MICROSCOPIC STRUCTURE AT THE SHOCK IN THE ASYMMETRIC SIMPLE EXCLUSION

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We study for a semi-infinite one dimensional initial distribution the asymptotic behaviour in the hydrodynamical limit at the shock. In this case the location of the shock is naturally identified by the position of the leftmost particle of the system for which we prove a central limit theorem. From this we deduce that at the shock local equilibrium does not hold.

KEY WORDS: Infinite particle systems, hydrodynamical limit, tagged particle.

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1. INTRODUCTION

The Asymmetric Simple Exclusion Process (ASEP) is one of the simplest particle models where shock waves can be observed. The hydrodynamic equation for its one dimensional nearest neighbor version is in fact the Burgers equation without viscosity:

$$\frac{\partial \rho}{\partial t} + (p-q)\frac{\partial}{\partial r}(\rho(1-\rho)) = 0 \tag{1.1}$$

where $\rho(r,t), r \in R, t>0$, is the macroscopic particle density and the parameter $p(q=1-p<1/2)$ denotes the intensity for a particle in the ASEP to attempt a jump to its right (left). The solution to (1.1) may develop singularities after a finite time even when the initial condition is smooth. Therefore (1.1) has to be interpreted in a weak sense. Furthermore uniqueness for (1.1) does not hold and the solution to which the limiting particle density of the exclusion process converges is the entropic solution to (1.1) which can be defined for instance by adding first a viscosity term $(\lambda \partial^2 \rho / \partial r^2)$ on the right of (1.1) and then taking the limit $\lambda \to 0$ of the corresponding solution.

For the definition of the process see [11], for instance; the hydrodynamical limit for the ASEP is studied in [14, 11, 4], cf. also [2, 3, 15] for the analysis of the hydrodynamic limit in a Zero Range Process (ZRP) closely related to the ASEP.

The shock waves are spatially non homogenous solutions to (1.1) stationary for an observer moving with constant speed, namely solutions $\rho(r,t)$ of the form $\rho(r-vt)$, where v is a constant to be interpreted as the velocity of the shock. It is known that for (1.1) these solutions are step functions with density ρ_- to the left of the singularity and ρ_+ to its right, $\rho_- < \rho_+$. The velocity v equals then $(p-q)(1-\rho_- - \rho_+)$.

A most interesting question at a physical level concerns the microscopic structure of the shock. There is no common wisdom on whether the discontinuity at the shock is a byproduct of the particular space renormalization involved in the hydrodynamic limit, so that by choosing suitably finer space units it disappears, the density profile in this new scale becoming smooth, or whether such an intermediate scale between microscopic (lattice) and macroscopic does not exist (this should correspond to absence of local equilibrium in the proximity of the shock. The question arises then to determine the nature of the typical configurations in this region. We refer to [13] and [9] for a discussion on these points and on the related problem of the dynamical stability of the shock.

In this paper we give a sharp and detailed answer to all the above questions in the particular case $\rho_- = 0$ thus extending the results of Wick, [15], who studied the problem in a ZRP. Translated into the ASEP these correspond to $\rho_- = 0$ and $p = 1$ (only jumps to the right). We extend Wick's results to any $p > 1/2$ (while keeping $\rho_- = 0$) and we establish some finer estimates on the evolution of the shock.

Our main conclusions (from the analysis of the ASEP) is that the shock profile at least in this case is stable and very rigid, that there is no intermediate space scaling where it would look smoother and that this extends down to the microscopic level. Recall that there is in general no microscopic definition of the shock, however for our process the location of the shock is linked to the location of the leftmost particle (since we are considering $\rho_- = 0$ there is a leftmost particle at time 0 as well as at all other times). In fact it was proved by Ferrari [7] that there are (explicit) stationary measures as seen from the first (i.e. leftmost) particle. They are characterized by an asymptotic density $\rho > 0$ to the right, which is reached exponentially fast so that, microscopically, far away from the first particle, the stationary state looks very similar to a Bernoulli measure (an invariant measure for the whole process). The analysis done in [7] constitutes the main technical ingredient for our studies. Our results are based on proving good ergodic properties of these measures. This enables us to prove a central limit theorem for the motion of the first particle from which the conclusions on the evolution of the shock easily follow. For the reader's convenience we briefly summarize the main results of this paper in a more precise way.

Let $\bar{\nu}_\rho, 0 < \rho < 1$, be the product measure on $\{0,1\}^{\mathbb{Z}}$ with average occupation number equal to 0 (resp. ρ) to the left (resp. right) of the origin. Denote by τ_x the space shift by x (to the left) as an operator acting on the probability measure on $\{0,1\}^{\mathbb{Z}}$ and let P_t be the Markov semigroup of the ASEP. Then for all $r \in R$ and $t > 0$

$$\lim_{\varepsilon \to 0} \tau_{v_\rho \varepsilon^{-1} t + r\varepsilon^{-1/2}} \bar{\nu}_\rho P_{\varepsilon^{-1} t} = \lambda(r,t)\nu_0 + (1-\lambda(r,t))\nu_\rho \tag{1.2}$$

where $v_\rho = (p-q)(1-\rho)$ and ν_a is the Bernoulli measure on $\{0,1\}^{\mathbb{Z}}$ with density a. Finally $\lambda(r,t) = P(B_t > r)$, where B_t stands for a Brownian motion with diffusion coefficient v_ρ. We also prove that

$$\frac{X_t - v_\rho t}{\sqrt{t}}$$

converges to the distribution of B_1, where X_t is the random position of the leftmost particle at time t. Our analysis shows indeed that in the diffusion scaling the process describing the evolution of the first particle converges to B_t.

We conclude this introduction with two remarks. The first one is that as a consequence of the rigidity of the shock profile if we look at a given space time point (as in (1.2)) we see either the empty state (if the leftmost particle is to the right of this point) or the equilibrium state with density ρ in the other case. This explains why the r.h.s. of (1.2) describes a superposition of the two states. When $0 < \rho_- < \rho_+$ and $\rho_- + \rho_+ = 1$ the shock velocity is 0, hence one would expect that the time asymptotic state for the ASEP starting from a product distribution with densities ρ_- (resp. ρ_+) to the right (resp. to the left) is a 1/2 1/2 superposition of the Bernoulli states with the two densities. This has in fact been proven by [1]; indeed the case $\rho_- > 0$ is very intriguing and interesting. In this case the same microscopic location of the shock is not as obvious to guess as in the $\rho_- = 0$ case. One can however (see [5]) relate the position of the shock to that of another special particle. Briefly speaking, one constructs the initial measure by putting a layer of ρ_- "normal" particles and adds extra particles to the right of the origin so that there the total density becomes ρ_+. Coupling the system so that the common layer is unaffected in its motion, one naturally gives the extra particles a "second class motion" in order that the all system has the correct description. There is clearly a first second class particle, but we know nothing about the existence (and even less properties) of the medium seen from the first second class particle. Although there are no rigorous results for the evolution of the shock in this case, a numerical analysis of the model seems to indicate that the results established here extend to $\rho_- > 0$.

2. PRELIMINARIES AND NOTATION

Let (η_t) be the asymmetric simple exclusion process on $\mathbb{Z}$, whose generator is:

$$Lf(\eta)=\sum_{x\in\mathbb{Z}}\{\eta(x)(1-\eta(x+1))p[f(\eta^{x,x+1})-f(\eta)]+\eta(x)(1-\eta(x-1))$$

$$\times q[f(\eta^{x,x-1})-f(\eta)]\}$$

where $q<p<1$, $p+q=1$ and f is a cylindrical function, i.e. depends on finitely many coordinates. As usual, we have

$$\eta^{x,y}(z)=\begin{cases}\eta(y) & \text{if } z=x\\ \eta(x) & \text{if } z=y\\ \eta(z) & \text{otherwise}\end{cases}$$

We take as initial distribution the product measure $\bar{\nu}_\rho$ whose marginals are:

$$\nu_\rho\{\eta(x)=1\}=0 \quad \text{for } x<0; \qquad \nu_\rho\{\eta(0)=1\}=1$$

and

$$\nu_\rho\{\eta(x)=1\}=\nu_\rho\{\eta(x)=1\}=\rho \quad \text{for } x>0,$$

where, for $0<\rho<1$, ν_ρ is the Bernoulli measure of parameter ρ.

We denote by X_t the position at time t of the last particle (with $X_0=0$).

We now recall two techniques that we will basically employ in this paper.

1) To study our system, we use the correspondence developed by Ferrari [7] between this simple exclusion process seen from the last particle and a zero range process (ξ_t) with infinite source at -1. Let us remind its main properties:

For $S=\mathbb{Z}_+$, the Markov process (ξ_t) is defined on $\mathbb{N}^s$ and has the pregenerator Λ:

$$\Lambda g(\xi)=\sum_{y=0}^{\infty}1_{\{\xi(y)>0\}}\{p[g(\xi^{y,y-1})-g(\xi)]+q[g(\xi^{y,y+1})-g(\xi)]\}$$

$$+q[g(\xi^{-1,0})-g(\xi)]$$

where g is a cylindrical function, $1_{\{\ \}}$ is the characteristic function and

$$\xi^{0,-1}(u)=\begin{cases}\xi(u) & \text{if } u\geqq 1\\ \xi(u)-1 & \text{if } u=0\end{cases}$$

$$\xi^{y,z}(u)=\begin{cases}\xi(y)-1 & \text{if } y=u\\ \xi(z)+1 & \text{if } u=z\\ \xi(u) & \text{otherwise}\end{cases} \qquad \xi^{-1,0}(u)=\begin{cases}\xi(u) & \text{if } u\geqq 1\\ \xi(u)+1 & \text{if } u=0\end{cases}$$

Notice that this process can be described by saying that, provided there are particles at site y (where $y\geqq 0$), one of them is removed from y and moves to $y+1$ (resp. $y-1$) with probability q (resp. p) after an exponential mean one holding time. Site $y=-1$ behaves as an infinite source of particles, which jump to site 0 with probability q.

Its extremal invariant measures λ_ρ are inhomogeneous product measures with marginals $\lambda_\rho\{\xi(y)=k\}=\rho_y^k(1-\rho_y)$ for $y\geqq 0$ and $k\geqq 0$ where

$$\rho_y=(1-\rho)\left[1-\left(\frac{q}{p}\right)^{y+1}\right]+\left(\frac{q}{p}\right)^{y+1}$$

The one to one correspondence between (η_t) and (ξ_t) is given by:

- If $x_0<x_1<\cdots<x_n<\cdots$ are the positions of the particles in the configuration η, then $\xi(i)=x_{i+1}-x_i-1\ (i\geqq 0)$ are the distances between two particles.
- $Lf(\eta(\xi))=\Lambda f^\eta(\xi)$ where $f^\eta(\xi)=f(\eta(\xi))$ is a cylindrical function on $\{0,1\}^{\mathbb{Z}}$.
- The images of λ_ρ are the extremal invariant measures μ_ρ for the simple exclusion process seen from the last particle. For μ_ρ, the distance between the m-th and the $(m+1)$-th particle has geometric distribution with parameter ρ_m.

2) To deal with the zero-range process (ξ_t), we use two auxiliary processes $(\xi_{1,t})$ and $(\xi_{2,t})$ such that $\xi_t=\xi_{1,t}+\xi_{2,t}$ (coordinatewise) and that the $\xi_{1,t}$-particles have priority on the $\xi_{2,t}$-particles; (they are called respectively first class and second class particles). It means that when a particle is removed from a site x where particles of the two classes are present, it is necessarily a first class particle; therefore, second class particles can only move if there are no first class particles at x. This interpretation of the evolution (which of course does not change the global process) was introduced in [2].

3. POSITION OF THE LAST PARTICLE

We prove here a law of large numbers and a central limit theorem for X_t.

THEOREM 1 *When t goes to infinity, X_t/t converges $\mathbb{P}_{\mu_\rho}$-a.s. to $(1-\rho)(p-q)=v_\rho$.*

Proof For the zero-range process (ξ_t), let N_t^+ (resp. N_t^-) be the number of jumps from -1 to 0 (resp. from 0 to -1) during $[0,t]$.

Then $X_t=N_t^- - N_t^+=N_t$ (with $N_0=0$). As $M_{1,t}=N_t^+-qt$ and $M_{2,t}=N_t^- - p\int_0^t 1_{\{\xi_s(0)<0\}}ds$ are two martingales without common jumps (thus orthogonal) w.r.t. the filtration $\{\mathcal{F}_t, t\geqq 0\}$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{\xi_s, s\leqq t\}$, we write

$$X_t=-M_{1,t}+M_{2,t}+(p-q)t-p\int_0^t 1_{\{\xi_s(0)=0\}}ds$$

Besides, since the martingales $M_{1,t}$ and $M_{2,t}$ are compensated sums of jumps of size one,

$$M^2_{1,t}-qt \text{ and } M^2_{2,t}-p\int_0^t 1_{\{\xi_s(0)<0\}}\,ds$$

are also martingales. Taking expectations with respect to $\mathbb{E}_{\lambda_\rho}$ yields

$$\mathbb{E}_{\lambda_\rho}(M^2_{1,t})=qt \text{ and } \mathbb{E}_{\lambda_\rho}(M^2_{2,t})=p\rho_0 t=t[(p-q)(1-\rho)+q].$$

Therefore, $M_{1,t}/t$ and $M_{2,t}/t$ converge a.s. to 0 when t tends to infinity (This convergence is proved in [10]).

Because of the extremality of λ_ρ, we can apply the ergodic theorem to

$$\frac{1}{t}\int_0^t 1_{\{\xi_s(0)=0\}}\,ds$$

which converges $\mathbb{P}_{\lambda_\rho}$-a.s. to

$$1-\rho_0=\rho\left(1-\frac{q}{p}\right).$$

This concludes the proof. □

We now compute the reversed process of (ξ_t), that we will use intensively in the proof of the central limit theorem.

LEMMA 2 *The adjoint in $\mathbb{L}^2(\lambda_\rho)$ of the zero range process (ξ_t) is another zero range process (ζ_t) on $\mathbb{N}^s$ whose generator Λ^* is*

$$\Lambda^* g(\zeta)=\sum_{y\geq 0} 1_{\{\zeta(y)<0\}}\left[p\frac{\rho_{y+1}}{\rho_y}g(\zeta^{y,y+1})+q\frac{\rho_{y-1}}{\rho_y}g(\zeta^{y,y-1})-g(\zeta)\right]$$

$$+p\rho_0[g(\zeta^{-1,0})-g(\zeta)]$$

where $\rho_{-1}=1$.

Proof Let f,g be two cylindrical functions on $\mathbb{N}^s$. The generator Λ^* can be easily seen to satisfy $\int g\Lambda f\,d\lambda_\rho=\int f\Lambda^* g\,d\lambda_\rho$, using the following identities:

For $y\geqq 1$

$$\int f(\xi^{y,y-1})g(\xi)1_{\{\xi(y)<0\}}\,d\lambda_\rho(\xi)=\frac{\rho_y}{\rho_{y-1}}\int f(\zeta)g(\zeta^{y-1,y})1_{\{\zeta(y-1)<0\}}\,d\lambda_\rho(\zeta)$$

For $y\geqq 0$

$$\int f(\xi^{y,y+1})g(\xi)1_{\{\xi(y)<0\}}\,d\lambda_\rho(\xi)=\frac{\rho_y}{\rho_{y+1}}\int f(\zeta)g(\zeta^{y+1,y})1_{\{\zeta(y+1)<0\}}\,d\lambda_\rho(\zeta)$$

$$\int f(\xi^{0,-1})g(\xi)1_{\{\xi(0)<0\}}\,d\lambda_\rho(\xi)=\rho_0\int f(\zeta)g(\zeta^{-1,0})\,d\lambda_\rho(\zeta)$$

and

$$\int f(\xi^{-1,0})g(\xi)\,d\lambda_\rho(\xi)=\frac{1}{\rho_0}\int 1_{\{\zeta(0)<0\}}f(\zeta)g(\zeta^{0,-1})\,d\lambda_\rho(\zeta)$$ □

THEOREM 3 *Under $\mathbb{P}_{\mu_\rho}$,*

$$\frac{X_t-(1-\rho)(p-q)t}{\sqrt{t}}$$

converges in distribution to a centered normal distribution with variance $(1-\rho)(p-q)$.

Proof We use the same argument as in [8]. The following proposition is stated for discrete time processes, but it applies to continuous time processes as well.

PROPOSITION (Newman–Wright, quoted in [8] and in [12]) *Let X_n be a strictly stationary finite variance sequence, and $S_n=X_1+\cdots+X_n$. It is a weakly negatively associated iff for every $k>n$, and f,g increasing,*

$$E[f(S_n)g(S_k-S_n)]\leqq E[f(S_n)]\,E[g(S_k-S_n)]$$

If X_n is weakly negatively associated and such that

$$\lim_{n\to+\infty}\frac{V(S_n)}{n}=\sigma^2,\ \sigma^2<+\infty,$$

then

$$\frac{1}{\sqrt{n}}(S_n-nE(X_1))\xrightarrow{\mathscr{L}}\mathcal{N}(0,\sigma^2).$$

Remark that by negative association, $E(X_1^2)<+\infty$ implies that $\sigma^2<+\infty$, but we still have to check that $\sigma^2>0$.

We prove here that $A_t=N_t^+-N_t$ is negatively associated for the zero-range process: let f and g be two increasing functions, $\phi(\xi)=\mathbb{E}_\xi[g(A_s)]$ and $\psi(\xi)=\mathbb{E}^*_\xi[f(-A_t)]$, where $\mathbb{E}^*$ is the expectation w.r.t. the reversed zero-range process i.e. the process with generator Λ^*, and $s,t\geqq 0$. The function ϕ is clearly decreasing in the occupation number of the positive sites, while ψ varies in the opposite sense. So,

$$\mathbb{E}_{\lambda_\rho}[f(A_t)g(A_{t+s}-A_t)]=\int\phi(\xi)\psi(\xi)\lambda_\rho(d\xi)$$

using the reversed process at time t

$$\leqq\int\phi(\xi)\lambda_\rho(d\xi)\int\psi(\xi)\lambda_\rho(d\xi)$$

by the lemma in Section 2 of [8].

The central limit theorem is now a consequence of the next proposition, in which we prove that the variance is strictly positive by computing it explicitly.

PROPOSITION 4 *When t goes to infinity, $\mathbb{E}_{\mu_\rho}[(X_t-(1-\rho)(p-q)t)^2]/t$ converges to $(1-\rho)(p-q)=v_\rho$.*

Proof Since λ_ρ is invariant, we can by stationarity extend the zero range process for $t \leqq 0$. It enables us to use the reversed zero range process: let $\mathscr{G}_t$ be the σ-algebra it generates for $s \leqq t$. Remember that we proved in Theorem 1 that

$$M_t^+ = N_t - \int_0^t [p\,1_{\{\xi_s(0)<0\}} - q]\,ds = N_t - \int_0^t \phi^+(\xi_s)\,ds$$

is a martingale w.r.t. $\{\mathscr{F}_t, t \geqq 0\}$ for the zero range process.

Similarly, using the formula for Λ^* in Lemma 2,

$$M_t^- = N_t^* - \int_0^t \left[\frac{q}{\rho_0} 1_{\{\xi_s(0)<0\}} - p\rho_0\right] ds = N_t^* - \int_0^t \phi^-(\xi_s)\,ds$$

is a martingale w.r.t. $\{\mathscr{G}_t, t \geqq 0\}$ for the reversed zero range process (where N_t^* is the number of jumps across the bond between 0 and -1, i.e. the number of jumps from 0 to -1 minus the number of jumps from -1 to 0).

Furthermore, as was already seen in Theorem 1 (we will use the shorthand $\langle\ \rangle_\rho$-resp. $\langle\ \rangle_\rho^*$-for expectations with respect to $\mathbb{E}_{\lambda_\rho}$-resp. the reversed process $\mathbb{E}_{\lambda_\rho}^*$-) we have:

$$\frac{1}{t}\langle (M_t^+)^2\rangle_\rho = p\rho_0 + q \tag{3.1}$$

We now prove the equality

$$\lim_{t\to+\infty} \frac{1}{t}\langle (N_t - t(1-\rho)(p-q))^2\rangle_\rho$$

$$= p\rho_0 + q - 2\int_0^{+\infty} [\langle \phi^-(\xi_0)\phi^+(\xi_s)\rangle_\rho + (1-\rho)^2(p-q)^2]\,ds \tag{3.2}$$

for which we need the absolute convergence of the integral

$$I = \int_0^{+\infty} [\langle \phi^-(\xi_0)\phi^+(\xi_s)\rangle_\rho + (1-\rho)^2(p-q)^2]\,ds.$$

We shall compute I after the proof of (3.2). (Remark however that (3.2) is contained in Theorem 2 of [6] but we include it here for the sake of completeness).

a) *Proof of (3.2)* Let us write $M_t^+ = N_t - v_\rho t - \int_0^t (\phi^+(\xi_s) - v_\rho)\,ds$.

We square both sides of this equality, we divide by t and take $\mathbb{E}_{\lambda_\rho}$-expectation. Recalling (3.1) we obtain

$$\frac{1}{t}\langle (M_t^+)^2\rangle_\rho = \frac{1}{t}\langle (N_t - v_\rho t)^2\rangle_\rho + \frac{1}{t}\int_0^t ds \int_0^t [\langle \phi^+(\xi_s)\phi^+(\xi_{s'})\rangle_\rho - v_\rho^2]\,ds'$$

$$- \frac{2}{t}\int_0^t \langle (N_t - v_\rho t)(\phi^+(\xi_s) - v_\rho)\rangle_\rho\,ds = p\rho_0 + q.$$

So

$$\frac{1}{t}\langle (N_t - v_\rho t)^2\rangle_\rho = p\rho_0 + q - \frac{2}{t}\int_0^t ds \int_s^t [\langle \phi^+(\xi_{s'})\phi^+(\xi_s)\rangle_\rho - v_\rho^2]\,ds'$$

$$+ \frac{2}{t}\int_0^t [\langle N_t \phi^+(\xi_s)\rangle_\rho - v_\rho^2 t]\,ds \tag{3.3}$$

because, since M_t^+ is a martingale,

$$\int_0^t ds \langle \phi^+(\xi_s)\rangle_\rho = \langle N_t\rangle_\rho = (p\rho_0 - q)t = (1-\rho)(p-q)t = v_\rho t.$$

On the other hand,

$$\frac{2}{t}\int_0^t [\langle N_t \phi^+(\xi_s)\rangle_\rho - v_\rho^2 t]\,ds = \frac{2}{t}\int_0^t [\langle (N_t - N_s)\phi^+(\xi_s)\rangle_\rho - v_\rho^2(t-s)]\,ds$$

$$- \frac{2}{t}\int_0^t [\langle (-N_s)\phi^+(\xi_s)\rangle_\rho + v_\rho^2 s]\,ds$$

$$= \frac{2}{t}\int_0^t ds \int_s^t [\langle \phi^+(\xi_s)\phi^+(\xi_{s'})\rangle_\rho - v_\rho^2]\,ds' -$$

$$\frac{2}{t}\int_0^t [\langle N_s^* \phi^+(\xi_0)\rangle_\rho^* + v_\rho^2 s]\,ds$$

where we have introduced the martingale M_t^+ to get the first term, and used the reversed process at time s to get the second term.

$$= -\frac{2}{t}\int_0^t ds \int_0^s [\langle \phi^-(\xi_{s'})\phi^+(\xi_0)\rangle_\rho^* + v_\rho^2]\,ds'$$

$$+ \frac{2}{t}\int_0^t ds \int_s^t [\langle \phi^+(\xi_s)\phi^+(\xi_{s'})\rangle_\rho - v_\rho^2]\,ds' \tag{3.4}$$

where we used M_t^- as M_t^+ previously.

From (3.3) and (3.4) we derive that, using again the reversed process at time s,

$$\frac{1}{t}\langle (N_t - v_\rho t)^2\rangle_\rho = p\rho_0 + q - 2\int_0^t \left(1-\frac{s}{t}\right)[\langle \phi^-(\xi_0)\phi^+(\xi_s)\rangle_\rho + v_\rho^2]\,ds.$$

b) *Computation of I* As $p\rho_0 - q = (p-q)(1-\rho)$ we have that

$$\langle \phi^+(\xi_s)\phi^-(\xi_0)\rangle_\rho + (p-q)^2(1-\rho)^2 = \frac{pq}{\rho_0}[\mathbb{P}_{\lambda_\rho}\{\xi_s(0)>0, \xi_0(0)>0\} - \rho_0^2]$$

$$= pq[\mathbb{P}_{\lambda_\rho}(\xi_s(0)>0/\xi_0(0)>0) - \mathbb{P}_{\lambda_\rho}(\xi_s(0)>0)]$$

An easy computation shows that $\lambda_\rho(\ \ /\xi_0(0)>0)$ is equal to the measure obtained by adding one particle at site 0 to λ_ρ. Thus in order to compare P_{λ_ρ} and $P_{\lambda_\rho(\ \ /\xi_0(0)<0)}$, we couple those two zero range processes $\{\xi_t\}$ and $\{\bar{\xi}_t\}$ as follows:

- At time 0, ξ_0 is distributed according to λ_ρ, and $\bar{\xi}_0$ is equal to ξ_0 plus a particle at site 0, that we call the extra particle. Therefore each particle of $\bar{\xi}_0$ corresponds to a particle of ξ_0, except the extra particle.
- Then the corresponding particles in the two processes move together, so that the extra particle behaves as a second class particle w.r.t. the other $\bar{\xi}_t$-particles (for more details on the coupling, see [11] chapter II).

Let Z_t be the position at time t of the extra particle, S_n be the moment when it performs its nth jump and $\mathbb{E}, \mathbb{P}$ be the expectation and probability w.r.t. the coupling; then (see [6] equality (2.13) which is obtained by the same way)

$$\int_0^{+\infty} \mathbb{E}[1_{\{\bar{\xi}_s(0)<0\}} - 1_{\{\xi_s(0)<0\}}]\,ds = \int_0^{+\infty} \mathbb{E}[Z_s = 0, \xi_s(0)=0]\,ds$$

$$= \int_0^{+\infty} ds \sum_{n=0}^{\infty} \mathbb{P}\{Z_{S_n}=0; S_n \leq s < S_{n+1}; \xi_s(0)=0\}$$

$$= \sum_{n=0}^{\infty} P(Z_{S_n}=0)$$

where P is the law of an asymmetric random walk (with $q(0,1)=q$ and $p=q(0,-1)$) with absorbing barrier at -1.

$$= E(\mathcal{N})$$

where $\mathcal{N}$ is the number of returns to the origin before absorbtion

$$= \sum_{k=1}^{\infty}(k+1)q^k p = \frac{1}{p}.$$

Finally

$$\int_0^{+\infty} [\langle \phi^+(\xi_s)\phi^-(\xi_0)\rangle_\rho + (p-q)^2(1-\rho)^2]\,ds = q. \tag{3.5}$$

It follows from (3.2) and (3.5) that

$$\lim_{t\to+\infty}\frac{1}{t}\langle (N_t - t(1-\rho)(p-q))^2\rangle_\rho = p\rho_0 + q - 2q = p\rho_0 - q = (p-q)(1-\rho) = v_\rho. \qquad \square$$

4. ASYMPTOTIC BEHAVIOUR OF THE SIMPLE EXCLUSION PROCESS

In order to prove (1.2) we start by proving the same result when the initial measure is μ_ρ instead of $\bar{v}_\rho$, so that we can take advantage of stationarity and reversibility. Besides, it is easy to see that far enough to the right of the last particle, μ_ρ is close to v_ρ.

THEOREM 5 *Suppose that we start the evolution with the last particle at zero and the distribution μ_ρ to the right of the origin. We denote by $\mathbb{E}_{\mu_\rho}$ the expectation w.r.t. this process.*

Let f be any cylindrical function, fix r and $t>0$. For $\varepsilon>0$ denote by $z(\varepsilon)$ the integer part of $\varepsilon^{-1}r + (1-\rho)(p-q)\varepsilon^{-2}t$. Then

$$\lim_{\varepsilon\to 0}\mathbb{E}_{\mu_\rho}[(\tau_{z(\varepsilon)}f)(\eta_{t\varepsilon^{-2}})] = \mathbb{P}(B_t<r)\int f(\eta)\,dv_\rho(\eta) + \mathbb{P}(B_t>r)f(\mathbf{0})$$

where

- *τ is the space shift operator defined by $(\tau_z f)(\eta) = f(\tau_z\eta)$ with $(\tau_z\eta)(u) = \eta(u+z)$ for every site u.*
- *B_t is a Brownian motion with variance $(1-\rho)(p-q)t$*
- *$\mathbf{0}$ is the configuration for which each site is empty.*

Proof Similarly to what we did for the zero range process, we extend by stationarity the simple exclusion process $(\bar{\eta}_t)$ seen from the last particle (with invariant measure μ_ρ) for $t \leq 0$. We then define the reversed simple exclusion process seen from the last particle as the image of the reversed zero range process with infinite source at -1 (via the same correspondence as before).

It means that for a cylindrical function g, we have, if we denote by $\bar{\mathbb{E}}_{\mu_\rho}$ (resp. $\bar{\mathbb{E}}^*_{\mu_\rho}$) the expectation w.r.t. the $(\bar{\eta}_t)$-process (resp. the reversed simple exclusion process):

$$\mathbb{E}_{\mu_\rho}[g(\eta_{t\varepsilon^{-2}})] = \bar{\mathbb{E}}_{\mu_\rho}[(\tau_{X_{t\varepsilon^{-2}}}g)(\bar{\eta}_{t\varepsilon^{-2}})]$$

by the definition of $(\bar{\eta}_t)$. Now since $X_{t\varepsilon^{-2}}$ can be computed in terms of the trajectories of $\bar{\eta}_s$ for $0 \leq s \leq t\varepsilon^{-2}$, using the reversed process at time t yields

$$\mathbb{E}_{\mu_\rho}[g(\eta_{t\varepsilon^{-2}})] = \bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{X_{t\varepsilon^{-2}}}g)(\bar{\eta}_0)]$$

so that

$$\mathbb{E}_{\mu_\rho}[(\tau_{z(\varepsilon)}f)(\eta_{t\varepsilon^{-2}})]=\bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{Z_t}f)(\bar\eta_0)]$$

where $Z_t=z(\varepsilon)+X_{t\varepsilon^{-2}}$.

Notice that while $z(\varepsilon)$ is a (deterministic) function of ε, t and r, the quantity Z_t is random. Besides, this new formulation has the advantage that all the randomness coming from the evolution is carried by Z_t.

Now, a central limit theorem holds for X_t under the reversed process (which can be proved along the lines of Theorem 3:

$$\frac{X_t+v_\rho t}{\sqrt{t}}$$

converges in distribution to a centered normal distribution with variance v_ρ) so that for any $0<\gamma<1$, $\bar{\mathbb{P}}^*_{\mu_\rho}\{|Z_t|\leqq\varepsilon^{-\gamma}\}$ converges to 0 when ε tends to 0. Therefore, $\mathbb{E}_{\mu_\rho}[(\tau_{z(\varepsilon)}f)(\eta_{t\varepsilon^{-2}})]$ has the same limit (when ε goes to 0) as $\bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{Z_t}f)(\bar\eta_0)\,1_{\{|Z_t|<\varepsilon^{-\gamma}\}}]$.

Recall that since f is cylindrical its value depends only on the configuration on a finite number of sites, say $\{-a,\dots,+a\}$. When ε is small enough and $Z_t<-\varepsilon^{-\gamma}$, the sites that matter for $\tau_{Z_t}f$ are all empty. So, for ε small enough,

$$\bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{Z_t}f)(\bar\eta_0)\,1_{\{Z_t>\varepsilon^{-\gamma}\}}]=f(\mathbf{0})\,\bar{\mathbb{P}}^*_{\mu_\rho}\{Z_t<\varepsilon^{-\gamma}\}$$

which converges to $f(0)\,\mathbb{P}(B_t>r)$, by the central limit theorem.

To deal with $\bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{Z_t}f)(\bar\eta_0)1_{\{Z_t<\varepsilon^{-\gamma}\}}]$ we use the next lemma, which we will prove later:

Lemma 6 *Write every element $\bar\eta$ of $\{0,1\}^{\mathbb{Z}}$ as (η^1,η^2), with η^1 (resp. η^2) belonging to the set $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) of configurations describing the occupation on sites in $\{x\leqq\frac{1}{2}\varepsilon^{-\gamma}\}$ (resp. in $\{x>\frac{1}{2}\varepsilon^{-\gamma}\}$). Denote by μ^1 and μ^2 the marginals of μ_ρ on $\mathcal{X}_1$ and $\mathcal{X}_2$. The set $A=\{\eta^1:\sum_{0\leqq x\leqq(1/2)\varepsilon^{-\gamma}}\eta^1(x)\geqq\varepsilon^{-\gamma/2}\}$ satisfies $\lim_{\varepsilon\to0}\mu^1(A)=1$ for $0<\rho<1$ (by the law of large numbers). Then for every element η^1 of A and any configuration η^2 in $\mathcal{X}_2$,*

$$\lim_{\varepsilon\to0}|\bar{\mathbb{P}}^*_{(\eta^1,\eta^2)}\{Z_t>\varepsilon^{-\gamma}\}-\bar{\mathbb{P}}^*_{(\eta^1,\bar0)}\{Z_t>\varepsilon^{-\gamma}\}|=0$$

where $\bar0$ is the configuration for which each site is empty in $\mathcal{X}_2$.

Now, on one hand $\tau_x\mu_\rho$ converges to v_ρ when x tends to infinity, thus it is almost a product measure; and on the other hand, for $Z_t>\varepsilon^{-\gamma}$, the support of the cylindrical function $\tau_{Z_t}f$ contains only sites in the configuration η^2 (if $\bar\eta_0=(\eta^1,\eta^2)$ as in Lemma 6) for ε small enough. Therefore, by the decomposition given in Lemma 6,

$$\bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{Z_t}f)(\bar\eta_0)\,1_{\{Z_t<\varepsilon^{-\gamma}\}}]$$

has the same limit as

$$\bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{Z_t}f)(\eta^2)\,\bar{\mathbb{P}}^*_{(\eta^1,\eta^2)}\{Z_t>\varepsilon^{-\gamma}\}]$$

which in its turn behaves like

$$\bar{\mathbb{E}}^*_{\mu_\rho}[(\tau_{Z_t}f)(\eta^2)\,1_A(\eta^1)\,\bar{\mathbb{P}}^*_{(\eta^1,\bar0)}\{Z_t>\varepsilon^{-\gamma}\}]$$

using Lemma 6.

As ε goes to 0 this last quantity has the same limit as

$$\bar{\mathbb{E}}^*_{\mu^1}[1_A(\eta^1)\,\bar{\mathbb{P}}^*_{(\eta^1,\bar0)}\{Z_t>\varepsilon^{-\gamma}\}\int(\tau_{Z_t}f)(\eta^2)\,d\mu^2(\eta^2)].$$

Since for $y>\varepsilon^{-\gamma}$, $\int(\tau_y f)(\eta^2)\,d\mu^2(\eta^2)$ is uniformly close to $\int f(\eta)\,dv_\rho(\eta)$ when ε tends to 0, and, by Lemma 6, $\bar{\mathbb{E}}^*_{\mu^1}[1_A(\eta^1)\,\bar{\mathbb{P}}^*_{(\eta^1,\bar0)}\{Z_t>\varepsilon^{-\gamma}\}]$ is close to $\bar{\mathbb{P}}^*_{\mu_\rho}\{Z_t>\varepsilon^{-\gamma}\}$ which converges to $P(B_t<r)$ when ε tends to 0 by the central limit theorem, the theorem is proved. □

Proof of Lemma 6 We use the correspondence of the reversed simple exclusion process seen from the last particle with the reversed zero range process. We connect the partition of $\{0,1\}^{\mathbb{Z}}$ with the introduction of a priority in the reversed zero range process (we detailed this notion in the preliminaries). If ξ is the image in $\mathbb{N}^{\mathbb{Z}}$ of $\bar\eta=(\eta^1,\eta^2)$, the ξ-particles in $\{y\leqq\varepsilon^{-\gamma/2}\}$ are first class particles, and the others are second class particles. Furthermore, we fix η^1 in A, so that the first class particles of ξ are only determined by η^1. We denote by a the site $[\varepsilon^{-\gamma/2}]+1$ (where [ ] means integer part). For every $y\in\mathbb{Z}$, the particles $\xi(y)$ jump to the right with probability

$$p_y=p\,\frac{\rho_{y+1}}{\rho_y},$$

and to the left with probability

$$q_y=q\,\frac{\rho_{y-1}}{\rho_y}.$$

Since p_y (resp. q_y) is an increasing (resp. decreasing) function of y, we fix a site b, $0<b<a$, with $p_b>1/2$, so that the ξ-particles on the right of b have a drift to the right. We take $b=[a/2]$.

First recall that $X_{+t\varepsilon^{-2}}$ is equal to the algebraic number of ξ particles that crossed the bond between -1 and 0. Therefore for any η^1 in $\mathcal{X}_2$,

$$\tfrac{1}{2}|\bar{\mathbb{P}}^*_{(\eta^1,\eta^2)}\{Z_t>\varepsilon^{-\gamma}\}-\bar{\mathbb{P}}^*_{(\eta^1,\bar0)}\{Z_t>\varepsilon^{-\gamma}\}|$$

is bounded by the probability that second class particles of ξ reach 0, and consequently ever cross b. Because the interaction can only slow down the particles and since the more time they have the further left they can reach, we get another upper bound by putting at time zero a Poisson number N of particles (with parameter $q_a t\varepsilon^{-2}$) at site a (so that we bound the total number of second class particles that will ever enter $[b, a]$ during the time interval $[0, t\varepsilon^{-2}]$) and letting them evolve independently, with probability to jump left (resp. right) q_b (resp. p_b, because of the monotonicities of p_y and q_y).

Then each of the N particles ever reaches b with a probability which is equal to $(q_b/p_b)^{b-a}$. Therefore the number of these particles ever reaching b has a Poisson distribution with parameter $q_a t\varepsilon^{-2}(q_b/p_b)^{b-a}$. Since $b-a$ goes to infinity as a power of ε^{-1} and $q_b/p_b < 1$, this parameter goes to 0 as ε tends to 0, and thus

$$\tfrac{1}{2}\left|\bar{\mathbb{P}}^*_{(\eta^1,\eta^2)}\{Z_t > \varepsilon^{-\gamma}\} - \bar{\mathbb{P}}^*_{(\eta^1,0)}\{Z_t > \varepsilon^{-\gamma}\}\right|$$

converges to 0. □

When we start with $\bar{\nu}_\rho$, the result stems from Theorem 5 and the following two easy results:

THEOREM 7 *Let ξ and ξ' in $\mathbb{N}^s$ be two initial configurations of the zero range process, such that for a finite number L, $\xi(x) = \xi'(x)$ for all $x \geq L$. Denote by $\mathbb{P}^t_\xi$ and $\mathbb{P}^t_{\xi'}$ the laws of the zero range processes of initial positions ξ and ξ' in the time interval $[t, +\infty[$. Then $\|\mathbb{P}^t_\xi - \mathbb{P}^t_{\xi'}\|$ tends to 0 in variation norm when t goes to infinity.*

Proof We call second class particles the particles in $[0, L[$. With probability 1, those particles will eventually reach -1 (remember that they jump to the left with probability p) and disappear. □

THEOREM 8 *For $y \in \mathbb{Z}$ and any cylindrical function f*

$$\lim_{t \to +\infty} \left|\mathbb{E}_{\mu_\rho}[(\tau_y f)(\eta_t)] - \mathbb{E}_{\nu_\rho}[(\tau_y f)(\eta)]\right| = 0.$$

Proof It is enough to notice that μ_ρ and $\bar{\nu}_\rho$ can be coupled by a measure ν such that

$$\nu\left\{(\eta, \xi) : \sum_{x \in \mathbb{Z}} |\eta(x) - \xi(x)| < +\infty\right\} = 1$$

and to use the preceeding result. □

Remark It follows from the proofs that the increments of $\varepsilon(X_{t\varepsilon^{-2}} - (1-\rho)(p-q)t\varepsilon^{-2})$ are asymptotically independent as ε tends to zero. Besides by negative correlation we obtain tightness so that we have convergence to Brownian motion (see [12] cor. 3 and th. 22 for a sketch of this proof).

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