetization is still typically less than ε^{Γ} can be studied as follows. We first write down the BBGKY hierarchy, i.e., the infinite set of equations which describes the evolution of the correlation functions $P_{v_0^\varepsilon}[\sigma(x_1, t) \cdots \sigma(x_n, t)]$, where $n \geqslant 1$ and $x_1,..., x_n$ are mutually distinct.

Such equations have the following features. The equation for the n-body correlation involves the $n, n-2, n+1$, and $n+2$ correlation functions. The $n+2$ correlation function enters with a minus sign. One knows *a priori* that the odd correlations are identically zero (by symmetry) while the even ones are positive, by ferromagnetic inequalities. One then finds an *a priori* upper bound on the correlation functions by truncating the hierarchy.

Up to times when the magnetization is small, $\leqslant \varepsilon^{\Gamma}$, the upper bound for the n-body correlation functions behaves like $\varepsilon^{n\Gamma}$, hence one controls the error made by truncating the hierarchy. In this way we prove Theorem 2.3.

We have also gained very precise information on the measure at the end of the "first time regime," since we control the n-body correlation functions for arbitrarily large values of n, in the limit of small ε. We can then prove that with large probability the spin configurations $\sigma = (\sigma_x)$, $x \in Z_\varepsilon$, are such that

$$\sup_{y \in Z} \varepsilon^{-\Gamma} \left| \varepsilon^{\lambda} \sum_{|x-y| \leqslant \varepsilon^{-\lambda}/2} \sigma(x) - m(\sigma) \right| \leqslant \varepsilon^{\eta}, \qquad \eta > 0 \tag{2.13}$$

where $m(\sigma)$ is the average magnetization of the configuration σ, which, because of what said above, is typically of order ε^{Γ}. λ in Eq. (2.13) is some, suitably fixed, positive number less than 1.

The magnetization starting from such configurations evolves according to Eq. (1.1) even at times of order $\log \varepsilon^{-1}$ and the microscopic state is approximately Bernoulli with the corresponding parameter. This is what we have described before as the "evolution in the second time regime." Technically this part requires very accurate estimates on the behavior of the simple exclusion process, obtained by extending some of the results established in Refs. 40 and 41.

3. PROOF OF THEOREM 2.3

We follow the strategy outlined at the end of Sec. 2. In this section we consider the "first time regime," namely times less than $T_\varepsilon(a)$ [cf. Eq. (2.2)] for any $a < 1$. We will prove Theorem 2.3 and establish properties of the state at time $T_\varepsilon(a)$ which will allow us to study the evolution for the remaining time, i.e., up to time T_ε. This will be done in Sec. 4.

The n-body correlation functions $V_n^\varepsilon(x_1,..., x_n, t)$ are defined for $\varepsilon > 0$, $n \geqslant 1$, $x_1 \cdots x_n$ mutually distinct and $t \geqslant 0$, as

$$V_n^\varepsilon(x_1,..., x_n, t) = \mu_t^\varepsilon\left[\prod_{i=1}^n \sigma(x_i, t)\right] \tag{3.1}$$

It is convenient to write $\underset{\sim}{x}$ for $(x_1,..., x_n)$ in the argument of V_n^ε and to set $V_0^\varepsilon = 1$.

3.1. Lower Bound for the Correlation Functions

The following holds for all $n \geqslant 0$ and $t \geqslant 0$

$$V_{2n+1}^\varepsilon(\underset{\sim}{x}, t) = 0 \tag{3.2}$$

$$V_{2n}^\varepsilon(\underset{\sim}{x}, t) \geqslant 0 \tag{3.3}$$

Proof. Equation (3.2) follows from the symmetry of the process under reversing all spins at all times [recall that the initial measure v_0^ε is also invariant under such transformation]. To prove Eq. (3.3) we first recall a useful representation of our process, namely the following:

Glauber–Stirring Process (GSP). The state space of the GSP is

$$\hat{\mathscr{X}} = (Z \times \{-1, 1\})^Z, \qquad \xi \in \hat{\mathscr{X}}, \qquad \xi = (\xi(u), u \in Z)$$

$$\xi(u) = (x(u), \hat{\sigma}(u)), \qquad x(u) \in Z, \qquad \hat{\sigma}(u) \in \{-1, 1\}$$

$$\hat{\sigma} = (\hat{\sigma}(u), u \in Z)$$

$x(u)$ denotes the position of the "stirring particle" u, and $\hat{\sigma}(u)$ denotes its spin. The initial state of the system will always be such that $x(u) = u$. The GSP process is the Markov process on $\hat{\mathscr{X}}$ whose gerator $\hat{L}^\varepsilon$ acts on cylinder functions $\hat{f}$ as

$$\hat{L}^\varepsilon \hat{f}(\xi) = \sum_u \{c_u(\xi)[\hat{f}(\xi^u) - \hat{f}(\xi)] + \tfrac{1}{2}\varepsilon^{-2}[\hat{f}(\xi^{u,+}) + \hat{f}(\xi^{u,-}) - 2\hat{f}(\xi)]\}$$

where

$$\xi^u(v) = \xi(v) \qquad \text{whenever } v \neq u$$

$$\xi^u(u) = (x(u), -\hat{\sigma}(u))$$

$$c_u(\xi) = 1 - \gamma\hat{\sigma}(u)[\hat{\sigma}(u^+) + \hat{\sigma}(u^-)] + \gamma^2\hat{\sigma}(u^+)\,\hat{\sigma}(u^-)$$

$$x(u^\pm) = x(u) \pm 1$$

$$\xi^{u,\pm}(v) = \xi(v) \qquad \text{whenever } v \neq u, u^\pm$$

$$\xi^{u,\pm}(u) = (x(u) \pm 1, \hat{\sigma}(u))$$

$$\xi^{u,\pm}(u^\pm) = (x(u), \hat{\sigma}(u^\pm))$$

We denote by $x(u, t)$ and $\hat{\sigma}(u, t)$ the canonical variables of the process, i.e., $x(u, t)$ is the random position of particle u at time t and $\hat{\sigma}(u, t)$ its spin. The following remarks are important for the next considerations:

(a) For any $y \in Z$, let $u_t(y)$ be such that $x(u_t(y), t) = y$. Then the variables

$$\sigma(y, t) = \hat{\sigma}(u_t(y), t)$$

have the same law as the canonical variables of the process with generator L^ε.

(b) The process of the $x(u, t)$ alone is the usual stirring process (speeded up by ε^{-2}).

(c) After conditioning on the paths of the stirring particles, the process of the $\hat{\sigma}(u, t)$ is a time-dependent ferromagnetic Glauber-type process.

We condition on the stirring particles' paths. By remark (b) above, the process $\hat{\sigma}(u, t)$ is ferromagnetic and by Theorems 2.2 of Chapter 3 and 2.14 of Chapter 2 of Liggett's book,[42] we have

$$\mathbb{E}^\varepsilon\left[\prod_{i=1}^{2n} \hat{\sigma}(u_i, t) \,\middle|\, \{x(v, s), \forall v \in Z, \forall s \geqslant 0\}\right] \geqslant 0$$
$$\text{for all } u_1,..., u_{2n} \text{ mutually distinct} \tag{3.4}$$

where $\mathbb{E}^\varepsilon$ denotes the expectation in the GSP process when initially the spins have law ν_0^ε and $x(u, 0) = u$ for all u in Z. In particular, we choose the u_i in Eq. (3.4) so that $x(u_i, t) = x_i$, $i = 1,..., 2n$, where $x = (x_1,..., x_{2n})$ is the set in Eq. (3.3). By remark (a) we therefore get that

$$\mathbb{E}^\varepsilon\left[\prod_{i=1}^{2n} \sigma(x_i, t) \,\middle|\, \{x(v, s), \forall v \in Z, \forall s \geqslant 0\}\right] \geqslant 0$$

hence Eq. (3.3) follows. ∎

3.2. The BBGKY Hierarchy

Denote by $U^\varepsilon(t)$ the semigroup with generator L^ε and by $U_e^\varepsilon(t)$ the semigroup of the Stirring Process with generator $\varepsilon^{-2}L_e$. Then

$$U^\varepsilon(t) = U_e^\varepsilon(t) + \int_0^t ds\, U^\varepsilon(t-s)\, L_G\, U_e^\varepsilon(s)$$

Therefore

$$V_n^\varepsilon(x_1,..., x_n, t)$$
$$= \sum_{z_1,z_n} P_t^\varepsilon(z_1,..., z_n \mid x_1,..., x_n)\, V_n^\varepsilon(z_1,..., z_n; 0)$$
$$+ \int_0^t ds \sum_{z_1,z_n} P_{t-s}^\varepsilon(z_1,..., z_n \mid x_1,..., x_n)\, \mathbb{E}_{\mu_0^\varepsilon}^\varepsilon\left[L_G \prod_{i=1}^n \sigma(z_i, s)\right] \quad (3.5)$$

where $P_t^\varepsilon(z_1,..., z_n \mid x_1,..., x_n)$ is the probability that n particles which start from $x_1,..., x_n$ and move according to the Stirring Process with intensity ε^{-2} (generator $\varepsilon^{-2}L_e$) are at $z_1,..., z_n$ at time t. $\mathbb{E}_{\mu_0^\varepsilon}^\varepsilon$ denotes the expectation w.r.t. the process when the initial measure on $\{-1, 1\}^{Z_\varepsilon}$ is μ_0^ε. The case of interest for us is when $\mu_0^\varepsilon = \nu_0^\varepsilon$. It is now easy to obtain from Eq. (3.5) closed equations for the V_n^ε, $n \geq 1$. This is what the physicists call the BBGKY hierarchy.

We use Eq. (2.1) with $f = \prod_{i=1}^n \sigma(z_i, s)$ to write the r.h.s. of Eq. (3.5) in terms of the V_n's. The structure of the equation when the initial measure is ν_0^ε becomes then the following (g, g_-, h, h_- $\hat{z}'$ appearing below are still to be defined):

$$V_{2n}^\varepsilon(\underline{x}, t) = \int_0^t ds \sum_{\underline{z} \in Z^{2n}} P_{t-s}^2(\underline{z} \mid \underline{x})$$
$$\times \Big\{ -4n V_{2n}^\varepsilon(\underline{z}, s) + \sum_{\underline{z}'} (g(\underline{z}, \underline{z}')\, 2\gamma V_{2n}^\varepsilon(\underline{z}', s)$$
$$+ g_-(\underline{z}, \underline{z}')\, 2\gamma V_{2n-2}^\varepsilon(\hat{\underline{z}}', s)) \Big\}$$
$$-\beta \int_0^t ds \sum_{\underline{z} \in Z^{2n}} P_{t-s}^\varepsilon(\underline{z} \mid \underline{x})$$
$$\cdot \sum_{\underline{z}' \in Z^{2n+2}} (h(\underline{z}, \underline{z}')\, V_{2n+2}(\underline{z}', s) + h_-(\underline{z}, \underline{z}')\, V_{2n-2}(\hat{\underline{z}}', s)) \quad (3.6)$$

By assumption, in fact $V_n^\varepsilon(\underline{x}, 0) = 0$ for all $\underline{x}$ and n and $V_{2n+1}^\varepsilon = 0$, by Eq. (3.2). g_-, h_- are characteristic functions which can be different from zero only if some pair z_i, z_j in $\underline{z}$ differ by 1. Hence, roughly speaking, they do not contribute much to Eq. (3.6).

We will now give the definition of g, g_-, h, h_-. We set $\underline{z} = (z_1,..., z_{2n}) \in Z^{2n}$, $z_i \neq z_j$ whenever $i \neq j$. We interpret the z_i in $\underline{z}$ as the sites where particles are.

For $\underline{z}' = (z_1',..., z_{2n}') \in Z^{2n}$, let us define

$$g(\underline{z}, \underline{z}') = \begin{cases} 1 & \text{if } \sum_{i=1}^{2n} |z_i' - z_i| = 1 \text{ and } z_i' \neq z_j' \text{ whenever } i \neq j \\ 0 & \text{otherwise} \end{cases} \quad (3.7a)$$

Namely $g(\underset{\sim}{z}, \underset{\sim}{z}')=1$ if and only if $\underset{\sim}{z}'$ is obtained from $\underset{\sim}{z}$ by letting only one particle move to an n.n. "allowed" site. An allowed site is a site which is not occupied by any other particle.

$$g_-(\underset{\sim}{z}, \underset{\sim}{z}') = \begin{cases} 1 & \text{if } \sum_{i=1}^{2n} |z_i' - z_i| = 1 \text{ and } z_i' = z_j' \text{ for some } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{3.7b}$$

As before, only one particle can move to an "allowed" n.n. site, which now means a site occupied by some other particle.

As a consequence of Eq. (3.5b), if $\underset{\sim}{z}'$ is such that $g_-(\underset{\sim}{z}, \underset{\sim}{z}')=1$, then $z_i' = z_j'$ for some $i \neq j$. We then define $\hat{\underset{\sim}{z}}' \in Z^{2n-2}$ as

$$\hat{z}_k' = z_k \qquad \forall k \neq i, j \tag{3.7c}$$

For $\underset{\sim}{z}' = (z_1', ..., z_{2n+2}') \in Z^{2n+2}$, let

$$h(\underset{\sim}{z}, \underset{\sim}{z}') = \begin{cases} 1 & \text{if } z_k' = z_k,\ k = 1, ..., 2n, \text{ and there is } i \leqslant 2n \text{ such} \\ & \text{that } z_{2n+1}' = z_i - 1,\ z_{2n+2}' = z_i + 1,\ z_i \pm 1 \neq z_j,\ \forall j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{3.7d}$$

Namely, $\underset{\sim}{z}'$ is obtained from $\underset{\sim}{z}$ by letting one particle of $\underset{\sim}{z}$ create two new particles at its n.n. sites. This is allowed only if both sites were not occupied. Finally, let

$$h_-(\underset{\sim}{z}, \underset{\sim}{z}') = \begin{cases} 1 & \text{if } z_k' = z_k,\ k = 1, ..., 2n, \text{ and there is } i \leqslant 2n \\ & \text{such that } z_{2n+1}' = z_i - 1,\ z_{2n+2}' = z_i + 1, \text{ and there} \\ & \text{are } j \text{ and } m \text{ such that } z_i + 1 = z_j \text{ and } z_i - 1 = z_m \\ 0 & \text{otherwise} \end{cases} \tag{3.7e}$$

If $\underset{\sim}{z}'$ is such that $h_-(\underset{\sim}{z}, \underset{\sim}{z}')=1$ and i, j, m are as in Eq. (3.7e), then we define $\hat{\underset{\sim}{z}}' \in Z^{2n-2}$ as

$$\hat{z}_k' = z_k \qquad k \neq i, j, m \tag{3.7f}$$

3.3. Upper Bounds for the Correlation Functions

There are constants[7] $c(2n)$ such that for any $a<1$, $t \in [0, T_\varepsilon(a)]$, $T_\varepsilon(a)$ being defined in Eq. (2.2), and any $n \geqslant 1$

$$\sup_{\underset{\sim}{x} \in Z^{2n}} V_{2n}^\varepsilon(\underset{\sim}{x}; t) \leqslant c(2n)\, \varepsilon^n \exp(2n\alpha t) \tag{3.8}$$

where the sup is over all $\underset{\sim}{x} = (x_1, ..., x_{2n})$ such that $x_i \neq x_j$ whenever $i \neq j$.

[7] Explicit bounds for $c(2n)$ are given in Eqs. (3.13) and (3.13b) below.

Proof. Let

$$a_{2n}^{\varepsilon}(t) = \sup_{\underset{\sim}{x} \in Z^{2n}} \sup_{s \leqslant t} V_{2n}^{\varepsilon}(\underset{\sim}{x}; s) \tag{3.9}$$

where $\underset{\sim}{x} = (x_1, ..., x_{2n})$ is such that $x_i \neq x_j$ whenever $i \neq j$. From Eq. (3.6) and the positivity of V_{2n}^{ε}, we then get

$$\begin{aligned} a_{2n}^{\varepsilon}(t) &\leqslant \int_0^t ds \left\{ 2n\alpha a_{2n}^{\varepsilon}(s) + 4\gamma \tfrac{1}{2} 2n(2n-1) \right. \\ &\quad \left. \times \sup_{x_1, x_2} \sum_{z_1 z_2} P_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2)\, 1(|z_1 - z_2| = 1)\, a_{2n-2}^{\varepsilon}(s) \right\} \\ &\leqslant \int_0^t ds \{ 2n\alpha a_{2n}^{\varepsilon}(s) + 4\gamma \tfrac{1}{2} 2n(2n-1)\, c_1 (t-s)^{-1/2}\, \varepsilon a_{2n-2}^{\varepsilon}(s) \} \\ &\quad + 4\gamma \tfrac{1}{2} 2n(2n-1)\, \varepsilon^2 a_{2n-2}^{\varepsilon}(t) \end{aligned} \tag{3.10}$$

because for a suitable constant c_1 (cf. Appendix A), $c_1 \geqslant 1$,

$$\begin{aligned} &\sup_{x_1, x_2} \sum_{z_1 z_2} P_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2)\, 1(|z_1 - z_2| = 1) \\ &\quad \leqslant c_1 (t-s)^{-1/2}\, \varepsilon 1(\varepsilon^{-2} |t-s| \geqslant 1) + 1(\varepsilon^{-2} |t-s| \leqslant 1) \end{aligned} \tag{3.11}$$

From Eq. (3.10)

$$a_2^{\varepsilon}(t) \leqslant c(2)\, \varepsilon \exp(2\alpha t) \tag{3.12a}$$

where

$$c(2) = \sup_{t > 0} \sup_{\varepsilon \leqslant 1} 4\gamma c_1 \int_0^t ds (s^{-1/2} + \varepsilon) \exp(-2\alpha s) \tag{3.12b}$$

Then for $n > 1$

$$\begin{aligned} a_{2n}^{\varepsilon}(t) &\leqslant \exp(2n\alpha t)\, \varepsilon^n 4\gamma \tfrac{1}{2} 2n(2n-1)\, c(2n-2) \\ &\quad \times c_1 n \int_0^t ds (s^{-1/2} + (2n-2)\, \alpha\varepsilon) \exp(-2\alpha s) \end{aligned}$$

so that

$$a_{2n}^{\varepsilon}(t) \leqslant c(2n)\, \varepsilon^n \exp(2n\alpha t) \tag{3.13a}$$

where

$$\begin{aligned} c(2n) &= c(2n-2)\, 4\gamma \tfrac{1}{2} 2n(2n-1) \\ &\quad n c_1 \sup_{t > 0} \sup_{\varepsilon \leqslant 1} \int_0^t ds \exp(-2\alpha s)(s^{-1/2} + (2n-2)\, \alpha\varepsilon) \end{aligned} \tag{3.13b}$$

We will next prove that the V_{2n}^{ε}'s converge, when suitably normalized, to the moments of some Gaussian measure. It is easy to guess what the limiting measure should be, so we first write down the equations for its moments and we then compare them with Eq. (3.6).

3.4. The Limiting Gaussian Measure

Let $\tilde{C}^{\varepsilon}(r, r'; t)$ be the solution of the following PDE on $[0, L]$ with periodic boundary conditions,

$$\partial_t \tilde{C}^{\varepsilon} = \tfrac{1}{2}(\partial_r^2 \tilde{C}^{\varepsilon} + \partial_{r'}^2 \tilde{C}) + 2\alpha \tilde{C}^{\varepsilon} + \varepsilon 8\gamma\, \delta(r - r') \tag{3.14a}$$

$$\tilde{C}^{\varepsilon}(r, r'; 0) = 0 \tag{3.14b}$$

By Fourier transforming Eq. (3.14) we easily prove that:

There exists c_2 such that for all r, r' in $[0, L]$

$$|\tilde{C}^{\varepsilon}(r, r'; t) - (2\alpha)^{-1}\, \varepsilon 8\gamma \exp(2\alpha t)| \leqslant c_2 \exp(-\hat{a} t)\, \varepsilon e^{2\alpha t} \tag{3.15a}$$

$$\hat{a} = \min((L^{-1} 2\pi)^2; 2\alpha) \tag{3.15b}$$

For what follows it is convenient to introduce a "discretized" version of $\tilde{C}$. For $x \neq y \in Z_{\varepsilon}$, let

$$C^{\varepsilon}(\varepsilon x, \varepsilon y; t) = \int_0^t ds \sum_{z_1 z_2} G_{t-s}^{\varepsilon}(z_1, z_2 \mid x, y) \times \{2\alpha C^{\varepsilon}(\varepsilon z_1, \varepsilon z_2; s) + 4\gamma 1(|z_1 - z_2| = 1)\} \tag{3.16a}$$

where

$$G_t^{\varepsilon}(z_1, z_2 \mid x, y) = (2\pi t)^{-1}\, \varepsilon^2 \exp(-(2t)^{-1}\, \{(\varepsilon z_1 - \varepsilon x)^2 + (\varepsilon z_2 - \varepsilon y)^2\} \tag{3.16b}$$

Then it easily follows that:

Let $C^{\varepsilon}(\varepsilon x, \varepsilon y; t)$ be the solution of Eq. (3.16); then for any $a < 1$ and $t \leqslant T_{\varepsilon}(a)$ there are c_3 and $\delta_1 > 0$ such that $\forall x, y$

$$|C^{\varepsilon}(\varepsilon x, \varepsilon y; t) - \tilde{C}^{\varepsilon}(\varepsilon x, \varepsilon y; t)| \leqslant c_3 \varepsilon \exp(2\alpha t)\, \varepsilon^{\delta_1} \tag{3.17}$$

We next define $C_2^{\varepsilon} = C^{\varepsilon}$ and

$$C_{2n}^{\varepsilon}(\underline{x}; t) = \sum_{l \in \mathscr{G}_n} \prod_{(i,j) \in l} C^{\varepsilon}(\varepsilon x_i, \varepsilon x_j; t), \qquad n > 1 \tag{3.18}$$

where $\mathscr{G}_n$ is the set of graphs l of $2n$ points, a graph l being a set of unordered pairs $l = \{(i_1, i_2),..., (i_{2n-1}, i_{2n})\}$ where $i_s \in \{1,..., 2n\}$ for any $s = 1,..., 2n$. Furthermore $i_s \neq i_{s'}$ if $s \neq s'$.

It is then easy to check that:

Let C^ε_{2n} be as in Eq. (3.18); pose $C^\varepsilon_0 = 1$; then

$$C^\varepsilon_{2n}(\underline{x}; t) = \int_0^t ds \sum_{\underline{z} \in Z^{2n}} G^\varepsilon_{t-s}(\underline{z} \mid \underline{x}) \left\{ 2n\alpha C^\varepsilon_{2n}(\underline{z}; s) + 4\gamma \frac{1}{2} \sum_{j \neq k} C^\varepsilon_{2n-2}(\underline{z}^{k,j}; s)\, 1(|z_k - z_j| = 1) \right\} \tag{3.19a}$$

where $\underline{z}^{k,j} = \underline{z} \backslash (\{z_k\} \cup \{z_j\})$ and

$$G^\varepsilon_t(\underline{z} \mid \underline{x}) = (2\pi t)^{-n/2}\, \varepsilon^n \exp\left[-\sum_{i=1}^{2n} (2t)^{-1} (\varepsilon z_i - \varepsilon x_i)^2 \right] \tag{3.19b}$$

Furthermore, for any $a < 1$ there are constants $c'(2n)$ and $c''(2n)$ such that for all $n \geqslant 1$ and all $t \leqslant T_\varepsilon(a)$

$$c'(2n)\, \varepsilon^n \exp(2n\alpha t) \leqslant C_{2n}(\underline{x}; t) \leqslant c''(2n)\, \varepsilon^n \exp(2n\alpha t) \tag{3.20}$$

3.5. Convergence to the Limiting Gaussian Measure

We only need to draw conclusions from what we have so far established. Firstly we will compare Eqs. (3.19) and (3.16): they have very similar structure and it will not be difficult to see that $\varepsilon^{-2n\Gamma(a)} |C^\varepsilon_{2n} - V^\varepsilon_{2n}|$ vanishes when ε goes to zero. By Eq. (3.18) the limiting values of the V^ε_{2n} will then be determined in terms of C^ε_2. The final result will then be the following:

3.6. Proposition

Let $V^\varepsilon_{2n}(x_1,..., x_{2n}; t)$ be as in Eq. (3.1). Then, for any $a < 1$, $n \geqslant 1$, there exist constants $b(2n)$ and $\delta > 0$ so that for any mutually distinct sites $x_1,..., x_{2n}$

$$|\varepsilon^{-2n\Gamma(a)} V^\varepsilon_{2n}(x_1,..., x_{2n}; T_\varepsilon(a)) - (n!\, 2^n)^{-1} (2n)!\, ((2\alpha)^{-1}\, 8\gamma)^n| \leqslant b(2n)\, \varepsilon^\delta \tag{3.21a}$$

$$\lim \varepsilon^{-2n\Gamma(a)} V^\varepsilon_{2n}(x_1,..., x_{2n}; T_\varepsilon(a) + t) = \int dx\, x^{2n} \mathscr{G}(x)\, e^{2n\alpha t} \tag{3.21b}$$

where $\mathscr{G}(x)$ is defined in Eqs. (2.8) and (2.9).

Before proving Proposition 3.6, let us remark that Theorem 2.3 is a consequence of Proposition 3.6, which actually proves that all the moments of the extensive fields $X_t^{\varepsilon,a}(\psi)$ are converging to the moments of the limiting measure. Such a derivation is quite standard and we omit it.

Proof of Proposition 3.6. We first prove that $D_2^\varepsilon(T_\varepsilon(a))$ vanishes, $D_{2n}^\varepsilon(t)$ being

$$D_{2n}^\varepsilon(t) = \sup_{\underline{x} \in Z^{2n}} |V_{2n}^\varepsilon(\underline{x}; t) - C_{2n}^\varepsilon(\underline{x}; t)| \tag{3.22}$$

From Eqs. (3.6) and (3.16), using Eqs. (3.8) and (3.20), we get that

$$\begin{aligned}
D_2^\varepsilon(t) = \int_0^t ds\, 2\alpha\, D_2^\varepsilon(s) \\
&+ \int_0^t ds \sup_{x_1 \neq x_2} 2\alpha \sum_{z_1 z_2} |P_{t-s}^\varepsilon(z_1, z_2 \mid x_1, x_2) \\
&- G_{t-s}^\varepsilon(z_1, z_2 \mid x_1, x_2)|\, c''(2)\, \varepsilon\, e^{2\alpha s} \\
&+ \int_0^t ds \sum_{z_1 z_2} \sup_{x_1 \neq x_2} P_{t-s}^\varepsilon(z_1, z_2 \mid x_1, x_2) \\
&\times \mathbb{1}(|z_1 - z_2| = 1)\, 4\gamma c(2)\, \varepsilon\, e^{2\alpha s} \\
&+ \int_0^t ds \sup_{x_1 \neq x_2} \Big| \sum_{z_1 z_2} P_{t-s}^\varepsilon(z_1, z_2 \mid x_1, x_2) \sum_{\underline{z}'} g(\underline{z}, \underline{z}') \\
&\times 2\gamma [V_2^\varepsilon(z_1', z_2'; s) - V_2^\varepsilon(z_1, z_2; s)] \Big| \\
&+ \int_0^t ds\, 4\gamma \sup_{x_1 \neq x_2} \Big| \sum_{z_1 z_2} \{P_{t-s}^\varepsilon(z_1, z_2 \mid x_1, x_2) \\
&- G_{t-s}^\varepsilon(z_1, z_2 \mid x_1, x_2)\}\, \mathbb{1}(|z_1 - z_2| = 1) \Big| \\
&+ \int_0^t ds\, \beta 4 c(4)\, \varepsilon^2\, e^{4\alpha s}
\end{aligned} \tag{3.23}$$

In Appendix A it is proven that there is $\delta_2 > 0$ ($\delta_2 < \frac{1}{2}$) such that for all $x_1 \neq x_2$

$$\begin{aligned}
\lim_{t \to \infty} t^{1/2 + \delta_2} \sum_{z_1 z_2} [P_t^1(z_1, z_2 \mid x_1, x_2) \\
&- G_t^1(z_1, z_2 \mid x_1, x_2)]\, \mathbb{1}(|z_1 - z_2| = 1) = 0
\end{aligned} \tag{3.24}$$

In Ref. 40 it is proven that there exists $\delta_3 > 0$ ($\delta_3 < 1$) so that, for all $n \geqslant 2$,

$$\lim_{t \to \infty} t^{\delta_3} \sup_{\underline{x} \in Z^n} \sum_{\underline{z} \in Z^n} |P_t^1(\underline{z} \mid \underline{x}) - G_t^1(\underline{z} \mid \underline{x})| = 0 \tag{3.25}$$

Therefore for suitable constants c_4 and c_5

$$\begin{aligned} &\sup_{x_1 \neq x_2} \left| \sum_{z_1 z_2} \{P_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2) - G_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2)\} \mathbb{1}(|z_1 - z_2| = 1) \right| \\ &\qquad \leqslant \mathbb{1}(|t-s| \geqslant \varepsilon^2)(t-s)^{-1/2-\delta_2} \varepsilon^{1+2\delta_2} 2c_4 + c_4 \mathbb{1}(|t-s| \leqslant \varepsilon^2) \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\sup_{x_1 \neq x_2} \sum_{z_1 z_2} |P_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2) - G_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2)| \\ &\qquad \leqslant c_5 \mathbb{1}(|t-s| > \varepsilon^2) |t-s|^{-\delta_3} \varepsilon^{2\delta_3} + c_5 \mathbb{1}(|t-s| \leqslant \varepsilon^2) \end{aligned} \tag{3.27}$$

By Eq. (3.27) we can control also the fourth term in the r.h.s. of Eq. (3.23). In fact, for $t - s > \varepsilon^2$,

$$\begin{aligned} &\left| \sum_{z_1 z_2} \sum_{z_1' z_2'} g(\underline{z}, \underline{z}') V_2^{\varepsilon}(z_1, z_2; s)[P_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2) - P_{t-s}^{\varepsilon}(z_1', z_2' \mid x_1, x_2) \right| \\ &\qquad \leqslant c(2)\, \varepsilon\, e^{2\alpha s 2} c_5 |t-s|^{-\delta_3} \varepsilon^{2\delta_3} \\ &\qquad\quad + \varepsilon\, e^{2\alpha s} \sum_{z_1 z_2} \sum_{z_1' z_2'} g(\underline{z}, \underline{z}') \, |G_{t-s}^{\varepsilon}(z_1, z_2 \mid x_1, x_2) - G_{t-s}^{\varepsilon}(z_1', z_2' \mid x_1, x_2)| \\ &\qquad \leqslant c(2) \{ \varepsilon\, e^{2\alpha s} 2c_5 |t-s|^{-\delta_3} \varepsilon^{2\delta_3} + \varepsilon\, e^{2\alpha s} c_6 |t-s|^{-1/2} \varepsilon \} \end{aligned} \tag{3.28}$$

where c_6 is a suitable constant.

From Eq. (3.23), using Eqs. (3.26), (3.27), and (3.11), we can conclude that there exists $\delta > 0$ and a constant $d(2)$ so that

$$D_2^{\varepsilon}(t) \leqslant \varepsilon \exp(2\alpha t)\, \varepsilon^{\delta}\, d(2), \qquad \forall t \leqslant T_{\varepsilon}(a) \tag{3.29a}$$

The estimate for D_{2n}^{ε}, $n > 1$, is

$$D_{2n}^{\varepsilon}(t) \leqslant \varepsilon^n \exp(2n\alpha t)\, \varepsilon^{\delta}\, d(2n), \qquad t \leqslant T_{\varepsilon}(a) \tag{3.29b}$$

We prove Eq. (3.29) by induction on n. The generic step in the induction is essentially the same as the proof of the estimate for D_2^{ε}. There are, however, two new terms which require some care. They are

$$-\int_0^t ds\, \beta \sum_{\underline{z} \in Z^{2n}} \sum_{\underline{z}' \in Z^{2n}} P_{t-s}^{\varepsilon}(\underline{z} \mid \underline{x})\, h_-(\underline{z}, \hat{\underline{z}}')\, V_{2n-2}^{\varepsilon}(\underline{x}; s) \tag{3.30}$$

$$\begin{aligned} &\int_0^t ds \sum_{\underline{z} \in Z^{2n}} \Bigg\{ P_{t-s}^{\varepsilon}(\underline{z} \mid \underline{x})\, 2\gamma \sum_{\underline{z}' \in Z^{2n}} g_-(\underline{z}, \hat{\underline{z}}')\, V_{2n-2}^{\varepsilon}(\hat{\underline{z}}'; s) \\ &\qquad - G_{t-s}^{\varepsilon}(\underline{z} \mid \underline{x})\, 4\gamma \frac{1}{2} \sum_{i \neq j} \mathbb{1}(|z_i - z_j| = 1)\, C_{2n-2}^{\varepsilon}(\underline{z}^{i,j}; s) \Bigg\} \end{aligned} \tag{3.31}$$

Equation (3.30). By the definition of h_- [cf. Eq. (3.7e)], the configurations $\underline{z}$ in Eq. (3.30) have necessarily three contiguous particles.

We then use the following estimate, proven in Ref. 40, Lemma 5.1, namely that for any $\delta_4 > 0$ there is a constant c_7 such that

$$\sup_{x_1 x_2 x_3} \sum_{z_1 z_2 z_3} P_t^1(z_1, z_2, z_3 \mid x_1, x_2, x_3)\, \mathbb{1}(|z_1 - z_2| = 1, |z_2 - z_3| = 1) \\ \leqslant c_7 t^{-1+1/4+\delta_4} \tag{3.32}$$

Equation (3.31). We can bound the integral in Eq. (3.31) as

$$\begin{aligned} &\int_0^t ds \sum_{\underline{z} \in Z^{2n}} P_{t-s}^{\varepsilon}(\underline{z} \mid \underline{x})\, 4\gamma \sum_{i \neq j} \mathbb{1}(|z_i - z_j| = 1)\, D_{2n-2}^{\varepsilon}(s) \\ &\quad + \int_0^t ds\, 4\gamma \left| \sum_{\underline{z}} [G_{t-s}^{\varepsilon}(\underline{z} \mid \underline{x}) - P_{t-s}^{\varepsilon}(\underline{z} \mid \underline{x})] \right. \\ &\quad \left. \times \frac{1}{2} \sum_{i \neq j} \mathbb{1}(|z_i - z_j| = 1)\, C_{2n-2}^{\varepsilon}(\underline{z}^{i,j}; s) \right| \\ &\quad + \int_0^t ds \sum_{\underline{z}} P_{t-s}^{\varepsilon}(\underline{z} \mid \underline{x})\, 2\gamma 2n(2n-1)(2n-2) \\ &\quad \times \mathbb{1}(|z_1 - z_2| = 1, |z_2 - z_3| = 1) \cdot c(2n-2)\, \varepsilon^{n-1} \exp((2n-2)\, \alpha s) \end{aligned} \tag{3.33}$$

The first term is estimated by means of Eq. (3.11) and the induction hypothesis, and the third one again by Eq. (3.32). For the second one we proceed as follows. By Eq. (3.17) we can rewrite Eq. (3.18) with $\tilde{C}^{\varepsilon}$ appearing in the r.h.s. in place of C^{ε}, the error being $\varepsilon^{\delta_1} \varepsilon \exp(2\alpha t)\, \varepsilon^{n-1} \exp(2n-2)\, \alpha t)$ times some constant factor. We are then left with the same expression having $\tilde{C}_{2n-2}^{\varepsilon}$ in place of C_{2n-2}^{ε} where the former is defined by Eq. (3.18) with $\tilde{C}_2^{\varepsilon} := \tilde{C}^{\varepsilon}$. We now use Eq. (3.15) to change $\tilde{C}_2^{\varepsilon}$ with $(2\alpha)^{-1} \varepsilon\, 8\gamma \exp(2\alpha s)$. The error is like the previous ones. What is left is again the same expression with $C_{2n-2}^{\varepsilon}(\underline{z}^{i,j}, s)$ replaced by a constant (w.r.t. $\underline{z}$). We can then use Eq. (3.26) to show that this term also gives a contribution compatible with Eq. (3.29). From Eqs. (3.15a), (3.17), and (3.20) we then obtain the proof of Proposition 3.6.

4. PROOF OF THEOREMS 2.1 AND 2.2

The main characteristics of the measure at time $T_{\varepsilon}(a)$ have been established in Proposition 3.6. Unfortunately the techniques of Sec. 3 are inadequate to study the evolution afterwards. When the n-body correlation functions become finite, they are no longer comparatively negligible w.r.t.

the $n-2$ ones. This is just what the theorems we want to prove say, and, as a matter of fact, most of the estimates of Sec. 3 are valid only for times less than $T_\varepsilon(a)$, with $a<1$.

It seems desperate to control the whole hierarchy for infinitely long times, so we attack the problem from a different point of view. The measure at time $T_\varepsilon(a)$, by Proposition 3.6, is concentrated on "regular" spin configurations having magnetic density $\sim \varepsilon^{\Gamma(a)}$. We shall see that the magnetic density starting from any such configuration evolves and reaches finite values, which, due to the stochastic nature of the evolution, fluctuates around their average. The point is that if $\varepsilon^{\Gamma(a)}$ is not too small, i.e., a close to 1, the fluctuations remain infinitesimal when ε goes to 0, even when the magnetic density has reached finite values.

We shall prove in fact that the magnetic density evolves following Eq. (1.1) up to $t \sim \log \varepsilon^{-1}$, uniformly in ε (namely up to times t when it becomes finite) and that at the same times, the microscopic measure converges to a pure Bernoulli measure. As a consequence the microscopic state at time $T_\varepsilon + t$ becomes an integral of pure Bernoulli states. Each of them is characterized by a magnetic density which is obtained from that of a "regular" configuration at time $T_\varepsilon(a)$ by the flow induced by Eq. (1.1) for a time $T_\varepsilon + t - T_\varepsilon(a)$.

The law $\lambda_t(dm)$ is therefore the image under such transformation of the law of the magnetization at time $T_\varepsilon(a)$, and this is just what Theorems 2.1 and 2.2 state. Moreover we see that only the "initial" fluctuations are responsible for the fact that the state becomes a nontrivial decomposition of Bernoulli measures. When the initial magnetization is $\varepsilon^{\Gamma(a)}$, a close enough to 1, then the fluctuations do not play any significant role. We guess that this should also happen when the initial value is $\sim \varepsilon^\gamma$, $\gamma < \frac{1}{2}$, but, for technical reasons, we need to require γ close enough to 0.

Our first result proves that the measure at time $T_\varepsilon(a)$ is "well" concentrated on suitably regular, flat, configurations. The average magnetization in the scale $\varepsilon^{-\lambda}$ is

$$m_\lambda(\sigma, x) = \varepsilon^\lambda \sum_{y \in I_\lambda(x)} \sigma(y), \qquad x \in Z \tag{4.1a}$$

$$I_\lambda(x) = \{ y \in Z : x \leqslant y \leqslant x + \varepsilon^{-\lambda} - 1 \} \tag{4.1b}$$

We set

$$m(\sigma) = (L\varepsilon^{-1})^{-1} \sum_{0 \leqslant x \leqslant L\varepsilon^{-1}} \sigma(x) \tag{4.2}$$

which is the average magnetic density. The configuration σ is "flat" or "regular" if $m_\lambda(\sigma, x)$ is suitably close to $m(\sigma)$ for all x in Z.

Closeness will be defined so that the difference between $m_\lambda(\sigma, x)$ and $m(\sigma)$ is much smaller than the value of $m(\sigma)$ itself. A key element for proving that the configurations at time $T_\varepsilon(a)$ are "flat," with large probability, is that the correlation functions at $T_\varepsilon(a)$ are "almost" independent of the sites, when ε is small.

From Proposition 3.6 it is in fact easy to prove that for any $\lambda > 0$

$$\lim_{\varepsilon \to 0} \sup_{x \neq y} \mu^{\varepsilon}_{T_\varepsilon(a)}[|m_\lambda(\sigma, x) - m_\lambda(\sigma, y)| > \varepsilon^{\Gamma(a)+\eta}] = 0 \tag{4.3}$$

where η is some small enough but positive number.

In order to prove that the configurations are flat, we would need to have in Eq. (4.3) the sup inside the expectation. In this case, however, Proposition 3.6 is not enough to ensure that the limit is still zero. What helps us is the smoothing effect of the Stirring Process which makes $m_\lambda(\sigma, x)$ approximately constant in regions of order $\varepsilon^{-\lambda}$, $\lambda < 1$. This happens after any finite time in the limit of small ε, whatever the initial configuration is, as we shall see below. Combining this and Eq. (4.3), we will obtain the desired property that configurations are with large probability flat if a, λ, and η (the accuracy entering in the definition of flat configurations) are properly chosen (i.e., a and λ close to 1 and η to 0). The following is the key estimate, which will frequently appear throughout the section.

4.1. Proposition (Factorization Property)

There is $\delta_0 > 0$ such that the following holds.

For any $k \geqslant 1$, $T > 0$, there are constants $e(k)$ (depending on δ_0, k, and T) such that for all configurations σ, all $x_1,..., x_k$ mutually distinct, all t in $[\varepsilon^{1/3}, T]$

$$\left| \mathbb{E}^{\varepsilon}_{\sigma}\left[\prod_{i=1}^{k} \tilde{\sigma}(x_i, t)\right]\right| \leqslant e(k)(\varepsilon^{-2}t)^{-\delta_0 k} \tag{4.4a}$$

$$e(k) \leqslant e(k+1), \qquad \forall k \geqslant 1 \tag{4.4b}$$

$$\tilde{\sigma}(x, t) = \sigma(x, t) - \mathbb{E}^{\varepsilon}_{\sigma}[\sigma(x, t)] \tag{4.5}$$

where $\mathbb{E}^{\varepsilon}_{\sigma}$ denotes the expectation w.r.t. the process with initial distribution concentrated on the single configuration σ.

The proof of Proposition 4.1 is technically quite involved, and we report it in Appendix B. The proof follows closely that concerning the analogous property for the simple exclusion process (cf. Ref. 40), and it is similar to that given in the appendix of Ref. 1b, where the corrections at order ε of the measure at time t are considered.

With the help of Propositions 4.1 and 3.6 we have the following:

4.2. Proposition (The Magnetic Density Profiles are Flat)

There exist $\lambda<1$, $a_0<1$, $\eta_1>0$, $\delta_5>0$, and c_8 so that for all $a>a_0$ (and $a<1$)

$$\mu^{\varepsilon}_{T_\varepsilon(a)}[\sup_x |m_\lambda(\sigma, x)-m(\sigma)|>\varepsilon^{\eta_1+\Gamma(a)}]\leqslant c_8\varepsilon^{\delta_5} \tag{4.6}$$

where $m(\sigma)$ and $m_\lambda(\sigma, x)$ are defined in Eqs. (4.2) and (4.1) respectively.

Remarks. From the proof of Proposition 4.2 it turns out that

$$\tfrac{1}{2}+(\Gamma(a_0)+\eta_1)<\lambda<1-\tfrac{3}{2}(\Gamma(a_0)+\eta_1) \tag{4.7}$$

$$2(\Gamma(a_0)+\eta_1)<\delta_0 \tag{4.8}$$

$$\delta-2\eta_1>1-\lambda \tag{4.9}$$

δ is defined in Proposition 3.6.

Proof of Proposition 4.2. We divide the whole interval $[0, \varepsilon^{-1}L]$ into intervals of length $\varepsilon^{-\lambda}$; we have chosen λ, a_0, η_1, δ_5 according to Eqs. (4.7), (4.8), and (4.9). For notational simplicity, assume that $\varepsilon^{-\lambda}$ is an integer as well as $N:=\varepsilon^{-1}L\varepsilon^{\lambda}$, which is then the number of consecutive intervals I_i, $i=1,..., N$, in which $[0, \varepsilon^{-1}L]$ can be divided.

Let $m_\lambda(\sigma, I_i)$ be the average magnetization in the interval I_i; then, for any i and j,

$$\begin{aligned}\mu^{\varepsilon}_{T_\varepsilon(a)}&[|m_\lambda(\sigma, I_i)-m_\lambda(\sigma, I_j)|>\varepsilon^{\Gamma(a)+\eta_1}]\\ &\leqslant c_9\max(\varepsilon^{\delta-2\eta_1}, \varepsilon^{\lambda-2(\Gamma(a)+\eta_1)})\end{aligned} \tag{4.10}$$

Equation (4.10) follows frm Proposition 3.6 using the Chebyshev inequality with the second moment.

Since

$$m(\sigma)=N^{-1}\sum_{i=1}^{N} m_\lambda(\sigma, I_i)$$

it follows that

$$\begin{aligned}\mu^{\varepsilon}_{T_\varepsilon(a)}&[\sup_i |m_\lambda(\sigma, I_i)-m(\sigma)|>\varepsilon^{\Gamma(a)+\eta_1}]\\ &\leqslant Nc_9\max(\varepsilon^{\delta-\eta_1}, \varepsilon^{\lambda-2(\Gamma(a)+\eta_1}) \end{aligned} \tag{4.11}$$

It is therefore enough to control the difference between $m_\lambda(\sigma, x)$ and $m_\lambda(\sigma, I_i)$ for any $i = 1, \ldots, N$ and x in I_i. We will prove that for any σ, x, and y such that $|x-y| \leqslant \varepsilon^{-\lambda}$,

$$E_\sigma^\varepsilon[|m_\lambda(\sigma_t, x) - m_\lambda(\sigma_t, y)| \geqslant \varepsilon^{\Gamma(a)+\eta_1}] \leqslant c_{10}\varepsilon^2 \tag{4.12}$$

where $t = \varepsilon^{2/3(1-\lambda)}$ (the reason for such choice will become clear in the sequel), c_{10} is a suitable constant. ε^2 is not the optimal estiate but it is enough for our purposes.

From Eq. (4.12) it follows that

$$\begin{aligned} &\mu^\varepsilon_{T_\varepsilon(a)}[\sup_i \sup_{x \in I_i} |m_\lambda(\sigma, x) - m_\lambda(\sigma, I_i)| \geqslant \varepsilon^{\Gamma(a)+\eta_1}] \\ &\quad \leqslant \varepsilon^{-1+\lambda} L \varepsilon^{-\lambda} \mu^\varepsilon_{T_\varepsilon(a)}[E^\varepsilon \sigma_{T_\varepsilon(a)-t}[|m_\lambda(\sigma_t, x) - m_\lambda(\sigma_t, I_i)| \geqslant \varepsilon^{\Gamma(a)+\eta_1}]] \\ &\quad \leqslant \varepsilon^{-1} L c_{10} \varepsilon^2 \end{aligned} \tag{4.13a}$$

and from Eqs. (4.13a) and (4.11), Eq. (4.6) follows with $2\varepsilon^{\Gamma(a)+\eta_1}$ instead of the required $\varepsilon^{\Gamma(a)+\eta_1}$. It is, however, trivial to modify our previous proof to get Eq. (4.6); we did not do that from the beginning for the sake of notational simplicity.

It only remains, therefore, to prove Eq. (4.12). We use the Chebyshev inequality with $2k$ moments; k has to be chosen large enough, as we shall see. We then need to estimate terms like

$$E_\sigma^\varepsilon\left[\prod_{i=1}^{2k} (\sigma(x_i, t) - \sigma(x_i + d, t))\right]$$

for given $x_1, \ldots, x_{2k}$ and with $d \leqslant \varepsilon^{-\lambda}$, $t = \varepsilon^{2/3(1-\lambda)}$. We write each $\sigma(x, t)$ in terms of $\tilde{\sigma}(x, t)$. The expectation of products of $\tilde{\sigma}(x, t)$ is controlled by means of Proposition 4.1.

For the differences $E_\sigma^\varepsilon[\sigma(x_i, t) - \sigma(x_i + d, t)]$ we have that

$$|E_\sigma^\varepsilon[\sigma(x_i, t) - \sigma(x_i + d, t)]| \leqslant 2\varepsilon^{2/3(1-\lambda)} + c_{11}(\varepsilon^{-2}t)^{-1/2}\, d \tag{4.13b}$$

Collecting all the above, we get an estimate whose ε dependence is

$$\max\{\varepsilon^{[(1/2)\lambda - (\Gamma(a)+\eta_1)]2k}, \varepsilon^{[\delta_0 - (\Gamma(a)+\eta_1)]2k}, \varepsilon^{[2/3(1-\lambda) - (\Gamma(a)+\eta_1)]2k}\}$$

which by Eqs. (4.7) and (4.8) is a positive power of ε. By choosing k large enough, this proves Eq. (4.12).

To study the evolution after time $T_\varepsilon(a)$, we condition on the state of the system at time $T_\varepsilon(a)$. By the Markov property and because of Proposition 4.2, it is like starting from a "flat" configuration at time zero.

We are disregarding sets whose probability vanishes when ε goes to zero. Since

$$L_G\sigma(x) = -2\sigma(x) + 2\gamma(\sigma(x+1)+\sigma(x-1)) \\ -2\gamma\sigma^2(x-1)\,\sigma(x+1) \tag{4.14}$$

the average magnetization does not obey closed equations. By Proposition 4.1, however, at least for finite times, the factorization property allows us to close the equation, with small error. Such property does not extend automatically to all times, as is clear from the analysis of Sec. 3. We shall, however, see that if the initial magnetic density is "large," $\sim\varepsilon^{\Gamma(a)}$, then the factorization property holds up to times $\sim\log\varepsilon^{-1}$, which is what is needed. We proceed by fixing λ, a_0, η_1 as in Eqs. (4.7), (4.8), and (4.9) and stating the following:

4.3. Corollary (of Proposition 4.1)

There is c_{12} so that for all configurations σ'

$$P^{\varepsilon}_{\sigma'}\left[\sup_y \left|\varepsilon^{\lambda}\sum_{x\in I_\lambda(y)}\tilde\sigma(x,1)\right| > \varepsilon^{\Gamma(a)+\eta_1}\right] \leqslant c_{12}\varepsilon \tag{4.15}$$

We have now the necessary tools for studying the evolution in the time interval $[T_\varepsilon(a), T_\varepsilon + t]$.

4.4. Definition

We fix $m' > 0$. For each $\varepsilon > 0$ σ will denote, in the sequel, any configuration such that

$$m'\varepsilon^{\Gamma(a)} \leqslant m(\sigma) \tag{4.16}$$

$$|m_\lambda(\sigma, x) - m(\sigma)| \leqslant \varepsilon^{\eta_1+\Gamma(a)} \tag{4.17}$$

with λ as in Eqs. (4.7) and (4.8). We proceed this way for the sake of definiteness. Analogous proofs apply in the cases when $m(\sigma) \leqslant -\varepsilon^{\Gamma(a)}m'$. We also need an uper bound for $m(\sigma)$. The tyical values of $m(\sigma)$ are of order $\varepsilon^{\Gamma(a)}$, but we need to iterate the estimates: our arguments will apply up to values m'' of $m(\sigma)$ such that $S_2 m'' = (3\beta)^{-1}\alpha^{1/2}$ (cf. Proposition 4.6 below). $\langle\cdot\rangle$ denotes the expectation for the process with initial distribution, which should be understood from the context.

4.5. Proposition

There exist c' and c'' so that the following holds.

Let σ be as in Definition 4.4; then, for all $x \in [0, \varepsilon^{-1}L]$,

$$m_1' \leqslant \langle \sigma(x, 1) \rangle \leqslant m_1'' \tag{4.18}$$

where

$$\int_{m(\sigma)+c''\varepsilon^{\delta_6}}^{m_1''} dm\, V'(m)^{-1} = 1 = \int_{m(\sigma)-c'\varepsilon^{\delta_6}}^{m_1'} dm\, V'(m)^{-1} \tag{4.19}$$

and

$$\delta_6 = \min\{\tfrac{2}{3}(1-\lambda), \eta_1 + \Gamma(a), \delta_0\} \tag{4.20}$$

Proof. Set

$$\eta_2 = \tfrac{2}{3}(1-\lambda)$$

By Eq. (4.14) we have, for $t \geqslant \varepsilon^{\eta_2}$,

$$\begin{aligned}\langle \sigma(x, t) \rangle \leqslant {} & \sum_y P_t^\varepsilon(y \mid x)\, \sigma(y) + \int_{\varepsilon^{\eta_2}}^{t-\varepsilon^{\eta_2}} \sum_y P_{t-s}^\varepsilon(y \mid x) \\ & \times [2\gamma(\langle \sigma(y+1, s) \rangle + \langle \sigma(y-1, s) \rangle) - 2\langle \sigma(y, s) \rangle \\ & - \beta \langle \sigma(y-1, s)\, \sigma(y, s)\, \sigma(y+1, s) \rangle] \\ & + 2(4\gamma + 2 + \beta)\, \varepsilon^{\eta_2}\end{aligned} \tag{4.21}$$

We have for any y and z, $|z| \leqslant \varepsilon^{-\lambda}$, $t \geqslant \varepsilon^{\eta_2}$, that

$$|P_t^\varepsilon(y \mid x) - P_t^\varepsilon(y+z \mid x)| \leqslant c_{13} \varepsilon^{-\lambda + 1/2(2-\eta_2)} \pi_t(y \mid x) \leqslant c_{13} \varepsilon^{\delta_6} \pi_t(y \mid x) \tag{4.22a}$$

$$\sum_y \pi_t(y \mid x) = 1 \tag{4.22b}$$

$$\begin{aligned}&\langle \sigma(y-1, t)\, \sigma(y, t)\, \sigma(y+1, t) \rangle - \langle \sigma(y-1, t) \rangle \langle \sigma(y, t) \rangle \langle \sigma(y+1, t) \rangle \\ &\quad \leqslant c_{14} \varepsilon^{\delta_6}\end{aligned} \tag{4.23}$$

Therefore, using Eq. (4.13b), we have

$$\begin{aligned}\langle \sigma(x, t) \rangle \leqslant {} & m(\sigma) + \varepsilon^{\eta_1 + \Gamma(a)} + 2c_{13} \varepsilon^{\delta_6} \\ & + \int_{\varepsilon^{\eta_2}}^{t-\varepsilon^{\eta_2}} ds \sum_y P_{t-s}^\varepsilon(y \mid x)[\alpha \langle \sigma(y, s) \rangle - \beta \langle \sigma(y, s) \rangle^3] \\ & + \beta c_{14} \varepsilon^{\delta_6} + (4\gamma + 3\beta)(2\varepsilon^{\eta_2} + c_{11} \varepsilon^{1-\eta_2}) + 2(4\gamma + 2 + \beta)\, \varepsilon^{\eta_2}\end{aligned} \tag{4.24}$$

Let

$$m_t'' = \sup_x \langle \sigma(x, t) \rangle \; m_t' = \inf_x \langle \sigma(x, t) \rangle \tag{4.25}$$

If m_s'', $s \leqslant t$, is such that $m_s'' \leqslant ((3\beta)^{-1} \alpha)^{1/2}$, then

$$\alpha \langle \sigma(y, s) \rangle - \beta \langle \sigma(y, s) \rangle^3 \leqslant \alpha m_t'' - \beta (m_t'')^3$$

and

$$m_t'' \leqslant m(\sigma) + \int_0^t ds(\alpha m_s'' - \beta (m_s'')^3) + c'' \varepsilon^{\delta_6} \tag{4.26}$$

We have therefore proven the first equality in Eq. (4.19). The second one is proven analogously and we omit the details.

By Proposition 4.5 and Corollary 4.3 we can find with large probability good "flat" configurations at time 1. We hen study the evolution for a unit time interval starting from any such configuration, so that we can apply again Proposition 4.5. By iterating this procedure we get the following:

4.6. Proposition

Let t^* be the largest integer so that $m_{t^*} := S_{t^*}(m(\sigma)) \leqslant ((3\beta)^{-1} \alpha)^{1/2}$, where $S_t(m)$ is the solution of Eq. (2.4) with initial value $m(\sigma)$ (σ being a configuration chosen as in Definition 4.4). Then

$$m_{t^*-1}' - c_{12} \varepsilon (t^* - 1) \leqslant \langle \sigma(x, t^* - 1) \rangle \leqslant m_{t^*-1}'' + c_{12} \varepsilon (t^* - 1) \tag{4.27}$$

c_{12} being the constant appearing in Eq. (4.15). m_{t^*-1}' and m_{t^*-1}'' are such that

$$\int_{m(\sigma) + c''(t^* - 1)\varepsilon^{\delta_6}}^{m_{t^*-1}''} dm \, V'(m)^{-1} = t^* - 1 = \int_{m(\sigma) - c'(t^* - 1)\varepsilon^{\delta_6}}^{m_{t^*-1}'} dm \, V'(m)^{-1} \tag{4.28}$$

Furthermore, there is a constant c_{15} such that

$$m_{t^*-1}'' - m_{t^*-1}' \leqslant c_{15} t^* \varepsilon^{\delta_6 - \Gamma(a)} \tag{4.29}$$

Remarks. Notice that t^* increases like $\log \varepsilon^{-1}$ when ε goes to zero (since $m(\sigma) \geqslant m_1' \varepsilon^{\Gamma(a)}$, according to Definition 4.4). Therefore, if we choose a so close to 1 that

$$\delta_7 := \delta_6 - \Gamma(a) > 0 \tag{4.30}$$

then $m_{t^*-1}'' - m_{t^*-1}'$ vanishes like $\log \varepsilon^{-1} \varepsilon^{\delta_7}$ when ε goes to zero.

Proof of Proposition 4.6. For $n \leqslant t^* - 1$, we define m'_n and m''_n so that

$$\int_{m(\sigma) - c'\varepsilon^{\delta_6} n}^{m'_n} dm\, V'(m)^{-1} = n = \int_{m(\sigma) + c''\varepsilon^{\delta_6} n}^{m''_n} dm\, V'(m)^{-1} \tag{4.31}$$

We are going to prove that, for all $n \leqslant t^* - 1$ and all x

$$m'_n \leqslant \langle \sigma(x, n) \rangle \leqslant m''_n \tag{4.32a}$$

for all $\sigma' \in \mathscr{G}_{n-1}$, where

$$\mathbb{P}^{\varepsilon}_{\sigma}[\mathscr{G}_n] \geqslant 1 - nc_{12}\varepsilon \tag{4.32b}$$

$$\mathscr{G}_n = \left\{ \text{for any } y\ m'_n - \varepsilon^{\eta_1 + \Gamma(a)} \leqslant \varepsilon^{\lambda} \sum_{x \in I_\lambda(y)} \sigma(x, n) \leqslant m''_n + \varepsilon^{\eta_1 + \Gamma(a)} \right\} \tag{4.32c}$$

The proposition will then follow after integrating Eq. (4.32a) w.r.t. the law at time $t^* - 1$ and by using Eq. (4.32b).

The proof of Eq. (4.32) is obtained by induction on n. We have already proved it for $n = 1$. So we assume that it holds for $n - 1 < t^* - 1$ and we are going to prove it for n. We condition on the configuration σ' at time $n - 1$, and we consider the case where σ' is in $\mathscr{G}_{n-1}$. We can then proceed as in Proposition 4.5; we only have to write m''_{n-1} instead of $m(\sigma)$ in the r.h.s. of Eq. (4.24) and in the successive equations. We then get that

$$\mathbb{E}^{\varepsilon}_{\sigma'}[\sigma(x, 1)] \leqslant \tilde{m}''_n \tag{4.33}$$

where

$$\int_{m''_{n-1} + c''\varepsilon^{\delta_6}}^{\tilde{m}''_n} dm\, V'(m)^{-1} = 1 \tag{4.34}$$

From the definition of m''_{n-1}, it then follows that

$$\begin{aligned}
&\int dm\, V'(m)^{-1}\, \mathbb{1}(m''_{n-1} \leqslant m \leqslant \tilde{m}''_n) \\
&\qquad + \int dm\, V'(m)^{-1}\, \mathbb{1}(m(\sigma) + c''(n-1)\,\varepsilon^{\delta_6} \leqslant m \leqslant m''_{n-1}) \\
&\quad = n + \int dm\, V'(m)^{-1}\, \mathbb{1}(m''_{n-1} \leqslant m \leqslant m''_{n-1} + c''\varepsilon^{\delta_6})
\end{aligned}$$

$$\begin{aligned}
&\int dm\, V'(m)^{-1}\, \mathbb{1}(m(\sigma) + c''n\varepsilon^{\delta_6} \leqslant m \leqslant \tilde{m}''_n) \\
&\quad = \int dm\, V'(m)^{-1}\, \mathbb{1}(m(\sigma) + c''n\varepsilon^{\delta_6} \leqslant m \leqslant m''_n) \\
&\qquad + \int dm\, V'(m)^{-1}\, \mathbb{1}(m''_{n-1} \leqslant m \leqslant m''_{n-1} + c''\varepsilon^{\delta_6}) \\
&\qquad - \int dm\, V'(m)^{-1}\, \mathbb{1}(m(\sigma) + c''(n-1)\,\varepsilon^{\delta_6} \leqslant m \leqslant m(\sigma) + c''n\varepsilon^{\delta_6})
\end{aligned}$$

Therefore $\tilde{m}_n'' \leqslant m_n''$. From this we get

$$\langle \mathbb{E}^{\varepsilon}_{\sigma'}[\sigma(x, 1)] \rangle \leqslant m_n''$$

and since analogous argument holds for the lower bound, Eq. (4.32a) is proven for n.

Denote by $\mathbb{P}^{\varepsilon}_{\sigma}$ the law when the initial measure is concentrated on σ and by $\mathscr{F}_{n-1}$ the σ-algebra generated by the variables $\sigma(x, t')$ for all x in Z and $t' \leqslant n-1$. We then have by Eq. (4.15), Corollary 4.3, that

$$\mathbb{P}^{\varepsilon}_{\sigma}\left[\sup_{y} \left| \varepsilon^{\lambda} \sum_{x \in I_{\lambda}(y)} \sigma(x, n) - \varepsilon^{\lambda} \sum_{x \in I_{\lambda}(y)} \mathbb{P}^{\varepsilon}_{\sigma}[\sigma(x, n) \mid \mathscr{F}_{n-1}] \right| > \varepsilon^{\eta_1 + \Gamma(a)}\right]$$
$$\leqslant c_{12}\varepsilon$$

Therefore, by Eq. (4.32a),

$$\mathbb{P}^{\varepsilon}_{\sigma}[\mathbb{P}^{\varepsilon}_{\sigma}[\mathscr{G}^{c}_{n} \mid \mathscr{F}_{n-1}] \; \mathscr{G}_{n-1}] \leqslant c_{12}\varepsilon$$

By the induction assumption, we estimate the probability of $\mathscr{G}^{c}_{n-1}$; so we prove Eq. (4.32b) for n. The proposition is therefore proven.

Proof of Theorems 2.1 and 2.2. By the Markov property, for any cylinder function f

$$\mu^{\varepsilon}_{T_{\varepsilon}+t}[f] = \mu^{\varepsilon}_{T_{\varepsilon}(a)}[\mathbb{E}^{\varepsilon}_{\sigma}[f_{T_{\varepsilon}+t-T_{\varepsilon}(a)}]] \tag{4.35}$$

We fix σ as in Definition 4.4 and we distinguish two cases: $t^* - 1$ smaller or larger than $T_{\varepsilon} - T_{\varepsilon}(a) + t$. To simplify the notation, let

$$\tau = T_{\varepsilon} - T_{\varepsilon}(a) + t \tag{4.36}$$

Case $\tau \leqslant t^* - 1$. For notational simplicity we assume that τ is an integer. Let $\bar{\mu}^{\varepsilon}_{t}$ be the measure at time t when the initial distribution is concentrated on a configuration $\bar{\sigma}$ as in Definition 4.4. Then, for any n and $x_1, \dots, x_n$ pairwise disjoint,

$$\bar{\mu}^{\varepsilon}_{\tau}\left[\prod_{i=1}^{n} \sigma(x_i)\right] = \bar{\mu}^{\varepsilon}_{\tau-1}\left[\mathbb{E}^{\varepsilon}_{\sigma}\left[\prod_{i=1}^{n} \sigma(x_i, 1)\right]\right] \tag{4.37}$$

and by Proposition 4.1

$$\left|\mathbb{E}^{\varepsilon}_{\sigma}\left[\prod_{i=1}^{n} \sigma(x_i, 1)\right] - \prod_{i=1}^{n} \mathbb{E}^{\varepsilon}_{\sigma}[\sigma(x_i, 1)]\right| \leqslant 2^n e(n)\, \varepsilon^{2\delta_0}$$

Therefore, from Eqs. (4.32b) and (4.29),

$$\left|\bar{\mu}^{\varepsilon}_{\tau}\left[\prod_{i=1}^{n}\sigma(x_i)\right]-(S_{\tau}(m(\bar{\sigma}))^n)\right|$$
$$\leqslant 2^n e(n)\,\varepsilon^{2\delta_0}+t^* c_{12}\varepsilon+nc_{15}t^*\varepsilon^{\delta_6-\Gamma(a)} \tag{4.38}$$

Case $\tau \geqslant t^*-1$. There is $\tau(m')$, m' as in Definition 4.4, such that for any $\bar{\sigma}$ as in Definition 4.4, $t^*-1+\tau(m')\geqslant\tau$ (t^* depends on $\bar{\sigma}$).

Just as before and with the same notation,

$$\left|\bar{\mu}^{\varepsilon}_{\tau}\left[\prod_{i=1}^{n}\sigma(x_i)\right]-\bar{\mu}^{\varepsilon}_{t^*-1}\left[\prod_{i=1}^{n}\mathbb{E}^{\varepsilon}_{\sigma}[\sigma(x_i,\tau-t^*-1)]\right]\right|\leqslant 2^n e(n)\,\varepsilon^{2\delta_0} \tag{4.39}$$

where the constants $e(n)$ of Proposition 4.1 depend on $\tau(m')$ and hence are fixed once m' is given. As before, we need to estimate, for $\sigma\in\mathscr{G}_{t^*-1}$,

$$\mathbb{E}^{\varepsilon}_{\sigma}[\sigma(x,\tau-t^*-1)]-S_{\tau}(m(\bar{\sigma}))$$

We have to modify our previous argument because we cannot say any longer that $\alpha\langle\sigma(y,t)\rangle-\beta\langle\sigma(y,t)\rangle^3$ is increasing. We proceed as follows. We assume that $\sigma_t\in\mathscr{G}_{t^*-1}$, $t\in[\varepsilon^{\eta_2},\tau(m')]$, and we set

$$\langle\sigma(x,t)\rangle := \mathbb{E}^{\varepsilon}_{\sigma}[\sigma(x,t)] \tag{4.40}$$

and similarly to Eq. (4.24), we get

$$\begin{aligned}\langle\sigma(x,t)\rangle \leqslant{}& m''_{t^*-1}+\varepsilon^{\Gamma(a)+\eta_1}+2c_{13}\varepsilon^{\delta_6}+\beta c_{14}\varepsilon^{\delta_6}\tau(m')\\ &+(4\gamma+3\beta)(2\varepsilon^{1/2}+c_{11}\varepsilon^{1-\eta_2})\,\tau(m')+2(4\gamma+2+\beta)\,\varepsilon^{\eta_2}\\ &+\int_{\varepsilon^{\eta_2}}^{t-\varepsilon^{\eta_2}} ds\sum_y P^{\varepsilon}_{t-s}(y\,|\,x)(\alpha\langle\sigma(y,s)\rangle-\beta\langle\sigma(y,s)\rangle^3)\end{aligned} \tag{4.41}$$

An analogous lower bound is obtained similarly. Let

$$d^{\varepsilon}_t=\sup\langle\sigma(x,t)\rangle-\inf\langle\sigma(x,t)\rangle \tag{4.42}$$

then, for a suitable constant c_{16},

$$d^{\varepsilon}_t\leqslant m''_{t^*-1}-m'_{t^*-1}+c_{16}\varepsilon^{\delta_6}+\int_0^t ds(\alpha\, d^{\varepsilon}_s+3\beta\, d^{\varepsilon}_s) \tag{4.43}$$

Therefore, posing

$$\tilde{m}''_t=\sup\langle\sigma(x,t)\rangle,\qquad \tilde{m}'_t=\inf\langle\sigma(x,t)\rangle \tag{4.44}$$

we get

$$\tilde{m}_t'' \leqslant m_{t^*-1}'' + c_{16}\varepsilon^{\delta_6} + \int_0^t ds(\alpha\tilde{m}_s'' - \beta(\tilde{m}_s'')^3) + \int_0^t ds\, 3\beta\, d_s^\varepsilon \tag{4.45}$$

An analogous expression gives the lower bound.

From Eqs. (4.29), (4.43), and (4.45) and because $t \leqslant \tau(m')$, we then prove that for, suitable $\delta_7 > 0$ and c_{17},

$$\sup_x |\mathbb{E}_\sigma^\varepsilon[\sigma(x, \tau - t^* - 1) - S_{\tau - t^* - 1}(m(\sigma))| \leqslant c_{17}\varepsilon^{\delta_7} \tag{4.46}$$

From here the same argument as in the case $\tau \leqslant t^* - 1$ applies. Summarizing, we have proven so far that, for any $n \geqslant 1$, $t \geqslant 0$, $x_1,..., x_n$ mutually distinct, any $m' > 0$,

$$\lim_{\varepsilon \to 0} \left| \mu_{T_\varepsilon + 1}^\varepsilon \left[\prod_{i=1}^n \sigma(x_i) \right] - \mu_{T_\varepsilon(a)}^\varepsilon \left[(S_{T_\varepsilon - T_\varepsilon(a) + t}(m(\sigma))^n \right] \right|$$
$$\leqslant \lim\sup \mu_{T_\varepsilon(a)}^\varepsilon [|m(\sigma)| \leqslant m' \varepsilon^{\Gamma(a)}]$$

By letting m' go to zero, using Proposition 3.6, we obtain the proof of Theorem 2.1 and 2.2.

APPENDIX A

Here we prove Eqs. (3.11) and (3.24). We start with the following definition.

A.1. Definition

We denote by $x_1(s), x_2(s)$ (resp. $x_1^0(s), x_2^0(s)$), $s \geqslant 0$, the positions at time s of two stirring particles (respectively, two independent random walks) moving with intensity one and such that

$$x_i(0) = x_i \quad x_1^0(0) = x_i \qquad i = 1, 2 \tag{A.1}$$

where $x_1 \neq x_2$ are in Z. We call interacting (resp. independent) i-particle the particle starting from x_i at time 0, $i = 1, 2$. We denote by P (resp. P^0) the law of the stirring (resp. random walk) process defined above.

A.2. Lemma

For any $x_1, x_2 \in Z$ and $t \geqslant 0$, the following holds.

Let

$$T_0 = \min\{s \geqslant 0: x_1^0(s) = x_2^0(s)\} \tag{A.2}$$

then

$$P(\{|x_1(t)-x_2(t)|=1\})=\tfrac{1}{2}P^0(\{|x_1^0(t)-x_2^0(t)|=1\}\cap\{T_0<t\})$$
$$+P^0(\{x_1^0(t)=x_2^0(t)\}\cap\{T_0<t\})$$
$$+P^0(\{|x_1^0(t)-x_2^0(t)|=1\}\cap\{T_0>t\}) \qquad \text{(A.3)}$$

Proof. We define a coupling between the two processes, i.e., we define a probability distribution Q on $(Z^4)^{[0,t]}$ such that its marginals are P and P^0, respectively. Such coupling is that introduced in Ref. 40. For the reader's convenience, we sketch here its main features. Q is such that the displacements of the independent and interacting particles are the same with one exception. When the independent particles have a displacement that would bring an interacting particle over the other, the interacting ones, instead, interchange their positions.

Let us define the following stopping times for the independent process.

$$T_0 \text{ is defined in Eq. (A.2)} \qquad \text{(A.4a)}$$

$$\tau_0=\inf\{s\leqslant T_0\colon |x_1^0(s)-x_2^0(s)|=1\} \qquad \text{(A.4b)}$$

$$T_n=\inf\{s\geqslant \tau_{n-1}\colon x_1^0(s)=x_2^0(s)\} \qquad \text{(A.4c)}$$

$$\tau_n=\inf\{s\geqslant T_n\colon |x_1^0(s)-x_2^0(s)|=1\} \qquad \text{(A.4d)}$$

Finally, let

$$d(s)=|x_1(s)-x_2(s)| \qquad 0\leqslant s\leqslant t \qquad \text{(A.5a)}$$

$$d^0(s)=|x_1^0(s)-x_2^0(s)| \qquad 0\leqslant s\leqslant t \qquad \text{(A.5b)}$$

From the definition of Q it follows that

$$d(s)\in\{d^0(s), d^0(s)+1\} \qquad \text{for all } s\in[0,t]$$

$$d(T_n)=1 \qquad \text{for all } n=0,1,\ldots$$

$$d(s)=d(\tau_n) \qquad \text{for all } \tau_n\leqslant s<T_{n+1}$$

On the other hand, by definition of Q and by Eq. (A.4d),

$$d(\tau_n)\in\{d^0(\tau_n), 2\}$$

with equal probability. Therefore we have that

$$d(t)=d^0(t)+\sum_{n=0}^{\infty}\xi(n)\,\mathbb{1}(\tau_n\leqslant t<T_{n+1})+\sum_{n=0}^{\infty}\mathbb{1}(T_n\leqslant t<\tau_n) \qquad \text{(A.6a)}$$

where the $\xi(n) \in \{0, 1\}$ are i.i.d. random variables (independent of $d^0(t)$) and such that

$$Q(\xi(0)=0)=\tfrac{1}{2}=Q(\xi(0)=1) \tag{A.6b}$$

From Eqs. (A.6b) and (A.4), the lemma follows. In fact,

$$\begin{aligned} P(\{d(t)=1\}) = Q(\{d^0(t)=1\} \cap \{T_0>t\}) \\ + \sum_{n=0}^{\infty} [\tfrac{1}{2}Q(\{\tau_n<t<T_{n+1}\} \cap \{d^0(t)=1)\} \\ + Q(\{T_n \leqslant t < \tau_n\} \cap \{d^0(t)=0\}) \end{aligned}$$

Proof of Eqs. (3.11) and (3.24). We will show that there exist $\delta>0$ and $t_\delta>0$ such that

$$\begin{aligned} \sup_{t \geqslant t_\delta} \sup_{x_1, x_2} \left| \sum_{z_1 z_2} \{P_t(z_1, z_2 \mid x_1, x_2) - G_t(z_1, z_2 \mid x_1, x_2)\} \mathbb{1}(|z_1-z_2|=1) \right| \\ \leqslant ct^{-(1/2+\delta)} \end{aligned} \tag{A.7}$$

where c is a suitable positive constant.

From Eq. (A.7), Eqs. (3.11) and (3.24) follow. We have that

$$\begin{aligned} |P^0(\{x_1^0(t)=x_2^0(t)\} \cap \{T_0<t\}) - \tfrac{1}{2}P^0(\{|x_1^0(t)-x_2^0(t)|=1\} \cap \{T_0<t\})| \\ \leqslant \int_0^t P^0(d\tau)\, |P^0(z_0(t-\tau)=1) - P^0(z_0(t-\tau)=0)| \end{aligned} \tag{A.8}$$

where $z_0(s)$ is the position at time s of a random walker starting from the origin and $P^0(d\tau)$ is the distribution of its first return to the origin. From the local central limit theorem,[43] we have that there exists $\hat{\delta}<\frac{1}{2}$ such that

$$\begin{aligned} \text{r.h.s. of Eq. (A.8)} \leqslant \text{const} \int_0^t d\tau (1+\tau^{1/2})^{-1} (1+(t-\tau)^{1/2})^{-1}, \\ \leqslant \text{const } t^{-1+\hat{\delta}} \end{aligned} \tag{A.9}$$

From Eqs. (A.3), (A.8), and (A.9) it follows that

$$|P(\{|x_1(t)-x_2(t)|=1\}) - P^0(\{|x_1^0(t)-x_2^0(t)|=1\})| \leqslant c' t^{-1+\hat{\delta}} \tag{A.10}$$

Therefore we can put P^0 instead of P in the l.h.s. of Eq (A.7). Then the estimate in Eq. (A.7) follows from the local central limit theorem.[43]

APPENDIX B

The proof of Proposition 4.1 is based on techniques introduced in Ref. 40 and developed in Ref. 41 to study the symmetric simple exclusion process. We shall follow Ref. 1b to adapt them to the present context.

Notation. In what follows, ε is a positive (small enough) number, $\sigma^* \in \{-1, 1\}^{Z_\varepsilon}$ is the initial (arbitrary) configuration for the process with generator L^ε, $n \geqslant 1$ and $x_1,..., x_n$ are mutually distinct sites in Z_ε. Constant $c_1, c_2,...$ and exponents $\delta_1, \delta_2,..., \gamma_1, \gamma_2,...$ are not the same as in the text.

Our estimates require initial measures with fast decaying correlations and an everage spin magnetization which varies smoothly in space. At time 0 we have the first but not the second property. Correlations increase with time while the average magnetization becomes smoother. A possible compromise is to choose a time $T = \varepsilon^{2/3}$ (recall that at time T the stirring has been effective for a time $\varepsilon^{-2+2/3}$) and then to consider the state at such time T as the initial state (in Proposition 4.1 we are interested in times $t > \varepsilon^{1/3}$, so the above procedure is justified by the Markov nature of the process).

B.1. Lemma

There is a constant c_1 and given $\delta_1 > \frac{2}{3}$ a constant c_2 so that the following holds.

Let v^ε be the distribution of the process at time $\varepsilon^{2/3}$ starting from σ^*. Let $a \in \{-1, 1\}$ and pose

$$1_a(x) = 1(\sigma(x) = a) \tag{B.1a}$$

$$\tilde{1}_a(x) = 1_a(x) - v^\varepsilon[1_a(x)] \tag{B.1b}$$

Set

$$\varepsilon_0 = \varepsilon^{2/3} \tag{B.2}$$

(i) Then, for $a = \pm 1$,

$$|v^\varepsilon[1_a(x)] - P_{\varepsilon,a}(\varepsilon_0 x)| \leqslant c_1 \varepsilon_0 \tag{B.3}$$

where

$$p_{\varepsilon,1}(r) = \varepsilon_0 \sum_y (2\pi)^{-1/2} \exp(-\tfrac{1}{2}(r - \varepsilon_0 y)^2)\, 1(\sigma^*(y) = 1) \tag{B.4a}$$

$$p_{\varepsilon,-1}(r) = 1 - p_{\varepsilon,1}(r) \tag{B.4b}$$

(ii) For any $n \geqslant 1$ and mutually distinct sites $x_1,..., x_n$ in Z_ε

$$\sum_{a_1,...,a_n} \left| \nu^\varepsilon \left[\tilde{1}_{a_1}(x_1) \prod_{i=2}^{n} 1_{a_i}(x_i) \right] \right|$$
$$\leqslant c_2 n \exp \left\{ -\varepsilon^{-(\delta_1 - 2/3)} \prod_{i=2}^{n} 1(|x_i - x_1| > \varepsilon^{-\delta_1}) \right\} \tag{B.5}$$

Notice that there are constants $b(n)$ so that

$$\sup_{\varepsilon > 0} \sup_{r \in \mathbb{R}} |p_{\varepsilon,\pm 1}^{(n)}(r)| \leqslant b(n) \tag{B.6}$$

Proof of Lemma B.1(i). We write Eq. (3.5) for $n=1$. The contribution of the integral in Eq. (3.5) goes like $\varepsilon^{2/3} = \varepsilon_0$. So it fits with the estimate in Eq. (B.3) and we are left with a single random walk: Lemma B.1(i) is then easily proven. We postpone the proof of Lemma B.1(ii) to set notation and definitions which will be used both in its proof and afterwards.

B.2. Definition (Stirring Process)

The Stirring Process in Z_ε is defined as follows. Recall that Z_ε is Z after identification of x and y if $|y - x| = [L\varepsilon^{-1}]$. For any x in Z define a Poisson point process with intensity 1, hereafter called "the process for the pair $x, x+1$." The processes at $x, x+1$ and $y, y+1$ with $|x - y| = [L\varepsilon^{-1}]$ are identical while the processes for x in $[0, [\varepsilon^{-1}L] - 1]$ are independent. On each site of Z there is a stirring particle. The stirring particle which at time 0 is at site x has label x. Its later positions are denoted by $Y^\varepsilon(x, t)$, $t \leqslant 0$. The evolution is determined by the Poisson processes as follows. Each time a Poisson event occurs, say time t at sites $x, x+1$, the stirring particles which at time t were at x and $x+1$ exchange their positions.

B.3. Definition [the Branching Labeled Stirring (BLS) Process]

B.3a. The Branching

Let

$$\lambda_1 = (1+\gamma)^2, \qquad \lambda_2 = 1 - \gamma^2, \qquad \lambda_3 = (1-\gamma)^2 \tag{B.7a}$$

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 3 + \gamma^2 \tag{B.7b}$$

Denote by $(t_n)_{n \geqslant 1}$ the process on R_+ such that the variables $(t_n - t_{n-1})$, $t_0 = 0$, are mutually independent and have Poisson distribution with intensity $n\lambda$.

Let $(\beta_n)_{n \geqslant 1}$ be the independent process such that for any $n \geqslant 1$ the probability that $\{\beta_n = j\}$ is $\lambda^{-1}\lambda_j$, $j = 1, 2, 3$. $(\underline{\alpha}_n)_{n \geqslant 1}$ is the following Markov process. $\underline{\alpha}_n$ has values on the subsets of $\{-1, 0, 1\}^n$. The probability of $\underline{\alpha}_{n+1}$ given $\underline{\alpha}_n$ is determined as follows. With equal probability, take any of the elements of $\underline{\alpha}_n$, say α (which is therefore a sequence in $\{-1, 0, 1\}^n$). Then $\underline{\alpha}_{n+1}$ is obtained by adding to any n-sequence $\beta \neq \alpha$ in $\underline{\alpha}_n$ an $n+1$ entry which is 0. Three more elements are obtained by adding to α, as $n+1$ entry, $-1, 0$, and 1, respectively. We shall say that the element with all zeros in $\underline{\alpha}_n$ is the ancestor of the family $\underline{\alpha}_n$. The position in a sequence of the last entry which is different from 0 denotes the age of that sequence.

The elements in $\underline{\alpha}_n$ are ordered according to their age, and if the age is the same the sequence having -1 as the last entry $\neq 0$ precedes that with 1.

We denote by $(\omega_n)_{n \geqslant 1}$ the process obtained by taking the direct product of the above three and by (Ω, P) its canonical probability space. ω denotes the generic element in Ω.

We shall often employ the following notation:

$$\underline{\alpha}(t) = \underline{\alpha}_n \qquad \text{whenever} \quad t_n \leqslant t < t_{n+1} \tag{B.8}$$

$$N_t(\omega) = \sup\{k: t_k \leqslant t\} \tag{B.9}$$

B.3b. Branching Particles

Given n particles $1, 2, ..., n$ and $(\omega(1), ..., \omega(n)) \in \Omega^n$, we define the following branching structure.

For particle 1 we look at $\omega(1)$. Particle 1 remains alone in the time interval $0 \leqslant s < t_1(1)$. At $t_1(1)$ two new particles are created, the "left" particle corresponding to -1 in $\underline{\alpha}_1(1)$, the right particle, $+1$ in $\underline{\alpha}_1(1)$, while the original particle has label 0. At time $t_1(2)$ one of the previous three particles generates two new ones. The old particles at time $t_1(2)$ have 0 as second entry, hence they are distinguished by their first entry. The two new particles have as second entry -1 and $+1$ and are, respectively, the left and right descendants of the particle determined by their first entry. Iterating this procedure we give labels to all the particles which descend from particle 1. (Notice that the ordering in $\underline{\alpha}_n(1)$ corresponds to the order of appearance of the particles). In a completely analogous fashion we define the descendants of particles $2, ..., n$. We write $\underline{\alpha}(i, t)$ for the variable $\underline{\alpha}(t)$

corresponding to $\omega(i)$, and $|\alpha(i, t)|$ for the number of particles in $\underline{\alpha}(i, t)$ ($|\underline{\alpha}(i, t)| = 2N_t(\omega(i)) + 1$). We order the labels in $\bigcup_{i=1}^{n} \underline{\alpha}(i, t)$ as follows: $\alpha < \beta$ if either (1) $\alpha \in \underline{\alpha}(i, t)$, $\beta \in \underline{\alpha}(j, t)$, and $i < j$, or (2) $\alpha \in \underline{\alpha}(i, t)$, $\beta \in \underline{\alpha}(i, t)$, and the age of α precedes that of β, according to Definition B.3a.

B.3c. Branching and Stirring

Let $n \geq 1$, $x_1, \ldots, x_n$ and $\omega(n)$ be given. Then $(\underline{x}_1(t), \ldots, \underline{x}_n(t))$ is the following process. $x_i(t) \in Z_\varepsilon^{|\underline{\alpha}(i,t)|}$, $i = 1, \ldots, n$, and it is defined as follows. Let $0 < \tau_1 \cdots \tau_k < \cdots$ be the ordered sequence of all the times $t_k(i)$, $k \geq 1$ and $i = 1, \ldots, n$ (assume strict inequality holds, as it is a.s.). Let $y_1, \ldots, y_m$ be the sites occupied by the particles at time τ_k. Then the particles will move like $Y^\varepsilon(y_i, t)$, $i = 1, \ldots, m$, for $\tau_k \leq t < \tau_{k+1}$, namely if the particle α is at y_i at time τ_k, then it will be at $Y^\varepsilon(y_i, t)$ at time t.

To complete the definition of the process we say that the particles are initially at $x_1, \ldots, x_n$ and that the left (right) descendants of a particle are created to its left (right) nearest-neighbor site, according to the specifications given by $\omega(1) \cdots \omega(n)$. Notice that it is possible that more particles stay on the same site; in that case they will remain stuck to each other thereafter. We shall denote by $(\underline{x}_1, \ldots, \underline{x}_n)$ the process $(\underline{x}_1(t), \ldots, \underline{x}_n(t))_{t \geq 0}$ and by $P_{\underline{\omega}}^\varepsilon(d\underline{\underline{x}}_1 \cdots d\underline{\underline{x}}_n)$ its law. We shall also denote by $x_i(\alpha, t)$ the position of the particle with label $\alpha \in \underline{\alpha}(i, t)$. When we do not want to specify the ancestor i, we shall simply write $x(\alpha, t)$. We shall then use the following notation: $\underline{\omega}$ for $(\omega(1) \cdots \omega(n))$ and $N_t(\underline{\omega})$ for $\sum_{i=1}^{n} N_t(\omega(i))$.

The following definition reports some results proven in Ref. 1b, Sec. 3, which we will often use in the sequel.

B.4. Definition (Duality)

Fix $t > 0$, $x_1, \ldots, x_n$ and let μ^ε be any probability on $\{-1, 1\}^{Z_\varepsilon}$; let $(a_1^*, \ldots, a_n^*) \in \{-1, 1\}^n$. Denote by $\underline{a}$ the elements in $\bigsqcup_{k \geq 1} \{-1, 1\}^k$ and let $|\underline{a}| = k$ if $\underline{a} \in \{-1, 1\}^k$. Then there are functions $F_{a_i^*}^{\{1, \ldots, n\}}$ $(\underline{a}_i, \omega(i), \underline{x}_i$ $i = 1 \cdots n; t)$ (which we frequently write as $F_{a_i^*}$) with the following properties:

(1) $F_{a_i^*}$ is either 0 or 1 and $F_1 + F_{-1} = 1$ $(i = 1, \ldots, n)$.

(2) Label the elements of $\underline{a}$ in $\{-1, 1\}^{|\underline{\alpha}|}$ as a_α set

$$1_{\underline{a}_i}(\underline{x}_i(t)) = \prod_{\alpha \in \underline{\alpha}(i, t)} 1_{a_\alpha}(x(\alpha, t)) \tag{B.10a}$$

Then, denoting by μ_t^ε the law of the process at time t which starts from μ^ε

$$\mu_t^\varepsilon\left[\prod_{i=1}^n 1_{a_i^*}(x_i)\right]=\int P(d\omega(1)\cdots P(d\omega(n))\,P_\omega^\varepsilon(d\underline{\underline{x}}_1\cdots d\underline{\underline{x}}_n)\sum_{\underline{a}_1\cdots\underline{a}_n}$$

$$\left\{\prod_{i=1}^n 1(|a_i|=|\underline{\alpha}(i,t)|)\,F_{a_i^*}\right\}\mu^\varepsilon\left[\prod_{i=1}^n 1_{a_i}(\underline{x}_i(t))\right] \tag{B.10b}$$

(3) Assume $\underline{\underline{x}}_1\cdots\underline{\underline{x}}_n$ are such that $x_i(\alpha,t)\neq x_i(\alpha',t)$ for any $\alpha\neq\alpha'$ and i in $\{1,\dots,n\}$ and that $\underline{x}_i(t)\cap\underline{x}_j(t)=0$ for any $i\neq j$. Then there are functions $G_{a^*}(\underline{a},\omega;t)$ so that in the above set

$$F_{a_i^*}^{\{1,\dots,n\}}\underline{a}_j,\omega(j),x_j\qquad j=1,\dots,n;t)=G_{a_i^*}(\underline{a}_i,\omega(i);t)$$

(4) Let $\underline{\underline{x}}_1\cdots\underline{\underline{x}}_n$ be such that there are two sets I and J, $I\cap J=\varnothing$, $I\sqcup J=\{1,\dots,n\}$, so that for $t>0$

$$x_i(\alpha,t)=x_j(\beta,t)\qquad \forall i\in I,\forall j\in J\text{ and }\alpha\in\underline{\alpha}(i,t),\beta\in\underline{\alpha}(j,t)$$

Then for all i in I

$$F_{a_i^*}^{\{1,\dots,n\}}(a_j,\omega(j),\underline{\underline{x}}_j,j=1\cdots n;t)=F_{a_i^*}^I(\underline{a}_j,\omega(j),j\in I;t)$$

so that also for all i in J

$$F_{a_i^*}^{\{1,\dots,n\}}(\underline{a}_j,\omega(j),\underline{\underline{x}}_j,j=1\cdots n;t)=F_{a_i^*}^J(\underline{a}_j,\omega(j),x_j,j\in J;t)$$

(5) Assume that given $\underline{\underline{x}}_1,\dots,\underline{\underline{x}}_n$ there are I and J mutually disjoint and covering $\{1\cdots n\}$ and furthermore that (a) $\underline{x}_i(\alpha,t)=x_i(\alpha',t)$ for any $\alpha\neq\alpha'$ and i in I, (b) $x_i(t),x_j(t)=\varnothing$ for any i in I and $j\neq i$. Then

$$F_{a_j^*}^{\{1,\dots,n\}}=F_{a_j^*}^J\qquad\text{for any}\quad j\text{ in }J$$

$$F_{a_i^*}^{\{1,\dots,n\}}=G_{a_i^*}\qquad\text{for any}\quad i\text{ in }I$$

We are now ready for the following:

Proof of (ii) *in Lemma B.1.* Assume $|x_i-x_1|>\varepsilon^{-\delta_i}$ for all $i>1$ [Eq. (B.5) is obviously true in the other case]. We write below T for $\varepsilon^{2/3}$ and ν^ε for the law of the process at time T. Then

$$\sum_{a_i^*\cdots a_n^*}\nu^\varepsilon\left[\tilde{1}_{a_1^*}(x_i)\prod_{i=2}^n 1_{a_i^*}(x_i)\right]$$

$$=\sum_{a_i^*\cdots a_n^*}\int P(d\omega(1))\cdots P(d\omega(n))\sum_{\underline{a}_i\cdots\underline{a}_n}\prod_{i=1}^n 1(|a_i|=|\alpha(i,t)|)$$

$$\times\left\{P_{\omega(i)\cdots\omega(n)}^\varepsilon(d\underline{\underline{x}}_1,\dots,d\underline{\underline{x}}_n)\prod_{i=1}^n F_{a_i^*}^{\{1,\dots,n\}}-P_{\omega(1)}^\varepsilon(d\underline{\underline{x}}_1)\right.$$

$$\left.\times P_{\omega(2)\cdots\omega(n)}^\varepsilon(d\underline{\underline{x}}_2,\dots,d\underline{\underline{x}}_n)\,F_{a_1^*}^{\{1\}}\prod_{i=2}^n F_{a_i^*}^{\{2,\dots,n\}}\right\}$$

We introduce the functions

$$\varphi(i)=1(|x_i(\alpha, s)-x_i|<\tfrac{1}{2}\varepsilon^{-\delta_i} \text{ for all } \alpha \text{ in } \underset{\sim}{\alpha}(i, s) \text{ and } 0\leqslant s\leqslant T)$$

and notice that after conditioning on $\prod_{i=1}^n \varphi(i)$ the two measures $P^{\varepsilon}_{\omega(1)\cdots\omega(n)}$ and $P^{\varepsilon}_{\omega(1)}P^{\varepsilon}_{\omega(2)\cdots\omega(n)}$ become the same since $|x_i-x_1|>\varepsilon^{-\delta_1}$, $i=2,..., n$. Furthermore, by (4) of Definition B.4, $F^{\{1,...,n\}}_{a_1^*}=F^{\{1\}}_{a_1^*}$ and $F^{\{1,...,n\}}_{a_i^*}=F^{\{2,...,n\}}_{a_i^*}$ $i=1,..., n$. We are therefore in the following setup. We have two probabilities ν and μ with the same conditional probabilities on a given set A. We have nonnegative bounded functions f_i and g_i equal on A and such that $\sum_i f_i=\sum_i g_i=1$. Therefore $\sum_i |\int \nu(dx) f_i(x)-\int \mu(dx)\, g_i(x)|\leqslant 2\int (\nu(dx)+\mu(dx))\, 1_{A^c}(x)$. Therefore

$$\sum_{a_1^*,...,a_n^*}\left|\nu^{\varepsilon}\left[\tilde{1}_{a_1^*}(x_1)\prod_{i=2}^{n} 1_{a_i^*}(x_i)\right]\right|$$

$$\leqslant 2\int P(d\omega(1))\cdots P(d\omega(n))\,[P^{\varepsilon}_{\omega(1)\cdots\omega(n)}(d\underset{=}{x}_1\cdots dx_n)$$

$$+P^{\varepsilon}_{\omega(1)}(d\underset{=}{x}_1)\,P^{\varepsilon}_{\omega(2)\cdots\omega(n)}(dx_2\cdots dx_n))\}$$

$$\sum_{i=1}^{n}(1-\varphi(i))\leqslant 4n\int P(d\omega(1))\,P^{\varepsilon}_{\omega(1)}(d\underset{=}{x}_i)(1-\varphi(1))$$

$$\leqslant 4n\int P(d\omega(1))\,|\underset{\sim}{\alpha}(1, t)|$$

$$\times P^0_t[|x(s)|\leqslant \tfrac{1}{2}\varepsilon^{-\delta_1}-N_t(\omega) \text{ for all } 0\leqslant s\leqslant \varepsilon^{-2}T]$$

where P^0_t is the law at time t of a random walk starting at time 0 from the origin. Since $\varepsilon^{-2}T=\varepsilon^{-4/3}$ and $\delta_i>\frac{2}{3}$, (ii) of Lemma B.1 becomes a consequence of classical estimates on random walks and of the following inequality [cf. Ref. 10, Eq. (A.21)]: for any $n\geqslant 1$,

$$P(N_t(\underset{\sim}{\omega})=N)\leqslant c_3 e^{-n\lambda t}(1-e^{-\lambda t})^N\, nN^{n/2-1} \tag{B.11}$$

So the proof of Lemma B.1 is completed. The process of n stirring particles is in several respects close to that of n independent particles. With this in mind we pose the following:

B.5. Definition [The Branching Labeled Independent (BLI) Process]

Given any $n\geqslant 1$, $x_1,..., x_n$, $\omega(1)\cdots\omega(n)$, we define the process $(\underset{=}{x}^0_1\cdots \underset{=}{x}^0_n)$ with the same procedure as in the BLS process, the only difference being that in each time interval $[\tau_k, \tau_{k+1})$ the particles move independently of each other. In particular, therefore, a particle created on an already occupied site will, eventually, separate from the other one, in contrast to what happens in the BLS process. We shall use for the BLI process the same notation as for the BLS process, except for the addition of a superscript 0.

Another preliminary step in the proof of Proposition 4.1 is the following.

B.6. Lemma

Let ν^ε be as in Lemma B.1 and denote by ν_t^ε the distribution of the process at time t when the initial law is ν^ε. Assume $t \in [\varepsilon^{1/3}, \tilde{T}]$, where $\tilde{T}$ is some arbitrarily fixed positive number. Then there exist $\delta_2 > 0$, $c_3 < \infty$ so that

$$|\nu_t^\varepsilon[1_{a^*}(x)] - q_{t,a^*}^\varepsilon(x)| \leqslant c_3 \varepsilon^{\delta_2} \tag{B.12a}$$

where

$$q_{t,a^*}^\varepsilon(x) = \int P(d\omega)\, P_\omega^{\varepsilon,0}(d\underline{x}^0) \sum_{\underline{a}} 1(|\underline{a}| = |\underline{\alpha}(t)|)\, G_a(\underline{a}, \omega, t) \times \prod_{\alpha \in \underline{\alpha}(t)} p_{\varepsilon,a}(\varepsilon_0 x^0(\alpha, t)) \tag{B.12b}$$

We postpone the proof of the above Lemma to the Remarks B.12 following Proposition B.1.

The following is just a useful rewriting of the duality relation established in Definition B.4.

B.7. Lemma

Fix any $n \geqslant 1$, $x_1 \cdots x_n$, ν^ε in $\{-1, 1\}^{Z_\varepsilon}$, $t > 0$, and $(a_1^*, ..., a_n^*)$ in $\{-1, 1\}^n$. Then [cf. Notes added in proof]

$$\begin{aligned} &\nu_t^\varepsilon\left[\prod_{i=1}^n \tilde{1}_{a_i^*}(x_i)\right] \\ &= \sum_{\Delta \subset \{1 \cdots n\}} \int P(d\omega(1)) \cdots P(d\omega(n))\, \varphi_\Delta \psi_{\Delta^c} \\ &\quad \times \sum_{\underline{a}_1 \cdots \underline{a}_n} 1(|a_i| = |\alpha_i(t)|,\, i = 1, ..., n) \left(\prod_{i \in \Delta} F_{a_i^*}^\Delta\right) \left(\prod_{i \notin \Delta} G_{a_i^*}\right) \\ &\quad \times \nu^\varepsilon\left[\prod_{i=1}^n \{1_{\underline{a}_i}(x_i(t)) - 1(|\alpha(i, t)| = 0)\, \nu_t^\varepsilon[1_{a_i^*}(x_i)]\}\right] \end{aligned} \tag{B.13}$$

where

$$\varphi_\Delta(\omega(1)\cdots\omega(n), \underline{x}_1\cdots\underline{x}_n; t) = 1(\forall i\in\Delta\ \exists\alpha, \beta, j: \alpha\neq\beta, j\in\Delta,$$
$$x(\alpha, t) = x(\beta, t) \quad \text{and} \quad \alpha\in\underline{\alpha}(i, T), \beta\in\underline{\alpha}(j, T)) \tag{B.14}$$

[so that φ_Δ only depends on $(\omega(i), \underline{x}_i, i\in\Delta)$]

$$\psi_\Gamma\Big(\omega(1)\cdots\omega(n), \underline{x}_1\cdots\underline{x}_n; t\Big) = 1(\text{for all } i \text{ in } \Gamma, \text{ for all } \alpha \text{ in } \underline{\alpha}(i, t),$$
$$\text{for all } \beta \text{ in } \bigsqcup_{j=1}^{n} \underline{\alpha}(j, t), \beta\neq\alpha, x(\alpha, t)\neq x(\beta, t)\Big) \tag{B.15}$$

We will now use the following decomposition of the identity:

$$\varphi_\Delta\psi_{\Delta^c} = \sum_k (-1)^k \sum_{J_1,\ldots,J_k} \varphi_\Delta(J_1,\ldots, J_k) \tag{B.16}$$

where $\varphi_\Delta(J_1,\ldots, J_k) = 0$ if there are $i\neq j$, $J_i\cap J_j\neq\varnothing$, or if there is i such that $J_i\cap\Delta\neq\varnothing$; otherwise $\varphi_\Delta(J_1,\ldots, J_k) = 1$ if and only if for all i in $\{1,\ldots, k\}$, $\alpha\in\bigsqcup_{j\in J_i}\underline{\alpha}(j, t)$ and β in $\underline{\alpha}(j, t)$, $j\in\Delta\cup J_1\cdots\cup J_{i-1}$, $x(\alpha, t) = x(\beta, t)$.

Next we rewrite the ν^ε expectations in the r.h.s. of Eq. (B.13) by adding and subtracting $\nu^\varepsilon[1_a(x)]$ to each factor $1_a(x)$. The purpose is to exploit Lemma B.1 so as to reduce the problem to the case where the initial measure is Bernoulli. To control the combinatorics we stop the expansion at h, which will be chosen later on to be equal to n. Namely,

$$\nu_t^\varepsilon\Big[\prod_{i=1}^{n}\tilde{1}_{a_i^*}(x_i)\Big]$$
$$= \prod_{i=1}^{n}\Big\{\prod_{\alpha\in\underline{\alpha}(i,t)}\nu^\varepsilon[1_{a_\alpha}(x(\alpha, t))] - 1(|\underline{\alpha}(i, t)| = 0)\,\nu_t^\varepsilon[1_{a_i^*}(x_i)]\Big\}$$
$$+\sum_{k=1}^{h-1}\sum_{i_1\leqslant\cdots\leqslant i_k}\sum_{\substack{\alpha(1)\in\underline{\alpha}(i_1,t)\cdot\alpha(k)\in\underline{\alpha}(i_k,t)\\ \alpha(1)<\cdots<\alpha(k)}}\nu^\varepsilon\Big[\prod_{j=1}^{k}\tilde{1}_{a_{\alpha(j)}}(x(\alpha(j), t))\Big]$$
$$\cdot\Big\{\prod_{\substack{\alpha\in\bigsqcup_{j=1}^{k}\underline{\alpha}(i_j,t)\\ \alpha\neq\alpha(1),\ldots,\alpha(k)}}\nu^\varepsilon[1_{a_\alpha}(x(\alpha, t))]\Big\}\prod_{i\neq i_1\cdots i_k}\Big(\prod_{\alpha\in\underline{\alpha}(i,t)}\nu^\varepsilon[1_{a_\alpha}(x(\alpha, t))]$$
$$-1(|\underline{\alpha}(i, t)| = 0)\,\nu_t^\varepsilon[1_{a_i^*}(x_i)]\Big)$$

$$+\sum_{i_1<\cdots<i_k}\ \sum_{\substack{\alpha(1)\in\underline{\alpha}(i_1,t)\cdots\alpha(k)\in\underline{\alpha}(i_h,t)\\ \alpha(1)<\cdots<\alpha(h)}} v^\varepsilon\Bigg[\prod_{i<i_1}\Bigg\{\prod_{\alpha\in\underline{\alpha}(i,t)} 1_{a_\alpha}(x(\alpha,t))$$

$$-1(|\alpha(i,t)|=0)\,v_t^\varepsilon[1_{a_i^*}(x_i)]\Bigg\}$$

$$\cdot\prod_{\substack{\alpha<\alpha(1)\\ \alpha\in\underline{\alpha}(i_1,t)}} 1_{a_\alpha}(x(\alpha,t))\Bigg]\prod_{\substack{i>i_1\\ i\neq i_1\cdots i_n}}\Bigg\{\prod_{\alpha\in\underline{\alpha}(i_1,t)} v^\varepsilon[1_{a_\alpha}(x(\alpha,t))]$$

$$-1(|\alpha(i,t)|=0)\,v_t^\varepsilon[1_{a_i^*}(x_i)]\Bigg\} \tag{B.17}$$

Next we will write Eq. (B.13) by means of Eqs. (B.16) and (B.17) and then we will use Lemma B.6 to reduce the problem to a comparison of the BLS and BLI processes. The main tool for that is the following coupling (very similar to those introduced in Refs. 40 and 1b).

B.8. Definition (The Coupling)

For any $n\geq 1$, $x_1,\dots,x_n$, $\omega(1)\cdots\omega(n)$, we define a process $(\underline{x}_1\cdots\underline{x}_n, \underline{x}_1^0\cdots\underline{x}_n^0)$ whose law Q_ω^ε is the following (we shall call "interacting" the particles in $\underline{x}_i$ and "free" those in $\underline{x}_i^0$, $i=1,\dots,n$). The branching structure of the $\underline{x}_i$ is that described in Definition B.3, and of the $\underline{x}_i^0$, that in Definition B.5. It remains to say how particles move in each interval $[\tau_k,\tau_{k+1})$. Fix $\underline{x}_i(\tau_k)$, $i=1,\dots,n$. Erase, at first instance, all interacting particles which at time τ_k sit together with another interacting particle having smaller label. Let $\mathcal{A}$ be the remaining labels. The interacting and free particles with label in $\mathcal{A}$ are coupled in each time interval $[\tau_k,\tau_{k+1})$ like in Ref. 40.

We briefly recall such definition.

If the free particle α moves by d ($|d|=1$), then also the interacting particle α moves from its position $x(\alpha)$ by d, unless $x(\alpha)+d=x(\beta)$. If $\alpha<\beta$, then the displacement is effective and at the same time particle β goes to $x(\alpha)$. If $\alpha>\beta$, then none of them moves. An interacting particle with label not in $\mathcal{A}$ moves stuck to the interacting particle with label in $\mathcal{A}$ with which it was at time τ_k. The free particles with label not in $\mathcal{A}$ move independently of all the other particles.

A direct consequence of the above definition is the following:

B.9. Lemma

For any $n\geq 1$, $x_1,\dots,x_n$, $\omega(1)\cdots\omega(n)$, the following holds.

(i) The marginals of Q_ω^ε are respectively P and P^0.

(ii) Fix any j in $\{1,..., n\}$, $t>0$, $\alpha \in \underset{\sim}{\alpha}(j, t)$. Then $x(\alpha, t)$ is measurable on the σ-algebra generated by $\{x^0(\beta, s),\ \beta \leqslant \alpha, \text{ and } s \leqslant t\}$.

B.10. Definition (The Variables D)

Let $n \geqslant 1$, $x_1,..., x_n$, $\omega(1) \cdots \omega(n)$ be given. Consider the process $(\underset{\approx}{x}_1 \cdots \underset{\approx}{x}_n, \underset{\approx}{x}_1^0 \cdots \underset{\approx}{x}_n^0)$ whose law is $Q^\varepsilon_{\underset{-}{\omega}}$. Let $\alpha \in \underset{\sim}{\alpha}(i, t)$ and assume $x(\alpha, t) = x(\alpha', t)$ for all $\alpha' \neq \alpha$. Let $\beta < \alpha$ be the label of some other particle existing at time $\tau_k < t$.

Case (i). There is γ: $x(\beta, \tau_k) = x(\gamma, \tau_k)$ and $\gamma < \beta$. Then $D(\alpha, \beta; \tau_k, \tau_{k+1}) = 0$.

Case (ii). Assume case (i) above is not verified. Then $D(\alpha, \beta; \tau_k, \tau_{k+1})$ is the algebraic sum of the jumps of $x(\alpha, s) - x^0(\alpha, s)$ at times s in $[\tau_k, \tau_{k+1})$, where either $x^0(\alpha, s)$ jumps by $x(\beta, s) - x(\alpha, s)$ or $x^0(\beta, s)$ jumps by $x(\alpha, s) - x(\beta, s)$.

B.11. Proposition (Probability Estimates on the Variables D)

Let $n \geqslant 1$, $x_1,..., x_n$, $\omega(1) \cdots \omega(n)$, $Q^\varepsilon_{\underset{-}{\omega}}$ be given. Let π be the measurable partition determined by fixing the paths $\underset{\approx}{x}_1 \cdots \underset{\approx}{x}_n$. Then

(i) Let $t > 0$ and $\alpha \in \bigsqcup_i \underset{\sim}{\alpha}(i, t)$. Then

$$1(\alpha, t)\left\{[x(\alpha, t) - x^0(\alpha, t)] - \sum_{k=0}^{N_t} \sum_{\beta < \alpha} D(\alpha, \beta; \tau_k, \tau_{k+1})\right\} = 0 \quad \text{(B.18)}$$

where $N_t = N_t(\omega)$, $\tau_{N_t+1} = t$, $\tau_0 = 0$, and

$$1(\alpha, t) = 1\left(x(\alpha, t) \neq x(\gamma, t),\ \forall \gamma \in \bigsqcup_{i=1}^{n} \underset{\sim}{\alpha}(i, t): \gamma < \alpha\right) \quad \text{(B.19)}$$

(ii) For any $k \geqslant 1$, α, β in $\bigsqcup_{i=1}^n \underset{\sim}{\alpha}(i, \tau_k)$, $m \geqslant 1$

$$\begin{aligned} &1(\alpha, \tau_k)\, Q^\varepsilon_{\underset{-}{\omega}}[D(\alpha, \beta; \tau_k, \tau_{k+1})^m \mid \pi] \\ &\quad = 1(\alpha, \tau_k)\, 1(\beta, \tau_k)\, \tilde{D}_m(\alpha, \beta; \tau_k, \tau_{k+1}) \end{aligned} \quad \text{(B.20)}$$

where $\tilde{D}_m(\alpha, \beta; \tau_k, \tau_{k+1})$ is measurable on $\{x(\alpha, s), x(\beta, s), \tau_k \leqslant s \leqslant \tau_{k+1}\}$. $\tilde{D}_1$, for instance, is explicitly given by the formula

$$\begin{aligned} \tilde{D}_1(\alpha, \beta; \tau_k, \tau_{k+1}) &= N^-(\alpha, \beta; \tau_k, \tau_{k+1}) - N^+(\alpha, \beta; \tau_k, \tau_{k+1}) \\ &\quad - \tfrac{1}{2} T^-(\alpha, \beta; \tau_k, \tau_{k+1}) + \tfrac{1}{2} T^+(\alpha, \beta; \tau_k, \tau_{k+1}) \end{aligned} \quad \text{(B.21)}$$

where $N^-[N^+]$ counts the number of times s when $x(\alpha, s)$ and $x(\beta, s)$ exchange their positions (with each other) and $x(\alpha, s^-) < x(\beta, s^-)$ $[x(\alpha, s^-) > x(\beta, s^-)]$. $T^-[T^+]$ is the Lebesgue measure of all the times s in $[\tau_{k,k+1})$ when $x(\alpha, s) = x(\beta, s) - 1$ $[x(\alpha, s) = x(\beta, s) + 1]$.

(iii) For any $m \geqslant 1$, let $D_1,..., D_m$ be mutually distinct D-variables. Then for any $k_1,..., k_m$ strictly positive integers

$$Q_\omega^\varepsilon\left[\prod_{i=1}^m D_i^{k_i} \,|\, \pi\right] = \prod_{i=1}^m Q_\omega^\varepsilon[D_i^{k_i} \,|\, \pi] \tag{B.22}$$

(iv) There are $\gamma_1 > 0$, c_4 and $c_5 > 0$ depending on γ_1 and constants $b'(m)$, $m \geqslant 1$, so that the following holds:

$$Q_\omega^\varepsilon[|\tilde{D}| > \varepsilon^{-1/2-\gamma_1}] \leqslant c_4 \exp(-c_5 \varepsilon^{-\gamma_1}) \tag{B.23}$$

$$Q_\omega^\varepsilon\left[\prod_{i=1}^m \tilde{D}_{k_i}\right] \leqslant b'(N)\, \varepsilon^{-(1/2)N}, \qquad N := \sum_{i=1}^m k_i \tag{B.24}$$

where the $\tilde{D}$ are derived fro some D as in Eq. (B.21) and are mutually distinct.

Proof. (i) It is a straight consequence of the definition of the variables D.

(ii) Recalling the Definition B.10, we have a contribution due to the jumps of $x^0(\beta, \cdot)$. Such a contribution is fixed in an atom of π since any such jump corresponds to an interchange of $x(\beta, \cdot)$ and $x(\alpha, \cdot)$. The contribution due to the jumps of $x^0(\alpha, s)$ is random even in an atom of π. Its distribution is, however, completely specified once we give the times T^- and T^+ when α and β are close, α to the right or to the left, respectively, of β.

(iii) The same arguments given in (ii) imply also (iii).

(iv) This is the crucial probability estimate of the Proposition: it is proven in Lemma 3.7 of Ref. 41.

Besides those in Lemma B.9 and Proposition B.10, we shall use the probability estimates contained in the following:

B.12. Proposition

Let $n \geqslant 1$ and $x_1,..., x_n$ be given.

(i) Then there is $\delta_3 > 0$ and $c_6 > 0$ so that [cf. Eq. (B.14) for notation]

$$\int P(d\omega(1))\cdots P(d\omega(n))\, P^{\varepsilon}_{\omega}(d\underline{x}_1\cdots d\underline{x}_n)\, \varphi_{\{1,\ldots,n\}} \leqslant c_6\varepsilon^{\delta_3 n} \tag{B.25a}$$

$$\int P(d\omega(1))\cdots P(d\omega(n))\, P^{\varepsilon}_{\omega}(d\underline{x}_1,\ldots, d\underline{x}_n)\, 1(\exists C \bigsqcup_{i=1}^{n} \underline{\alpha}_i(t)\colon |C| \geqslant n,$$
$$\forall\alpha\in C\ \exists\beta\neq\alpha;\ x(\alpha, t) = x(\beta, t)) \leqslant c_6\varepsilon^{\delta_3 n} \tag{B.25b}$$

(ii) Let Δ and Γ be in $\{1,\ldots, n\}$ and let $\Delta\cup\Gamma = \{1,\ldots, n\}$. For any $\omega(1)\cdots\omega(n)$ set

$$\chi_\Gamma = \Big\{\text{for all } i \text{ in } \Gamma \text{ there is } \alpha\in\underline{\alpha}(i, t) \text{ and } \beta\neq\alpha \text{ in } \bigsqcup_{j=1}^{n} \underline{\alpha}(j, t)\colon$$
$$|x(\alpha, t) - x(\beta, t)| \leqslant \varepsilon^{-\delta_1}\Big\}$$

Then there is c_7 so that

$$\int P(d\omega(1))\cdots P(d\omega(n))\, P^{\varepsilon}_{\omega}(d\underline{x}_1,\ldots, d\underline{x}_n)\, \varphi_\Delta \chi_\Gamma \leqslant c_7\varepsilon^{\delta_3 n} \tag{B.26}$$

Proof.

Proof of Eq. (B.25). Fix $\omega(1)\cdots\omega(n)$. Then the r.h.s. in Eqs. (B.25a), (B.25b) can be bounded by a sum of terms like the following:

$$f := \prod_{i=1}^{n} 1(|x(\alpha_i, \tau_{k_i}) - x(\beta_i, \tau_{k_i})| = 1)$$

where $k_1 < k_2 < \cdots < k_n$ and α_i, β_i are in $\bigsqcup_j \alpha(j, \tau_{k_i-1})$; $i = 1,\ldots, n$.

Fix any $\gamma_2 > 0$; then f is bounded by the product over only those values of i for which $\tau_{k_i} - \tau_{k_i-1} \geqslant \varepsilon^{1/4+\gamma_2}$. It is not difficult to see that given m there is c_8 so that

$$P[\exists j_1\cdots j_m\colon \tau_{j_1} - \tau_{j_i-1} \leqslant \varepsilon^{1/4+\gamma_2}] \leqslant c_8 N_t(\omega)^{2m}\, \varepsilon^{m/4}$$

then by Eqs. (3.11) and (B.11), Eq. (B.25) follows.

Equation (B.26) is derived analogously. We use the fact that for any $t > 0$, $\gamma_3 > 0$, and $m > 1$ there is c_9 so that

$$\sup_{x_1\cdots x_m} \sum_{z_1\cdots z_m} P_t(z_1\cdots z_m \mid x_1\cdots x_m)\, 1(\text{for all } i \text{ there is } j\neq i\colon |z_i - z_j| \leqslant d)$$
$$\leqslant c_9(t^{-1/2}(d + t^{1/4+\gamma_3})^{(1/2)m}$$

which is proven introducing the coupling Q of Ref. 40 between the simple exclusion and the independent processes and using Eq. (3.6) of Ref. 40 to reduce it to an estimate for independent particles.

The key point in the proof of Proposition 4.1 is the following:

B.13. Proposition

Let ν^ε be the measure introduced in Lemma B.1, and let t be as in Lemma B.6. Then there are $\delta_4 > 0$, c_{10}, $u > 0$, and for any $\Delta \subset \{1,..., n\}$ functions $h_\Delta^{(1)}, h_\Delta^{(2)}, h_\Delta^{(3)}$ so that the following holds.

Let f be any bounded measurable function which only depends on $\underline{x}_1 \cdots \underline{x}_{n-1}$, $\omega(1) \cdots \omega(n-1)$; then

$$\int P(d\omega(1)) \cdots P(d\omega(n))\, P_{\underline{\omega}}^\varepsilon(d\underline{x}_1 \cdots d\underline{x}_n)\, f \sum_{\underline{a}_n} 1(|a_n| = |\underline{\alpha}(n,t)|)\, G_{a_n^*}$$

$$\times \Big\{ \prod_{\alpha \in \underline{\alpha}(n,t)} \nu^\varepsilon[1_{a_\alpha}(x(\alpha, t))] - 1(|\underline{\alpha}(n,t)| = 0)\, q_{t,a^*}^\varepsilon(x_n) \Big\}$$

$$= \int P(d\omega(1)) \cdots P(d\omega(n))\, P_{\underline{\omega}}^\varepsilon(d\underline{x}_1 \cdots d\underline{x}_n)\, f \sum_{\underline{a}_n} 1(|a_n| = |\underline{\alpha}(n,t)|)\, G_{a_n^*}$$

$$\cdot \sum_{\Delta \ni n} \Big\{ h_\Delta^{(1)} + \varphi_\Delta (1 + \psi_{\Delta^c})\, h_\Delta^{(2)} + \sum_{\Gamma \ni n} \varphi_\Delta \psi_{\Delta^c} h_\Gamma^{(3)} \Big\} + R \tag{B.27a}$$

where

$$|R| \leqslant \|f\|\, c_{10} \varepsilon^{\delta_4 n} \tag{B.27b}$$

$h_\Delta^{(1)}, h_\Delta^{(2)}, h_\Delta^{(3)}$ depend on ε, t and $\{\underline{x}_i, \omega(i)$, for i in $\Delta\}$ and $\underline{a}_n$. Furthermore, there is u such that

$$\sum_{a_n} (\|h_\Delta^{(1)}\| + \|h_\Delta^{(3)}\|) \leqslant c_{10} N_t(\underline{\omega})^u\, (\varepsilon_0 \varepsilon^{-(1/2) - \gamma_i})^{|\Delta|} \tag{B.28}$$

while $\sum_{a_n} \|h_\Delta^{(2)}\| \leqslant c_{10} N_t(\underline{\omega})^u$.

Before proving Proposition B.13, we discuss its consequences in the following:

B.14. Remarks

Remark 1. Lemma B.6 is a straight consequence of Proposition B.13. We have, in fact, by Eq. (B.10) [$1(\alpha, t)$ below was defined in Eq. (B.19)],

$$\nu_t^\varepsilon[1_{a^*}(x)] = \int P(d\omega)\, P_\omega^\varepsilon(d\underline{x}) \sum_{\underline{a}} 1(|\underline{a}| = |\underline{\alpha}(t)|)\, F_{a^*} \nu^\varepsilon[1_{\underline{a}}(\underline{x}(t))]$$

$$\cdot \Big\{ 1(\exists \alpha \neq \beta\colon x(\alpha, t) = x(\beta, t)) + \prod_{\alpha \in \underline{\alpha}(t)} 1(\alpha, t) \Big\} \tag{B.29}$$

The first term in the r.h.s. of Eq. (B.29) is bounded by first writing 1 in place of F_{a^*}, then by performing the sum over $\underline{a}$, which is normalized to 1 because of $v^\varepsilon[1_{\underline{a}}(\underline{x}(t))]$. Hence the first term is bounded by $c_6\varepsilon^{\delta_3}$ [cf. Eq. (B.25)]. For the second term we recall (3) of Definition B.4 and then use Eq. (B.17) with $h=n=1$. Therefore

$$\left|v_t^\varepsilon[1_{a^*}(x)] - \int P(d\omega)\, P_\omega^\varepsilon(d\underline{x}) \sum_{\underline{a}} 1(|\underline{a}| = |\underline{\alpha}(t)|)\, G_{a^*} \prod_{\alpha\in\underline{\alpha}(t)} v^\varepsilon[1_{a_\alpha}(x(\alpha, t))]\right|$$

$$\leqslant c_6\varepsilon^{\delta_3} + \int P(d\omega)\, P_\omega^\varepsilon(d\underline{x}) \sum_{\underline{a}} 1(|\underline{a}| = |\underline{\alpha}(t)|)\, G_{a^*} \prod_{\alpha\in\underline{\alpha}(t)} 1_{a_\alpha}(x(\alpha, t))$$

$$\times\, 1(\exists\alpha \neq \beta\colon x(\alpha, t) = x(\beta, t))$$

$$+ \sum_{\alpha\in\underline{\alpha}(t)} \left| v^\varepsilon\left[\prod_{\beta<\alpha} 1_{a_\beta}(x(\beta, t))\, \tilde{1}_{a_\alpha}(x(\alpha, t)\right]\right| \prod_{\beta>\alpha} v^\varepsilon[1_{a_\beta}(x(\beta, t)]$$

$$\leqslant 2c_6\varepsilon^{\delta_3} + \int P(d\omega)\, P_\omega^\varepsilon(d\underline{x})\, |\underline{\alpha}(t)|\, 1(\exists\alpha\neq\beta\colon |x(\alpha, t) - x(\beta, t)| \leqslant \varepsilon^{-\delta_1})$$

$$+ c_2 \exp(-\varepsilon^{-(\delta_1 - 2/3)}) \int P(d\omega)\, |\underline{\alpha}(t)|^2$$

where we used (ii) of Lemma B.1. We use Eqs. (B.26) and (B.11), so that Lemma B.6 follows from Proposition B.13.

Remark. Proposition 4.1 follows from Proposition B.13. Let us sketch the proof of this statement. We use Eq. (B.16). In the terms where $\bigsqcup_{i=1}^k J_i \cup \Delta = \{1\cdots n\}$ we bound $F_{a_i^*}$ with 1. We then perform the sum over $\underline{a}_1\cdots\underline{a}_n$; we use Eq. (B.25) and we get the bound $c_6\varepsilon^{n\delta_3}$. For the other terms, i.e., when $\Gamma = \bigsqcup_{j=1}^k J_j \cup \Delta$ is strictly contained in $\{1\cdots n\}$, we use Eq. (B.17) with $h=n$. The contribution from the last term in Eq. (B.17) goes like $\varepsilon^{\delta_3 n}$, by Lemma B.1(ii) and Eq. (B.26). We are therefore left with terms like

$$v^\varepsilon\left[\prod_{j=1}^k 1_{a_{\alpha(j)}}(x(\alpha(j), t)\right]$$

Let $\Lambda \subset \{1\cdots n\}$ be the minimal set such that $\alpha(j) \in \bigsqcup_{i\in\Lambda} \underline{\alpha}(i, t)$. If $\Lambda \cup \Gamma = \{1\cdots n\}$, we use again Lemma B.1(ii) and Eq. (B.26) to get a contribution which goes like $\varepsilon^{\delta_3 n}$. If $\Lambda\cup\Gamma \subset \{1\cdots n\}$ (strictly), then there is an index, say n, which is not in $\Lambda\cup\Gamma$. The structure of such a term is that appearing on the r.h.s. of Eq. (B.27a) (use Lemma B.6 to change $v_t^\varepsilon[1_a(x)]$ into $q_{t,a}^\varepsilon(x)$). Using Eq. (B.27a), we obtain a sum of terms which involve Λ, Γ, and Δ. If $\Lambda\cup\Gamma\cup\Delta = \{1\cdots n\}$, we use Proposition B.12 to get an estimate which goes like $\varepsilon^{\delta_3 n}$. Otherwise we rewrite the ψ_{Λ^c} term in Eq. (B.27) using Eq. (B.16). If the whole $\{1 \cdots n\}$ is not yet fully covered, we again use Proposition B.13, so that finally we obtain the proof of Proosition 4.1.

Proof of Proposition B.13. We have

$$\begin{aligned}
&\text{r.h.s. of Eq. (B.27a)} \\
&\quad \leqslant \int P(d\omega(1)) \cdots P(d\omega(n))\, Q_{\omega}^{\varepsilon}(d\underline{x}_1 \cdots d\underline{x}_n) \\
&\quad\quad \times \sum_{\underline{a}_n} 1(|a_n| = |\underline{\alpha}(n,t)|)\, G_{a_n^*} f \\
&\quad\quad \times \left\{ \prod_{\alpha \in \underline{\alpha}(n,t)} v^{\varepsilon}[1(x(\alpha,t))] - \prod_{\alpha \in \underline{\alpha}(n,t)} p_{\varepsilon, a_\alpha}(\varepsilon_0 x^0(\alpha,t)) \right\} \\
&\quad\quad \cdot \left\{ A_\phi + \sum\nolimits^* A_{\alpha_1 \cdots \alpha_k} \right\}
\end{aligned} \tag{B.30}$$

where $\sum^*$ is a sum over all $k \geqslant 1$ and $\alpha_1 \cdots \alpha_k$ in $\underline{\alpha}(n,t)$ and

$$A_\phi = \prod_{\alpha \in \underline{\alpha}(n,t)} 1(\alpha, t)$$

$$A_{\alpha_1 \cdots \alpha_k} = \left(\prod_{i=1}^{k} (1 - 1(\alpha_i, t)) \right) \left(\prod_{\alpha \neq \alpha_1 \cdots \alpha_k} 1(\alpha, t) \right)$$

Analysis of the Term with A_Φ in Eq. (B.30)

We use the Taylor Lagrange expansion up to order $n+1$ for the function

$$\prod_{\alpha} p_{\varepsilon, a_\alpha}(\varepsilon_0 x^0(\alpha, t))$$

around its value with x in place of x^0. The contribution to Eq. (B.30) due to the remainder term is bounded by

$$\begin{aligned}
&\int P(d\omega(1)) \cdots P(d\omega(n))\, Q_{\omega}^{\varepsilon}(d\underline{x}_1 \cdots d\underline{x}_n) \\
&\quad \times \sum_{\underline{a}_n} \|f\| \sum_{\alpha_1 \cdots \alpha_{n+1} \in \underline{\alpha}(n,t)} \left(\prod_{\alpha \in \underline{\alpha}(n,t)} p_{\varepsilon, a_\alpha}(r) \right)^{(\alpha_1 \cdots \alpha_{n+1})} \\
&\quad \cdot \varepsilon_0^{n+1} \prod_{j=1}^{n+1} |x^0(\alpha_j, t) - x(\alpha_j, t)|\, A_\phi
\end{aligned}$$

$$\leqslant 2^{n+1} b(n+1) \|f\| \int P(d\omega(1)) \cdots P(d\omega(n))\, Q^{\varepsilon}_{\underline{\omega}}(d\underline{x}_1 \cdots d\underline{x}_n)$$

$$\times \sum_{\alpha_1 \cdots \alpha_{n+1} \in \underline{\alpha}(n,t)} \sum_{k_1 \cdots k_{n+1}} \sum_{\beta_1 \cdots \beta_{n+1}} \prod_{j=1}^{n+1} |D(\alpha_j, \beta_j; \tau_{k_j}, \tau_{k_j+1})|$$

$$\leqslant c_{11} \varepsilon^{(2/3 - 1/2)(n+1)} \tag{B.31}$$

where $(\prod_{\alpha \in \underline{\alpha}(n,t)} p_{\varepsilon, a_\alpha}(r))^{(\alpha_1 \cdots \alpha_{n+1})}$ denotes the $(n+1)$ derivative w.r.t. $(\varepsilon_0 x^0_{\alpha_1}) \cdots (\varepsilon_0 x^0_{\alpha_{n+1}})$ of $\prod_{\alpha \in \underline{\alpha}(n,t)} p_{\varepsilon, a_\alpha}(\varepsilon_0 x^0(\alpha, t))$ computed at some intermediate value between $\varepsilon_0 x$ and $\varepsilon_0 x^0$. c_{11} is a suitable coefficient. The first inequality in Eq. (B.31) is obtained using Eq. (B.6) while the second follows from Eqs. (B.24) and (B.11).

The zeroth-order term of the expansion contributes to $h^{(1)}$ because of (i) of Lemma B.1. The kth-order term can be written as

$$\int P(d\omega(1)) \cdots P(d\omega(n))\, Q^{\varepsilon}_{\underline{\omega}}(d\underline{x}_1 \cdots d\underline{x}_n)$$

$$\times f \sum_{\underline{a}_n} G_{a_n^*} \sum_k \sum_{\alpha_1 < \cdots < \alpha_k \in \underline{\alpha}(n,t)} \prod_{\alpha \neq \alpha_1 \cdots \alpha_k}$$

$$\times \left(\frac{\partial}{\partial x(\alpha_1, t)} \cdot \frac{\partial}{\partial x(\alpha_k, t)} \right) (\varepsilon_0 x(\alpha_1, t)) \cdots p_{\varepsilon, a_{\alpha_k}}(\varepsilon_0 x(\alpha_k, t)))$$

$$\times \sum_{h_1 \cdots h_k} \sum_{\beta_1 \cdots \beta_k} \prod_{j=1}^{k} Q^{\varepsilon}_{\underline{\omega}}[D(\alpha_j, \beta_j; \tau_{h_j}, \tau_{h_j+1}) | \pi]\, A_\phi \tag{B.32}$$

Each D in Eq. (B.32) is written as

$$D = D\, 1(|\tilde{D}| \leqslant \varepsilon^{-(1/2) - \gamma_1}) + D\, 1(|\tilde{D}| > \varepsilon^{-(1/2) - \gamma_1}) \tag{B.33}$$

Terms with $1(|\tilde{D}| > \varepsilon^{-(1/2) - \gamma_1})$ give a contribution which is estimated by means of Eq. (B.23) and contribute to R in Eq. (B.27a). When all the $|\tilde{D}|$ are less than $\varepsilon^{-(1/2) - \gamma_1}$, we have by Eq. (B.24) the bounds required by Eq. (B.28). However, we still have A_ϕ and terms with $1(\alpha, \tau_k)\, 1(\beta, \tau_k)$ which come together with $D(\alpha, \beta; \tau_k, \tau_{k+1})$ [cf. Eq. (B.20)]. They depend on other $\underline{x}$ than those allowed by Eq. (B.27a). We proceed as in Eq. (B.17). We write [cf. Eq. (B.19)]

$$1(\alpha, \tau_k) = \prod_{\gamma < \alpha} \{1 - 1(x(\alpha, \tau_k) = x(\gamma, \tau_k))\}$$

$$\times \sum_{h=1}^{n-1} \sum_{\gamma_1 < \cdots < \gamma_n} \prod_{i=1}^{h} 1(x(\alpha, \tau_k) = x(\gamma_i, \tau_k))$$

$$+ (-1)^n \sum_{\gamma_1 < \cdots < \gamma_n} \prod_{\gamma < \gamma_1} 1(x(\alpha, \tau_k) \neq x(\gamma, \tau_k))$$

$$\times \prod_{i=1}^{n} 1(x(\alpha, \tau_k) = x(\gamma_i, \tau_k))$$

The last term contributes to R, by Eq. (B.25b). Given $\alpha, \gamma_1 \cdots \gamma_h$, let $\Delta \subset \{1 \cdots n\}$ be the minimal set such that α and all the γ_i belong to $\bigsqcup_{i \subset \Delta} \underline{\alpha}(i, t)$. Then obviously,

$$\prod_{i=1}^{h} 1(x(\alpha, \tau_k) = x(\gamma_i, \tau_k)) = \prod_{i=1}^{h} 1(x(\alpha, \tau_k) = x(\gamma_i, \tau_k))\, \varphi_\Delta$$

so that such terms have the desired form: they contribute to $h^{(2)}$ in Eq. (B.27a). A_ϕ can be written as $A_\phi = 1 - \sum^* A_{\alpha_1 \cdots \alpha_k}$. The term with 1 is all right, for what has been said so far, while the other can be analyzed just like the terms with $A_{\alpha_1 \cdots \alpha_k}$ in Eq. (B.30).

Analysis of the Terms with $A_{\alpha_1 \cdots \alpha_k}$ in Eq. (B.30)

We have that

$$\sum\nolimits^* A_{\alpha_1 \cdots \alpha_k} = \sum_{\Delta \ni n} \varphi_\Delta \psi_{\Delta^c}$$

when such an expression multiplies $\prod_{\alpha \in \underline{\alpha}(n,t)} \nu^\varepsilon[1(x(\alpha, t))]$, it gives rise to an $h^{(2)}$ term [cf. Eq. (B.27a)]. We are then left with

$$\begin{aligned}\int & P(d\omega(1)) \cdots P(d\omega(n))\, Q^\varepsilon_{\underline{\omega}}(d\underline{\underline{x}}_1 \cdots d\underline{\underline{x}}^0_n) \\ & \times f \sum_{\underline{a}_n} G_{\underline{a}^*_n} \prod_{\alpha \in \underline{\alpha}(n,t)} p_{\varepsilon, a_\alpha}(\varepsilon_0 x^0(\alpha, t)) \sum\nolimits^* A_{\alpha_1 \cdots \alpha_k}\end{aligned} \tag{B.34}$$

We denote by C subsets of labels in $\bigsqcup_{i=1}^n \underline{\alpha}(i, t)$ and by

$$\begin{aligned} 1(C) = {} & 1(\forall \alpha \in C\ \exists \beta \in C{:}\ x(\alpha, t) = x(\beta, t)) \\ & \times 1(\forall \alpha \notin C\ \forall \gamma{:}\ (x(\alpha, t) \neq x(\gamma, t)) \end{aligned} \tag{B.35}$$

Therefore

$$A_{\alpha_1 \cdots \alpha_k} = \sum_C 1(C)\, 1(C \cap \underline{\alpha}(n, t) = \{1 \cdots n\}) \tag{B.36}$$

Denote by γ_i the smallest label in C and by τ_i the smallest time s when $x(\alpha_i, s) = x(\gamma_i, s)$. We can disregard sets C with cardinality larger than n: they contribute to R, as already noticed. Therefore, from now on we shall only consider $|C| \leqslant n$. We fix the paths of all the interacting and free particles except for the $x^0(\alpha_i, s)$, $i = 1,\ldots, k$ and $\tau_i \leqslant s \leqslant t$. Call ζ the partition obtained in such way. Conditioned on ζ the variables $\{x^0(\alpha_i, s) - x^0(\alpha_i, \tau_i),$

$\tau_i \leqslant s \leqslant t\}$ are independent symmetric random walks starting from 0 at times τ_i. We then have

$$1(C)\left\{Q^{\varepsilon}_{\omega}\left[\prod_{i=1}^{k} p_{\varepsilon,a_{\alpha_i}}(\varepsilon_0 x^0(\alpha_i, t))\,|\,\zeta\right] - \prod_{i=1}^{k} \tilde{p}_{\varepsilon,a_{\alpha_i}}(\varepsilon_0 x^0(\alpha, \tau_i))\right\} = 0 \qquad \text{(B.37)}$$

where $\tilde{p}_{\varepsilon,a_\alpha}$ is some smooth function which satisfies Eq. (B.6) with some $b'(n)$ instead of $b(n)$. Hence the terms with $A_{\alpha_1 \cdots \alpha_k}$ in Eq. (B.34) can be rewritten as

$$\begin{aligned}
&\int P(d\omega(1)) \cdots P(d\omega(n))\, Q^{\varepsilon}_{\omega}(d\underline{x}_1 \cdots d\underline{x}^0_n) \\
&\quad \times f \sum_{a_n} G_{a^*_n} \sum_{C} 1(C)\, 1(C \cap \underline{\alpha}(n, c) = \{\alpha_1 \cdots \alpha_k\}) \\
&\quad \times \prod_{\substack{\alpha \neq \alpha_1 \cdots \alpha_k \\ \alpha \in \underline{\alpha}(n,t)}} p_{\varepsilon,a_\alpha}(\varepsilon_0 x^0(\alpha, t)) \prod_{j=1}^{k} \tilde{p}_{\varepsilon,a_\alpha}(\varepsilon_0 x^0(\alpha_j, \tau_j))
\end{aligned} \qquad \text{(B.38)}$$

We can now expand in $\varepsilon_0(x^0(\alpha, t) - x(\alpha, t))$, $\alpha \neq \alpha_1 \cdots \alpha_k$ and in $\varepsilon_0(x^0(\alpha_i, \tau_i) - x(\alpha_i, \tau_i))$, $i = 1,..., k$. As before, we stop the expansion at order $n+1$. Because of the definition of the τ_i, we can write any $x - x^0$ in terms of sums of variables D and proceed as before.

For $|C| \leqslant n$, we write

$$\begin{aligned}
1(C) &= 1(\forall \alpha \in C\, \exists \beta \in C\colon x(\alpha, t) = x(\beta, t)) \\
&\quad \times \prod_{\beta \notin C} (1 - 1(\exists \alpha \in C\colon x(\alpha, t) = x(\beta, t)) \\
&= 1(\forall \alpha \in C\, \exists \beta \in C\colon x(\alpha, t) = x(\beta, t)) \\
&\quad \times \sum_{k=1}^{n-1} \sum_{\beta_1 < \cdots < \beta_k} \prod_{i=1}^{k} 1(\exists \alpha \in C\colon x(\beta_i, t) = x(\alpha, t) \\
&\quad + \sum_{\beta_1 < \cdots < \beta_n} \left(\prod_{\beta < \beta_1} [1 - 1(x(\beta, t) = x(\alpha, t))]\right) \\
&\quad \times \prod_{i=1}^{n} 1(\exists \alpha \in C\colon x(\beta_i, t) = x(\alpha, t))
\end{aligned} \qquad \text{(B.39)}$$

The terms arising from the last term in Eq. (B.39) contribute to R [in Eq. (B.27a)], the others to $h^{(3)}$. Computations are straightforward but tedious, and we omit the details.

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