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Potts models in the continuum. Uniqueness and exponential decay in the restricted ensembles

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Abstract

In this paper we study a continuum version of the Potts model, where particles are points in $\mathbb{R}^d$, $d \geq 2$, with a spin which may take $S \geq 3$ possible values. Particles with different spins repel each other via a Kac pair potential of range γ^{-1}, $\gamma > 0$. In mean field, for any inverse temperature β there is a value of the chemical potential λ_β at which $S+1$ distinct phases coexist. We introduce a restricted ensemble for each mean field pure phase which is defined so that the empirical particles densities are close to the mean field values. Then, in the spirit of the Dobrushin Shlosman theory, [9], we prove that while the Dobrushin high-temperatures uniqueness condition does not hold, yet a finite size condition is verified for γ small enough which implies uniqueness and exponential decay of correlations. In a second paper, [8], we will use such a result to implement the Pirogov-Sinai scheme proving coexistence of $S+1$ extremal DLR measures.

Contents

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1 Introduction

In this paper we consider a continuum version of the classical Potts model, namely a system of point particles in $\mathbb{R}^d$ where each particle has a spin $s \in \{1, .., S\}$, $S > 1$, and particles with different spins repel each other, this being the only interaction present. When $S = 2$ this is a simple version of the famous Widom-Rowlinson model which has been the first system where phase transitions in the continuum have been rigorously proved, [15], and for $S \geq 2$ and at very low temperature, a phase coexistence between the S symmetric phases for continuum Potts models was established in [11].
The mean field version of the continuum Potts model has been recently studied in [10]. The phase diagram has an interesting structure. In the (β, λ)-plane, β the inverse temperature, λ the chemical potential, there is a critical curve, see Figure 1, above which (i.e. λ "large"), there is segregation, namely there are S pure phases, each one characterized by having "a most populated species" (of particles with same spin). Instead, below the critical curve there is only one phase, the disordered one where the spin densities are all equal. The behavior on the critical curve depends on S. If $S = 2$ there is only the disordered phase while if $S > 2$ there is coexistence, namely there are $S + 1$ phases, the "ordered phases" where there is a spin density larger than all the others and the disordered phase as well.
An analogous phenomenon occurs in the mean field lattice Potts model where at a critical temperature there is a first order phase transition with coexistence of $S+1$ phases if $S > 2$, but in the continuum there is an extra phenomenon occurring at the transition, namely the total particles density has a strictly positive jump when going from the disordered to an ordered phase. This can be seen as an example of interplay between magnetic and elastic properties and interpreted as a magneto-striction effect, as the appearance of a net magnetization is accompanied by an increase of density and thus a decrease of inter-particles distances.
Our purpose is to prove that the above picture remains valid if mean field is replaced by a finite range interaction. Let $q = (..., r_i, s_i, ...)$, $i = 1, .., n$, $r_i \in \mathbb{R}^d$, $s_i \in \{1, .., S\}$, a finite configuration of particles. We suppose that their energy is

$$H_\lambda(q) = \frac{1}{2} \sum_{i \neq j} V_\gamma(r_i, r_j) \mathbf{1}_{s_i \neq s_j} - \lambda n \tag{1.1}$$

where

$$V_\gamma(r, r') = \int_{\mathbb{R}^d} J_\gamma(r, z) J_\gamma(z, r') \tag{1.2}$$

$J_\gamma(r,r') = \gamma^d J(0,\gamma(r'-r))$, $\gamma > 0$ a Kac scaling parameter, $J(0,r)$ a smooth probability kernel supported by $|r| \le 1/2$. (Observe that $H_\lambda(q)$ is independent of the particles labeling). To motivate the above choice recall that the mean field energy density (mean field energy over volume) is

$$e_\lambda(\rho(\cdot)) := \frac{1}{2}\sum_{s\neq s'} \rho(s)\rho(s') - \lambda\rho_{\rm tot}, \quad \rho_{\rm tot} = \sum_s \rho(s) \tag{1.3}$$

where $\rho(s)$ is the density of particles with spin s. Then

$$H_\lambda(q) = \int e_\lambda(\rho_{q,r}(\cdot)), \quad \rho_{q,r}(s) = \sum_i \mathbf{1}_{s_i=s} J_\gamma(r,r_i) \tag{1.4}$$

Thus $H_\lambda(q)$ is the integral of the mean field free energy density, where the latter is computed using the empirical averages $\rho_{q,r}(s)$. If γ is small one may think that (1.1) "simulates mean field". Indeed we will prove in [8] that

Theorem 1.1. *For any $d \ge 2$, $S > 2$ and $\beta > 0$ there is $\gamma^* > 0$ such that for any $\gamma \le \gamma^*$ there exist $\lambda_{\beta,\gamma}$ and $S+1$ mutually distinct, extremal DLR measures at $(\beta, \lambda_{\beta,\gamma})$.*

To keep the statement simple we have not reported all the information we have on the structure of the DLR measures referring to [8] for the full result. In particular we know that the particles densities are close to their mean field values (for γ small). The proof of Theorem 1.1 follows the Pirogov-Sinai strategy which is based on the introduction of "restricted ensembles" where the original phase space of the system is restricted by constraints which impose local closeness to one of the putative pure phases, in our case local closeness of empirical averages to the mean field values in a pure phase. We need a full control of such "restricted ensembles" and then a general machinery applies giving the desired phase transition. As a difference with the classical Pirogov-Sinai theory, here the small parameter is the inverse interaction range γ instead of the temperature, as we are "perturbing" mean field instead of the ground states, see for instance the LMP model, [13], where these ideas have been applied to prove phase transitions for particles systems in the continuum with Kac potentials.
In the typical applications of Pirogov-Sinai, restricted ensembles are studied using cluster expansion which yields a complete analyticity (in the Dobrushin-Shlosman sense, [9]) characterization of the system. Namely constraining the system into a restricted ensemble raises the effective temperature and the state enjoys the characteristics of high temperature systems. An analogous effect has been found in the Ising model with Kac potentials, [7], [5], and in the LMP model, in both the high-temperatures Dobrushin uniqueness condition has been proved to hold. This is a "finite size" condition, and the Dobrushin uniqueness theorem states that if such a condition is verified, then there is a unique DLR state. The importance of the result is that the condition involves only the analysis of the system in a finite box: loosely speaking it is a contraction property which states that compared with the variations of the boundary conditions, the Gibbs measure has strictly smaller changes, all this being quantified using the

Wasserstein distance. *Dobrushin's high temperatures* means that the size of the box [where the conditional measures are compared] can be chosen small (a single spin in the Ising case) or a small cube in LMP so that there is no self interaction in Ising or a negligible interaction among particles of the box (in LMP) and the main part of the energy is due to the interaction with the boundary conditions. The measure and its variations are then quite explicit and it is possible to check the validity of the above contraction property.
As explained by Dobrushin and Shlosman, one expects that when lowering the temperature the above high temperature property eventually fails, the point however being that it could be regained if we look at systems still in a finite box but with a larger size, eventually divergent as approaching the critical temperature. The problem is that if the finite size condition involves a large box then self interactions are important and it is difficult to check whether the condition is verified.
While it is generally believed that the above picture is correct, there are however not many examples where it has been rigorously established. Unlike Ising with Kac potentials and LMP, in an interval of values of the temperature, where the high temperature Dobrushin condition is valid in restricted ensembles, in the continuum Potts model we are considering there is numerical evidence (at least) that it is not verified. We will prove here that a finite size condition (involving some large boxes where self interaction is important) is verified in our restricted ensembles and then prove using the disagreement percolation techniques introduced in [2], [3], that our finite size condition implies uniqueness and exponential decay of correlations and all the properties needed to implement Pirogov-Sinai, a task accomplished in [8].

Part I
Model and main results

2 Mean field

The "multi-canonical" mean field free energy is

$$F^{\mathrm{mf}}(\rho) = \frac{1}{2}\sum_{s\neq s'}\rho_s\rho_{s'} + \frac{1}{\beta}\sum_s \rho_s[\log\rho_s - 1], \quad \rho = \{\rho_1, .., \rho_S\} \in \mathbb{R}_+^S \tag{2.1}$$

where ρ_s represents the density of particles with spin s and β the inverse temperature, to underline dependence on β we may add it as a subscript. The "canonical" mean field free energy is instead

$$f^{\mathrm{mf}}(x) = \inf \big\{ F^{\mathrm{mf}}(\rho); \sum_s \rho_s = x \big\}, \qquad x > 0 \tag{2.2}$$

and the mean field free energy $CEf^{\mathrm{mf}}(x)$ is the convex envelope of $f^{\mathrm{mf}}(x)$. $F^{\mathrm{mf}}_\lambda(\rho)$, $f^{\mathrm{mf}}_\lambda(x)$ and $CEf^{\mathrm{mf}}_\lambda(x)$, $\lambda \in \mathbb{R}$ the chemical potential, are defined by adding the term $-\lambda x$, where in the case of $F^{\mathrm{mf}}_\lambda(\rho)$, $x = \sum_s \rho_s$.

Observe that for any $a > 0$,

$$F^{\mathrm{mf}}_{\beta,\lambda}(\rho) = a^{-2} F^{\mathrm{mf}}_{\beta/a,\lambda'}(a\rho), \quad \lambda = a^{-1}\lambda' - \frac{\log a}{\beta} \tag{2.3}$$

so that if the graph of $CEf^{\mathrm{mf}}_{\beta,\lambda}(x)$ has a horizontal segment, then for any β', $CEf^{\mathrm{mf}}_{\beta',\lambda'}(x)$ has also a horizontal segment when $\lambda' = a\lambda + \beta^{-1} a \log a$, $a = \beta/\beta'$, which reduces the analysis of phase transitions to a single temperature, object of the following considerations.

As shown in [12] (see the proof of Theorem A.1 therein), the variational problem (2.2) is actually reduced to a two-dimensional problem because:

Lemma 2.1.

$$f^{\mathrm{mf}}(x) = \inf \big\{ F^{\mathrm{mf}}(\rho); \sum_s \rho(s) = x; \rho_1 \ge \rho_2 = \cdots = \rho_S \big\} \tag{2.4}$$

The analysis of (2.4) yields:

Theorem 2.2. *Let $S > 2$ and $\beta > 0$. Then there are $0 < x_- < x_+$ such that $CEf^{\mathrm{mf}}_\beta(x)$ coincides with $f^{\mathrm{mf}}_\beta(x)$ in the complement of (x_-, x_+) and it is a straight line in $[x_-, x_+]$. As a consequence there is λ_β such that $CEf^{\mathrm{mf}}_{\beta,\lambda_\beta}(x)$ has the whole interval $[x_-, x_+]$ as minimizers, it is strictly convex in the complement and $D^2 f^{\mathrm{mf}}_{\beta,\lambda_\beta}(x_\pm) > 0$.*

By using the scaling property (2.3) we then obtain the phase diagram in Figure 1.

We will next discuss the structure of the minimizers of $F^{\mathrm{mf}}_{\beta,\lambda_\beta}(\rho)$.

Theorem 2.3. *Let $S > 2$, $\beta > 0$ and λ_β as in Theorem 2.2. Then $F^{\mathrm{mf}}_{\beta,\lambda_\beta}(\rho)$ has $S+1$ minimizers denoted by $\rho^{(k)}$, $k = 1, .., S+1$. For $k \le S$, $\rho^{(k)}_k > \rho^{(k)}_s, s \ne k$ and $\rho^{(k)}_s = \rho^{(k)}_{s'}$ for all s, s' not equal to k. Instead $\rho^{(S+1)}_s = \rho^{(S+1)}_1$ for all s and*

$$\sum_s \rho^{(1)}_s > \sum_s \rho^{(S+1)}_s \tag{2.5}$$

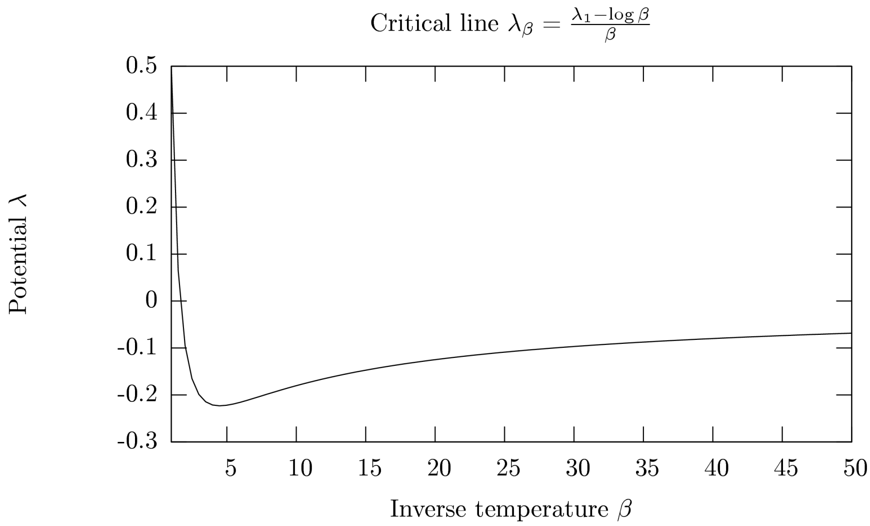

Figure 1: Phase Diagram of the Mean field Potts gas

Finally for any k the Hessian matrix $L^{(k)} := D^2 F^{\mathrm{mf}}_{\beta,\lambda_\beta}(\rho^{(k)})$ is strictly positive, namely there is $\kappa^* > 0$ such that for any vector $v = v(s), s \in \{1, .., S\}$,

$$\langle v, L^{(k)} v\rangle = \sum_{s,s'} L^{(k)}(s,s')v(s)v(s') \geq \kappa^* \langle v, v\rangle \tag{2.6}$$

The proof of Theorems 2.2 and 2.3 is given in Appendix C.

The minimizers satisfy the mean field equation

$$\rho^{(k)}_s = \exp\Big\{ -\beta\{\sum_{s'\neq s} \rho^{(k)}_{s'} - \lambda_\beta\}\Big\} \tag{2.7}$$

The Hessian $L^{(k)}$ has the explicit form:

$$L^{(k)}(s,s') = \frac{\partial^2 F^{\mathrm{mf}}_{\beta,\lambda_\beta}}{\partial\rho_s \partial\rho_{s'}}\Big|_{\rho=\rho^{(k)}} = \frac{1}{\beta\rho^{(k)}_s}\mathbf{1}_{s=s'} + \mathbf{1}_{s\neq s'} \tag{2.8}$$

3 Restricted ensembles

The purpose of this paper is to study the system in restricted ensembles defined by restricting the phase space to particles configurations which are "close to a mean field equilibrium phase".

Unfortunately the requests from the Pirogov-Sinai theory will complicate the picture, but let us do it gradually and start by defining notions as local equilibrium and "coarse grained" variables, adapted to the present context.

3.1 Geometrical notions

We discretize $\mathbb{R}^d$ by introducing cells of size $\ell > 0$, the mesh parameter ℓ will be specified in the next paragraph.

The partition $\mathcal{D}^{(\ell)}$

- $\mathcal{D}^{(\ell)}$, $\ell > 0$, denotes the partition $\{C_x^{(\ell)},\, x \in \ell\mathbb{Z}^d\}$ of $\mathbb{R}^d$ into the cubes $C_x^{(\ell)} = \{r \in \mathbb{R}^d : x_i \le r_i < x_i + \ell, i = 1,..,d\}$ (r_i and x_i the cartesian components of r and x), calling $C_r^{(\ell)}$ the cube which contains r.
- A set Λ is $\mathcal{D}^{(\ell)}$-measurable if it is union of cubes in $\mathcal{D}^{(\ell)}$ and $\delta^\ell_{\rm out}[\Lambda]$ denotes the union of all $\mathcal{D}^{(\ell)}$ cubes in Λ^c(the complement of Λ) which are connected to Λ, two sets being connected if their closures have non empty intersection. Analogously, $\delta^\ell_{\rm in}[\Lambda]$ is the union of all $\mathcal{D}^{(\ell)}$ cubes in Λ which are connected to Λ^c.
- A function $f : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{D}^{(\ell)}$-measurable if its inverse images are $\mathcal{D}^{(\ell)}$-measurable sets.

The basic scales

There are four main lengths in our analysis: $\ell_0 \ll \ell_{-,\gamma} \ll \gamma^{-1} \ll \ell_{+,\gamma}$. More precisely let α_+, α_- and a verify

$$\frac{1}{2} \gg \alpha_+ > \alpha_- \gg a > 0 \tag{3.1}$$

(the precise meaning of the inequality will become clear in the course of the proofs), then

$$\lim_{\gamma\to 0} \frac{\ell_0}{\gamma^{-1/2}} = \lim_{\gamma\to 0} \frac{\ell_{-,\gamma}}{\gamma^{-(1-\alpha_-)}} = \lim_{\gamma\to 0} \frac{\ell_{+,\gamma}}{\gamma^{-(1+\alpha_+)}} = 1 \tag{3.2}$$

with the additional request that $\ell_{+,\gamma}$ is an integer multiple of γ^{-1} which is an an integer multiple of $\ell_{-,\gamma}$ which is an integer multiple of ℓ_0. The partition $\mathcal{D}^{(\ell)}$ is coarser than $\mathcal{D}^{(\ell')}$ if each cube of the former is union of cubes of the latter, we will then also say that $\mathcal{D}^{(\ell')}$ is finer than $\mathcal{D}^{(\ell)}$. This happens if and only if ℓ is an integer multiple of ℓ', thus $\mathcal{D}^{(\ell_0)}$ is finer than $\mathcal{D}^{(\ell_{-,\gamma})}$ which is finer than $\mathcal{D}^{(\gamma^{-1})}$ which is finer than $\mathcal{D}^{(\ell_{+,\gamma})}$.
We will need that

$$\frac{(\alpha_+ + \alpha_-)d}{2(1-\alpha_-)} < \frac{1}{1000}, \quad 8\alpha_+ + 9\alpha_- < \frac{1}{2} \tag{3.3}$$

Eventually we define, for any $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable region Λ, :

$$N_\Lambda := \frac{|\Lambda|}{\ell^d_{+,\gamma}} \tag{3.4}$$

where $|\Lambda|$ is the volume of the region Λ, thus N_Λ is the number of blocks $C^{(\ell_{+,\gamma})}$ inside Λ.

The accuracy parameter ζ

Finally, the parameter a in (3.1) is not related to a length, it defines an "accuracy parameter"

$$\zeta = \gamma^a \tag{3.5}$$

whose role will be specified next.

3.2 Local equilibrium

A particles configuration q is a sequence $(...r_i, s_i....)$ such that for any compact set Λ and any $s \in \{1, .., S\}$,

$$n(x,s) := |q(s) \cap \Lambda| < \infty, \quad q(s) = \{r_i, s_i \in q : s_i = s\} \tag{3.6}$$

We then associate to any such q the empirical densities

$$\rho^{(\ell)}(q; r, s) := \frac{|q(s) \cap C_r^{(\ell)}|}{\ell^d}, \quad s \in \{1, \dots, S\} \tag{3.7}$$

as functions on $\mathbb{R}^d \times \{1, .., S\}$ and the "local phase indicators" first for any $\rho \in L^1(\mathbb{R}^d \times \{1, .., S\})$ ($\rho^{(k)}$ below as in Theorem 2.3)

$$\eta^{(\zeta,\ell)}(\rho; r) = \begin{cases} k & \text{if } |\fint_{C_r^{(\ell)}} [\rho(r', s) - \rho_s^{(k)}]| \le \zeta, \text{ for all } s \in \{1, .., S\} \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

and then for any particles configuration q as above,

$$\eta^{(\zeta,\ell)}(q; r) = \eta^{(\zeta,\ell)}\big(\rho^{(\ell)}(q; \cdot); r\big) \tag{3.9}$$

With ζ and $\ell_{-,\gamma}$ as in (3.5) and (3.2), we then define

$$\mathcal{X}^{(k)} := \Big\{q : \eta^{(\zeta, \ell_{-,\gamma})}(q; r) = k, \text{ for all } r \in \mathbb{R}^d\Big\} \tag{3.10}$$

$\mathcal{X}^{(k)}$ is the restricted phase space and the configurations in $\mathcal{X}^{(k)}$ are said to be in local equilibrium in the phase k. Their restrictions to a $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable set Λ is denoted by $\mathcal{X}_\Lambda^{(k)}$ and we will study (in the simplest case) the Gibbs measure with Hamiltonian H_λ as in (1.4) on the phase space restricted to $\mathcal{X}^{(k)}$. As mentioned in the beginning of this section to apply Pirogov-Sinai we will need to complicate the picture, by adding a "polymer structure" to the phase space and by modifying the Hamiltonian H_λ.

3.3 Polymer configurations

A polymer is a pair $\Gamma = (\text{sp}(\Gamma), \eta_\Gamma)$, $\text{sp}(\Gamma)$, the spatial support of Γ, is a bounded, connected $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable region and η_Γ, its specification, a $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable function on $\text{sp}(\Gamma)$ with values in $\{0, 1, .., S+1\}$. In the applications of Pirogov-Sinai, Γ will be contours and η_Γ

not as general as above, to keep it simple we skip all that sticking to the above definition. We tacitly fix in the sequel $k \in \{1,..,S+1\}$ and the corresponding phase space $\mathcal{X}^{(k)}$ and define:

Polymer weights

The weight of Γ is a function $w(\Gamma;q)$, $q \in \mathcal{X}^{(k)}$, (dependence on k is not made explicit in w) which depends on the restriction of q to $\delta_{\rm out}^{\gamma^{-1}}[{\rm sp}(\Gamma)]$ and which satisfies the bound

$$\sup_{q\in\mathcal{X}^{(k)}} |w(\Gamma;q)| \le e^{-c_{\rm pol}\zeta^2\ell_{-,\gamma}^d N_\Gamma}, \quad N_\Gamma = \frac{|{\rm sp}(\Gamma)|}{\ell_{+,\gamma}^d} \tag{3.11}$$

Polymer configurations and weights

We denote by $\underline{\Gamma}$ sequences $...\Gamma_i...$ of polymers with the restriction that any two polymers Γ_i and Γ_j, $i \ne j$, are mutually disconnected (i.e. the closures of their spatial supports do not intersect and they are therefore at least at mutual distance $\ell_{+,\gamma}$). The collection of all such sequences is denoted by $\mathcal{B}$ and $\mathcal{B}_\Lambda$, Λ a $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable region, the subset of $\mathcal{B}$ made by sequences whose elements Γ have all ${\rm sp}(\Gamma)$ in Λ; $\mathcal{B}^0_\Lambda$ subset of $\mathcal{B}_\Lambda$ with the further request that ${\rm sp}(\Gamma)$ is not connected to Λ^c. If $\underline{\Gamma} \in \mathcal{B}$ is a finite sequence we define its weight as

$$w(\underline{\Gamma};q) = \prod_{\Gamma\in\underline{\Gamma}} w(\Gamma;q) \tag{3.12}$$

3.4 The interpolated Hamiltonian

Pirogov-Sinai applications also require to change the Hamiltonian. Let Λ be a bounded, $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable region, $q_\Lambda \in \mathcal{X}_\Lambda^{(k)}$, then the "reference Hamiltonian" in Λ is

$$h_\Lambda(q_\Lambda) = \sum_s \Big[\big(\sum_{s'\ne s} \rho_{s'}^{(k)}\big) - \lambda_\beta\Big]\, \ell_0^d \sum_{x\in\ell_0\mathbb{Z}^d\cap\Lambda} \rho^{(\ell_0)}(q_\Lambda;x,s) \tag{3.13}$$

where λ_β is the chemical potential introduced in Theorem 2.2, ℓ_0 is defined in Subsection 3.1, $\rho^{(\ell_0)}$ in (3.7).
For any $t \in [0,1]$ we then define the "interpolated Hamiltonian"

$$H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c}) = tH_\Lambda(q_\Lambda|\bar q_{\Lambda^c}) + (1-t)h_\Lambda(q_\Lambda) \tag{3.14}$$

where $q_\Lambda \in \mathcal{X}_\Lambda^{(k)}$, $\bar q_{\Lambda^c} \in \mathcal{X}_{\Lambda^c}^{(k)}$ and

$$H_\Lambda(q_\Lambda|\bar q_{\Lambda^c}) = H(q_\Lambda \cup \bar q_{\Lambda^c}) - H(\bar q_{\Lambda^c}) \tag{3.15}$$

H as in (1.1) with λ such that $|\lambda - \lambda_\beta| \le c\gamma^{1/2}$. Since $H_{\Lambda,1}(q_\Lambda|\bar q_{\Lambda^c}) = H_\Lambda(q_\Lambda|\bar q_{\Lambda^c})$ and $H_{\Lambda,0}(q_\Lambda|\bar q_{\Lambda^c}) = h_\Lambda(q_\Lambda)$, $H_{\Lambda,t}$ interpolates between the true and the reference Hamiltonians. As we will see in [8], $H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})$ enters in the analysis of the finite volume corrections to the pressure, a key step in the implementation of the Pirogov-Sinai strategy.

3.5 DLR measures

The finite volume Gibbs measure in Λ, Λ a bounded, $\mathcal{D}^{(\ell_+,\gamma)}$-measurable region, with boundary condition $\bar q_{\Lambda^c}$, is the following probability on $\mathcal{X}^{(k)}_\Lambda \times \mathcal{B}^0_\Lambda$

$$dG_\Lambda(q_\Lambda, \underline{\Gamma}|\bar q_{\Lambda^c}) := \frac{w(\underline{\Gamma};q)e^{-\beta H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})}}{Z_\Lambda(\bar q_{\Lambda^c})} d\nu_\Lambda(q_\Lambda) \tag{3.16}$$

where the free measure $d\nu_\Lambda(q_\Lambda)$ is

$$\int_{\mathcal{X}^{(k)}_\Lambda} f(q_\Lambda) d\nu_\Lambda(q_\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{s_1,\dots,s_n} \int_{\Lambda^n} f(r_1,s_1,\dots,r_n,s_n) dr_1 \cdots dr_n \tag{3.17}$$

and where the partition function $Z_\Lambda(\bar q_{\Lambda^c})$ is the normalization factor which makes the above a probability. In (3.16) the boundary conditions only involve particles configurations, to define the DLR measures we also need to condition on the outside polymers.

DLR measures

Given $\underline{\Gamma} \in \mathcal{B}$, $\underline{\Gamma} = (\Gamma_1, \Gamma_2, \dots)$, we call $\underline{\Gamma}_{\Lambda^c}$ the collection of all pairs $(\mathrm{sp}(\Gamma_i) \cap \Lambda^c, \eta_{\mathrm{sp}(\Gamma_i)\cap\Lambda^c})$ where $\eta_{\mathrm{sp}(\Gamma_i)\cap\Lambda^c}$ denotes the restriction of η_Γ to $\mathrm{sp}(\Gamma) \cap \Lambda^c$. We then define the probability $dG(q_\Lambda, \underline{\Gamma}|\bar q_{\Lambda^c}, \bar{\underline{\Gamma}}_{\Lambda^c})$ on $\mathcal{X}^{(k)}_\Lambda \times \mathcal{B}$ by

$$dG_\Lambda(q_\Lambda, \underline{\Gamma}|\bar q_{\Lambda^c}, \bar{\underline{\Gamma}}_{\Lambda^c}) := \frac{\mathbf{1}_{\underline{\Gamma}_{\Lambda^c} = \bar{\underline{\Gamma}}_{\Lambda^c}}}{Z_\Lambda(\bar q_{\Lambda^c}, \bar{\underline{\Gamma}}_{\Lambda^c})} e^{-\beta H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})} \Big\{ \prod_{\Gamma \in \underline{\Gamma}:\mathrm{sp}(\Gamma)\cap\Lambda\neq\emptyset} w(\underline{\Gamma};q) \Big\} d\nu_\Lambda(q_\Lambda) \tag{3.18}$$

A probability μ on $\mathcal{X}^{(k)} \times \mathcal{B}$ is DLR if the two properties below hold.

- it verifies the Peierls bound: for any $\Gamma_1,..,\Gamma_k$,

$$\mu\Big(\{\underline{\Gamma} \ni \Gamma_1\} \cap \dots \cap \{\underline{\Gamma} \ni \Gamma_k\}\Big) \le e^{-c_{\mathrm{pol}}\zeta^2 \ell^d_{-,\gamma}(N_{\Gamma_1}+..+N_{\Gamma_k})} \tag{3.19}$$

- for any bounded, $\mathcal{D}^{(\ell_+,\gamma)}$-measurable region Λ the conditional probability of μ given that the particles configurations in Λ^c is $\bar q_{\Lambda^c}$ and that $\underline{\Gamma}_{\Lambda^c} = \bar{\underline{\Gamma}}_{\Lambda^c}$ is $dG_\Lambda(q_\Lambda, \underline{\Gamma}|\bar q_{\Lambda^c}, \bar{\underline{\Gamma}}_{\Lambda^c})$ as given by (3.18).

A few remarks on the above definitions: the Gibbs measures $dG_\Lambda(q_\Lambda, \underline{\Gamma}|\bar q_{\Lambda^c})$ satisfy the Peierls bound (3.19). Indeed given any $\Gamma_1,..,\Gamma_k$ in $\mathcal{B}^0_\Lambda$ such that $\mathrm{sp}(\Gamma_i)$ is not connected to $\mathrm{sp}(\Gamma_j)$ for any $i \neq j$, then, for any q_Λ,

$$\begin{aligned}\sum_{\underline{\Gamma}\in\mathcal{B}_\Lambda:\Gamma_1,..\Gamma_k\in\underline{\Gamma}} w(\underline{\Gamma}, q_\Lambda) &= \{\prod_{i=1}^k w(\Gamma_i, q_\Lambda)\} \sum_{\underline{\Gamma}\in\mathcal{B}_\Lambda:\Gamma_1,..\Gamma_k\in\underline{\Gamma}} \prod_{\Gamma\in\underline{\Gamma},\Gamma\neq\Gamma_i,i=1,..,k} w(\Gamma, q_\Lambda) \\ &\le \{\prod_{i=1}^k w(\Gamma_i, q_\Lambda)\} \sum_{\underline{\Gamma}\in\mathcal{B}_\Lambda} w(\Gamma, q_\Lambda)\end{aligned}$$

and (3.19) follows from (3.11). On the other hand we have not specified all the properties of the weights as they arise in the applications (to the continuum Potts model) so that in the present context wild things may happen. For instance weights still compatible with (3.11) may be such that whenever $\mathrm{sp}(\Gamma)$ contains $\delta_{\mathrm{out}}^{\ell_+,\gamma}[\Delta]$, Δ a bounded, simply connected $\mathcal{D}^{(\ell_+,\gamma)}$ measurable set, then $w(\Gamma, q) = 0$ unless $\mathrm{sp}(\Gamma) \supset \Delta$. If the weights had such a property then there are sequences of finite volume Gibbs measures whose limits are not supported by $\underline{\Gamma} \in \mathcal{B}$. Thus a support property like (3.19) is necessary in the present context.

3.6 Main result

We fix $k \in \{1,..,S+1\}$, the statements below being valid for any such k and for all γ small enough. We will employ the following notion: $(q, \underline{\Gamma})$ agrees with $(q', \underline{\Gamma}')$ in Δ (Δ a $\mathcal{D}^{(\ell_+,\gamma)}$-measurable set) if all $\Gamma \in \underline{\Gamma}$ such that the closure of $\mathrm{sp}(\Gamma)$ intersects Δ are also in $\underline{\Gamma}'$ and viceversa and moreover

$$q \cap \Delta^* = q' \cap \Delta^*, \quad \Delta^* := \Delta \bigcup_{\Gamma \in \underline{\Gamma}} \{\mathrm{sp}(\Gamma) \cup \delta_{\mathrm{out}}^{(\ell_+,\gamma)}[\mathrm{sp}(\Gamma)]\} \tag{3.20}$$

Theorem 3.1. *For all γ small enough there is a unique DLR measure μ and there are constants c_1 and c_2 such that the following holds. For any bounded, $\mathcal{D}^{(\ell_+,\gamma)}$-measurable regions Λ and $\Lambda' \supset \Lambda$ and any boundary conditions $\bar{q}'_{\Lambda^c}$ and $\bar{q}''_{\Lambda'^c}$ there is a coupling dQ of $dG_\Lambda(q_\Lambda, \underline{\Gamma}|\bar{q}'_{\Lambda^c})$ and $dG_{\Lambda'}(q_{\Lambda'}, \underline{\Gamma}|\bar{q}''_{\Lambda'^c})$ such that if Δ is any $\mathcal{D}^{(\ell_+,\gamma)}$-measurable subset of Λ:*

$$Q\Big(\{\, (q'_\Lambda, \underline{\Gamma}') \text{ and } (q''_{\Lambda'}, \underline{\Gamma}'') \text{ agree in } \Delta\}\Big) \ge 1 - c_1 e^{-c_2 \frac{\mathrm{dist}(\Delta, \Lambda^c)}{\ell_{+,\gamma}}} \tag{3.21}$$

3.7 A finite size condition

The proof of Theorem 3.1 follows the Dobrushin Shlosman approach: we first introduce and verify a finite size condition and then prove that this implies uniqueness and exponential decay. In this subsection we describe the former step. Let Λ be a $\mathcal{D}^{(\ell_+,\gamma)}$-measurable, connected region contained in Λ^* where Λ^* is obtained by taking a cube $C \in \mathcal{D}^{(\ell_+,\gamma)}$, then considering $A := C \cup \delta_{\mathrm{out}}^{\ell_+,\gamma}[C]$ and finally $\Lambda^* = A \cup \delta_{\mathrm{out}}^{\ell_+,\gamma}[A]$. All the bounds we will write must be uniform in such a class. Notice that the diameter of Λ is $> \ell_{+,\gamma}$ which for γ small is much larger than the interaction range, in this sense Λ is "large" and we are away from the Dobrushin's high temperatures uniqueness scenario.

Our finite size condition involves only Gibbs measures without polymers: namely the probability on $\mathcal{X}_\Lambda^{(k)}$ defined for any given $\bar{q}_\Lambda \in \mathcal{X}_{\Lambda^c}^{(k)}$ as follows

$$dG_\Lambda^0(q_\Lambda|\bar{q}_{\Lambda^c}) := \frac{e^{-\beta H_{\Lambda,t}(q_\Lambda|\bar{q}_{\Lambda^c})}}{Z_\Lambda^0(\bar{q}_{\Lambda^c})} d\nu_\Lambda(q_\Lambda) \tag{3.22}$$

We want to compare two such measures with different boundary conditions $\bar q'_{\Lambda^c}$ and $\bar q''_{\Lambda^c}$, thus introducing the product space $\mathcal{X}^{(k)}_\Lambda \times \mathcal{X}^{(k)}_\Lambda$ whose elements are denoted by $(q'_\Lambda, q''_\Lambda)$. The finite size condition requires that there is a coupling dQ of $dG^0_\Lambda(q_\Lambda|\bar q'_{\Lambda^c})$ and $dG^0_\Lambda(q_\Lambda|\bar q''_{\Lambda^c})$ with the property that the event we define below has a "large probability".

Notation

Let $\bar m = 2^d + 2$ and $c_{\rm acc} = 2c^*$ with c^* as in Theorem 5.1 below. Call $\zeta_n := c_{\rm acc}^{-n}\zeta$ and define a partition of $\mathbb{R}_+$ into the intervals $[0,\zeta_{\bar m})$, $[\zeta_{\bar m},\zeta_{\bar m-1})$,...,$[\zeta_3,\zeta_2)$, $[\zeta_2,\infty)$.

Definition 3.2. The function $K_\Lambda(\cdot)$ and the set $\Theta_\Lambda(\cdot)$.

We denote by

$$A_x := B_x(10^{-10}\ell_{+,\gamma}) \cap \Lambda^c, \qquad B_x(R) \text{ the ball of center } x \text{ and radius } R \tag{3.23}$$

Given $\bar q'_{\Lambda^c}$ and $\bar q''_{\Lambda^c}$, we define the function $K_\Lambda(\bar q'_{\Lambda^c}, \bar q''_{\Lambda^c}; x)$, $x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Lambda$ as follows.
If $A_x = \emptyset$ then $K_\Lambda(\bar q'_{\Lambda^c}, \bar q''_{\Lambda^c}; x) = \bar m + 1$.
If $A_x \neq \emptyset$ and $\bar q'_{\Lambda^c} \cap A_x \neq \bar q''_{\Lambda^c} \cap A_x$, then $K_\Lambda(q'_{\Lambda^c}, q''_{\Lambda^c}; x) = 0$.
If $A_x \neq \emptyset$ and $q'_{\Lambda^c} \cap A_x = q''_{\Lambda^c} \cap A_x$, call $b := \max\limits_{r\in A_x, s\in\{1,..,S\}} |\rho^{(\ell_{-,\gamma})}(\bar q'_{\Lambda^c}; r, s) - \rho^{(k)}_s|$, then if $b \in [\zeta_{m+1},\zeta_m)$ for some $m \geq 2$, we set $K_\Lambda(\bar q'_{\Lambda^c}, \bar q''_{\Lambda^c}; x) = m$, otherwise we set $K_\Lambda(\bar q'_{\Lambda^c}, \bar q''_{\Lambda^c}; x) = 0$.

The set $\Theta_\Lambda(x) = \Theta_\Lambda(\bar q'_{\Lambda^c}\bar q''_{\Lambda^c}; x)$, $x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Lambda$, is defined as the whole space $\{q'_\Lambda, q''_\Lambda\}$ if $K(\cdot; x) = K_\Lambda(\bar q'_{\Lambda^c}, \bar q''_{\Lambda^c}; x) = 0$ and otherwise by

$$\Theta_\Lambda(x) = \Big\{ q'_\Lambda, q''_\Lambda : \;\; q'_\Lambda \cap C^{(\ell_{-,\gamma})}_x = q''_\Lambda \cap C^{(\ell_{-,\gamma})}_x, \\ \max_{s\in\{1,..,S\}} |\rho^{(\ell_{-,\gamma})}(q'_\Lambda; x, s) - \rho^{(k)}_s| \leq \zeta_{K(\cdot;x)-1} \Big\} \tag{3.24}$$

In section 7.4, we will use Theorem 3.3 below with $n = 5^d - 1$ and $\Lambda \subset \Lambda^*$. Recalling the definition of N_Λ in (3.4), we state:

Theorem 3.3. *For any integer $n > 0$ there exist $\gamma_n > 0$ and $\epsilon_n < 1$ such that for all $\gamma < \gamma_n$ and for any Λ with $N_\Lambda \leq n$, for any $\bar q'_{\Lambda^c}$ and $\bar q''_{\Lambda^c}$ as above, there is a coupling dQ_Λ of $dG^0_\Lambda(q_\Lambda|\bar q'_{\Lambda^c})$ and $dG^0_\Lambda(q_\Lambda|\bar q''_{\Lambda^c})$ such that with $K(\cdot; x) = K_\Lambda(\bar q'_{\Lambda^c}, \bar q''_{\Lambda^c}; x)$ and $\Theta_\Lambda(x) = \Theta_\Lambda(\bar q'_{\Lambda^c}\bar q''_{\Lambda^c}; x)$ defined above,*

$$Q_\Lambda\Big(\bigcap_{x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda} \Theta_\Lambda(x)\Big) \geq 1 - \epsilon_n \tag{3.25}$$

The proof of Theorem 3.3 is given in Part II of this paper. It consists of three parts, in the first one we use a step of the renormalization group to describe the marginal of dG^0_Λ over the variables $\{\rho^{(\ell_{-,\gamma})}(x,s), x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Lambda, s \in \{1,..,S\}\}$. Their distribution is proved to be Gibbsian with an effective Hamiltonian at the inverse effective temperature $\beta\ell_{-,\gamma}$. In a second part we study the ground states of the effective Hamiltonians, proving exponential decay from the boundary conditions. In a third and final part we bound the Wasserstein distance between the Gibbs measures by approximating the latter to Gaussian distributions describing fluctuations around the ground states characterized in the previous step.

3.8 Disagreement percolation

The finite size condition established in Theorem 3.3 is used to construct the coupling Q of Theorem 3.1. The proof uses the ideas introduced by van der Berg and Maes in their disagreement percolation paper, [3]. The proof given in Part III of this paper consists of two steps. In the first one we introduce set-valued stopping times, called stopping sets, and prove that monotone sequences of stopping sets define couplings of the Gibbs measures and that if the sequence stops, then in the last set there is agreement. In the second and last step we prove that the probability that the sequence stops late is related to a percolation event which is then shown to have exponentially small probability.

Part II
The finite size condition

4 Effective Hamiltonians

We will use the following notations.

4.1 General notation for Part II

• By default in this section Λ is a connected, $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable region contained in Λ^*, see Subsection 3.7, and regions in $\mathbb{R}^d$ are all $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable. To discretize $\mathbb{R}^d$ we will use the lattice $\ell_{-,\gamma}\mathbb{Z}^d$. Thus in the sequel $\ell_{-,\gamma}$ is the basic mesh. We define

$$J_\gamma^{(\ell)}(x,y) = \fint_{C_x^{(\ell)}} \fint_{C_y^{(\ell)}} J_\gamma(r,r'), \quad x,y \in \ell\mathbb{Z}^d, \;\; \ell = \ell_{-,\gamma} \tag{4.1}$$

• The basic variables are the densities $\rho_\Delta = \{\rho_\Delta(x,s) \ge 0, x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Delta, \, s \in \{1,..,S\}\}$, $\Delta \subset \mathbb{R}^d$, (by default variables denoted by ρ are non negative densities). Call $X_\Delta^{(k)}$ the set of all ρ_Δ such that $n_\Delta := \ell_{-,\gamma}^d \rho_\Delta$ has integer values, so that $X_\Delta^{(k)}$ is the range of values of the densities $\rho_\Delta^{(\ell_{-,\gamma})}(q_\Delta; x, s)$ when $q_\Delta \in \mathcal{X}_\Delta^{(k)}$, $x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Delta$, $s \in \{1,..,S\}$; $\rho_\Delta^{(\ell)}$ being defined in (3.7).
• To have lighter notation we will use the label i for a pair (x,s), $x \in \ell_{-,\gamma}\mathbb{Z}^d, s \in \{1,..,S\}$, writing $x(i) = x$, $s(i) = s$ if $i = (x,s)$ and sometimes shorthand $|i-j|$ for $|x(i) - x(j)|$ and $i \in \Lambda$ for $x(i) \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Lambda$.
• $\mathcal{H}$ denotes the Euclidean space of vectors $u = \big(u(i), i \in \Lambda\big)$ with the usual scalar product $(u,v) = \sum_i u(i)v(i)$. By an abuse of notation we also denote by $\mathcal{H}$ the Hilbert space with Λ above replaced by $\mathbb{R}^d$.

4.2 The effective Hamiltonian

The effective Hamiltonian $H_\Lambda^{\rm eff}(\rho_\Lambda|\bar q_{\Lambda^c})$, $\rho_\Lambda \in X_\Lambda^{(k)}$, $\bar q_{\Lambda^c} \in \mathcal{X}_{\Lambda^c}^{(k)}$, is defined by the equality

$$e^{-\beta \ell_{-,\gamma}^d H_\Lambda^{\rm eff}(\rho_\Lambda|\bar q_{\Lambda^c})} := \int_{\{\rho(q_\Lambda;\cdot)=\rho_\Lambda\}} e^{-\beta H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})} \nu_\Lambda(dq_\Lambda) \tag{4.2}$$

$H_{\Lambda,t}$ as in (3.14), so that $\beta\ell_{-,\gamma}^d$ is the effective inverse temperature. The Gibbs measure with Hamiltonian $H_\Lambda^{\rm eff}(\rho_\Lambda|\bar q)$, inverse temperature $\beta\ell_{-,\gamma}^d$ and free measure the counting measure on $X_\Lambda^{(k)}$ is then the marginal over the variables $\{\rho_\Lambda \in X_\Lambda^{(k)}\}$ of the Gibbs measure $dG_\Lambda^0(q_\Lambda|\bar q_{\Lambda^c})$ defined in (3.22).
Since $\ell_{-,\gamma} = \gamma^{-1+\alpha_-}$ and α_- is small, the effective temperature vanishes as $\gamma \to 0$, and the analysis of the Gibbs measure becomes intimately related to the study of the ground states of $H_\Lambda^{\rm eff}$. This will be the argument of the next section, in this one we determine $H_\Lambda^{\rm eff}$. In this subsection we describe its main terms and state the main theorem; in the successive ones we give the proof.

The LP term.

The main contribution to the effective Hamiltonian will be the Lebowitz-Penrose free energy functional, the LP term in the title of the paragraph. This is

$$F_\Lambda(\rho_\Lambda|\bar\rho_{\Lambda^c}) = t\Big\{\frac{1}{2}(\rho_\Lambda, \bar V_\gamma \rho_\Lambda) + (\rho_\Lambda, \bar V_\gamma \bar\rho_{\Lambda^c})\Big\} - \frac{1}{\beta}(1_\Lambda, \mathcal{I}(\rho_\Lambda)) + (1-t)(\rho^{(k)}1_\Lambda, \rho_\Lambda) \tag{4.3}$$

where we employ the usual vector notation: if $A(i,j)$ is a matrix, $u(i)$ a vector in $\mathcal{H}$,

$$\big(u,v\big) = \sum_i u(i)v(i), \quad Au(i) = \sum_j A(i,j)u(j) \tag{4.4}$$

calling 1_Λ the vector $1_\Lambda(i) = 1$ if $i \in \Lambda$ and $= 0$ otherwise. In (4.3)

$$\bar{V}_\gamma(i,j) = \ell_{-,\gamma}^d \sum_{y \in \ell_{-,\gamma}\mathbb{Z}^d} J_\gamma^{(\ell_{-,\gamma})}(x(i),y) \ell_{-,\gamma}^d J_\gamma^{(\ell_{-,\gamma})}(y,x(j)) \mathbf{1}_{s(i)\neq s(j)} \tag{4.5}$$

The normalization is such that $\bar{V}_\gamma$ is a probability kernel. The term $(1_\Lambda, \mathcal{I}(\rho_\Lambda))$ in (4.3) is "the entropy minus the chemical potential energy":

$$\mathcal{I}(\rho_\Lambda)(i) = \mathcal{I}^*(\rho_\Lambda(i)), \quad \mathcal{I}^*(b) := -b(\log b - 1) + \beta\lambda_\beta b \tag{4.6}$$

When $t = 1$, F_Λ is just the usual LP free energy and for this reason we call F_Λ the LP term. Notice that if $\rho_\Lambda(i) = \rho_{s(i)}^{(k)} 1_\Lambda(i)$, then the bulk terms of F_Λ which are proportional to t cancel, this will play an important role in the study of the ground states.

The one body effective potential.

This term is due to second order terms in the Stirling formula when computing the entropy contribution. It has the form:

$$H_\Lambda^{(1)}(\rho_\Lambda) = \frac{\ell_{-,\gamma}^{-d}}{\beta}\big(1_\Lambda, \log\sqrt{2\pi \ell_{-,\gamma}^d \rho_\Lambda} + t[\lambda_\beta - \lambda]\rho_\Lambda\big) \tag{4.7}$$

The many-body effective potential.

This term denoted by $H_\Lambda^{(2)}(\rho_\Lambda|\bar{q}_{\Lambda^c})$, takes into account variations of the potential energy inside the elementary cells $C_x^{(\ell_{-,\gamma})} \in \mathcal{D}^{(\ell_{-,\gamma})}$ which have been neglected in the LP term. The dependence of $H_\Lambda^{(2)}$ on ρ_Λ is very simple, it is in fact a polynomial of order $< N$, N a suitable positive integer. The coefficients of the polynomial are described next, they have a simpler form once we use Poisson polynomials. We denote by $\pi_k(n) = n(n-1)\cdots(n-k+1)$, $k \in \mathbb{N}_+$, $n \in \mathbb{N}_+$, the Poisson polynomial of order k and, by an abuse of notation we write

$$\pi_k^*(\rho) = \ell_{-,\gamma}^{-dk}\pi_k(n), \quad \rho = \frac{n}{\ell_{-,\gamma}^d} \tag{4.8}$$

We shorthand $\underline{i} = (i_1, .., i_n)$, $n < N$, and call $n = n(\underline{i})$; $\underline{i} \cap \Lambda \neq \emptyset$ meaning that there is $i_h \in \underline{i}$ such that $i_h \in \Lambda$. Given $\underline{i}$ we denote by $k(\underline{i}) = (k(i_1), .., k(i_n))$, with $k(i_h)$ positive integers, calling $|k(\underline{i})| = \sum_{h=1}^{n(\underline{i})} k(i_h)$. We finally call $\bar{\rho}_{\Lambda^c}(i) := \rho^{(\ell_{-,\gamma})}(\bar{q}_{\Lambda^c}; i)$ and denote by $\rho(i)$ the function equal to $\rho_\Lambda(i)$ and to $\bar{\rho}_{\Lambda^c}(i)$ when $i \in \Lambda$, respectively $i \in \Lambda^c$; a_0 below is a positive number < 1. Then $H_\Lambda^{(2)}$ has the form:

$$H_\Lambda^{(2)}(\rho_\Lambda|\bar{q}_{\Lambda^c}) = \sum_{\underline{i}\cap\Lambda\neq\emptyset} \sum_{k(\underline{i}): 2 \le |k(\underline{i})| < N} (\gamma\ell_{-,\gamma})^{a_0|k(\underline{i})|} \Phi(\underline{i}, k(\underline{i}), \bar{q}_{\Lambda^c,\underline{i}}) \prod_{h=1}^{n(\underline{i})} \pi_{k(i_h)}^*(\rho(i_h)) \tag{4.9}$$

Φ are coefficients which may depend on $\bar{q}_{\Lambda^c}$ but only if $\underline{i} \cap \Lambda^c \neq \emptyset$, in such a case they only depend on $\bar{q}_{\Lambda^c,\underline{i}} := \bigcup_{i \in \underline{i}: x(i) \in \Lambda^c} \{\bar{q}_{\Lambda^c} \cap C_{x(i)}^{(\ell_{-\gamma})}\}$. The main features of the coefficients Φ (whose dependence on t is not made explicit) is that:

$$\Phi(\underline{i}, k(\underline{i}), \bar{q}_{\Lambda^c,\underline{i}}) = 0 \quad \text{if } \mathrm{diam}(x(i_1), ..., x(i_n)) \ge 2N\gamma^{-1} \tag{4.10}$$

and

$$\sum_{\underline{i}\ni i_0} \sum_{k(\underline{i}):2\le|k(\underline{i})|<N} \Phi(\underline{i}, k(\underline{i}), \bar q_{\Lambda^c,\underline{i}}) \le c, \quad \text{for any } i_0 \tag{4.11}$$

where $c > 0$ is a constant independent of $\bar q_{\Lambda^c}$ and t.

Theorem 4.1. *For any $a_0 < 1$ there are c, N and coefficients Φ as above such that for all γ small enough*

$$H^{\rm eff}_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) = F_\Lambda(\rho_\Lambda|\bar\rho_{\Lambda^c}) + H^{(1)}_\Lambda(\rho_\Lambda) + H^{(2)}_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) + R_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) \tag{4.12}$$

with the remainder $R_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) = R^{(1)} + R^{(2)}$

$$|R^{(i)}| \le c\gamma^\tau,\ i = 1, 2 \tag{4.13}$$

with $\tau = (3 - 5\alpha_- - 2\alpha_+)\frac{d}{2} > 0$ (see (4.18) and (4.30)).

Recall that in this section Λ is a subset of Λ^* thus $|\Lambda| \le c\ell^d_{+,\gamma}$, c a constant, if we wanted larger volumes we would have to increase N, namely to include more body-potentials and longer interaction range, the expansion in Theorem 4.1 being highly non uniform in Λ. The proof which follows closely the one in [13] of a similar result, is given in the remaining subsections.

4.3 Derivation of the LP term

We fix arbitrarily $\rho_\Lambda \in X^{(k)}_\Lambda$, call $n_\Lambda(i) = \ell^d_{-,\gamma}\rho_\Lambda(i)$, introduce a set of labels $\mathcal{L}$ whose elements are denoted by $\xi = (i, \ell)$, where $i = (x, s) \in \Lambda$, $\ell \in \{1, .., n_\Lambda(\xi)\}$; the coordinate functions on $\mathcal{L}$ are $x(\xi)$, $s(\xi)$ and $\ell(\xi)$ respectively equal to the first, second and third entry in ξ. We then define for $\xi \in \Lambda$ (meaning $x(\xi) \in \Lambda$) the probability measures on $\Lambda \times \{1, .., S\}$ as $dp_\xi(r, s) = \mathbf{1}_{r\in C^{(\ell_{-,\gamma})}_{x(\xi)}}\mathbf{1}_s \dfrac{dr}{\ell^d_{-,\gamma}}$ and call $dp_\Lambda = \prod_{\xi\in\mathcal{L}} dp_\xi$, remembering that this measure as well as the index set $\mathcal{L}$ depend on the initial choice of ρ_Λ, as this is momentarily fixed we are not making it explicit. We obviously have:

$$e^{-\beta\ell^d_{-,\gamma}H^{\rm eff}_\Lambda(\rho_\Lambda|\bar q)} = \Big(\prod_{i\in\Lambda}\frac{\ell^{dn_\Lambda(i)}_{-,\gamma}}{n_\Lambda(i)!}\Big)\int e^{-\beta H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})}dp_\Lambda \tag{4.14}$$

where q_Λ on the r.h.s. should be thought of as a ξ-labeled configuration of particles (the label specifying also the cube where the particle is) which is identified to the integration variable relative to the measure dp_Λ: thus the dependence on ρ_Λ is hidden in the structure of the probability dp_Λ. The bracket on the r.h.s. is equal to

$$\prod_{i\in\Lambda}\frac{\ell^{dn_\Lambda(i)}_{-,\gamma}}{n_\Lambda(i)!} = e^{\ell^d_{-,\gamma}\big(1_\Lambda, S(\rho_\Lambda)\big)}, \quad S(\rho_\Lambda)(i) = \ell^{-d}_{-,\gamma}\big(n\log\ell^d_{-,\gamma} - \log n!\big),\ \ n = n_\Lambda(i) = \ell^d_{-,\gamma}\rho_\Lambda(i) \tag{4.15}$$

Then, recalling the Stirling formula:

$$n! = n^{n+1/2} e^{-n} \sqrt{2\pi} \Big(1 + 0 \left(\frac{1}{\sqrt{n}} \right) \Big) \tag{4.16}$$

we can estimate $\big(1_\Lambda, S(\rho_\Lambda)\big)$ as follows

$$\big(1_\Lambda, S(\rho_\Lambda)\big) = \big(1_\Lambda, S^{\rm app}(\rho_\Lambda)\big) - \beta H^{(1,0)} - \beta R^{(1)} \tag{4.17}$$

where $S^{\rm app}(\rho) = -\rho(\log\rho - 1)$ and $H^{(1,0)}$ is equal to the r.h.s. of (4.7) with $t = 0$.

Proof of (4.13) **for** $R^{(1)}$**.**
We now show that $R^{(1)}$ defined in (4.17) above satisfies the bound (4.13):

$$\begin{aligned} \ell^d_{-,\gamma} \left[S^{\rm app}(\rho_\Lambda)(i) - S(\rho_\Lambda)(i) \right] &= -n_\Lambda(i) \left(\log n_\Lambda(i) - 1 \right) + \log n_\Lambda(i)! \\ &= \frac{1}{2} \log n_\Lambda(i) + \log\sqrt{2\pi} + 0 \left(\frac{1}{\sqrt{n_\Lambda(i)}} \right), \end{aligned}$$

$$\big(1_\Lambda, S^{\rm app}(\rho_\Lambda)\big) - \big(1_\Lambda, S(\rho_\Lambda)\big) = \beta H^{(1,0)} + \sum_{i\in\Lambda} 0 \left(\ell^{-d/2}_{-,\gamma} \right)$$

where we used the fact that $n_\Lambda(i) \geq c\ell^d_{-,\gamma}$, since $\rho_\Lambda \in X^{(k)}_\Lambda$. From this, we get

$$\begin{aligned} |\beta R^{(1)}| &\leq \#\{i \in \Lambda\} \cdot \ell^{-3d/2}_{-,\gamma} \\ &\leq S N_\Lambda \left(\frac{\ell_{+,\gamma}}{\ell_{-,\gamma}} \right)^d \ell^{-3d/2}_{-,\gamma} \quad \Box \end{aligned} \tag{4.18}$$

Call $\bar H_\Lambda(q_\Lambda|\bar q_{\Lambda^c})$ the energy $H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})$ defined with J_γ replaced by $J^{(\ell_{-,\gamma})}_\gamma$, then $\bar H_\Lambda(q_\Lambda|\bar q_{\Lambda^c})$ depends only on the densities $\rho^{(\ell_{-,\gamma})}(q_\Lambda; i)$ and $\rho^{(\ell_{-,\gamma})}(\bar q_{\Lambda^c}; i)$ which in (4.14) are fixed equal to $\rho_\Lambda(i)$ and $\bar\rho_{\Lambda^c}(i)$, hence

$$\bar H_\Lambda(q_\Lambda|\bar q_{\Lambda^c}) = \ell^d_{-,\gamma} \Big\{ t \Big(\frac{1}{2} \big(\rho_\Lambda, \bar V_\gamma \rho_\Lambda\big) + \big(\rho_\Lambda, \bar V_\gamma \bar\rho_{\Lambda^c}\big) - \lambda(1_\Lambda, \rho_\Lambda) \Big) + (1-t)(1_\Lambda[\rho^{(k)} - \lambda_\beta], \rho_\Lambda) \Big\} \tag{4.19}$$

Collecting all the above terms we thus identify in (4.14)

$$e^{-\beta\ell^d_{-,\gamma}\{H^{(2)}_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) + R^{(2)}\}} = \int e^{-\beta\{H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c}) - \bar H_\Lambda(q_\Lambda|\bar q_{\Lambda^c})\}} dp_\Lambda \tag{4.20}$$

4.4 Cluster expansion

To estimate the r.h.s. of (4.20) we use cluster expansion. Call $\mathcal{E}$ a set of unordered pairs (ξ, ξ'), $\xi \neq \xi'$, then $\mathcal{E}$ defines a graph structure $(\mathcal{L}, \mathcal{E})$ with vertices $\xi \in \mathcal{L}$ and edges $(\xi, \xi') \in \mathcal{E}$. We call diagrams the connected sets θ in $(\mathcal{L},\mathcal{E})$, $\underline{\theta}=(\theta_1,..,\theta_n)$ their collection. Call Θ and $\Theta_{\rm dsc}$ the spaces of all possible diagrams and of all possible $\underline{\theta}$ which appear when varying $\mathcal{E}$. Let

$$w(\theta)=\int\Big(\prod_{(\xi,\xi')\in\theta,s(\xi)\neq s(\xi')}\{e^{-\beta t\{V_\gamma(x(\xi),x(\xi'))-\ell_{-,\gamma}^{-d}\bar V_\gamma(x(\xi),x(\xi'))\}}-1\}\Big)dp_\Lambda \tag{4.21}$$

then, since dp_Λ is a product measure,

$$\int e^{-\beta\{H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})-\bar H_\Lambda(q_\Lambda|\bar q_{\Lambda^c})\}}dp_\Lambda=\sum_{\underline{\theta}\in\Theta_{\rm dsc}}\prod_{\theta\in\underline{\theta}}w(\theta) \tag{4.22}$$

(4.22) is derived from (4.20) by writing

$$e^{-\beta\{H_{\Lambda,t}(q_\Lambda|\bar q_{\Lambda^c})-\bar H_\Lambda(q_\Lambda|\bar q_{\Lambda^c})\}}=\prod_{(\xi,\xi'):s(\xi)\neq s(\xi')}\{e^{-\beta t\{V_\gamma(r_\xi,r_{\xi'})-\ell_{-,\gamma}^{-d}\bar V_\gamma(x(\xi),x(\xi'))\}}-1+1\}$$

where the labels ξ include both the particles in Λ and those of $\bar q_{\Lambda^c}$ outside Λ. After expanding the product we then get (4.22), details are omitted.
The basic condition for cluster expansion which we have in the present context, involves the elementary diagrams namely $\theta=(\xi,\xi')$ and states that given any $a>0$

$$\sum_{\xi'}|w\big((\xi,\xi')\big)|\gamma^{-\alpha_-+a}<1,\quad\text{for any }\gamma\text{ small enough} \tag{4.23}$$

(4.23) is proved by observing that the densities $\rho_\Lambda(i)$ are bounded and that (4.21) yields for $\theta=(\xi,\xi')$

$$|w\big((\xi,\xi')\big)|\le c\gamma^d(\gamma\ell_{-,\gamma})\mathbf{1}_{{\rm dist}(C_{x(\xi)}^{(\ell_{-,\gamma})},C_{x(\xi')}^{(\ell_{-,\gamma})})\le\gamma^{-1}} \tag{4.24}$$

"Cluster expansion" then applies for any γ small enough and the following holds (for any $\rho_\Lambda\in X_\Lambda^{(k)}$).

Notation. We give Θ a graph structure by calling vertices the diagrams $\theta\in\Theta$ and edges the pairs θ and θ' which have non empty intersection, as sets in $\mathcal{L}$.
Denote by $m(\theta)$, $\theta\in\Theta$, positive, integer valued functions, calling $m(\theta)$ "the multiplicity" of θ. We restrict to $m\in\mathcal{M}$ where

$$m\in\mathcal{M}\text{ if and only if }{\rm sp}(m):=\{\xi:\xi\in\theta,m(\theta)>0\}\text{ is a connected set} \tag{4.25}$$

and shorthand $\xi\in m$ when $\xi\in{\rm sp}(m)$.

Cluster expansion tells us that given any $a_0<1$ for all γ small enough there are coefficients $\omega(m)$, $m\in\mathcal{M}$, such that

$$\log Z(\{w(\cdot)\}):=\log\{\sum_{\underline{\theta}\in\Theta_{\rm dsc}}\prod_{\theta\in\underline{\theta}}w(\theta)\}=\sum_{m\in\mathcal{M}}\omega(m) \tag{4.26}$$

and, for any $\xi\in\mathcal{L}$,

$$\sum_{m\in\mathcal{M}:m\ni\xi}|\omega(m)|\{\prod_{\theta:m(\theta)>0}(\gamma\ell_{-,\gamma})^{a_0|\theta|_{\rm edg}m(\theta)|}\}<1 \tag{4.27}$$

where $|\theta|_{\rm edg}$ is the number of edges in θ. The coefficients $\omega(m)$ have the following explicit expression:

$$\omega(m) = C_m \prod_{\theta : m(\theta)>0} w(\theta) \tag{4.28}$$

where thinking of $Z(\{w(\cdot)\})$ in (4.26) as a function of the weights $\{w(\theta), \theta \in \Theta\}$,

$$C_m = \prod_{\theta:m(\theta)>0} \frac{1}{m(\theta)!} \{ \prod_{\theta:m(\theta)>0} \frac{\partial^{m(\theta)}}{\partial w(\theta)^{m(\theta)}} \} \ \log Z_\Lambda(w(\cdot)) \Big|_{w(\theta)=0} \tag{4.29}$$

(C_m being bounded coefficients independent of Λ). As said, all the above follows from the general theory (of cluster expansion) using the condition (4.23), see for instance [16].

4.5 Identification of the many body potential

We will next use (4.27) to truncate the sum in (4.26) identifying the remainder with the term $R^{(2)}$ and recognizing in the finite sum the Hamiltonian $H^{(2)}_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c})$, for this we will use the explicit representation of the terms of the expansion provided by (4.28)–(4.29).
Calling $|m| = \sum_{\theta\in\Theta} |\theta|_{\rm edg} m(\theta)$, by (4.27), for any $N > 0$,

$$\sum_{m\in\mathcal{M}:|m|\ge N} |\omega(m)| \le \sum_{\xi\in\mathcal{L}} \sum_{m\in\mathcal{M}:m\ni\xi,|m|\ge N} |\omega(m)|$$
$$\le |\mathcal{L}|(\gamma\ell_{-,\gamma})^{a_0 N} \sum_{m\in\mathcal{M}:m\ni\xi} |\omega(m)|\{ \prod_{\theta:m(\theta)>0} (\gamma\ell_{-,\gamma}))^{-a_0|\theta|_{\rm edg} m(\theta)|} \} \le |\mathcal{L}|(\gamma\ell_{-,\gamma})^{a_0 N}$$

Since $\Lambda \subset \Lambda^*$, there is $c > 0$ such that $|\mathcal{L}| \le c\ell^d_{+,\gamma}$ and we can then choose N so large that

$$-\beta\ell^d_- R^{(2)} := \sum_{m\in\mathcal{M}:|m|\ge N} \omega(m), \qquad |\sum_{m\in\mathcal{M}:|m|\ge N} \omega(m)| \le \ell^{-d/2}_{-,\gamma} \tag{4.30}$$

thus (4.13) is satisfied and

$$-\beta\ell^d_{-,\gamma} H^{(2)}(\rho_\Lambda|\bar q_{\Lambda^c}) := \sum_{m\in\mathcal{M}:|m|<N} \omega(m) \tag{4.31}$$

The dependence on ρ_Λ is hidden in the space Θ, on which the functions m are defined. Theorem 4.1 will be proved once we show that the r.h.s. of (4.31) can be written as the r.h.s. of (4.9).
We rewrite the r.h.s. of (4.31) by first summing over all m in "the same equivalence class" and then summing over all equivalence classes. Before defining the equivalence $m \sim m'$ we observe that if ψ is a one to one map of $\mathcal{L}$ onto itself, then ψ extends naturally to a map of Θ onto itself by letting $\psi(\theta)$ be the diagram with vertices $\psi(\xi)$, $\xi \in \theta$, and edges $(\psi(\xi), \psi(\xi'))$, (ξ, ξ') the edges of θ. We then call $m \sim m'$ if there is a one to one map ϕ from $\mathcal{L}$ onto $\mathcal{L}$ such that • $x(\phi(\xi)) = x(\xi)$, $s(\phi(\xi)) = s(\xi)$ for all ξ; • $m'(\phi(\theta)) = m(\theta)$ for all $\theta \in \Theta$.

Calling $[m]$ the equivalence class of m, i.e. the set of all $m' : m' \sim m$, we define the average weight

$$\omega^*(m) := \frac{1}{\mathrm{card}([m])} \sum_{m' \in [m]} \omega(m') \tag{4.32}$$

Notice that if $\mathrm{sp}(m)$ consists only of ξ such that $x(\xi) \in \Lambda$ then $\omega(m) = \omega(m') = \omega^*(m)$ for all $m' \in [m]$. If instead there are labels ξ in $\mathrm{sp}(m)$ such that $x(\xi) \in \Lambda^c$ then $\omega^*(m)$ is a non trivial average. Actually the averages involve the labels ℓ in each triple (x, s, ℓ), $x \in \Lambda^c$, with $m(x, s, \ell) > 0$. Calling $K(i; m)$ the number of $\xi \in m$ such that $i(\xi) = i$,

$$\mathrm{card}([m]) = \prod_i \pi_{K(i;m)}(n(i)) \tag{4.33}$$

where $\pi_k(n)$ is the Poisson polynomial and $n(i) = \rho(i)\ell^d_{-,\gamma}$. We then have

$$-\beta\ell^d_- H^{(2)}(\rho_\Lambda|\bar{q}_{\Lambda^c}) := \sum_{[m],|m|<N} \omega^*(m)\{\prod_i \pi_{K(i;m)}(n(i))\} \tag{4.34}$$

We next interchange the sums: for any sequence $K(i) \in \mathbb{N}_+$, $\sum_i K(i) < N$, let

$$\Psi(K(\cdot)) := \ell^{-d}_{-,\gamma} \sum_{[m],m:K(\cdot;m)=K(\cdot)} \omega^*(m) \prod_i \ell^{dK(i)}_{-,\gamma} \tag{4.35}$$

then

$$-\beta H^{(2)}(\rho_\Lambda|\bar{q}_{\Lambda^c}) := \sum_{K(\cdot)} \Psi(K(\cdot))\{\prod_i \ell^{-dK(i)}_{-,\gamma} \pi_{K(i)}(n(i))\} \tag{4.36}$$

thus identifying Φ in Theorem 4.1 in terms of Ψ:

$$\Psi(K(\cdot)) = (\gamma\ell_{-,\gamma})^{a_0|K(\underline{i})|}\Phi(\underline{i}, K(\underline{i}), \bar{q}_{\Lambda^c,\underline{i}}) \tag{4.37}$$

recalling the remark before (4.33), indeed the l.h.s. depends on $\bar{q}_{\Lambda^c}$ only via $\bar{q}_{\Lambda^c,\underline{i}}$.
Of course we still need to prove that the function Φ defined via (4.37) satisfies the bounds stated in (4.10)–(4.11). Since the coefficients C_m in (4.28), are bounded, say

$$\max_{m:|m|<N} |C_m| \le c_N \tag{4.38}$$

we just need to bound $|w(\theta)|$. The definition of $w(\theta)$ involves product of terms $w\big((\xi, \xi')\big)$ for each edge of the diagram which we bound using (4.24). The bound obtained in this way is the same for all $m' \in [m]$ so that the bound for $\omega^*(m)$ is the same as for $\omega(m)$. To fix up the combinatorics, we proceed as follows. For any m we define a graph structure $G(m)$ on $\mathrm{sp}(m)$ introducing a node for each element ξ of $\mathrm{sp}(m)$ which is then given the label $i = (x(\xi), s(\xi))$, thus different nodes may have the same label. Edges in $G(m)$ are the union of all the edges present in all the diagrams θ such that $m(\theta) > 0$. Each edge is then given a multiplicity equal to the sum of all $m(\theta)$ over the diagrams θ which contain the given edge. With this

definition any $m' \in [m]$ gives rise to the same $G(m)$ as we are only recording the coordinates $x(\xi)$ and $s(\xi)$ of ξ.
To proceed with the bound we assign a "weight" $\ell_{-,\gamma}^d$ to any node in $G(m)$. Having (4.24)in mind, we assign to each edge a weight $\Big(c\gamma^d(\gamma\ell_{-,\gamma})\mathbf{1}_{\mathrm{dist}(C_{x(\xi)}^{(\ell_{-,\gamma})},C_{x(\xi')}^{(\ell_{-,\gamma})})\le\gamma^{-1}}\Big)^p$, where p the multiplicity of the edge. We have thus assigned a weight $W(G(m))$ to $G(m)$ equal to the product of the weights of its nodes and of its edges and, with reference to (4.35) and recalling (4.38)

$$|\Psi(K(\cdot))| \le c_N\ell_{-,\gamma}^{-d} \sum_{[m],m:K(\cdot;m)=K(\cdot)} W(G(m)) \tag{4.39}$$

Recalling that $K(i;m)$ is the number of $\xi \in m$ such that $i(\xi)=i$, $K(i;m)$ is also the number of nodes in $G(m)$ with label i. Thus, calling $K(i,G)$ the number of nodes in G with label i, $\underline{i}=\{i, i\in G\}$, and $K(\underline{i},G)=\{K(i,G), i\in\underline{i}\}$,

$$|\Psi(K(\underline{i}))| \le c_N\ell_{-,\gamma}^{-d} \sum_{G:K(\underline{i};G)=K(\underline{i})} W(G) \tag{4.40}$$

(4.37) then yields

$$|\Phi(\underline{i},K(\underline{i}),\bar{q}_{\Lambda^c,\underline{i}})| \le c_N\ell_{-,\gamma}^{-d}(\gamma\ell_{-,\gamma})^{-a_0|K(\underline{i})|} \sum_{G:K(i;G)=K(i)} W(G) \tag{4.41}$$

By (4.35) the terms to consider have $\underline{i}$ such that $\sum_{i\in\underline{i}} K(i) < N$. Then $\Phi(\underline{i},K(\underline{i}),\bar{q}_{\Lambda^c,\underline{i}})=0$ if $\mathrm{diam}(\underline{x}) \ge 2\gamma^{-1}N$, $\underline{x}$ being the sites appearing in $\underline{i}$, because the weight of the edges in G are proportional to $\mathbf{1}_{\mathrm{dist}(C_{x(\xi)}^{(\ell_{-,\gamma})},C_{x(\xi')}^{(\ell_{-,\gamma})})\le\gamma^{-1}}$.
To prove (4.11) we fix i_0 and restrict the sum in (4.41) to $G: K(i_0;G)>0$. For each such G we can then define a tree structure $T_{i_0}(m)$ in $G(m)$ with root i_0, a first generation made by all nodes connected to the root, second generation made by the nodes connected to those of the first generation and so forth. To recover the original graph we may also have to add edges connecting individuals of the same generation and also attribute to each edge its multiplicity, as explained earlier. We then have

$$\text{l.h.s. of (4.11)} \;\le\; \sum_{\underline{i}\ni i_0} \sum_{K(\underline{i}):|K(\underline{i})|<N} \ell_{-.\gamma}^{-d}(\gamma\ell_{-,\gamma})^{-a_0|K(\underline{i})|} \sum_{T_{i_0}:K(\underline{i};T_{i_0})=K(\underline{i})} W(T_{i_0}) \tag{4.42}$$

Define a new weight $W^*(T)$ by changing the weights of the edges into

$$\Big(c(\gamma\ell_{-,\gamma})^{1-a_0}\gamma^d\mathbf{1}_{|x-x'|\le 2\gamma^{-1}}\Big)^p, \quad p \text{ the multiplicity of the edge}$$

while the weights of the node are unchanged. Then

$$\text{l.h.s. of (4.11)} \;\le\; \ell_{-,\gamma}^{-d} \sum_{\underline{i}\ni i_0} \sum_{K(\underline{i}):|K(\underline{i})|<N} \sum_{T_{i_0}:K(\underline{i};T_{i_0})=K(\underline{i})} W^*(T_{i_0}) \tag{4.43}$$

The weight of the root of the tree cancels with the prefactor $\ell_{-,\gamma}^{-d}$. We upper bound the sum on the r.h.s. if we regard a multiple edge with multiplicity k as k distinct edges originating from a same node and also regard edges between nodes in the same generation as edges into the next generation (thus dropping the constraint that the arrival node is the same as the arrival node of another edge), each node added in this way getting an extra weight $\ell_{-,\gamma}^{d}$. In this way we have an independent branching and since

$$\lim_{\gamma\to 0}\sum_{x'}(\gamma\ell_{-,\gamma})^{a_0}\gamma^d \mathbf{1}_{|x'|\le 2\gamma^{-1}}\ell_-^d = 0$$

we then get (4.11), details are omitted. Theorem 4.1 is proved. □

5 Ground states of the effective Hamiltonian

In this section we study the ground states of the main term in the effective Hamiltonian $H_\Lambda^{\rm eff}(\rho_\Lambda|\bar q_{\Lambda^c})$, which, with reference to (4.12), is

$$f(\rho_\Lambda;\bar q_{\Lambda^c}) := H_\Lambda^{\rm eff}(\rho_\Lambda|\bar q_{\Lambda^c}) - R_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) \tag{5.1}$$

While originally $\rho_\Lambda = \big(\rho_\Lambda(i), i=(x,s), x\in \ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda, s\in\{1,..,S\}\big)\in X_\Lambda^{(k)}$ defined in Subsection 4.1, it is convenient here to extend the range of values of $\rho_\Lambda(i)$ to an interval of the real line. We thus call

$$Y_\Lambda^{(k)} = \Big\{\rho_\Lambda : \rho_\Lambda(x,s)\in[\rho_s^{(k)}-\zeta,\rho_s^{(k)}+\zeta], \forall x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda, \forall s\in\{1,..,S\}\Big\}$$

The ground states in the title are then the minimizers of $f(\rho_\Lambda;\bar q_{\Lambda^c})$ as a function on $Y_\Lambda^{(k)}$ with $\bar q_{\Lambda^c}$ regarded as a parameter.

Let $\hat K_\Lambda(x)\equiv \hat K_\Lambda(\bar q'_{\Lambda^c},\bar q''_{\Lambda^c};x)$ be the function defined as $K_\Lambda(x)$ in Definition 3.2 but with the set A_x in (3.23) replaced with the set

$$\hat A_x = B_x(10^{-30}\ell_{+,\gamma})\cap\Lambda^c \tag{5.2}$$

Our main result is the following theorem:

Theorem 5.1. *There are* c^* *and* $\hat\omega$ *positive such that for any* $a_0<1$ *and for all* γ *small enough the following holds. For any* $\bar q_{\Lambda^c}\in\mathcal{X}_{\Lambda^c}^{(k)}$ *there is a unique minimizer* $\hat\rho_\Lambda$ *of* $\{f(\rho_\Lambda;\bar q_{\Lambda^c}),\rho_\Lambda\in Y_\Lambda^{(k)}\}$. *Let* $\hat K(x)$ $x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda$, *be as above and* $\hat\rho'_\Lambda$ *and* $\hat\rho''_\Lambda$ *the minimizers with* $\bar q'_{\Lambda^c}$ *and* $\bar q''_{\Lambda^c}$, *then for any* $s\in\{1,..,S\}$:

- *(i) If* $\hat K(x)>0$, $|\hat\rho'_\Lambda(x,s)-\hat\rho''_\Lambda(x,s)|\le ce^{-10^{-30}(\gamma\ell_{+,\gamma})\hat\omega}$.

- *(ii) If $\hat{K}(x) = m > 0$, $|\hat\rho'_\Lambda(x,s) - \rho_s^{(k)}| \le c^*(\zeta_m + (\gamma\ell_{-,\gamma})^{a_0} + e^{-10^{-30}(\gamma\ell_{+,\gamma})\hat\omega})$, with same bound for $\hat\rho''_\Lambda(x,s)$.*

Existence of a minimizer follows from f being a smooth function on a compact set of the Euclidean space. Uniqueness and exponential decay are more difficult and the proof will take the whole section. The basic ingredient is that $D^2 f$ (the Hessian matrix of the derivatives w.r.t. the variables $\rho_\Lambda(i)$) computed on the minimizer in the constraint space $Y_\Lambda^{(k)}$ is positive and "quasi diagonal", which would then give the required uniqueness and exponential decay if we had $Df = 0$. This is however not necessarily the case because the minimum could be reached on the boundaries of the domain of definition, which, on the other hand, is necessary to ensure convexity. We will solve the problem by relaxing the constraint and then studying the limit when the cutoff is reconstructed.

5.1 Extra notation and definitions

The basic notation are those established in Subsection 4.1, here we add a few new ones specific to this section:

- We will write $f(\rho_\Lambda;\bar q_{\Lambda^c}) = F(\rho_\Lambda;\bar\rho_{\Lambda^c}) + g(\rho_\Lambda;\bar q_{\Lambda^c})$ where, recalling (4.12),

$$g(\rho_\Lambda;\bar q_{\Lambda^c}) = H_\Lambda^{(1)}(\rho_\Lambda) + H_\Lambda^{(2)}(\rho_\Lambda|\bar q_{\Lambda^c}) \tag{5.3}$$

- To evidentiate some of the variables in ρ_Λ, say those in $\Delta \subset \Lambda$, we write $\rho_\Lambda = (\rho_\Delta, \rho_{\Lambda\setminus\Delta})$, where ρ_Δ and $\rho_{\Lambda\setminus\Delta}$ are the restrictions of ρ_Λ to Δ and respectively to $\Lambda\setminus\Delta$.
- It will be convenient to relax the constraint $\rho_\Lambda \in Y_\Lambda^{(k)}$ by enlarging $Y_\Lambda^{(k)}$ into $W_\Lambda^{(k)}$

$$W_\Lambda^{(k)} = \Big\{\rho_\Lambda : \rho_\Lambda(x,s) \in [\rho_s^{(k)} - b, \rho_s^{(k)} + b], \forall x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Lambda, \forall s \in \{1,..,S\}\Big\} \tag{5.4}$$

where $b := \min\limits_{k_1\neq k_2} \dfrac{\|\rho^{(k_1)} - \rho^{(k_2)}\|_\infty}{2}$ has been chosen such that

$$W_\Lambda^{(k)} \cap \{\rho^{(1)},..,\rho^{(S+1)}\} = \{\rho^{(k)}\}$$

We then introduce a cutoff parameter $\epsilon \in (0,1)$ (which will eventually vanish), call $(a)_+ = a\,\mathbf{1}_{a>0}$, $(a)_- = a\,\mathbf{1}_{a<0}$ and define for any $\epsilon > 0$, the function f_ϵ on $W_\Lambda^{(k)}$ as

$$\begin{aligned} f_\epsilon(\rho_\Lambda;\bar q_{\Lambda^c}) := f(\rho_\Lambda;\bar q_{\Lambda^c}) + \frac{\epsilon^{-1}}{4}\sum_{i\in\Lambda}\Big(&\{(\rho_\Lambda(i) - [\rho_{s(i)}^{(k)} + \zeta])_+\}^4 \\ &+\{(\rho_\Lambda(i) - [\rho_{s(i)}^{(k)} - \zeta])_-\}^4\Big) \end{aligned} \tag{5.5}$$

• Since f $[f_\epsilon]$ is a continuous function of ρ_Λ which varies on a compact set, it has a minimizer denoted by $\hat\rho_\Lambda$ $[\hat\rho_{\Lambda,\epsilon}]$, and we will later see that this minimizer is unique. We call $\hat\rho$ its extension to the whole $\ell_{-,\gamma}\mathbb{Z}^d \times \{1,..,S\}$, by setting $\hat\rho = \bar\rho_{\Lambda^c}$ on Λ^c. Here $\bar\rho_{\Lambda^c}$ is the density associated to $\bar q_{\Lambda^c}$ via (3.7) with $\ell = \ell_{-,\gamma}$, thus $\hat\rho$ of course depends on $\bar q_{\Lambda^c}$.
• For any $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable set B we write for any differentiable and $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable function $\psi(\rho)$

$$D_B\psi \;=\; \Big\{ \frac{\partial\psi}{\partial\rho(i)},\; x(i)\in \ell_{-,\gamma}\mathbb{Z}^d\cap B\Big\} \tag{5.6}$$

5.2 A-priori estimates

In this subsection we prove some a-priori bounds on $\hat\rho_{\Lambda,\epsilon}(i)$. When $\epsilon>0$ we loose the bound $|\hat\rho_\Lambda(i)-\rho^{(k)}_{s(i)}|\le\zeta$ valid at $\epsilon=0$ but, as we will see, we have the great simplification to know that for ϵ small enough, minimizers are critical points, thus satisfying $D_\Lambda f_\epsilon = 0$, and $|\hat\rho_{\Lambda,\epsilon}(i)-\rho^{(k)}_{s(i)}|\le 2\zeta$.

Lemma 5.2. *There is a constant $c>0$ such that for all $\epsilon>0$ and for any minimizer $\hat\rho_{\Lambda,\epsilon}\in W^{(k)}_\Lambda$ of f_ϵ the following holds: for all $x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda$ and all $s\in\{1,..,S\}$,*

$$\big|\hat\rho_{\Lambda,\epsilon}(x,s)-\rho^{(k)}_s\big|\le\zeta+c(\frac{\ell_{+,\gamma}}{\ell_{-,\gamma}})^{d/4}\epsilon^{1/4} \tag{5.7}$$

In particular, if $\zeta<b/2$ then for all $\epsilon>0$ small enough, any minimizer $\hat\rho_{\Lambda,\epsilon}\in W^{(k)}_\Lambda$ of f_ϵ is also a critical point.

Proof. We denote by

$$\psi(\rho_\Lambda)=\sum_{i\in\Lambda}\{(\hat\rho_{\Lambda,\epsilon}(i)-[\rho^{(k)}_{s(i)}+\zeta])_+\}^4+\{(\hat\rho_{\Lambda,\epsilon}(i)-[\rho^{(k)}_{s(i)}-\zeta])_-\}^4$$

Then for all $\rho_\Lambda\in W^{(k)}_\Lambda$,

$$\frac{1}{4\epsilon}\psi(\hat\rho_{\Lambda,\epsilon})\le f(\rho_\Lambda;\bar q_{\Lambda^c})-f(\hat\rho_{\Lambda,\epsilon};\bar q_{\Lambda^c})+\frac{1}{4\epsilon}\psi(\rho_\Lambda)$$

and since Ψ vanishes on $Y^{(k)}_\Lambda$:

$$\frac{1}{4\epsilon}\psi(\hat\rho_{\Lambda,\epsilon})\le\inf_{\rho_\Lambda\in Y^{(k)}_\Lambda}f(\rho_\Lambda;\bar q_{\Lambda^c})-f(\hat\rho_{\Lambda,\epsilon};\bar q_{\Lambda^c})$$

and, calling $\phi'=\min\limits_{\rho_\Lambda\in Y^{(k)}_\Lambda}f(\rho_\Lambda;\bar q_{\Lambda^c})$, $\phi''=\min\limits_{\rho_\Lambda\in W^{(k)}_\Lambda}f(\rho_\Lambda;\bar q_{\Lambda^c})$

$$\frac{1}{4\epsilon}\psi(\hat\rho_{\Lambda,\epsilon})\le\phi'-\phi''$$

and in conclusion

$$|\hat\rho_{\Lambda,\epsilon}(x,s)-\rho^{(k)}_s|\le \Big(4\epsilon(\phi'-\phi'')\Big)^{1/4}+\zeta \tag{5.8}$$

and (5.7) follows because ϕ' and ϕ'' are bounded proportionally to the cardinality of $\{x: x\in \ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda\}$.
By choosing ϵ so small that $\zeta+c\gamma^{-(\alpha_++\alpha_-)d/4}\epsilon^{1/4}<2\zeta<b$, we conclude that $\hat\rho_{\Lambda,\epsilon}$ is in the interior of $W^{(k)}_\Lambda$ and is thus a critical point. □

Lemma 5.3. *$\hat\rho_{\Lambda,\epsilon}$ converges by subsequences and any limit point $\hat\rho_\Lambda$ is a minimizer of f.*

Proof. Convergence by subsequences follows from compactness and by (5.7) any limit point $\hat\rho_\Lambda$ is in $Y^{(k)}_\Lambda$. Now for any $\rho_\Lambda\in Y^{(k)}$, we get $f(\rho_\Lambda)=f_\epsilon(\rho_\Lambda)\ge f_\epsilon(\hat\rho_{\Lambda,\epsilon})\ge f(\hat\rho_{\Lambda,\epsilon})$ and by taking $\epsilon\to 0$ along a convergent subsequence $f(\rho_\Lambda)\ge f(\hat\rho_\Lambda)$. □

A minimizer $\hat\rho_\Lambda$ of f is not necessarily a critical point, i.e. $D_\Lambda f=0$, the equality may fail if the minimizer is on the boundary of the constraint. In such a case however, the gradient if different from zero "must be directed along the normal pointing toward the interior".

Lemma 5.4. *Any minimizer $\hat\rho_\Lambda$ of $\{f(\rho_\Lambda,\bar q_{\Lambda^c}),\ \rho_\Lambda\in Y^{(k)}_\Lambda\}$ is "a critical point" in the following sense:*

- *If for some $i\in\Lambda$, $|\hat\rho_\Lambda(i)-\rho^{(k)}_{s(i)}|<\zeta$ (strictly!), then*

$$\frac{\partial}{\partial\rho_\Lambda(i)}f(\hat\rho_\Lambda,\bar q_{\Lambda^c})=0 \tag{5.9}$$

- *If instead $\hat\rho_\Lambda(i)=\rho^{(k)}_{s(i)}\pm\zeta$, then*

$$\frac{\partial}{\partial\rho_\Lambda(i)}f(\hat\rho_\Lambda,\bar q_{\Lambda^c})\le 0,\quad \textit{respectively} \ge 0 \tag{5.10}$$

5.3 Convexity and uniqueness

Convexity is a key ingredient in our analysis:

Theorem 5.5. *Given any $\kappa\in(0,\kappa^*)$ (κ^* as in (2.6)), for all γ small enough the following holds. Let $\rho_\Lambda\in W^{(k)}_\Lambda$ be such that $\big|\rho_\Lambda(i)-\rho^{(k)}_{s(i)}\big|\le 4\zeta$, then the matrix $A:=D^2_\Lambda f_\epsilon(\rho_\Lambda,\bar q_{\Lambda^c})$ is strictly positive, as an operator on $\mathcal{H}$, namely (recall the definitions in Subsection 4.1)*

$$\big(u,Au\big)\ge\kappa(u,u),\quad \textit{for all } u\in\mathcal{H} \tag{5.11}$$

Same inequality holds when $\epsilon = 0$.

Proof. Recalling (5.3) and denoting by ρ_Λ^{-1} below the diagonal matrix with entries $\rho_\Lambda(i)^{-1}$

$$(u, Au) = t(u, \bar{V}_\gamma u) + \frac{1}{\beta}(u, \rho_\Lambda^{-1} u) + (u, [D^2_\Lambda g]u) + (u, [D^2_\Lambda (f_\epsilon - f)]u)$$

and get a lower bound by dropping the last term thus reducing the proof to the case $\epsilon = 0$. Extend u and A as equal to 0 outside Λ and set

$$U(x,s) = \ell^d_{-,\gamma} \sum_{y \in \ell_{-,\gamma}\mathbb{Z}^d} J_\gamma^{(\ell_{-,\gamma})}(x,y) u(y,s), \quad x \in \ell_{-,\gamma}\mathbb{Z}^d$$

where $J_\gamma^{(\ell)}$ is defined in (4.1). Then,

$$\begin{aligned}
(u, Au) &\geq t \sum_{s \neq s'} \sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d} U(x,s)U(x,s') + \frac{1}{\beta}(u, \rho_\Lambda^{-1} u) + (u, [D^2_\Lambda g]u) \\
&= \Big\{ t \sum_{s \neq s'} \sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d} U(x,s)U(x,s') + \sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d, s} [\frac{1}{\beta \rho^{(k)}(s)} - \kappa^*] U(x,s)^2 \Big\} \\
&\quad - \sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d, s} [\frac{1}{\beta \rho^{(k)}(s)} - \kappa^*] U(x,s)^2 + \frac{1}{\beta}(u, \rho_\Lambda^{-1} u) + (u, [D^2_\Lambda g]u)
\end{aligned}$$

recalling(2.8), by (2.6) the curly bracket is non negative as well as $\dfrac{1}{\beta \rho_s^{(k)}} - \kappa^*$.
Since for each s

$$\sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d} U(x,s)^2 \leq \sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d} u(x,s)^2$$

then

$$\sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d, s} [\frac{1}{\beta \rho_s^{(k)}} - \kappa^*] U(x,s)^2 \leq (u, [\frac{1}{\beta \rho^{(k)}} - \kappa^*] u)$$

Thus

$$(u, Au) \geq \Big(u, [\kappa^* + \frac{1}{\beta \rho_\Lambda} - \frac{1}{\beta \rho^{(k)}}] u \Big) + \big(u, [D^2_\Lambda g]u\big)$$

Recalling (A.2), (A.3) and using (4.9)–(4.11) we get

$$\|D^2_\Lambda g\| \leq \sup_i \sum_j |\frac{\partial^2 g}{\partial \rho_\Lambda(i) \partial \rho_\Lambda(j)}| \leq (\gamma \ell_{-,\gamma})^{a_0}$$

Thus

$$\big(u, [D^2_\Lambda g]u\big) \leq [\gamma \ell_{-,\gamma}]^{2a_0} (u,u)$$

(5.11) is then proved recalling the assumption $|\rho_\Lambda(i) - \rho^{(k)}_{s(i)}| \leq 4\zeta$.

□

Theorem 5.6. *Given any $\kappa \in (0,\kappa^*)$ (κ^* as in (2.6)), for all γ small enough the following holds. Let $\hat\rho_{\Lambda,\epsilon}$ be a minimizer of f_ϵ and for $\epsilon=0$ of f, then for both $\epsilon>0$ small enough and $\epsilon=0$*

$$f_\epsilon(\rho_\Lambda,\bar q_{\Lambda^c}) \ge f_\epsilon(\hat\rho_{\Lambda,\epsilon},\bar q_{\Lambda^c}) + \frac{\kappa}{2}\big(\rho_\Lambda-\hat\rho_{\Lambda,\epsilon},\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\big) \tag{5.12}$$

for all ρ_Λ such that $|\rho_\Lambda(i)-\rho^{(k)}_{s(i)}|\le 2\zeta$ for all $i\in\Lambda$. (5.12) remains valid if $\hat\rho_{\Lambda,\epsilon}$ is a critical point, $D_\Lambda f_\epsilon=0$, and $|\hat\rho_{\Lambda,\epsilon}-\rho^{(k)}\big|\le 2\zeta$ as well as when $\epsilon=0$ and $\hat\rho_{\Lambda,0}$ a "critical point" of f in the sense of Lemma 5.4.

Proof. We interpolate by setting $\rho_\Lambda(\theta)=\theta\rho_\Lambda+(1-\theta)\hat\rho_{\Lambda,\epsilon}$, $\theta\in[0,1]$, then calling $\psi_\epsilon(\theta):=f_\epsilon(\rho_\Lambda(\theta),\bar q_{\Lambda^c})$ we have

$$\begin{aligned}\psi_\epsilon(1)-\psi_\epsilon(0) &= \int_0^1 \big(D_\Lambda\psi_\epsilon(\theta),\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\big)\\ &= \int_0^1\int_0^\theta \big(D^2_\Lambda\psi_\epsilon(\theta')\{\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\},\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\big)+\big(D_\Lambda\psi_\epsilon(0),\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\big)\end{aligned}$$

By (5.7) for $\epsilon>0$ small enough and for $\epsilon=0$ as well, $|\rho_\Lambda(\theta)-\rho^{(k)}\big|\le 4\zeta$ so that by (5.11)

$$\int_0^1\int_0^\theta \big(D^2_\Lambda\psi_\epsilon(\theta')\{\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\},\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\big)\ge\frac{\kappa}{2}\big(\rho_\Lambda-\hat\rho_{\Lambda,\epsilon},\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\big)$$

Moreover $\big(D_\Lambda\psi_\epsilon(0),\rho_\Lambda-\hat\rho_{\Lambda,\epsilon}\big)\ge 0$. In fact, if $\epsilon>0$ and $\hat\rho_{\Lambda,\epsilon}$ is a minimizer of f_ϵ, by Lemma 5.2 (for $\epsilon>0$ small enough) $\hat\rho_{\Lambda,\epsilon}$ is also a critical point and $D_\Lambda\psi_\epsilon(0)=0$. If $\epsilon=0$ and $\hat\rho_\Lambda$ a minimizer of f then by Lemma 5.4, $\big(D_\Lambda\psi_0(0),\rho_\Lambda-\hat\rho_\Lambda\big)\ge 0$ which, for the same reason, holds if $\hat\rho_\Lambda$ is a critical point of f in the sense of Lemma 5.4.

□

Corollary 5.7. *For any γ and $\epsilon>0$ small enough the minimizer of f_ϵ is unique, same holds at $\epsilon=0$ for f. For $\epsilon>0$ (and small enough) there is a unique critical point in the space $\{|\rho_\Lambda-\rho^{(k)}\big|\le 2\zeta\}$; such a critical point minimizes f_ϵ. Analogously, when $\epsilon=0$ there is a unique critical point in the sense of Lemma 5.4. Such a critical point minimizes f. The minimizer of f_ϵ, $\epsilon>0$, converges as $\epsilon\to0$ to the minimizer of f.*

Proof. From Lemma 5.2 it follows that any minimizer $\hat\rho_{\Lambda,\epsilon}$ of f_ϵ is also a critical point and verifies (5.7), so for all $x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda$ and all $s\in\{1,..,S\}$, $\big|\hat\rho_{\Lambda,\epsilon}(x,s)-\rho^{(k)}_s\big|\le 2\zeta$ and we can apply Theorem 5.6 to the matrix $D^2_\Lambda f_\epsilon(\hat\rho_{\Lambda,\epsilon};\bar q_{\Lambda^c})$. If we assume that there are two minimizers, then (5.12) gives a contradiction. The proofs in the case $\epsilon=0$ follows by using Lemma 5.3. □

5.4 Perfect boundary conditions

In this subsection we restrict to "perfect boundary conditions", by this meaning that we study

$$f^{\rm pf}(\rho_\Lambda;\bar q_{\Lambda^c}) = F_\Lambda(\rho_\Lambda|\rho^{(k)}1_{\Lambda^c}) + g(\rho_\Lambda;\bar q_{\Lambda^c}) \tag{5.13}$$

namely we replace in the LP term of the effective Hamiltonian, see (4.12), $\bar\rho_{\Lambda^c}$ by the mean field equilibrium value. $f_\epsilon^{\rm pf}$ is then defined by adding to $f^{\rm pf}$ the term $f_\epsilon - f$ given by (5.5). All the previous considerations obviously apply to $f^{\rm pf}$ and $f_\epsilon^{\rm pf}$.

Theorem 5.8. *For any γ small enough and for all $\epsilon > 0$ small enough, the minimizer $\hat\rho^{\rm pf}_{\Lambda,\epsilon}$ of $f_\epsilon^{\rm pf}$ minimizes $f^{\rm pf}$ as well and it is such that*

$$|\hat\rho^{\rm pf}_{\Lambda,\epsilon}(i) - \rho^{(k)}_{s(i)}| \le c(\gamma\ell_{-,\gamma})^{a_0}, \qquad \text{for all } i\in\Lambda \tag{5.14}$$

$c > 0$ a constant.

Proof. Since $\hat\rho^{\rm pf}_{\Lambda,\epsilon}$ is a minimizer of $f_\epsilon^{\rm pf}$, $D_\Lambda f_\epsilon^{\rm pf}(\hat\rho^{\rm pf}_{\Lambda,\epsilon}) = 0$. Then if (5.14) holds, $D_\Lambda f^{\rm pf}(\hat\rho^{\rm pf}_{\Lambda,\epsilon}) = D_\Lambda f_\epsilon^{\rm pf}(\hat\rho^{\rm pf}_{\Lambda,\epsilon}) = 0$ and by Corollary 5.7 $\hat\rho^{\rm pf}_{\Lambda,\epsilon}$ is a minimizer of $f^{\rm pf}$. We thus have only to prove (5.14) for all $\epsilon > 0$ small enough. Consider first the simplified problem with $g = 0$.

Case $g = 0$

Recalling (4.3), if $D_\Lambda F_\Lambda(\rho_\Lambda|\rho^{(k)}1_{\Lambda^c}) = 0$, then by an explicit computation, for all $i\in\Lambda$,

$$\rho_\Lambda(i) = \exp\Big\{ -\beta\big[\sum_{j\in\ell_{-,\gamma}\mathbb{Z}^d} t\bar V_\gamma(i,j)\rho(j) + (1-t)\rho^{(k)}_{s(i)} - \lambda_\beta\big]\Big\} \tag{5.15}$$

where $\rho(j) = \rho_\Lambda(j)$ if $j\in\Lambda$ and $= \rho^{(k)}_{s(j)}$ if $j\in\Lambda^c$. $\rho_\Lambda(i) = \rho^{(k)}_{s(i)}$ is a solution of (5.15) and therefore also a solution of $D_\Lambda f_\epsilon^{\rm pf} = 0$ (with $g = 0$). By Corollary 5.7 it is then the unique minimizer of $f_\epsilon^{\rm pf}$ and (5.14) is proved (for $g = 0$).

Proof of (5.14).

Call

$$f_{\epsilon,\theta}(\rho_\Lambda) = F_\Lambda(\rho_\Lambda|\rho^{(k)}1_{\Lambda^c}) + \theta g(\rho_\Lambda;\bar q_{\Lambda^c},t) + (f_\epsilon - f)$$

$\theta\in[0,1]$; for all $\epsilon > 0$ small enough denote by $\hat\rho_{\Lambda,\epsilon,\theta}$ the minimizer of $f_{\epsilon,\theta}$, so that $D_\Lambda f_{\epsilon,\theta}(\hat\rho_{\Lambda,\epsilon,\theta}) = 0$. Suppose that

$$\frac{d\hat\rho_{\Lambda,\epsilon,\theta}}{d\theta} \text{ exists for all } \theta\in[0,1] \text{ and depends continuously on } \theta \tag{5.16}$$

Obviously $\hat\rho_{\Lambda,\epsilon,1} = \hat\rho_{\Lambda,\epsilon}$ while $\hat\rho_{\Lambda,\epsilon,0} = \rho^{(k)}1_\Lambda$ because of the above analysis with $g = 0$. Then

$$\hat\rho_{\Lambda,\epsilon,\theta} = \rho^{(k)}1_\Lambda + \int_0^1 \frac{d\hat\rho_{\Lambda,\epsilon,\theta}}{d\theta} \tag{5.17}$$

On the other hand by differentiating $D_\Lambda f_{\epsilon,\theta}(\hat\rho_{\Lambda,\epsilon,\theta}) = 0$ we get

$$D^2_\Lambda f_{\epsilon,\theta}(\hat\rho_{\Lambda,\epsilon,\theta}) \frac{d\hat\rho_{\Lambda,\epsilon,\theta}}{d\theta} = -D_\Lambda g(\hat\rho_{\Lambda,\epsilon,\theta}) \tag{5.18}$$

By Lemma 5.2 and Theorem 5.5 for all $\epsilon > 0$ small enough, $D^2_\Lambda f_{\epsilon,\theta}(\hat\rho_{\Lambda,\epsilon,\theta})$ is symmetric and positive definite, then by Theorem A.3 the inverse $(D^2_\Lambda f_{\epsilon,\theta}(\hat\rho_{\Lambda,\theta}))^{-1}$ is well defined and bounded as an operator on L^∞, and we thus get from (5.18)

$$|\frac{d\hat\rho_{\Lambda,\epsilon,\theta}}{d\theta}| \le c\|D_\Lambda g(\hat\rho_{\Lambda,\epsilon,\theta})\|_\infty \le c'(\gamma\ell_{-,\gamma})^{a_0} \tag{5.19}$$

which by (5.17) yields (5.14). (5.19) also implies that $|\hat\rho_{\Lambda,\epsilon,\theta} - \rho^{(k)}1_\Lambda| \le c'(\gamma\ell_{-,\gamma})^{a_0}$. Notice that (5.19) implies (5.16), but unfortunately the argument is circular as it started by supposing the validity of (5.16). To avoid the impasse we start from the equation in the unknown u_Λ

$$D^2_\Lambda f_{\epsilon,\theta}(\rho_\Lambda)u_\Lambda = -D_\Lambda g(\rho_\Lambda) \tag{5.20}$$

where ρ_Λ is considered as a "known term" such that $|\rho_\Lambda(i) - \rho^{(k)}_{s(i)}| \le 2\zeta$ for all $i \in \Lambda$. From what said before, (5.20) has a unique solution called $\dot\rho_\Lambda(i|\rho_\Lambda)$ and

$$|\dot\rho_\Lambda(i|\rho_\Lambda)| \le c(\gamma\ell_{-,\gamma})^{a_0}, \quad \text{for all } i \in \Lambda \tag{5.21}$$

Since $\dot\rho_\Lambda(\cdot|\rho_\Lambda)$ is Lipschitz in ρ_Λ (we omit the details) the ordinary differential equation

$$\frac{d\rho_\Lambda(\theta)}{d\theta} = \dot\rho_\Lambda(\cdot|\rho_\Lambda(\theta)), \quad \rho_\Lambda(0) = \rho^{(k)}1_\Lambda \tag{5.22}$$

has a unique solution $\tilde\rho_\Lambda(\theta)$. Then, by (5.20),

$$\frac{d}{d\theta} D_\Lambda f_{\epsilon,\theta}(\tilde\rho_\Lambda(\theta)) = 0, \text{ and hence } D_\Lambda f_{\epsilon,\theta}(\tilde\rho_\Lambda(\cdot;\theta)) = D_\Lambda f_{\epsilon,0}(\rho^{(k)}1_\Lambda) = 0 \tag{5.23}$$

Since $|\tilde\rho_\Lambda(\theta) - \rho^{(k)}1_\Lambda| \le c'(\gamma\ell_{-,\gamma})^{a_0}$, $D_\Lambda f_{0,\theta}(\tilde\rho_\Lambda(\cdot;\theta)) = 0$ as well, hence by Corollary 5.7, $\tilde\rho_\Lambda(\cdot;\theta) = \hat\rho_{\Lambda,\epsilon,\theta}(\cdot)$ and by (5.22) it is differentiable with continuous derivative. (5.16) thus holds and the theorem proved. □

5.5 Exponential decay

This subsection concludes our analysis with the following main theorem, Theorem 5.1 will be proved in the Subsection 5.6 as a corollary, taking Λ^c_1 as a neighborhood of x in Λ^c and $\Lambda^c_2 = \Lambda^c \setminus \Lambda^c_1$.

Theorem 5.9. *There are $\hat{\omega}$ and c positive such that the following holds. Let $\hat{\rho}'_\Lambda$ and $\hat{\rho}''_\Lambda$ be the minimizers of $f(\rho_\Lambda, \bar{q}'_{\Lambda^c})$, respectively $f(\rho_\Lambda, \bar{q}''_{\Lambda^c})$, with $\bar{q}'_{\Lambda^c}, \bar{q}''_{\Lambda^c} \in \mathcal{X}_{\Lambda^c}$. Then for any partition of Λ^c into two $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable sets Λ^c_1 and Λ^c_2,*

$$|\hat{\rho}''_\Lambda(i) - \hat{\rho}'_\Lambda(i)| \le c\Big(\min\big\{ \mathbf{1}_{\bar{q}''_{\Lambda^c_1} \neq \bar{q}'_{\Lambda^c_1}} ; \max_{j\in\Lambda^c_1} \big((\gamma\ell_{-,\gamma})^{a_0} + \ |\rho^{(\ell_{-,\gamma})}(\bar{q}''_{\Lambda^c}; j) - \rho^{(\ell_{-,\gamma})}(\bar{q}'_{\Lambda^c}; j)| \big) \big\}$$
$$+ \sum_{j\in\Lambda^c_2} e^{-\omega\gamma|x(i)-x(j)|} \ \mathbf{1}_{\bar{q}''_{C_j^{(\ell_{-,\gamma})}} \neq \bar{q}'_{C_j^{(\ell_{-,\gamma})}}} \Big), \qquad \forall i \in \Lambda \tag{5.24}$$

Proof. We follow the interpolation strategy used in the proof of Theorem 5.8. To this end we separate the "interaction part" in f_ϵ writing $f_\epsilon = f^0_\epsilon + f^1_\epsilon$ where $f^0_\epsilon = f^0_\epsilon(\rho_\Lambda)$ is independent of the boundary conditions while

$$f^1_\epsilon(\rho_\Lambda, \bar{q}_{\Lambda^c}) = t(\rho_\Lambda, \bar{V}_\gamma \bar{\rho}_{\Lambda^c}) + g_1(\rho_\Lambda, \bar{q}_{\Lambda^c}) \tag{5.25}$$

where g_1 is given by the r.h.s of (4.9) with the sum over $\underline{i}$ restricted to the set $\underline{i} \cap \Lambda^c \neq \emptyset$. We then interpolate between the two boundary conditions

$$f_{\theta,\epsilon}(\rho_\Lambda) := f^0_\epsilon(\rho_\Lambda) + \theta f^1_\epsilon(\rho_\Lambda, \bar{q}''_{\Lambda^c}) + (1-\theta) f^1_\epsilon(\rho_\Lambda, \bar{q}'_{\Lambda^c}), \quad \theta \in [0,1] \tag{5.26}$$

The analysis done in the previous subsections, applies to $f_{\theta,\epsilon}(\rho_\Lambda)$ as well. Thus the minimizer $\hat{\rho}_{\Lambda.\epsilon,\theta}$ of $f_{\theta,\epsilon}$ is unique, is a critical point, namely $D_\Lambda f_{\theta,\epsilon}(\hat{\rho}_{\Lambda.\epsilon,\theta}) = 0$ and satisfies for all $x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Lambda$ and all $s \in \{1,..,S\}$, $\big|\hat{\rho}_{\Lambda,\epsilon,\theta}(x,s) - \rho^{(k)}_s\big| \le 2\zeta$.
We can apply the same proof as the one given in Theorem 5.8. In fact by Theorem 5.5, for all $\epsilon > 0$ small enough, and for all ρ_Λ such that $\big|\rho_\Lambda(x,s) - \rho^{(k)}_s\big| \le 2\zeta$, we have that $D^2_\Lambda f_{\theta,\epsilon}(\rho_\Lambda)$ is symmetric and positive definite, then by Theorem A.3 the inverse $(D^2_\Lambda f_{\theta,\epsilon}(\rho_\Lambda))^{-1}$ is well defined and bounded as an operator on L^∞. Thus the equation

$$(D^2_\Lambda f_{\theta,\epsilon}(\rho_\Lambda)) u_\Lambda = -\frac{\partial D_\Lambda f_{\theta,\epsilon}(\rho_\Lambda)}{\partial\theta} \tag{5.27}$$

has a unique solution that we call $u_\Lambda(\cdot,\rho_\Lambda)$ that is Lipschitz in ρ_Λ. This implies that the equation

$$\frac{d\rho_{\Lambda,\epsilon,\theta}}{d\theta} = u_\Lambda(\cdot, \rho_{\Lambda,\epsilon,\theta}), \qquad \rho_{\Lambda,\epsilon,0} = \hat{\rho}_{\Lambda,\epsilon,0}$$

has a unique solution that coincides with the minimizer $\hat{\rho}_{\Lambda,\epsilon,\theta}$. Thus $\hat{\rho}_{\Lambda.\epsilon,\theta}$ is differentiable in θ and $d\hat{\rho}_{\Lambda.\epsilon,\theta}/d\theta$ satisfies

$$(D^2_\Lambda f_{\theta,\epsilon}(\hat{\rho}_{\Lambda,\epsilon,\theta})) \frac{d\hat{\rho}_{\Lambda,\epsilon,\theta}}{d\theta} = -\frac{\partial D_\Lambda f_{\theta,\epsilon}(\hat{\rho}_{\Lambda,\epsilon,\theta})}{\partial\theta} \tag{5.28}$$

By Corollary 5.7, $\hat{\rho}_{\Lambda.\epsilon,\theta}$ converges by subsequences as $\epsilon \to 0$ to a limit $\tilde{\rho}_{\Lambda,\theta}$ which minimizes f_θ, so that

$$|\hat{\rho}''_\Lambda(i) - \hat{\rho}'_\Lambda(i)| \le \lim_{\epsilon\to 0} \int_0^1 |\frac{d\hat{\rho}_{\Lambda.\epsilon,\theta}(i)}{d\theta}| \tag{5.29}$$

We now estimate $|\frac{d\hat{\rho}_{\Lambda.\epsilon,\theta}(i)}{d\theta}|$ uniformly in ϵ and θ to prove (5.24) as a consequence of (5.29).

Equations for $d\hat\rho_{\Lambda.\epsilon,\theta}/d\theta$.

we let

$$u := \frac{d}{d\theta}\hat\rho_{\Lambda.\epsilon,\theta}, \qquad v = -\frac{d}{d\theta'} D_\Lambda f_{\theta',\epsilon}(\hat\rho_{\Lambda.\epsilon,\theta})\Big|_{\theta'=\theta}, \qquad A := D^2_\Lambda f_{\theta,\epsilon}(\hat\rho_{\Lambda.\epsilon,\theta}) \tag{5.30}$$

so that (5.28) becomes

$$Au = v$$

We also define:

$$A_0 := D^2_\Lambda f_{\theta,0}(\hat\rho_{\Lambda.\epsilon,\theta}), \quad \alpha := A - A_0 \tag{5.31}$$

α is a diagonal matrix whose diagonal elements are

$$\alpha(i) := 3\epsilon^{-1}\Big(\{(\hat\rho_{\Lambda.\epsilon,\theta}(i) - [\rho^{(k)}_{s(i)} + \zeta])_+\}^2 + \{(\hat\rho_{\Lambda.\epsilon,\theta}(i) - [\rho^{(k)}_{s(i)} - \zeta])_-\}^2\Big) \tag{5.32}$$

To distinguish among large and non large (called small) values of $\alpha(i)$, we introduce a large positive number b which will be specified later and, calling $\mathcal{H}$ the Hilbert space of vectors $u = \big(u(i), i \in \Lambda\big)$,

$$G = \big\{(i) : \alpha(i) \geq b\big\}, \quad \mathcal{H}_G = \big\{u \in \mathcal{H} : u(i) = 0, \text{ for all } i \in G^c\big\} \tag{5.33}$$

Let Q be the orthogonal projection on $\mathcal{H}_G$ and $P = 1 - Q$, thus Q selects the sites where α is large and P those where it is small.
Our strategy will be the following: rewrite Pu, Qu as linear expressions of Pv, Qv to get bounds on Pu, Qu (and therefore on u) using knowledge on v.

Rewriting Pu, Qu in terms of Pv, Qv.

Since the matrices α, P, Q are diagonal they commute, giving for instance $Q\alpha P = \alpha PQ = 0$, i.e.:

$$QAP = QA_0P, \tag{5.34}$$

and symetrically:

$$PAQ = PA_0Q. \tag{5.35}$$

Using $Q^2 = Q$ together with (5.34) we get:

$$\begin{aligned} QAQQu &= QAQu = QA(u - Pu) \\ QAQQu &= Qv - QAPu \\ Qu &= (QAQ)^{-1}\{Qv - QA_0Pu\} \end{aligned} \tag{5.36}$$

where QAQ is invertible on the range of Q since A is a positive matrix.
Using $P^2 = P$ together with (5.36) and (5.35)we get:

$$\begin{aligned} PAPu + PAQu &= PAu = Pv \\ PAPu + PA_0(QAQ)^{-1}\{Qv - QA_0Pu\} &= Pv \\ \Big(PAP - PA_0(QAQ)^{-1}QA_0\Big)Pu &= Pv - PA_0(QAQ)^{-1}Qv \end{aligned}$$

Let

$$B = PAP - PA_0(QAQ)^{-1}QA_0 \tag{5.37}$$

so that if B is invertible on the range of P (as we will prove), then

$$Pu = B^{-1}\{Pv - PA_0(QAQ)^{-1}Qv\} \tag{5.38}$$

A decomposition of v.

Recalling (5.30) and (5.26), after expanding the Poisson polynomials in (4.9) we get,

$$\begin{aligned} v(i) = &-t \sum_{j\in\Lambda^c} \bar V_\gamma(i,j)\Big(\bar\rho''_{\Lambda^c}(j) - \bar\rho'_{\Lambda^c}(j)\Big) \\ &- \sum_n (\gamma\ell_{-,\gamma})^{a_0 n} \sum_{i_1,k_{i_1},..i_n,k_{i_n}:i_1=i} k_{i_1}\Big(d_n\big(i_1,k_{i_1},..,i_n,k_{i_n};\bar q''_{\Lambda^c};t\big)\rho''(i_1)^{k_{i_1}-1}\cdots\rho''(i_n)^{k_{i_n}} \\ &\qquad -d_n\big(i_1,k_{i_1},..,i_n,k_{i_n};\bar q'_{\Lambda^c};t\big)\rho'(i_1)^{k_{i_1}-1}\cdots\rho'(i_n)^{k_{i_n}}\Big) \end{aligned} \tag{5.39}$$

where $\rho''(i) = \rho'(i) = \hat\rho_{\Lambda,\epsilon,\theta}(i)$ if $x(i)\in\Lambda$ and $\rho''(i) = \bar\rho''_{\Lambda^c}(i)$, $\rho'(i) = \bar\rho'_{\Lambda^c}(i)$ when $x(i)\in\Lambda^c$. The coefficients d_n satisfy the same bounds as the coefficients Φ of (4.9) (with maybe a different constant).

Shorthand by $\{x_j\}$ the sites in $\{x(i_1),..,x(i_n)\}$ which are in Λ^c, noticing that by definition of g_1 there are not terms with $\{x_j\}=\emptyset$.

We then call $v^{(1)}$ the sum of $-t\sum_{j\in\Lambda_1^c}\bar V_\gamma(i,j)\Big(\bar\rho''_{\Lambda^c}(j)-\bar\rho'_{\Lambda^c}(j)\Big)$ minus the second sum on the r.h.s. of (5.39) restricted to sets $(i_1,...,i_n)$ such that: $\{x_j\}\neq\emptyset$ and any $x_j\in\{x_j\}$ is either in Λ_1^c or $\bar q''_{C^-_{x_j}} = \bar q'_{C^-_{x_j}}$, $C^-_x = C^{(\ell_{-,\gamma})}_x$, (or both). $v^{(2)} := v - v^{(1)}$.

By linearity $u = u^{(1)} + u^{(2)}$ where $u^{(1)}$ and $u^{(2)}$ are defined with v replaced by $v^{(1)}$ and $v^{(2)}$ and we will bound differently $u^{(1)}$ and $u^{(2)}$ using $\|\cdot\|_\infty$ norms for the former and $\|\cdot\|$ norms for the latter.

Bounds on $u^{(1)}$.

By Theorem A.1 if b is large enough and $c\ge\|A_0\|$,

$$\|PA_0(QAQ)^{-1}QA_0\| \le \frac{2c^2}{b} =: \delta, \quad \|PA_0(QAQ)^{-1}QA_0\|_\infty \le \frac{2c^2}{b}e^{2c'} \tag{5.40}$$

Moreover by (A.5)

$$\sup_i \sum_j |B(i,j)|e^{\gamma|i-j|} \le \sup_{i\in G^c}\sum_j |A(i,j)|e^{\gamma|i-j|} + \frac{2c^2e^{2c'}}{b} \le c'''b =: a \tag{5.41}$$

Then applying Theorem A.2,A.3 with B as in (5.37) and $R_1 = PA_0(QAQ)^{-1}QA_0$, B is invertible and there is a constant $c>0$ such that $\|B^{-1}\|_\infty\le c$. Therefore there is a new constant c such that

$$|Pu^{(1)}(i)| \le c\max_j |v^{(1)}(j)| \tag{5.42}$$

If $\bar q''_{\Lambda_1^c}=\bar q'_{\Lambda_1^c}$, $v^{(1)}=0$ and $u^{(1)}=0$ as well, let us then suppose $\bar q''_{\Lambda_1^c}\neq\bar q'_{\Lambda_1^c}$. Then (5.39) yields

$$|Pu^{(1)}(i)|\le c\Big(\max_{j\in\Lambda_1^c}|\bar\rho''_{\Lambda^c}(j)-\bar\rho'_{\Lambda^c}(j)|+(\gamma\ell_{-,\gamma})^{a_0}\Big) \tag{5.43}$$

To bound $|Qu^{(1)}(i)|$ we go back to (5.36), the same arguments used before prove that $\|(QAQ)^{-1}\|_\infty\le c$ as well, so that $|Qu^{(1)}(i)|$ is bounded as on the r.h.s. of (5.43) (with a new constant c) and $|u^{(1)}(i)|$ is therefore bounded as the first term on the r.h.s. (5.24), we will prove next that $|u^{(2)}(i)|$ is bounded as the second term on the r.h.s. (5.24) which will then be proved.

Bounds on $u^{(2)}$.

Recalling the definition of $v^{(2)}$

$$|v^{(2)}(i)|\le\sum_{j\in\Lambda_2^c}K_\gamma(i,j)\mathbf{1}_{\bar q''_{C_j^-}\neq\bar q'_{C_j^-}} \tag{5.44}$$

where $\sum_i K_\gamma(i,j)\le c_K$ and $K_\gamma(i,j)=0$ if $|x(i)-x(j)|\ge c'\gamma^{-1}$, c and c' suitable constants. By Theorem A.2

$$|B^{-1}(i,j)|\le(\frac1a+\frac1{\kappa'})\exp\Big\{-\frac{\kappa'\gamma|i-j|}{a+\kappa'}\Big\},\quad \kappa'=\kappa-\delta,\;\;\delta\text{ as in (5.40)} \tag{5.45}$$

By (5.45) and (5.44), calling $c''=1/a+1/\kappa'$ and $\omega=\kappa'/(a+\kappa')$,

$$|B^{-1}Pv^{(2)}(i)|\le\sum_{j\in\Lambda_2^c}\mathbf{1}_{\bar q''_{C_j^-}\neq\bar q'_{C_j^-}}\{c_Kc''e^{c'\omega}\}e^{-\omega\gamma|x(i)-x(j)|} \tag{5.46}$$

By (A.5)

$$\sum_i|(QAQ)^{-1}(i,j)|e^{\gamma|i-j|}\le\frac{c_Q}{b} \tag{5.47}$$

and since $A_0(i,j)=0$ if $|i-j|\ge c'\gamma^{-1}$ and $\sum_i|A_0(i,j)|\le c_{A_0}$,

$$\begin{aligned}|B^{-1}PA_0(QAQ)^{-1}Qv^{(2)}(i)|&\le\sum_{j'}\sum_{j''}\sum_{j'''\in\Lambda_2^c}\mathbf{1}_{\bar q''_{C_{j'''}^-}\neq\bar q'_{C_{j'''}^-}}\{c''e^{-\omega\gamma|x(i)-x(j')|}c_{A_0}e^{c'\omega}\}\\&\times e^{-\gamma|x(j'')-x(j')|}|(QAQ)^{-1}(j',j'')|e^{\gamma|x(j'')-x(j')|}K_\gamma(j'',j''')\\&\le\{c''c_{A_0}e^{c'\omega}\}\sum_{j'''\in\Lambda_2^c}\mathbf{1}_{\bar q''_{C_{j'''}^-}\neq\bar q'_{C_{j'''}^-}}e^{-\omega\gamma|x(i)-x(j''')|}e^{\omega c'}\;(\frac{c_Q}{b})\;c_K\end{aligned}$$

Thus supposing $\omega\le1$, we get from (5.38)

$$|Pu^{(2)}(i)|\le\sum_{j\in\Lambda_2^c}\mathbf{1}_{\bar q''_{C_j^-}\neq\bar q'_{C_j^-}}ce^{-\omega\gamma|x(i)-x(j)|} \tag{5.48}$$

To bound $Qu^{(2)}$ (recall (5.36)) we use (5.47) to get

$$|(QAQ)^{-1}Qv^{(2)}(i)| \le \sum_{j\in\Lambda_2^c} \mathbf{1}_{\bar q''_{C_j^-}\neq \bar q'_{C_j^-}} \frac{c_K e^{c'} c_Q}{b} e^{-\gamma|x(i)-x(j)|} \tag{5.49}$$

while, using (5.48) and (5.47),

$$\begin{aligned}|(QAQ)^{-1}QA_0Pu^{(2)}(i)| &\le \sum_{j''}\sum_{j'}\sum_{j'''\in\Lambda_2^c} \mathbf{1}_{\bar q''_{C_{j'''}^-}\neq \bar q'_{C_{j'''}^-}} ce^{-\omega\gamma|(x(j'')-x(j''')y|}\\ &\times|A_0(j',j'')|e^{-\gamma|x(j'')-x(i)|}e^{c'}|(QAQ)^{-1}(i,j')|e^{\gamma|(x(j')-x(i)|}\\ &\le ce^{c'}\sum_{j'''\in\Lambda_2^c} \mathbf{1}_{\bar q''_{C_{j'''}^-}\neq \bar q'_{C_{j'''}^-}} e^{-\omega\gamma|x(i)-x(j''')|}c_{A_0}\,(\frac{c_Q}{b})\end{aligned}$$

hence

$$|Qu^{(2)}(i)| \le \frac{c}{b}\sum_{j\in\Lambda_2^c} \mathbf{1}_{\bar q''_{C_j^-}\neq \bar q'_{C_j^-}} e^{-\gamma|x(i)-x(j)|} \tag{5.50}$$

□

5.6 Proof of Theorem 5.1

The proof is a corollary of Theorem 5.9. Indeed given any $x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda$, call Λ_1^c the union of all $C_y^{(\ell_{-,\gamma})}$, $y\in\ell_{-,\gamma}\mathbb{Z}^d\cap\big(\Lambda^c\cap B_x(10^{-30}\ell_{+,\gamma})\big)$. Then if $\hat K(x)>0$, same notation as in Theorem 5.9, $\bar q''_{\Lambda_1^c}=\bar q'_{\Lambda_1^c}$ and by (5.24) we are reduced to a sum over $j\in\Lambda_2^c$. We split the exponent $-\gamma\omega|x(i)-x(j)|$ into two equal terms and get

$$\begin{aligned}|\hat\rho''_\Lambda(x,s)-\hat\rho'_\Lambda(x,s)| &\le c\{e^{-(\omega/2)\gamma[10^{-30}\ell_{+,\gamma}-\ell_{-,\gamma}]}\}\{\sum_{j\notin\Lambda_1^c} e^{-(\omega/2)\gamma|x-x(j)|}\}\\ &\le c'e^{-(\omega/2)\gamma[10^{-30}\ell_{+,\gamma}-\ell_{-,\gamma}]}\end{aligned} \tag{5.51}$$

The exponent $\hat\omega$ in Theorem 5.1 is thus going to be half the ω of Theorem 5.9. Using Theorem 5.9 with $\bar\rho''_{\Lambda^c}$ replaced by $\rho^{(k)}1_{\Lambda^c}$, and calling $\hat\rho^{\rm pf}_\Lambda$ the corresponding minimizer,

$$|\hat\rho^{\rm pf}_\Lambda(x,s)-\hat\rho'_\Lambda(x,s)| \le c'e^{-(\omega/2)\gamma[10^{-30}\ell_{+,\gamma}-\ell_{-,\gamma}]} + \big(c_1(\gamma\ell_{-,\gamma})^{a_0}+\zeta_m\big)$$

and using (5.14)

$$|\hat\rho'_\Lambda(x,s)-\rho_s^{(k)}| \le c'e^{-(\omega/2)\gamma[10^{-30}\ell_{+,\gamma}-\ell_{-,\gamma}]} + \big([c_1+c](\gamma\ell_{-,\gamma})^{a_0}+\zeta_m\big)$$

6 Local Couplings

In this section we prove Theorem 3.3, thus we fix a region Λ, union of a finite number N_Λ, of cubes of $\mathcal{D}^{(\ell_+)}$ and two boundary conditions $\bar q_{i,\Lambda^c} \in \mathcal{X}^{(k)}_{\Lambda^c}$, $i=1,2$. We also fix a $t\in(0,1]$ and we consider the two Gibbs measures $dG^0_\Lambda(q_\Lambda|\bar q_{i,\Lambda^c})$ $i=1,2$ defined in (3.22) and with state space $\mathcal{X}^{(k)}_\Lambda$. The aim is to construct a coupling Q_Λ of these two probabilities such that (3.25) holds. Q_Λ, being a joint distribution, is defined on the product space $\mathcal{X}^{(k)}_\Lambda \times \mathcal{X}^{(k)}_\Lambda$ whose elements are denoted by $(q'_\Lambda, q''_\Lambda)$.

6.1 Definitions and main results

Recalling that $K_\Lambda(\cdot;x) := K_\Lambda(\bar q_{1,\Lambda^c}, \bar q_{2,\Lambda^c}; x)$ is defined in Definition 3.2 we denote by

$$\Delta_0 \equiv \Delta_0(\bar q_{1,\Lambda^c}, \bar q_{2,\Lambda^c}) := \Big\{x \in \ell_{-\gamma}\mathbb{Z}^d \cap \Lambda : K_\Lambda(\bar q_{1,\Lambda^c}, \bar q_{2,\Lambda^c}; x) > 0\Big\} \tag{6.1}$$

In order to prove Theorem 3.3 we have to find a coupling Q_Λ so that there is ϵ_g such that

$$\sum_{x\in\Delta_0} Q_\Lambda(\Theta_\Lambda(x)^c) \le \epsilon_g \tag{6.2}$$

We define (recall that $B_x(R)$ is the ball of center x and radius R),

$$\Delta_1 = \bigcup_{x\in\Delta_0} B_x(10^{-20}\ell_{+,\gamma}) \cap \Lambda \tag{6.3}$$

and we observe that $\Delta_1 \supset \Delta_0$, $\mathrm{dist}(\Delta_0, \Delta_1^c) > 10^{-20}\ell_+,\gamma$.

We denote by

$$\underline{n} \equiv n_\Lambda = \Big\{n(x,s)\in\mathbb{N}, x\in \ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda, s\in\{1,..,S\}\Big\} \tag{6.4}$$

and in the sequel we will consider only those $\underline{n}$ such that for all $x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Lambda$ and $s\in\{1,..,S\}$,

$$\Big|\frac{n(x,s)}{\ell_-^d} - \rho^{(k)}(s)\Big| \le \zeta$$

Given $\underline{n}$ and any subset $\Delta\subset\Lambda$ we will call n_Δ the restriction to Δ of $\underline{n}$.

Given a subset $\Delta\subset\Lambda$, we call d_Δ the following metric on $\mathcal{X}^{(k)}_\Lambda \times \mathcal{X}^{(k)}_\Lambda$:

$$d_\Delta(q'_\Lambda, q''_\Lambda) = \sum_{x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\Delta} d_x(q'_\Lambda, q''_\Lambda) \tag{6.5}$$

$$d_x(q'_\Lambda, q''_\Lambda) = \begin{cases} 0 & \text{if } q'_\Lambda\cap C_x^{(\ell_{-,\gamma})} = q''_\Lambda \cap C_x^{(\ell_{-,\gamma})} \\ 1 & \text{otherwise}\end{cases} \tag{6.6}$$

We call $R_\Delta(\mu,\mu')$ the corresponding Wasserstein distance between two measures μ and μ' in $\mathcal{X}_\Lambda^{(k)} \times \mathcal{X}_\Lambda^{(k)}$:

$$
\begin{aligned}
R_\Delta(\mu,\mu') &= \inf_Q \int d_\Delta(q'_\Lambda, q''_\Lambda) dQ(q'_\Lambda, q''_\Lambda) \\
&= \inf_Q \sum_{x\in \ell_{-,\gamma}\mathbb{Z}^d\cap\Delta} Q\big(q'_\Lambda \cap C_x^{(\ell_{-,\gamma})} \neq q''_\Lambda \cap C_x^{(\ell_{-,\gamma})}\big)
\end{aligned} \tag{6.7}
$$

where the inf runs over all possible joint distributions (couplings) of μ and μ'.

In Subsection 6.3 we prove the following Theorem.

Theorem 6.1. *Given Λ union of N_Λ cubes of $\mathcal{D}^{(\ell_+)}$ there is $\epsilon_0 = \epsilon_0(N_\Lambda)$ such that for all $\bar q_{i,\Lambda^c} \in \mathcal{X}^{(k)}_{\Lambda^c}$, $i=1,2$, the following holds.*
Given any $\underline{n}'$, $\underline{n}''$ such that $n'_{\Delta_1} = n''_{\Delta_1} =: n_{\Delta_1}$ (Δ_1 defined in (6.3)*), the following holds.*
Calling $\mathring{\Delta}_1 = \Delta_1 \setminus \delta_{\rm in}^{\gamma^{-1}}[\Delta_1]$, for any two configurations $\bar q_{i,\Lambda\setminus\mathring{\Delta}_1}$, $i=1,2$ on $\mathcal{X}^{(k)}_{\Lambda\setminus\mathring{\Delta}_1}$, we denote by $\bar q_{i,\mathring{\Delta}_1^c} = \bar q_{i,\Lambda\setminus\mathring{\Delta}_1} \cup \bar q_{i,\Lambda^c}$, $i=1,2$.
Let $dG^0_\Lambda(q_{\mathring{\Delta}_1}|q_{i,\mathring{\Delta}_1^c}, n_{\Delta_1})$, $i=1,2$ be the probabilities $dG^0_\Lambda(\cdot|\bar q_{i,\Lambda^c})$, $i=1,2$ conditioned to have the configuration in $\mathring{\Delta}_1^c$ equal to $\bar q_{i,\mathring{\Delta}_1^c}$ and occupation numbers in Δ_1 given by n_{Δ_1}.
Then for Δ_0 defined in (6.1)

$$
R_{\Delta_0}\big(dG^0_\Lambda(\cdot|q_{1,\mathring{\Delta}_1^c}, n_{\Delta_1}), dG^0_\Lambda(\cdot|q_{2,\mathring{\Delta}_1^c}, n_{\Delta_1})\big) \le \epsilon_0 \tag{6.8}
$$

The next result, proved at the end of Subsection 6.6, deals with the Wasserstein distance R_{Δ_1} of the distributions of the occupation numbers $\underline{n}$ that in Theorem 6.1 have been set equal to each other inside Δ_1. For these variables the metric d_x defined in (6.6) is replaced by

$$
d_x(\underline{n}', \underline{n}'') = \begin{cases} 0 & \text{if } n'(x,s) = n''(x,s), \forall s \\ 1 & \text{otherwise} \end{cases}
$$

Theorem 6.2. *Given Λ union of N_Λ cubes of $\mathcal{D}^{(\ell_+)}$ there is $\epsilon_1 = \epsilon_1(N_\Lambda)$ such that the following holds. Let $G^0_\Lambda(n_\Lambda|q_{i,\Lambda^c})$, $i=1,2$ be the marginals of $dG^0_\Lambda(q_\Lambda|\bar q_{i,\Lambda^c})$, $i=1,2$ on the variables n_Λ defined in* (6.4)*.*
Then

$$
R_{\Delta_1}\big(dG^0_\Lambda(n_\Lambda|\bar q_{1,\Lambda^c}), dG^0_\Lambda(n_\Lambda|\bar q_{2,\Lambda^c})\big) \le \epsilon_1 \tag{6.9}
$$

In Subsection 6.7 we show that Theorem 3.3 is a consequence of Theorems 6.1 and 6.2.

6.2 Two properties of the Wasserstein distance in an abstract setting

Let Ω be a complete, separable metric space with distance $d(\omega,\omega')$ and let $R(\mu_1,\mu_0)$ be the corresponding Wasserstein distance between two measures μ_1 and μ_0. Thus

$$R(\mu_1,\mu_0) = \inf_Q \int d(\omega,\omega')Q(d\omega,d\omega') \tag{6.10}$$

where the inf runs over all possible joint distributions of μ_1 and μ_0.

Theorem 6.3. *Let ν be a given positive measure on Ω. Let h and v be such that for all $t \in [0,1]$,*

$$Z_t = \int e^{-[h(\omega)+tv(\omega)]}\nu(d\omega) < \infty, \tag{6.11}$$

Set

$$m_t(\omega) = Z_t^{-1} e^{-[h(\omega)+tv(\omega)]}, \quad \mu_t(d\omega) = m_t(\omega)\nu(d\omega) \tag{6.12}$$

Then

$$R(\mu_1,\mu_0) \le \sup_{0\le t\le 1} \Big(\mu_t(|\omega|\,|v|) + \mu_t(|\omega|)\mu_t(|v|)\Big) \tag{6.13}$$

where, after fixing arbitrarily an element $\omega_0 \in \Omega$, we have called $|\omega| = d(\omega,\omega_0)$.
In particular,

$$R(\mu_1,\mu_0) \le 2\big(\sup|\omega|\big)\,\big(\sup|v(\omega)|\big) \tag{6.14}$$

Proof. Let

$$m(\omega) = \min\{m_1(\omega), m_0(\omega)\}, \qquad C = 1 - \int m(\omega)\nu(d\omega)$$

$$P(d\omega d\omega') = \{m(\omega)\delta_{\omega-\omega'} + \frac{1}{C}[m_1(\omega)-m(\omega)][m_0(\omega')-m(\omega')]\}\nu(d\omega)\nu(d\omega')$$

P is a coupling of μ_1 and μ_0 and therefore

$$\begin{aligned} R(\mu_1,\mu_0) &\le \int_{\Omega\times\Omega} d(\omega,\omega')P(d\omega d\omega') \le \int_\Omega |\omega|\big([m_1(\omega)-m(\omega)] + [m_0(\omega)-m(\omega)]\big)\nu(d\omega) \\ &= \int_\Omega |\omega|\,|m_1(\omega)-m_0(\omega)|\nu(d\omega) \end{aligned}$$

having bounded $d(\omega,\omega') \le |\omega| + |\omega'|$ and integrated over the missing variable.
(6.13) is then obtained by writing $m_1(\omega) - m_0(\omega) = \displaystyle\int_0^1 \frac{d}{dt} m_t(\omega)$.

□

The following estimate is taken from [14]:

Theorem 6.4. *Let $A \subset \Omega$ be a measurable set, μ a probability on Ω and μ_A the probability μ conditioned to A. Then*

$$R(\mu, \mu_A) \leq 2 \sup_{\omega \in \Omega} |\omega| \; \mu(A^c) \tag{6.15}$$

Proof. Let

$$Q(d\omega, d\omega') = \mathbf{1}_{\omega \in A} \mu(d\omega) \delta_\omega(d\omega') + \mathbf{1}_{\omega \in A^c} \mu(d\omega) \mu_A(d\omega')$$

where $\delta_\omega(d\omega')$ is the probability supported by ω. Let f be any bounded, measurable function on Ω, then

$$\int f(\omega) Q(d\omega, d\omega') = \int_A f(\omega)\mu(d\omega) + \int_{A^c} f(\omega)\mu(d\omega) \int \mu_A(d\omega') = \mu(f)$$

$$\begin{aligned} \int f(\omega') Q(d\omega, d\omega') &= \int_A f(\omega)\mu(d\omega) + \mu(A^c) \int f(\omega')\mu_A(d\omega') \\ &= \mu_A(f)\mu(A) + \mu_A(f)\mu(A^c) = \mu_A(f) \end{aligned}$$

Hence Q is a coupling and

$$R(\mu, \mu_A) \leq \int d(\omega, \omega') Q(d\omega, d\omega') \leq \int \mathbf{1}_{\omega \in A^c} (|\omega| + |\omega'|) \mu(d\omega) \mu_A(d\omega')$$

which proves (6.15). □

Eventually, we mention the following elementary property:

Proposition 6.5. *Assume that the distance d satisfies $m(d) := \inf_{\omega \neq \omega' \in \Omega} d(\omega, \omega') > 0$. Then for all probability measures μ, ν and for all $A \subset \Omega$*

$$m(d) \cdot |\mu(A) - \nu(A)| \quad \leq \quad R(\mu, \nu) \tag{6.16}$$

Proof. Without loss of generality we assume $\mu(A) \geq \nu(A)$. Remarking that $\mathbf{1}_{\omega \neq \omega'} \geq \mathbf{1}_{\omega \in A} - \mathbf{1}_{\omega' \in A}$, we get for any coupling Q of μ, ν

$$m(d)(\mu(A) - \nu(A)) \leq \int d(\omega, \omega') \mathbf{1}_{\omega \neq \omega'} G(d\omega, d\omega') \tag{6.17}$$

and the proposition is proved by taking the infimum over all possible couplings Q. □

Remark 6.6. *The proposition above states that Wasserstein distances associated to very particular distances d are finer than the* total variation *distance $d_{TV}(\mu, \nu) := \sup_{A \subset \Omega} |\mu(A) - \nu(A)|$. In the following, we will use this property for R_Δ, remarking that $m(d_\Delta) = 1$.*

6.3 Couplings of multi-canonical measures

Here we prove Theorem 6.1. Recalling that $\mathring{\Delta}_1 = \Delta_1 \setminus \delta_{\rm in}^{\gamma^{-1}}[\Delta_1]$, we fix two boundary conditions $\bar q_{i,\mathring{\Delta}_1^c} = \bar q_{i,\Lambda\setminus\Delta_1} \cup \bar q_{i,\Lambda^c}$, $i=1,2$. We have to compare the marginal distributions of $dG^0_\Lambda(q_{\bar\Delta_1}|q_{i,\mathring{\Delta}_1^c}, n_{\Delta_1})$, $i=1,2$ over the configurations in Δ_0 (i.e. well inside $\mathring{\Delta}_1$). Since the probabilities $dG^0_\Lambda(q_{\bar\Delta_1}|q_{i,\mathring{\Delta}_1^c}, n_{\Delta_1})$, $i=1,2$ depend only on the restrictions of $q_{i,\mathring{\Delta}_1^c}$ to $\delta_{\rm out}^{\gamma^{-1}}[\Delta_1]$ where $n'(x,s)=n''(x,s)$ the corresponding occupation numbers in the two measures are all equal to each other. We will thus study couplings of multi-canonical measures, hence the title of the Subsection.

It is now convenient to label the particles. To this purpose we use a multi-index $p=(C_x,s,j)$, where C_x is the cube of $\mathcal{D}^{(\ell_-)}$ where the particle is; s is its spin and $j\in\{1,..,n(x,s)\}$ distinguishes among the particles in the same cube with same spin. We call $\mathcal{L}_{\mathring{\Delta}_1}$ the set of labels

$$\mathcal{L}_{\mathring{\Delta}_1} = \{p=(C_x,s,j), x\in\mathring{\Delta}_1, s=1,\dots,S, j\in\{1,..,n(x,s)\}\}$$

Observe that $\mathcal{L}_{\mathring{\Delta}_1}$ is determined by $\underline{n}_{\Delta_1}$ and we thus have the same labels for the two measures. Given $p=(C_x,s,j)\in\mathcal{L}_{\mathring{\Delta}_1}$ we denote by r_p a vector configuration $r_p=(r_j,s)$ with $r_j\in C_x$. We then denote by $r_{\mathcal{L}_{\mathring{\Delta}_1}} = \{r_p, p\in\mathcal{L}_{\mathring{\Delta}_1}\}$ a vector configuration in $\mathring{\Delta}_1$. Analogously we define $r_{\mathcal{L}_{\mathring{\Delta}_1^c}}$. We then call $H_{\mathcal{L}_{\mathring{\Delta}_1}}(r_{\mathcal{L}_{\mathring{\Delta}_1}}|r_{\mathcal{L}_{\mathring{\Delta}_1^c}})$ the energy $H_{\mathring{\Delta}_1,t}$ defined in (3.14) and with n_{Δ_1} fixed as above.
Calling

$$d\nu_p(r) = \mathbf{1}_{r\in C_x}dr \tag{6.18}$$

we define

$$P_{\mathcal{L}_{\mathring{\Delta}_1}}(dr_{\mathcal{L}_{\mathring{\Delta}_1}}|r_{\mathcal{L}_{\mathring{\Delta}_1^c}}) = Z(r_{\mathcal{L}_{\mathring{\Delta}_1^c}})^{-1} e^{-\beta H_{\mathcal{L}_{\mathring{\Delta}_1}}(r_{\mathcal{L}_{\mathring{\Delta}_1}}|r_{\mathcal{L}_{\mathring{\Delta}_1^c}})} \prod_{p\in\mathcal{L}_{\mathring{\Delta}_1}} \nu_p(dr) \tag{6.19}$$

Remark 6.7. *If A is a $\mathcal{D}^{(\ell_-)}$ measurable subset of $\mathring{\Delta}_1$, then $\mathcal{L}_A$ denotes all labels (C,s,j) with $C\subset A$ and the marginal of $P_{\mathcal{L}_A}(dr_{\mathcal{L}_A}|r_{\mathcal{L}_{A^c}})$ over the unlabeled configurations is the original multi-canonical measure in A.*
We will thus prove Theorem 6.1 if we can compare

$$P' = P_{\mathcal{L}_{\mathring{\Delta}_1}}(\cdot|r'_{\mathcal{L}_{\mathring{\Delta}_1^c}}) \text{ and } P'' = P_{\mathcal{L}_{\mathring{\Delta}_1}}(\cdot|r''_{\mathcal{L}_{\mathring{\Delta}_1^c}}) \tag{6.20}$$

by evaluating the Wasserstein distance $R_{\Delta_0}(P',P'')$.

We will use the Dobrushin high-temperature techniques which allow to reduce to a comparison of the conditional probabilities of a single variable r_p.

Proposition 6.8 (Dobrushin high-temperature theorem). *There is c such that the following holds. For all $p_0=(C_{x_0},s_0,j_0)$, $C_{x_0}\subset\mathring{\Delta}_1$, all $p_1=(C_{x_1},s_1,j_1)$ and all r'_{p_1} and r''_{p_1}*

$$\sup_{\underline{r}} R_{\Delta_0}\Big(P_{\mathcal{L}_{p_0}}\big(\,\cdot\,|\underline{r},r'_{p_1}\big), P_{\mathcal{L}_{p_0}}\big(\,\cdot\,|\underline{r},r''_{p_1}\big)\Big) \le c\gamma^{d+\alpha_-}\mathbf{1}_{\mathrm{dist}(C_{x_0},C_{x_1})\le\gamma^{-1}} \tag{6.21}$$

where $\underline{r} = (r_p)_{p \neq p_0, p_1}$

Proof. The probabilities to compare have the form

$$P_{\mathcal{L}_{p_0}}\big(dr|\underline{r}, r'_{p_1}\big) = \frac{1}{Z(\underline{r}, r'_{p_1})} e^{W_\gamma(r)} \mathbf{1}_{r \in C_{x_0}}\, dr$$

while

$$P_{\mathcal{L}_{p_0}}\big(dr|\underline{r}, r''_{p_1}\big) = \frac{1}{Z(\underline{r}, r''_{p_1})} e^{W_\gamma(r) + W'_\gamma(r)} \mathbf{1}_{r \in C_{x_0}}\, dr$$

where $W_\gamma(r) = -\beta V_\gamma(r, r'_{j_1})$ and $W'_\gamma(r) = -\beta\{V_\gamma(r, r''_{j_1}) - V_\gamma(r, r'_{j_1})\}$ hence

$$|W'_\gamma(r)| \leq \beta \sup_{r' \in C_{x_1}} |\nabla V_\gamma(r, r')| \ell_- \leq c' \gamma^{d+\alpha_-} \mathbf{1}_{\mathrm{dist}(C_{x_0}, C_{x_1}) \leq \gamma^{-1}}$$

Proposition 6.8 then follows from Theorem 6.3. □

Remark 6.9. *From the proof above, we see that the r.h.s of* (6.21) *is actually proportionnal to* $\beta\gamma^{d+\alpha_-}$. *In other terms, the* effective temperature *of the system is of order* $\gamma^{-d-\alpha_-}$ *and thus* very high *indeed.*

Corollary 6.10. *With* P', P'' *defined by* (6.20)*, there is* ϵ_0 *such that for all* γ *small enough the following holds:*

$$R_{\Delta_0}(P', P'') \leq \epsilon_0 \tag{6.22}$$

Proof. For p_0 and p_1 as in Proposition 6.8 we call $\delta(p_0, p_1) = c\gamma^{d+\alpha_-} \mathbf{1}_{\mathrm{dist}(C_{x_0}, C_{x_1}) \leq \gamma^{-1}}$ (which is the r.h.s. of (6.21)). Then there is $\varsigma > 0$ such that for all γ small enough the following holds:

$$\begin{aligned} R_{\Delta_0}(P', P'') &\leq \sum_{p_0 \in \mathcal{L}_{\Lambda_0}} \sum_n \sum_{p_1, .., p_n \in \mathcal{L}_\Delta} \sum_{p \notin \mathcal{L}_\Delta} \delta(p_0, p_1) \cdots \delta(p_n, p) \\ &\leq e^{-\varsigma \mathrm{dist}(\Delta_0, \bar{\Delta}_1^c)} \end{aligned}$$

The first inequality follows from the Dobrushin high-temperature theorem (Proposition 6.8) while the second one is obvious once $\sum_{p' \neq p} \delta(p, p') \leq c\gamma^{\alpha_-} < 1$ (which is satisfied for all γ small enough). □

In view of Remark 6.7, the Theorem 6.1 is a straightforward consequence of 6.10. □

6.4 Taylor expansion

In this subsection we consider the marginal of $dG^0_\Lambda(q_\Lambda|\bar q_{\Lambda^c})$ on the variables $\rho_\Lambda = \ell_-^{-d} n_\Lambda$, $n_\Lambda = \{n(x,s), x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Lambda,\ s \in \{1,..,S\}$. By an abuse of notation we denote also the marginal with $G^0_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c})$.
Recalling (4.2) we get

$$G^0_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) = \frac{1}{Z^{\rm eff}(\bar q_{\Lambda^c})} e^{-\beta \ell^d_{-,\gamma} H^{\rm eff}_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c})} \tag{6.23}$$

Recalling (5.1) we also define

$$G^\star_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) = \frac{1}{Z^*(\bar q_{\Lambda^c})} e^{-\beta \ell^d_{-,\gamma} f(\rho_\Lambda;\bar q_{\Lambda^c})} \tag{6.24}$$

The following holds:

Proposition 6.11. *For all* $\bar q_{1,\Lambda^c}, \bar q_{2,\Lambda^c} \in \mathcal{X}^{(k)}_{\Lambda^c}$,

$$R_{\Delta_1}\big(G^0(\cdot|\bar q_{1,\Lambda^c}), G^0(\cdot|\bar q_{2,\Lambda^c})\big) \le R_{\Delta_1}\big(G^\star(\cdot|\bar q_{1,\Lambda^c}), G^\star(\cdot|\bar q_{2,\Lambda^c})\big) + 2c\gamma^\tau \tag{6.25}$$

with τ *given in* (4.13).

Proof. By (4.13) there is $c = c(N_\Lambda)$ such that

$$|H^{\rm eff}_\Lambda(\rho_\Lambda|\bar q_{\Lambda^c}) - f(\rho_\Lambda;\bar q_{\Lambda^c})| \le c\gamma^\tau \tag{6.26}$$

By (6.26) and Theorem 6.3, there is a (different) constant $c > 0$ such that

$$R_{\Delta_1}\big(G^0(\cdot|\bar q_{\Lambda^c}), G^\star(\cdot|\bar q_{\Lambda^c})\big) \le c\gamma^\tau \tag{6.27}$$

Hence the triangular inequality implies (6.25). □

We will bound $R_{\Delta_1}\big(G^\star(\cdot|\bar q_{1,\Lambda^c}), G^\star(\cdot|\bar q_{2,\Lambda^c})$ by using the triangular inequality to replace the two measures by their Taylor approximants.

We first prove the following result true for any $\mathcal{D}^{(\ell_+)}$-measurable region Λ.

Theorem 6.12. *For any* $\bar q_{\Lambda^c} \in \mathcal{X}^{(k)}_{\Lambda^c}$*, calling* $\mu = G^\star_\Lambda(\cdot|\bar q_{\Lambda^c})$*, the following holds.*
There are $c > 0$ *and* $\delta < 1/2$ *that verifies* (6.29) *below, so that, calling* $\hat\rho_\Lambda$ *the minimizer of* $f(\rho_\Lambda;\bar q_{\Lambda^c})$

$$\mu\Big(\{\exists x \in \Lambda, \exists s : |\rho_\Lambda(x,s) - \hat\rho_\Lambda(x,s)| \ge \ell_-^{-d/2+\delta}\}\Big) \le e^{-c\ell_-^{2\delta}} \tag{6.28}$$

Proof. Denoting simply $A := \{\exists x \in \Lambda, \exists s : |\rho_\Lambda(x,s) - \hat\rho_\Lambda(x,s)| \geq \ell_-^{-d/2+\delta}\}$ we have

$$\mu\Big(\{\exists x \in \Lambda, \exists s : |\rho_\Lambda(x,s) - \hat\rho_\Lambda(x,s)| \geq \ell_-^{-d/2+\delta}\}\Big) = \frac{1}{Z^*(\bar q_{\Lambda^c})} \sum_{\rho_\Lambda \in \mathcal{X}_\Lambda^{(k)}} e^{-\beta \ell_-^d f(\rho_\Lambda;\bar q_{\Lambda^c})} \mathbf{1}_A(\rho_\Lambda)$$

By Theorem 5.6 we have that

$$f(\rho_\Lambda, \bar q_{\Lambda^c}) \geq f(\hat\rho_\Lambda, \bar q_{\Lambda^c}) + \frac{\kappa}{2}\left(\rho_\Lambda - \hat\rho_\Lambda, \rho_\Lambda - \hat\rho_\Lambda\right)$$

Thus calling $C = \big(\sum_{n=0}^{\infty} e^{-\beta\frac{\kappa}{2}n^2}\big)^{SN_\Lambda}$ we get

$$\begin{aligned}
&\sum_{\rho_\Lambda \in \mathcal{X}_\Lambda^{(k)}} e^{-\beta \ell_-^d f(\rho_\Lambda;\bar q_{\Lambda^c})} \mathbf{1}_A(\rho_\Lambda) \\
&\qquad \leq e^{-\beta \ell_-^d f(\hat\rho_\Lambda;\bar q_{\Lambda^c})} \sum_{\rho_\Lambda \in \mathcal{X}_\Lambda^{(k)}} \exp\big\{ -\beta \ell_-^d \frac{\kappa}{2} \sum_{y,s} [\rho_\Lambda(y,s) - \hat\rho_\Lambda(y,s)]^2 - \beta\frac{\kappa}{2}\ell_-^{2\delta}\big\} \\
&\qquad \leq e^{-\beta \ell_-^d f(\hat\rho_\Lambda;\bar q_{\Lambda^c})} e^{-\beta\frac{\kappa}{2}\ell_-^{2\delta}} \Big[\big(\sum_{n=0}^{\infty} e^{-\beta\frac{\kappa}{2}n^2}\big)^S\Big]^{|\Lambda|/\ell_-^d} \\
&\qquad \leq e^{-\beta \ell_-^d f(\hat\rho_\Lambda;\bar q_{\Lambda^c})} e^{-\beta\frac{\kappa}{2}\ell_-^{2\delta}}\, C^{(\ell_+/\ell_-)^d}
\end{aligned}$$

We bound the partition function as follows, with $0 < \epsilon$ a small constant to be chosen later:

$$\begin{aligned}
Z^*(\bar q_{\Lambda^c}) &\geq \sum_{\rho_\Lambda \in \mathcal{X}_\Lambda^{(k)}} e^{-\beta \ell_-^d f(\rho_\Lambda;\bar q_{\Lambda^c})} \mathbf{1}_{\{|\rho_\Lambda(x,s) - \hat\rho_\Lambda(x,s)| \leq \epsilon \ell_-^{-d/2+\delta} \forall x, \forall s\}} \\
&\geq e^{-\beta \ell_-^d f(\hat\rho_\Lambda;\bar q_{\Lambda^c})} e^{-\beta\frac{C'\epsilon^2}{2}\ell_-^{2\delta}} (\epsilon \ell_-^{-d/2+\delta})^{S(\ell_+/\ell_-)^d},
\end{aligned}$$

so that

$$\mu\Big(\{|\rho_\Lambda(x,s) - \hat\rho_\Lambda(x,s)| \geq \ell_-^{-d/2+\delta}\}\Big) \leq \exp\{-\left[\beta\frac{\kappa - C'\epsilon^2}{2} - \ \ell_-^{-2\delta}\left(\frac{\ell_+}{\ell_-}\right)^d \log(C\epsilon^{-1}\ell_-^{d/2-\delta})\right]\ell_-^{2\delta}\}.$$

Remark now that

$$\ell_-^{-2\delta}\left(\frac{\ell_+}{\ell_-}\right)^d \log(C\epsilon^{-1}\ell_-^{d/2-\delta}) = a\gamma^b (\log\gamma)^c$$

with $a = (d/2 - \delta)(1-\alpha_-) > 0$, $b = (1-\alpha_-)2\delta - (\alpha_+ + \alpha_-)d$ and $c = C\epsilon^{-1} > 0$. Choosing δ such that $b > 0$, i.e.

$$\delta > \frac{(\alpha_+ + \alpha_-)d}{2(1-\alpha_-)} \tag{6.29}$$

which is always possible (see (3.3)), we get $\gamma^b(\log\gamma)^c \to 0$ as $\gamma \to 0$. The Theorem is now proved with $0 < c < \beta\frac{\kappa - C'\epsilon^2}{2}$, which is always possible for ϵ small enough. □

We call $\hat\rho_{\Lambda,i}$ the minimizer of $f(\cdot;\bar q_{i,\Lambda^c})$, $i=1,2$. We then let

$$A_{\le,i}=\big\{\rho_\Lambda\in\mathcal X^{(k)}:|\rho_\Lambda(x,s)-\hat\rho_{\Lambda,i}(x,s)|\le \ell_-^{-d/2+\delta},\forall x,\forall s\big\},\qquad i=1,2 \tag{6.30}$$

Proposition 6.13. *For all $\bar q_{i,\Lambda^c}\in\mathcal X^{(k)}_{\Lambda^c}$, $i=1,2$,*

$$R_{\Delta_1}\big(G^\star_\Lambda(\rho_\Lambda|\bar q_{1,\Lambda^c}),G^\star_\Lambda(\rho_\Lambda|\bar q_{2,\Lambda^c})\big)\le R_{\Delta_1}\big(G^\star_\Lambda(\rho_\Lambda|\bar q_{1,\Lambda^c},A_{\le,1}),G^\star_\Lambda(\rho_\Lambda|\bar q_{2,\Lambda^c},A_{\le,2})\big)+2ce^{-c\ell_-^{2\delta}} \tag{6.31}$$

where $G^\star_\Lambda(\rho_\Lambda|\bar q_{i,\Lambda^c},A_{\le,i})$ $i=1,2$ are the probabilities $G^\star_\Lambda(\cdot|\bar q_{i,\Lambda^c})$ conditioned to $A_{\le,i}$, $i=1,2$.

Proof. (6.31) follows from Theorem 6.12 and Theorem 6.4. □

Analogously to (6.3) we define the following subset of Λ.

$$\Delta_2=\bigcup_{x\in\Delta_1}B_x(10^{-30}\ell_{+,\gamma})\cap\Lambda \tag{6.32}$$

and we observe that $\Delta_2\supset\Delta_1$, $\mathrm{dist}(\Delta_1,\Delta_2^c)>10^{-30}\ell_+$. We also have

Lemma 6.14. *Let $\hat K$ be as in Theorem 5.1. Then $\hat K(x)>0$ for all $x\in\Delta_2$.*

Proof. Let $x\in\Delta_2$, by definition of $\hat K(x)$, if $\hat A_x=B_x(10^{-30}\ell_{+,\gamma})\cap\Lambda^c=\emptyset$ then $\hat K(x)=\bar m+1>0$. Assume then that $\hat A_x\neq\emptyset$. By (6.32) and (6.3) there is $x_0\in\Delta_0$ such that $|x-x_0|\le(1+10^{-10})10^{-20}\ell_{+,\gamma}$, thus $\hat A_x\subset A_{x_0}=B_x(10^{-10}\ell_{+,\gamma})\cap\Lambda^c$ and therefore $A_{x_0}\neq\emptyset$. By definition of Δ_0 we then have that $q'_{\Lambda^c}\cap\hat A_x=q''_{\Lambda^c}\cap\hat A_x$ and also that $K(x_0)=m+1>0$ with $m\ge2$ where m is given by $\max_{r\in A_{x_0},s\in\{1,..,S\}}|\rho^{(\ell_-,\gamma)}(\bar q'_{\Lambda^c};r,s)-\rho_s^{(k)}|\in[\zeta_{m+1},\zeta_m)$. Then $\max_{r\in\hat A_x,s\in\{1,..,S\}}|\rho^{(\ell_-,\gamma)}(\bar q'_{\Lambda^c};r,s)-\rho_s^{(k)}|<\zeta_m$, that implies that $\hat K(x)>0$. □

Recalling that $\hat\rho_{i,\Lambda}$ is the minimizer of $f(\cdot;\bar q_{i,\Lambda^c})$, $i=1,2$, we observe that in general the gradient of $D_\Lambda f$ (see (5.6) for notation), evaluated at $\hat\rho_{i,\Lambda}$ does not vanishes in all Λ. However, by Theorem 5.1 and Lemmas 5.4, 6.14 it follows that $D_{\Delta_2}f(\hat\rho_{i,\Lambda};\bar q_{i,\Lambda^c})=0$.
N being defined by Theorem 4.1, we set $\bar\Delta_2=\Delta_2\cup\delta_{\rm out}^{\gamma^{-1}N}[\Delta_2]$ and define

$$\rho^*_i(x,s)=\begin{cases}\hat\rho_{1,\Lambda}(x,s) & \text{if } x\in\ell_{-,\gamma}\mathbb Z^d\cap\bar\Delta_2\\ \hat\rho_{i,\Lambda}(x,s) & \text{if } x\in\ell_{-,\gamma}\mathbb Z^d\cap(\Lambda\setminus\bar\Delta_2)\end{cases} \tag{6.33}$$

Thus $\rho^*_2=\hat\rho_{1,\Lambda}$ in $\bar\Delta_2$ while $\rho^*_1(x,s)=\hat\rho_{1,\Lambda}(x,s)$ for all $x\in\ell_-\mathbb Z^d\cap\Lambda$ and $\forall s$. We denote by ρ^* the common value, thus

$$\rho^*(x,s)=\rho^*_1(x,s)=\rho^*_2(x,s),\qquad\forall x\in\ell_{-,\gamma}\mathbb Z^d\cap\bar\Delta_2,\forall s \tag{6.34}$$

We also define the matrix $B_{i,\Lambda}$ with entries:

$$B_{i,\Lambda}(x,s,x',s') = \begin{cases} D^2_\Lambda f(\hat\rho_{1,\Lambda};\bar q_{1,\Lambda^c})(x,s,x',s') & \text{if } x,x' \in \ell_{-,\gamma}\mathbb{Z}^d \cap \bar\Delta_2 \\ D^2_\Lambda f(\hat\rho_{i,\Lambda};\bar q_{i,\Lambda^c})(x,s,x',s') & \text{otherwise} \end{cases} \tag{6.35}$$

Observe that $B_{1,\Lambda} = D^2_\Lambda f(\hat\rho_{1,\Lambda};\bar q_{1,\Lambda^c})$. We denote by B the two matrices restricted to $\Delta_2 \cup \delta_{\rm out}^{\gamma^{-1}N}[\Delta_2]$ which are then equal; their common entries are then

$$B(x,s,x',s') = B_{1,\Lambda}(x,s,x',s') = B_{2,\Lambda}(x,s,x',s') \qquad \forall x,x' \in \ell_{-,\gamma}\mathbb{Z}^d \cap (\bar\Delta_2), \forall s \tag{6.36}$$

We define for $i=1,2$

$$\varphi_i(\rho_\Lambda;\bar q_{i,\Lambda^c}) = \Big(D_\Lambda f(\hat\rho_{i,\Lambda};\bar q_{i,\Lambda^c}),[\rho_\Lambda-\rho^*_i]\Big) + \frac{1}{2}\Big([\rho_\Lambda-\rho^*_i], B_{i,\Lambda}[\rho_\Lambda-\rho^*_i]\Big) \tag{6.37}$$

and the probabilities

$$\mu_i(\rho_\Lambda) := \frac{1}{Z_{i,\Lambda}} e^{-\beta\ell_-^d\,\varphi_i(\rho_\Lambda;\bar q_{i,\Lambda^c})}\chi_{A_{\le,i}}(\rho_\Lambda), \qquad Z_{i,\Lambda} = \sum_{\rho_\Lambda} e^{-\beta\ell_-^d\,\varphi_i(\rho_\Lambda;\bar q_{i,\Lambda^c})}\chi_{A_{\le,i}}(\rho_\Lambda) \tag{6.38}$$

where χ_A is the characteristic function of the set A:

The following holds:

Proposition 6.15. *For all $\bar q_{i,\Lambda^c} \in \mathcal{X}^{(k)}_{\Lambda^c}$, $i=1,2$, and for all $\epsilon_2>0$ if γ is small enough the following holds:*

$$R_{\Delta_1}(G^\star_\Lambda(\rho_\Lambda|\bar q_{1,\Lambda^c},A_{\le,1}), G^\star_\Lambda(\rho_\Lambda|\bar q_{2,\Lambda^c},A_{\le,2}) \le R_{\Delta_1}(\mu_1,\mu_2) + 2c\gamma^{d/4} + \epsilon_2 \tag{6.39}$$

Proof. We Taylor expand $f(\rho_\Lambda;\bar q_{i,\Lambda^c})$ and we call $\mathcal{R}_i$ the third order.

$$\begin{aligned}\mathcal{R}_i := f(\rho_\Lambda;\bar q_{i,\Lambda^c}) - f(\hat\rho_{i,\Lambda};\bar q_{i,\Lambda^c}) - \Big(D_\Lambda f(\hat\rho_{i,\Lambda};\bar q_{i,\Lambda^c}),[\rho_\Lambda-\hat\rho_{i,\Lambda}]\Big)\\ -\frac{1}{2}\Big([\rho_\Lambda-\hat\rho_{i,\Lambda}], D^2_\Lambda f(\hat\rho_{i,\Lambda};\bar q_{i,\Lambda^c})[\rho_\Lambda-\hat\rho_{i,\Lambda}]\Big)\end{aligned} \tag{6.40}$$

Observe that in $A_{\le,i}$ and for a suitable constant c_1

$$\beta\ell_-^d|\mathcal{R}_i| \le c_1\beta\ell_-^d\sum_{x,s}|\rho_\Lambda(x,s)-\hat\rho_{i,\Lambda}(x,s)|^3 \le c_1\ell_-^d\big(\frac{\ell_+}{\ell_-}\big)^d\ell_-^{3\delta-3d/2}$$

and conclude that the right hand side of the above inequality is estimated by $c\gamma^{d/4}$ as soon as δ satisfies

$$\delta < \frac{d}{6}\left[\frac{1}{2} - 3\alpha_- - 2\alpha_+\right] \tag{6.41}$$

which is compatible with (6.29), see (3.3).

Since $B_{1,\Lambda} = D^2_\Lambda f(\hat\rho_{1,\Lambda};\bar q_{1,\Lambda^c})$ and $\rho^*_1 = \hat\rho_{1,\Lambda}$, by applying Theorem 6.3 with $v=\beta\ell_-^d\mathcal{R}_1$ and $h=\beta\ell_-^d(f(\rho_\Lambda;\bar q_{i,\Lambda^c})-\mathcal{R}_1)$ we get that

$$R_{\Delta_1}\Big(G^\star_\Lambda(\rho_\Lambda|\bar q_{1,\Lambda^c},A_{\le,1}),\mu_1\Big) \le c\gamma^{d/4} \tag{6.42}$$

From Lemma 6.14 and (i) of Theorem 5.1 we get that given any ϵ_2 for γ small enough.

$$\Big|\frac{\beta\ell_-^d}{2}\Big([\rho_\Lambda-\hat\rho_{2,\Lambda}],D^2_\Lambda f(\hat\rho_{2,\Lambda};\bar q_{2,\Lambda^c})[\rho_\Lambda-\hat\rho_{2,\Lambda}]\Big)-\frac{\beta\ell_-^d}{2}\Big([\rho_\Lambda-\rho_2^*],B_{2,\Lambda}[\rho_\Lambda-\rho_2^*]\Big)\Big|$$
$$\le\Big|\frac{\beta\ell_-^d}{2}\Big([\rho_\Lambda-\hat\rho_{2,\Lambda}],\big(D^2_\Lambda f(\hat\rho_{2,\Lambda};\bar q_{2,\Lambda^c})-B_{2,\Lambda}\big)[\rho_\Lambda-\hat\rho_{2,\Lambda}]\Big)_{\bar\Delta_2}\Big|$$
$$+\Big|\frac{\beta\ell_-^d}{2}\Big([\hat\rho_{1,\Lambda}-\hat\rho_{2,\Lambda}],B_{2,\Lambda}[\hat\rho_{1,\Lambda}-\hat\rho_{2,\Lambda}]\Big)_{\bar\Delta_2}\Big|$$
$$\le\Big|\frac{\beta\ell_-^d}{2}\Big([\rho_\Lambda-\hat\rho_{2,\Lambda}],\big(D^2_\Lambda f(\hat\rho_{2,\Lambda};\bar q_{2,\Lambda^c})-D^2_\Lambda f(\hat\rho_{1,\Lambda};\bar q_{2,\Lambda^c})\big)[\rho_\Lambda-\hat\rho_{2,\Lambda}]\Big)_{\bar\Delta_2}\Big|$$
$$+\Big|\frac{\beta\ell_-^d}{2}\Big([\hat\rho_{1,\Lambda}-\hat\rho_{2,\Lambda}],B_{2,\Lambda}[\hat\rho_{1,\Lambda}-\hat\rho_{2,\Lambda}]\Big)_{\bar\Delta_2}\Big|$$
$$\le\frac{\beta\ell_-^d}{2}(\ell_-^{2\delta}+1)\sum_{x\in\ell_{-,\gamma}\mathbb{Z}^d\cap\bar\Delta_2}ce^{-10^{-30}(\gamma\ell_+)\hat\omega}\le\epsilon_2$$

By applying Theorem 6.3 with $v=\beta\ell_-^d[\mathcal{R}_2-\frac{1}{2}([\rho_\Lambda-\rho_2^*],B_{2,\Lambda}[\rho_\Lambda-\rho_2^*])]$ and $h=\beta\ell_-^d(f(\rho_\Lambda;\bar q_{2,\Lambda^c})-v)$ we get that

$$R_{\Delta_1}\Big(G^\star_\Lambda(\rho_\Lambda|\bar q_{2,\Lambda^c},A_{\le,2}),\mu_2\Big)\le c\gamma^{d/4}+\epsilon_2 \tag{6.43}$$

By using the triangular inequality we then get (6.39). □

6.5 Quadratic approximation in continuous variables

In this subsection we consider the conditional probabilities $\mu_i(\cdot|\bar\rho_{i,\Lambda\setminus\Delta_2})$, $\bar\rho_{i,\Lambda\setminus\Delta_2}\in A_{\le,i}$, $i=1,2$. Since $D_{\Delta_2}f(\hat\rho_{i,\Lambda};\bar q_{i,\Lambda^c})=0$, and recalling (6.34) and (6.36), we have that

$$\mu_i(\rho_{\Delta_2}|\bar\rho_{i,\Lambda\setminus\Delta_2}):=\frac{e^{-\beta\ell_-^d\left[\frac{1}{2}\left([\rho_{\Delta_2}-\rho^*],B_{\Delta_2}[\rho_{\Delta_2}-\rho^*]\right)+\left([\rho_{\Delta_2}-\rho^*],B[\bar\rho_{i,\Lambda\setminus\Delta_2}-\rho^*]\right)\right]}\chi_{A_{\le,i}}(\rho_{\Delta_2})}{Z_{i,\Delta_2}(\bar\rho_{i,\Lambda\setminus\Delta_2})} \tag{6.44}$$

where B_{Δ_2} is the matrix B restricted to Δ_2 and where, as usual, $Z_{i,\Delta_2}(\bar\rho_{i,\Lambda\setminus\Delta_2})$ is the sum over ρ_{Δ_2} of the numerator on the right hand side of (6.44).

We compare the probabilities $\mu_i(\cdot|\bar\rho_{i,\Lambda\setminus\Delta_2})$ with measures p_i with the same energy but with continuous state space. To define these measures we start by setting some notations.

By convenience we consider the variables $n_{\Delta_2}=\ell_-^d\rho_{\Delta_2}$, thus $n_{\Delta_2}=(n(x,s),x\in\ell_-\mathbb{Z}^d\cap\Delta_2,s\in\{1,..,S\})$. Since μ_i, $i=1,2$ defined in (6.44) have support on $A_{\le,i}$, the variables n_{Δ_2} are such that

$$[n(x,s)-a^*(x,s)]\in\Big\{-M,-M+1,\dots,M\Big\},\qquad a^*(x,s)=\ell_-^d\rho^*(x,s) \tag{6.45}$$

where M is the integer part of $\ell_-^{d/2+\delta}$ (δ as in Theorem 6.12).
We call $\xi=(\xi(x,s),x\in\ell_-\mathbb{Z}^d\cap\Delta_2,s\in\{1,..,S\})$ with

$$\xi(x,s)=\ell_-^{-d/2}[n(x,s)-a^*(x,s)] \tag{6.46}$$

and we denote by $X_M = \Big\{\xi : \xi(x,s) \in \{-M, -M+1, \dots, M\}\Big\}$. In this new variables the boundary conditions become

$$\xi_i^* = \ell_-^{-d/2} B[\bar n_{i,\Lambda\setminus\Delta_2} - a^*], \qquad \bar n_{i,\Lambda\setminus\Delta_2} = \ell_-^d \bar\rho_{i,\Lambda\setminus\Delta_2} \tag{6.47}$$

By an abuse of notation we call $\mu_i(\xi|\xi_i^*)$ the distribution of the variables ξ under the probabilities $\mu_i(\cdot|\bar\rho_{i,\Lambda\setminus\Delta_2})$ defined in (6.44), thus

$$\mu_i(\xi|\xi_i^*) = \frac{1}{Z(\xi_i^*)} e^{-\beta\left[\frac{1}{2}(\xi, B_{\Delta_2}\xi) + (\xi,\xi_i^*)\right]} \tag{6.48}$$

where $Z(\xi_i^*)$ is the sum over $\xi \in X_M$ of the numerator.
We next introduce variables $\underline{r} = (r(x,s), x \in \Delta_2, s \in \{1,..,S\})$ which take values in the interval of the real line:

$$r(x,s) \in \ell_-^{-d/2}[-M, M+1] \tag{6.49}$$

and we call

$$Y_M = \Big\{\underline{r} : r(x,s) \in \ell_-^{-d/2}[-M, M+1], \forall x \in \Delta_2, s \in \{1,..,S\})\Big\} \tag{6.50}$$

We next define the probabilities measures on Y_M as

$$dp_i(\underline{r}|\xi_i^*) = \frac{1}{Z_M(\xi_i^*)} e^{-\beta\left[\frac{1}{2}(\underline{r}, B_{\Delta_2}\underline{r}) + (\underline{r},\xi_i^*)\right]} \chi_{Y_M}(\underline{r}) d\underline{r}, \qquad i = 1,2 \tag{6.51}$$

where $d\underline{r} = \prod_{x,s} dr(x,s)$ and $Z_M(\xi_i^*)$ is the integral of the numerator.

Proposition 6.16. *For all $\bar\rho_{i,\Lambda\setminus\Delta_2} \in A_{\le,i}$, recalling (6.47) the following holds:*

$$R_{\Delta_1}\Big(\mu_1(\cdot|\xi_1^*)), \mu_2(\cdot|\xi_2^*))\Big) \le R_{\Delta_1}\Big(p_1(\cdot|\xi_1^*), p_2(\cdot|\xi_2^*)\Big) + 2c\gamma^{d/4} \tag{6.52}$$

Proof. Given $\xi \in X_M$ we call $C(\xi) = \{\underline{r} : 0 \le r(x,s) - \xi(x,s) < \ell_-^{-d/2}, \forall x \in \Delta_2, \forall s\}$ we define $H'(\underline{\xi}|\xi_i^*)$ as

$$e^{-H'(\xi|\xi_i^*)} := \int_{C(\xi)} e^{-\beta\ell_-^{-d}\left[\frac{1}{2}(\underline{r}, B_{\Delta_2}\underline{r}) + (\underline{r},\xi_i^*)\right]} d\underline{r} \tag{6.53}$$

and the following probabilities m_i on X_M

$$m_i(\underline{\xi}) = \frac{e^{-H'(\underline{\xi}|\xi_i^*)}}{\sum_{\underline{\xi}\in X_M} e^{-H'(\underline{\xi}|\xi_i^*)}}, \qquad i = 1,2 \tag{6.54}$$

By continuity there is a point $\underline{r}_\xi \in C(\xi)$ such that

$$H'(\xi|\xi_i^*) = \beta\Big[\frac{1}{2}(\underline{r}_\xi, B_{\Delta_2}\underline{r}_\xi) + (\underline{r}_\xi, \xi_i^*)\Big] \tag{6.55}$$

Therefore

$$\Big|H'(\xi)-\beta\big[\frac{1}{2}(\xi,B_{\Delta_2}\xi)+(\xi,\xi_i^*)\big]\Big| \le \sup_{r\in C(\xi)}\|\nabla\{(\underline{r},B_{\Delta_2}\underline{r})/2+\xi_i^*)\}\|\ell_-^{-d/2} \tag{6.56}$$

where $\nabla\psi(\underline{r})$ is the vector defined as the gradient of ψ with respect to the variables $r(x,s)$ and $\|\cdot\|$ is the norm of the vector $\cdot$.
Since $\|B_{\Delta_2}\|\le c^*\frac{|\Delta_2|}{\ell_-^d}$ then

$$\Big|H'(\xi)-\beta\big[\frac{1}{2}(\xi,B_{\Delta_2}\xi)+(\xi,\xi_i^*)\big]\Big| \le c^*\frac{|\Delta_2|}{\ell_-^d}S\ell_-^\delta\ell_-^{-d/2}\le c^*SN_\Lambda\Big(\frac{\ell_+}{\ell_-}\Big)^d\ell_-^{-d/2+\delta} \tag{6.57}$$

For γ small $\left(\frac{\ell_+}{\ell_-}\right)^d\ell_-^{-d/2+\delta}\le\gamma^{d/4}$, thus by Theorem 6.3 and the triangular inequality we get

$$R_{\Delta_1}\Big(\mu_1(\cdot|\bar\rho_{1,\Lambda\setminus\Delta_2}),\mu_2(\cdot|\bar\rho_{2,\Lambda\setminus\Delta_2})\Big)\le R_{\Delta_1}(m_1,m_2)+2c\gamma^{d/4} \tag{6.58}$$

We now observe that at any coupling Q of p_1 and p_2 we can associate a coupling Q^* of m_1 and m_2 by setting

$$Q^*(\xi',\xi'')=Q\big(C(\xi')\times C(\xi'')\big)$$

To prove that Q^* is indeed a coupling of m_1 and m_2 we compute for any function ψ on X_M

$$\begin{aligned}\sum_{\xi''}\sum_{\xi'}\psi(\xi')Q^*(\xi',\xi'') &= \sum_{\xi'}\psi(\xi')p_1(C(\xi'))\\ &= \frac{1}{Z_M(\xi_i^*)}\sum_{\xi'}\psi(\xi')\int_{C(\xi')}e^{-\beta\ell_-^{-d}\left[\frac{1}{2}(\underline{r},B_{\Delta_2}\underline{r})+(\underline{r},\xi_i^*)\right]}d\underline{r}\\ &= \sum_{\xi'}\psi(\xi')m_1(\xi')\end{aligned}$$

Thus

$$\forall Q,\quad R_{\Delta_1}(m_1,m_2)\le\sum_{\xi'',\xi'}d_{\Delta_1}(\xi',\xi')Q^*(\xi',\xi'') \tag{6.59}$$

We next observe that

$$\begin{aligned}\sum_{\xi'',\xi'}d_{\Delta_1}(\xi',\xi')Q^*(\xi',\xi'') &= \sum_{\xi',\xi'}\int_{C(\xi')\times C(\xi'')}d_{\Delta_1}(\xi',\xi')dQ(\underline{r}',\underline{r}'')\\ &\le\sum_{\xi',\xi'}\int_{C(\xi')\times C(\xi'')}d_{\Delta_1}(\underline{r}',\underline{r}'')dQ(\underline{r}',\underline{r}'')\end{aligned}$$

Taking the inf over the coupling Q in the above inequality and using (6.59), we get that $R_{\Delta_1}(m_1,m_2)\le R_{\Delta_1}\Big(p_1(\cdot|\xi_1^*),p_2(\cdot|\xi_2^*)\Big)$, thus (6.58) implies (6.52). □

6.6 Gaussian approximation

We now extend the measures $p_i(\cdot|\xi_1^*)$ on Y_M to a measures P_i, $i=1,2$, on the full Euclidean space, thus P_i, $i=1,2$ are the Gaussian measure defined by the r.h.s. of (6.51) without the last characteristic function.
Thus letting $\underline{r}=(r(x,s)\in\mathbb{R}^d : x\in\Delta_2, s\in\{1,\dots S\})$,

$$dP_i(\underline{r}|\xi_i^*)=\frac{1}{Z(\xi_i^*)}e^{-\beta\left[\frac{1}{2}(\underline{r},B_{\Delta_2}\underline{r})+(\underline{r},\xi_i^*)\right]}d\underline{r} \tag{6.60}$$

with $Z(\xi_i^*)$ the integral of the numerator.

The following holds:

Proposition 6.17. *There is $\delta^*>0$ such that the following holds:*

$$R_{\Delta_1}\Big(p_1(\cdot|\xi_1^*),p_2(\cdot|\xi_2^*)\Big)\le R_{\Delta_1}\Big(P_1(\cdot|\xi_1^*),P_2(\cdot|\xi_2^*)\Big)+2\gamma^{\delta^*} \tag{6.61}$$

Proof. By the Chebischev's inequality, and recalling that $\mathrm{Var}P_i(\cdot|\xi_i^*)=\|B_{\Delta_2}\|^{-1}$, there is c such that

$$P_i\big(\{|r(x,s)|\ge\ell_-^{\delta}\}\big)\le c\ell_-^{-2\delta}(\frac{\ell_+}{\ell_-})^{-d},\qquad i=1,2$$

By (6.29) there is $\delta^*>0$ such that

$$P_i\big(Y_M^c\big)\le\sum_s\sum_{x\in\ell_-\cap\mathbb{Z}^d\cap\Delta_2}P_i\big(\{|r(x,s)|\ge\ell_-^{\delta}\}\big)\le\gamma^{\delta^*} \tag{6.62}$$

Since p_i is equal to the probability P_i conditioned to the set Y_M, by using Theorem 6.4 and the triangular inequality, we get (6.61). □

We are thus left with the estimate of $R_{\Delta_1}(P_1,P_2)$ that we do next.

Proposition 6.18. *There is $\epsilon_3>0$ such that the following holds:*

$$R_{\Delta_1}\Big(P_1(\cdot|\xi_1^*),P_2(\cdot|\xi_2^*)\Big)\le\epsilon_3 \tag{6.63}$$

Proof. We first observe that from the definition of the Wasserstein distance

$$R_{\Delta_1}\Big(P_1(\cdot|\underline{b}_1),P_2(\cdot|\underline{b}_2)\Big)=\inf_Q Q\big(r_{\Delta_1}\ne r'_{\Delta_1}\big) \tag{6.64}$$

where r_{Δ_1} is the restriction of $\underline{r}$ to Δ_1, namely $r_{\Delta_1}\in\mathcal{Y}_{\Delta_1}:=\{r(x,s)\in\mathbb{R}^d, x\in\Delta_1, s=1,\dots S\}$. Thus the inf on the r.h.s. of (6.64) can be restricted to all couplings of the marginals P_{i,Δ_1} on the set $\mathcal{Y}_{\Delta_1}$ of the probabilities P_i, $i=1,2$.
Recalling (6.47) we define

$$b_i=B_{\Delta_2}^{-1}\xi_i^*=\ell_-^{-d/2}B_{\Delta_2}^{-1}\big(B[\bar{n}_{i,\Lambda\setminus\Delta_2}-a^*]\big),\qquad i=1,2 \tag{6.65}$$

We call b_{i,Δ_1} the restriction of the vector b_i to the set Δ_1.
We next call C the matrix with entries $C_{i,j} = (B_{\Delta_2})^{-1}_{i,j}$, $i = (x, s)$, $j = (x', s')$, $x, x' \in \Delta_2$, $s, s' \in \{1, \dots S\}\}$, $C^{-1}_{\Delta_1}$ denotes the restriction to Δ_1 of C^{-1}.
Then remark that marginals of Gaussian variables are Gaussian themselves, so we get:

$$dP_{i,\Delta_1}(r_{\Delta_1}) = \psi(r_{\Delta_1} - b_{i,\Delta_1})dr_{\Delta_1}, \qquad \psi(r_{\Delta_1} - b_{i,\Delta_1}) = Z_i^{-1} e^{-\frac{1}{2}(r_{\Delta_1} - b_{i,\Delta_1}, C^{-1}_{\Delta_1}(r_{\Delta_1} - b_{i,\Delta_1}))} \tag{6.66}$$

We use that the Wasserstein distance is related to the variational distance via the following relation

$$2R_{\Delta_1}\big(P_{1,\Delta_1}, P_{2,\Delta_1}\big) = \|P_{1,\Delta_1} - P_{2,\Delta_1}\| \tag{6.67}$$

where

$$\|P_{1,\Delta_1} - P_{2,\Delta_1}\| := \int |\psi(r_{\Delta_1} - b_{1,\Delta_1}) - \psi(r_{\Delta_1} - b_{2,\Delta_1})| dr_{\Delta_1} \tag{6.68}$$

We now prove that

$$\|P_{1,\Delta_1} - P_{2,\Delta_1}\| \le 2\|C^{-1}_{\Delta_1}\| \|b_{1,\Delta_1} - b_{2,\Delta_1}\|_{L^2} \Big(\sum_{i=(x,s), x \in \ell_- \mathbb{Z}^d \cap \Delta_1} C_{ii} \Big)^{1/2} \tag{6.69}$$

To prove (6.69) we interpolate defining $M(t) = tb_{1,\Delta_1} + (1 - t)b_{2,\Delta_1}$, $t \in [0, 1]$. Then, shorthanding $M = M(t)$,

$$\text{l.h.s. of(6.69)} \le 2 \int_0^1 \int |\Big(b_{1,\Delta_1} - b_{2,\Delta_1}, C^{-1}_{\Delta_1}(r_{\Delta_1} - M)\Big)| \psi(r_{\Delta_1} - M) dr_{\Delta_1}\, dt \tag{6.70}$$

Using Cauchy-Schwartz the r.h.s. is bounded by

$$\le 2\|C^{-1}_{\Delta_1}\| \|b_{1,\Delta_1} - b_{2,\Delta_1}\|_{L^2} \int_0^1 \int \Big(\sum_{s,x \in \Delta_1} (r(x, s) - M(x, s))^2 \Big)^{1/2} \psi(r_{\Delta_1} - M) dr_{\Delta_1}\, dt \tag{6.71}$$

hence (6.69).
To estimate $\|b_{1,\Delta_1} - b_{2,\Delta_1}\|_{L^2}$, we apply Theorem A.1 with $C' = C'' = I$, I the identity matrix, and with $A = B_{\Delta_2}$, observing that $B_{\Delta_2}(x, s, x's') = 0$ whenever $|x - x'| > \gamma^{-1}N$. Thus from (A.10) and (A.5), using that $\bar\rho_{i,\Lambda\setminus\Delta_2} \in A_{\le,i}$, $i = 1, 2$, (6.47) and (6.45) we get that there are c and c', such that for all $x \in \Delta_1$ and since $\text{dist}(\Delta_1, \Delta_2^c) > 10^{-30}\ell_+$

$$\begin{aligned}\big|b_{1,\Delta_1}(x, s) - b_{2,\Delta_1}(x, s)\big| &= \Big| \sum_{s',y \in \Lambda\setminus\Delta_2} B^{-1}_{\Delta_2}(x, s, y, s') B(\ell_-^{-d/2} \bar n_{1,\Lambda\setminus\Delta_2}(y, s) - \bar n_{2,\Lambda\setminus\Delta_2}(y, s))\Big| \\ &\le \|B\| \ell_-^\delta \sum_{s',y \in \Lambda\setminus\Delta_2} e^{-c|x-y|\gamma} \le c' \ell_-^\delta e^{-c\gamma 10^{-30}\ell_+}\end{aligned}$$

Thus this inequality together with (6.67) and (6.69) implies (6.63). □

Proof of Theorem 6.2. Recalling the definition (6.38) of the probabilities μ_i, and the conditional probabilities defined in (6.44), from Propositions 6.16, 6.17, 6.18 we get that for all $\bar\rho_{i,\Lambda\setminus\Delta_2}\in A_{\le,i}$, $i=1,2$,

$$R_{\Delta_1}\big(\mu_1(\cdot|\bar\rho_{1,\Lambda\setminus\Delta_2}),\mu_2(\cdot|\bar\rho_{2,\Lambda\setminus\Delta_2})\big)\le 2c\gamma^{d/4}+2\gamma^{\delta^*}+\epsilon_3=\epsilon_4$$

Thus, there is a coupling $\hat Q\big(n'_{\Delta_2},n''_{\Delta_2}|\bar\rho_{1,\Lambda\setminus\Delta_2},\bar\rho_{2,\Lambda\setminus\Delta_2}\big)$ of the conditional probabilities $\mu_i(\cdot|\bar\rho_{i,\Lambda\setminus\Delta_2})$, $i=1,2$ such that

$$\hat Q\big(n'_{\Delta_1}\neq n''_{\Delta_1}|\bar\rho_{1,\Lambda\setminus\Delta_2},\bar\rho_{2,\Lambda\setminus\Delta_2}\big)\le 2\epsilon_4 \tag{6.72}$$

We define for all $\bar\rho_{i,\Lambda\setminus\Delta_2}$

$$Q\big(n'_{\Delta_2},n''_{\Delta_2}|\bar\rho_{1,\Lambda\setminus\Delta_2},\bar\rho_{2,\Lambda\setminus\Delta_2}\big)=\begin{cases}\hat Q(n'_{\Delta_2},n''_{\Delta_2}|\bar\rho_{1,\Lambda\setminus\Delta_2},\bar\rho_{2,\Lambda\setminus\Delta_2}) & \text{if } \bar\rho_{i,\Lambda\setminus\Delta_2}\in A_{\le,i}, i=1,2\\ dG^0_\Lambda(n'_{\Delta_2}|\bar\rho_{1,\Lambda\setminus\Delta_2},\bar q_{1,\Lambda^c})dG^0_\Lambda(n''_{\Delta_2}|\bar\rho_{2,\Lambda\setminus\Delta_2},\bar q_{2,\Lambda^c}) & \text{otherwise}\end{cases}$$

We then define a coupling Q of the measures μ_i by letting

$$Q\big(n'_{\Delta_2},n''_{\Delta_2}\big)=Q\big(n'_{\Delta_2},n''_{\Delta_2}|\bar\rho_{1,\Lambda\setminus\Delta_2},\bar\rho_{2,\Lambda\setminus\Delta_2}\big)dG^0_\Lambda\big(\bar\rho_{1,\Lambda\setminus\Delta_2}|\bar q_{1,\Lambda^c}\big)dG^0_\Lambda\big(\bar\rho_{2,\Lambda\setminus\Delta_2}|\bar q_{2,\Lambda^c}\big) \tag{6.73}$$

From (6.27), (6.72) and Theorem 6.12 it follows that

$$Q\big(n'_{\Delta_1}\neq n''_{\Delta_1}\big)\le 2\epsilon_4+2c\gamma^\tau+2e^{-c\ell_-^{2\delta}}=\epsilon_5 \tag{6.74}$$

Observe that (6.74) implies that

$$R_{\Delta_1}(\mu_1,\mu_2)\le\epsilon_5 \tag{6.75}$$

Then, (6.75), Propositions 6.11, 6.13, 6.15 implies (6.9). □

6.7 Proof of Theorem 3.3

We need to construct a coupling Q_Λ such that (6.2) holds.

Recall $\mathring\Delta_1=\Delta_1\setminus\delta_{\rm in}^{\gamma^{-1}}[\Delta_1]$ and that for any two configurations $\bar q_{i,\Lambda\setminus\mathring\Delta_1}$, $i=1,2$ on $\mathcal X^{(k)}_{\Lambda\setminus\mathring\Delta_1}$ we denote by $\bar q_{i,\mathring\Delta_1^c}=\bar q_{i,\Lambda\setminus\mathring\Delta_1}\cup\bar q_{i,\Lambda^c}$, $i=1,2$. From Theorem 6.1 we have that, for any n_{Δ_1}, there is a coupling $Q_{\mathring\Delta_1}\big(q'_{\bar\Delta_1},q''_{\mathring\Delta_1}|\bar q_{1,\mathring\Delta_1^c},\bar q_{2,\bar\Delta_1^c},n_{\Delta_1}\big)$ of the two conditional Gibbs measures $dG^0_\Lambda(q_{\mathring\Delta_1}|q_{i,\mathring\Delta_1^c},n_{\Delta_1})$, $i=1,2$ such that

$$\sum_{x\in\ell_{-,\gamma}\mathbb Z^d\cap\Delta_0}Q_{\mathring\Delta_1}\big(q'_\Lambda\cap C_x^{(\ell_-,\gamma)}\neq q''_\Lambda\cap C_x^{(\ell_-,\gamma)}|\bar q_{1,\mathring\Delta_1^c},\bar q_{2,\bar\Delta_1^c},n_{\Delta_1}\big)\le 2\epsilon_0 \tag{6.76}$$

Given $\underline{n}'$ and $\underline{n}''$, we define a coupling $\hat Q_{\mathring\Delta_1}\equiv\hat Q_{\mathring\Delta_1}\big(q'_{\mathring\Delta_1}q''_{\mathring\Delta_1}|\bar q_{1,\mathring\Delta_1^c},\bar q_{2,\mathring\Delta_1^c},\underline{n}',\underline{n}''\big)$ of $dG^0_\Lambda(\cdot|\bar q_{1,\mathring\Delta_1^c},\underline{n}')$, $dG^0_\Lambda(\cdot|,\bar q_{2,\mathring\Delta_1^c},\underline{n}'')$, by setting

$$\hat Q_{\mathring\Delta_1}=\begin{cases}Q_{\mathring\Delta_1} & \text{if } n'_{\Delta_1}=n''_{\Delta_1}\\ dG^0_\Lambda(\cdot|\bar q_{1,\mathring\Delta_1^c},\underline{n}')dG^0_\Lambda(\cdot|,\bar q_{2,\mathring\Delta_1^c},\underline{n}'') & \text{otherwise}\end{cases}$$

From Theorem 6.2 there is a coupling Q^* of $G^0_\Lambda(n_\Lambda|q_{i,\Lambda^c})$, $i=1,2$ such that

$$Q^*(n'_{\Delta_1}\neq n''_{\Delta_1})\le 2\epsilon_1 \tag{6.77}$$

Then the final coupling Q_Λ is defined as follows:

$$Q_\Lambda(q'_\Lambda, q''_\Lambda) = \hat{Q}_{\mathring{\Delta}_1}\big(q'_{\mathring{\Delta}_1} q''_{\mathring{\Delta}_1} | \bar{q}_{1,\mathring{\Delta}_1^c}, \bar{q}_{2,\mathring{\Delta}_1^c}, \underline{n}', \underline{n}''\big)$$
$$dG^0_\Lambda(q'_{\Lambda\setminus\mathring{\Delta}_1} | \bar{q}_{1,\Lambda^c}, \underline{n}') dG^0_\Lambda(q''_{\Lambda\setminus\mathring{\Delta}_1} | \bar{q}_{2,\Lambda^c}, \underline{n}'') Q^*(n', n'') \tag{6.78}$$

Thus from (6.76) and (6.77) we get

$$\sum_{x \in \ell_{-,\gamma}\mathbb{Z}^d \cap \Delta_0} Q_\Lambda\big(q'_\Lambda \cap C_x^{(\ell_-,\gamma)} \neq q''_\Lambda \cap C_x^{(\ell_-,\gamma)}\big) \le \epsilon_6 \tag{6.79}$$

To complete the proof of (6.2) we need to show that

$$\sum_{s=1}^{S} \sum_{x\in\Delta_0} Q\big(q'_\Lambda \cap C_x^{(\ell_-,\gamma)} = q''_\Lambda \cap C_x^{(\ell_-,\gamma)}, |\rho^{(\ell_-,\gamma)}(q'_\Lambda; x, s) - \rho_s^{(k)}| > \zeta_{K(\cdot;x)-1}\big) \le \epsilon \tag{6.80}$$

Since in the set on the l.h.s. of (6.80), $q'_\Lambda = q''_\Lambda$, by using (6.27) we have

$$\begin{aligned}
Q_\Lambda\big(q'_\Lambda \cap C_x^{(\ell_-,\gamma)} &= q''_\Lambda \cap C_x^{(\ell_-,\gamma)},\ |\rho^{(\ell_-,\gamma)}(q'_\Lambda; x, s) - \rho_s^{(k)}| > \zeta_{K(\cdot;x)-1}\big) \\
&\le G^0_\Lambda\big(|\rho^{(\ell_-,\gamma)}(q'_\Lambda; x, s) - \rho_s^{(k)} > \zeta_{K(\cdot;x)-1}; \bar{q}'_\Lambda\big) \\
&\quad + G^0_\Lambda\big(|\rho^{(\ell_-,\gamma)}(q''_\Lambda; x, s) - \rho_s^{(k)}| > \zeta_{K(\cdot;x)-1}; \bar{q}''_\Lambda\big) \\
&\le G^*_\Lambda(|\rho^{(\ell_-,\gamma)}(q'_\Lambda; x, s) - \rho_s^{(k)}| > \zeta_{K(\cdot;x)-1} | \bar{q}'_{\Lambda^c}) \\
&\quad + G^*_\Lambda(|\rho^{(\ell_-,\gamma)}(q''_\Lambda; x, s) - \rho_s^{(k)}| > \zeta_{K(\cdot;x)-1} | \bar{q}''_{\Lambda^c}) + 2c\gamma^\tau
\end{aligned} \tag{6.81}$$

From Theorem 6.12 and (ii) of Theorem 5.1 it follows that for all $x \in \Delta_0$ and for $\bar{q}_{\Lambda^c} = \bar{q}'_{\Lambda^c}$ or $\bar{q}''_{\Lambda^c}$),

$$G^*_\Lambda(|\rho^{(\ell_-,\gamma)}(q'_\Lambda; x, s) - \rho_s^{(k)}| > \zeta_{K(\cdot;x)-1} | \bar{q}_{\Lambda^c}) \le e^{-c\ell_-^{2\delta}} \tag{6.82}$$

which together with (6.81) proves Theorem 3.3. □

Part III
Disagreement percolation

In this part we fix $t \in [0, 1]$, a bounded $\mathcal{D}^{\ell_+,\gamma}$-measurable region Λ, $k \in \{1, \dots, S+1\}$; μ' and μ'' stand for the measures $dG_\Lambda(q_\Lambda, \underline{\Gamma} | \bar{q}'_{\Lambda^c}, \bar{\underline{\Gamma}}'_{\Lambda^c})$ and $dG_\Lambda(q_\Lambda, \underline{\Gamma} | \bar{q}''_{\Lambda^c}, \bar{\underline{\Gamma}}''_{\Lambda^c})$. They are obtained by conditioning measures ν' and ν'' which could be either DLR measures or Gibbs measures $dG_{\Lambda'}(q_{\Lambda'}, \underline{\Gamma} | \bar{q}_{(\Lambda')^c})$ with $\Lambda' \supseteq \Lambda$. We will first construct a coupling of μ' and μ'' and, with the help of such a coupling, we will then define a coupling of ν' and ν'' proving that it satisfies the requirements of Theorem 3.1. The notation which are most used in this part are reported below.

Main notation and definitions.

We call

$$\xi = (q, \underline{\Gamma}) \in \mathcal{X}_\Lambda^{(k)} \times \mathcal{B}_\Lambda \tag{6.83}$$

Given a $\mathcal{D}^{(\ell_+,\gamma)}$ measurable subset Δ of Λ and $\xi = (q, \underline{\Gamma})$, we call $\xi_\Delta = (q_\Delta, \underline{\Gamma}_\Delta)$ its restriction to Δ. Namely if $\underline{\Gamma} = (\Gamma(1), \dots \Gamma(n))$, then

$$\Gamma_\Delta(i) = \big(\mathrm{sp}[\Gamma(\mathrm{i})] \cap \Delta, \eta_{\mathrm{sp}[\Gamma(\mathrm{i})]\cap\Delta}\big), \qquad \underline{\Gamma}_\Delta = (\Gamma_\Delta(1), \dots \Gamma_\Delta(\mathrm{n})) \tag{6.84}$$

We will say that we vary ξ in Δ^c if we change ξ leaving ξ_Δ invariant.
We denote by Ω_Λ the product space,

$$\Omega_\Lambda = (\mathcal{X}^{(k)}_\Lambda \times \mathcal{B}_\Lambda)^2, \qquad\qquad \omega = (\xi, \xi') \in \Omega_\Lambda \tag{6.85}$$

Given a subset $\Delta \subset \Lambda$ and $\omega = (\xi, \xi') \in \Omega_\Lambda$, we call $\omega_\Delta = (\xi_\Delta, \xi'_\Delta) \in \Omega_\Delta$ its restriction to Δ. We call $\mathcal{F}_\Lambda$ the σ-algebra of all Borel sets in Ω_Λ and for any $\mathcal{D}^{(\ell_+,\gamma)}$ measurable set Δ in Λ we call $\mathcal{F}_\Delta$ the σ-algebra of all Borel sets A such that $\mathbf{1}_A(\omega)$ does not vary when we change ω in Δ^c.

7 Construction of the coupling

The target of this section is to construct a "good" coupling Q of μ' and μ''. The basic idea is to implement the disagreement percolation technique used in van der Berg and Maes, [3], Butta et al., [6], Lebowitz et al,[13]. The first step is to introduce a sequence of random sets Λ_n, which is done in the next subsection. We will then introduce the notion of "stopping sets" and "strong Markov couplings" showing that the sets Λ_n are indeed stopping sets and, using the strong Markov coupling property, we will finally get the desired coupling of μ' and μ''.

7.1 The sequence Λ_n

We will define here for each $\omega = (\xi', \xi'') \in \Omega_\Lambda$ a decreasing sequence of $\mathcal{D}^{(\ell_+,\gamma)}$-measurable sets Λ_n, which are therefore set valued random variables. We set $\Lambda_0 = \Lambda$ and for $n \geq 0$, define $\Lambda_{n+1} = \Lambda_n \setminus \Sigma_{n+1}$, thus the sequence is defined once we specify the "screening sets" Σ_n. Screening sets are defined iteratively with the help of the notion of "good" and "bad cubes".
After defining in an arbitrary fashion an order among the $\mathcal{D}^{(\ell_+,\gamma)}$ cubes of $\delta^{\ell_+,\gamma}_{\mathrm{out}}[\Delta]$, for any $\mathcal{D}^{(\ell_+,\gamma)}$-measurable set $\Delta \subset \Lambda$, we start the definition by calling bad all the cubes of $\delta^{\ell_+,\gamma}_{\mathrm{out}}[\Lambda_0]$. We then select among these the first one (according to the pre-definite order) which intersects a polymer (i.e. either $\mathrm{sp}(\underline{\Gamma}') \cap C \neq \emptyset$, or $\mathrm{sp}(\underline{\Gamma}'') \cap C \neq \emptyset$), if there is no such cube we then take the first cube in $\delta^{\ell_+,\gamma}_{\mathrm{out}}[\Lambda_0]$. Call C_1 the cube selected with such a rule. We then define $\Sigma_1 = \delta^{\ell_+,\gamma}_{\mathrm{out}}[C_1] \cap \Lambda_0$ and call bad all cubes of Σ_1 if C_1 intersects a polymer. If not, we say that a $\mathcal{D}^{(\ell_+,\gamma)}$ cube $C \in \Sigma_1$ is good if $\mathrm{sp}(\underline{\Gamma}') \cap C = \mathrm{sp}(\underline{\Gamma}'') \cap C = \emptyset$ and if

$$\omega \in \bigcap_{x \in \ell_{-\gamma}\mathbb{Z}^d \cap C} \Theta_{\Lambda_0}(x), \quad \Theta_{\Lambda_0} \text{ has been defined in (3.24),} \tag{7.1}$$

otherwise $C \in \Sigma_1$ is called bad. In this way each cube of Σ_1 is classified as good or bad and therefore all cubes of $\delta_{\rm out}^{\ell_+,\gamma}[\Lambda_1]$ are classified as good or bad. We then select C_2 in $\delta_{\rm out}^{\ell_+,\gamma}[\Lambda_1]$ in the same way we had selected C_1 in $\delta_{\rm out}^{\ell_+,\gamma}[\Lambda_0]$, $\Sigma_2 = \delta_{\rm out}^{\ell_+,\gamma}[C_2] \cap \Lambda_1$ and the cubes of Σ_2 are then classified as good or bad by the same rule used for those of Σ_1. By iteration we then define a sequence which becomes eventually constant, as it stops changing at Λ_n if $\delta_{\rm out}^{\ell_+,\gamma}[\Lambda_n]$ has no bad cube or if Λ_n is empty. Since Λ has $N^* := |\Lambda|/\ell_{+,\gamma}^d$ cubes, Λ_n is certainly constant after N^*, but maybe even earlier. In Appendix B we will prove:

Theorem 7.1. *If the sequence $\{\Lambda_n\}$ stops at $n = N$ and Λ_N is non empty, then*

$$q'_\Lambda \cap \delta_{\rm out}^{\gamma^{-1}}[\Lambda_N] = q''_\Lambda \cap \delta_{\rm out}^{\gamma^{-1}}[\Lambda_N] \tag{7.2}$$

and

$$\mathrm{sp}(\underline{\Gamma}') \cap \delta_{\rm out}^{\gamma^{-1}}[\Lambda_N] = \mathrm{sp}(\underline{\Gamma}'') \cap \delta_{\rm out}^{\gamma^{-1}}[\Lambda_N] = \emptyset \tag{7.3}$$

7.2 Stopping sets

The random variables Λ_n are "stopping sets" and the sequence Λ_n is decreasing, $\Lambda_{n+1} \preccurlyeq \Lambda_n$, in the following sense.

- $\mathcal{F}_{\Delta^c}$, Δ a $\mathcal{D}^{(\ell_+,\gamma)}$ measurable subset of Λ, is the σ algebra of all Borel sets A such that $\mathbf{1}_A(\omega)$ does not change if we vary ω in Δ.
- A random variable $\mathcal{R}$ with values in the $\mathcal{D}^{(\ell_+,\gamma)}$ measurable subsets of Λ is called a stopping set if for all Δ,

$$\{\omega \in \Omega : \mathcal{R}(\omega) = \Delta\} \in \mathcal{F}_{\Delta^c} \tag{7.4}$$

- Two stopping sets $\mathcal{R}'$ and $\mathcal{R}$ are such that $\mathcal{R}' \preccurlyeq \mathcal{R}$ if

$$\mathcal{R}'(\omega) \subset \mathcal{R}(\omega), \quad \text{for all } \omega \in \Omega_\Lambda$$
$$\{\omega : \mathcal{R}'(\omega) = \Delta'\} \cap \{\omega : \mathcal{R}(\omega) = \Delta\} \in \mathcal{F}_{\Delta^c}, \quad \text{for all } \Delta' \subset \Delta$$

7.3 Strong Markov couplings

A coupling $Q(d\omega)$ of μ' and μ'' is called strong Markov in $\mathcal{R}$, $\mathcal{R}$ a stopping set, if the measure

$$d\tilde{Q}(\omega) := \sum_{\Delta \subset \Lambda} \mathbf{1}_{\{\mathcal{R}(\omega)=\Delta\}} d\pi_\Delta(\omega_\Delta|\bar{\omega}_{\Delta^c}) dQ(\bar{\omega}_{\Delta^c}) \tag{7.5}$$

is also a coupling of μ' and μ'' for all couplings $d\pi_\Delta(\omega_\Delta|\bar{\omega}_{\Delta^c})$ of $d\mu'(\xi_\Delta|\bar{\xi}_\Delta)$, and $d\mu''(\xi'_\Delta|\bar{\xi}'_\Delta)$.

Theorem 7.2. *Given any stopping set $\mathcal{R}$, let Q be a coupling of μ' and μ'' which is strong Markov in $\mathcal{R}$, Then any coupling $\tilde{Q}$ defined by (7.5) is strong Markov in $\mathcal{R}'$ provided the stopping set $\mathcal{R}'$ is such that $\mathcal{R}' \preccurlyeq \mathcal{R}$,*

Proof. We have to prove that for any family of couplings $\{\hat{\pi}_\Delta(d\omega_\Delta|\bar{\omega}_{\Delta^c}), \Delta \subset \Lambda, \bar{\omega}_{\Delta^c} \in \Omega_{\Delta^c}\}$, the probability $\hat{Q}(d\omega)$ defined as

$$d\hat{Q}(\omega) := \sum_{A\subset\Lambda} \mathbf{1}_{\{\mathcal{R}'(\omega)=A\}} d\hat{\pi}_\Delta(\omega_A|\bar{\omega}_{A^c}) d\tilde{Q}(\omega_{A^c}) \tag{7.6}$$

is a coupling of μ' and μ''. We thus take a function $f(\xi)$ and we prove that $\hat{Q}(f) = \mu'(f)$, where $\hat{Q}(f)$, $\mu'(f)$, is the expectation of f under Q, respectively μ'.
Using that $\mathcal{R}'$ is a stopping set we get

$$\begin{aligned}
\hat{Q}(f) &= \sum_{A\subset\Lambda} \int_{\Omega_{A^c}} \mathbf{1}_{\{\mathcal{R}'(\omega)=A\}} d\tilde{Q}(\omega_{A^c}) \int_{\Omega_A} f(\xi) d\hat{\pi}_A(\omega_A|\bar{\omega}_{A^c}) \\
&= \sum_{A\subset\Lambda} \int_{\Omega_{A^c}} \mathbf{1}_{\{\mathcal{R}'(\omega)=A\}} d\tilde{Q}(\omega_{A^c}) \mu'(f|\xi_{A^c}) \\
&= \sum_{A\subset\Lambda} \int_{\Omega} \mathbf{1}_{\{\mathcal{R}'(\omega)=A\}} d\tilde{Q}(\omega) \mu'(f|\xi_{A^c})
\end{aligned}$$

We now rewrite $d\tilde{Q}(\omega)$ by using its definition (7.5) and since $\mathcal{R}' \preccurlyeq \mathcal{R}$ we get

$$\hat{Q}(f) = \sum_{\Delta\subset\Lambda} \sum_{A\subset\Delta} \int_{\Omega_{\Delta^c}} \mathbf{1}_{\{\mathcal{R}(\omega)=\Delta\}} \mathbf{1}_{\{\mathcal{R}'(\omega)=A\}} dQ(\bar{\omega}_{\Delta^c}) \int_{\Omega_\Delta} d\pi_\Delta(\omega_\Delta|\bar{\omega}_{\Delta^c}) \mu'(f|\xi_{A^c}) \tag{7.7}$$

Observe that (recalling $A \subset \Delta$)

$$\int_{\Omega_\Delta} d\pi_\Delta(\omega_\Delta|\bar{\omega}_{\Delta^c}) \mu'(f|\xi_{A^c}) = \int d\mu'(\xi_\Delta|\xi_{\Delta^c}) \mu'(f|\xi_{\Delta^c}, \xi_{\Delta\setminus A}) = \mu'(f|\xi_{\Delta^c}) \tag{7.8}$$

We insert (7.8) in (7.7) and we get

$$\hat{Q}(f) = \sum_{\Delta\subset\Lambda} \int_{\Omega_{\Delta^c}} \mathbf{1}_{\{\mathcal{R}(\omega)=\Delta\}} dQ(\omega_{\Delta^c}) \mu'(f|\xi_{\Delta^c}) = \mu'(f)$$

The Theorem is proved. □

7.4 Construction of couplings

We use the sequence $\{\Lambda_n\}$ of decreasing stopping sets (in the order $\preccurlyeq$) and Theorem 7.2 to construct a sequence $\{Q^n\}$ of couplings of μ' and μ'', the desired coupling will then be Q^{N^*}, where $N^* = |\Lambda|/\ell^d_{+,\gamma}$. The sequence $\{Q^n\}$ is defined iteratively by setting Q^0 equal to the product coupling: $Q^0 = \mu' \times \mu''$ which, as it can be easily checked, is strong Markov in Λ_0. Then for any $n \geq 0$ we set

$$dQ^{n+1}(\omega_\Lambda) = \sum_{\Delta \neq \emptyset} \mathbf{1}_{\{\Lambda_n(\omega_\Lambda)=\Delta\}} d\pi_\Delta(\omega_\Delta|\omega_{\Lambda\setminus\Delta}, \bar{\omega}_{\Lambda^c}) dQ^n(\omega_{\Lambda\setminus\Delta}) + \mathbf{1}_{\{\Lambda_n(\omega_\Lambda)=\emptyset\}} dQ^n(\omega_\Lambda) \quad (7.9)$$

where $dQ^n(\omega_{\Delta^c})$ is the marginal of dQ^n over $\{\omega_{\Delta^c}\}$ and π_Δ, $\Delta \neq \emptyset$, is the coupling of $d\mu'(\xi'_\Delta|\bar{\xi}'_{\Delta^c})$, and $d\mu''(\xi''_\Delta|\bar{\xi}''_{\Delta^c})$ defined next. We distinguish three cases according to the values of $\bar{\omega}_{\Delta^c} = (\bar{\xi}'_{\Delta^c}, \bar{\xi}''_{\Delta^c})$.

- If $\bar{\omega}_{\Delta^c}$ is such that either $\mathrm{sp}(\underline{\Gamma}') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] \neq \emptyset$, or $\mathrm{sp}(\underline{\Gamma}'') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] \neq \emptyset$, or both, then π_Δ is the product coupling: $d\pi_\Delta(\xi'_\Delta, \xi''_\Delta|\bar{\omega}_{\Delta^c}) = d\mu'(\xi'_\Delta|\bar{\xi}'_{\Delta^c}) d\mu'(\xi''_\Delta|\bar{\xi}''_{\Delta^c})$.

- If $\bar{\omega}_{\Delta^c}$ is such that $\mathrm{sp}(\underline{\Gamma}') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] = \mathrm{sp}(\underline{\Gamma}'') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] = \emptyset$ and $q' \cap \delta^{\gamma^{-1}}_{\text{out}}[\Delta] = q'' \cap \delta^{\gamma^{-1}}_{\text{out}}[\Delta]$ then $d\pi_\Delta(\xi'_\Delta, \xi''_\Delta|\bar{\omega}_{\Delta^c}) = d\mu'(\xi'_\Delta|\bar{\xi}'_{\Delta^c}) \delta(\xi'_\Delta - \xi''_\Delta) d\xi''_\Delta$, namely $d\pi_\Delta$ is the coupling supported by the diagonal.

- Finally let $\bar{\omega}_{\Delta^c}$ be such that $\mathrm{sp}(\underline{\Gamma}') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] = \mathrm{sp}(\underline{\Gamma}'') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] = \emptyset$ but $q' \cap \delta^{\gamma^{-1}}_{\text{out}}[\Delta] \neq q'' \cap \delta^{\gamma^{-1}}_{\text{out}}[\Delta]$. Call $T = \Sigma_{n+1} \cup \left(\delta^{\ell+,\gamma}_{\text{out}}[\Sigma_{n+1}] \cap \Delta\right)$, $U = \Delta \setminus T$. Let $dP(q'_U, q''_U, \underline{\Gamma}', \underline{\Gamma}'') = d\mu'(q'_U, \underline{\Gamma}'|\bar{\xi}'_{\Delta^c}) d\mu''(q''_U, \underline{\Gamma}''|\bar{\xi}''_{\Delta^c})$ be the product of the marginal distributions of $d\mu'(\cdot|\bar{\xi}'_{\Delta^c})$ and $d\mu''(\cdot|\bar{\xi}''_{\Delta^c})$ over $\mathcal{X}^{(k)}_U \times \mathcal{B}_\Delta$. Let Q_T be the coupling defined in Theorem 3.3 and letting $\Xi = \{\omega_{\Delta^c} : \underline{\Gamma}' \cap (T \cup \delta^{\ell+,\gamma}_{\text{out}}[T]) = \underline{\Gamma}'' \cap (T \cup \delta^{\ell+,\gamma}_{\text{out}}[T]) = \emptyset\}$, we denote by $\mathbf{1}_\Xi$ the characteristic function of the set Ξ.

  Then we define

$$\begin{aligned} d\pi_\Delta(\omega_\Delta \,|\, \bar{\omega}_{\Delta^c}) = \mathbf{1}_\Xi(\bar{\omega}_{\Delta^c})\, dQ_T\left(q'_T, q''_T | q'_U, \bar{q}'_{\Delta^c}, q''_U, \bar{q}''_{\Delta^c}\right) dP(q'_U, q''_U, \underline{\Gamma}', \underline{\Gamma}'') \\ + [1 - \mathbf{1}_\Xi(\bar{\omega}_{\Delta^c})] d\mu'(q'_\Delta, \underline{\Gamma}'|\bar{\xi}'_{\Delta^c}) d\mu''(q''_\Delta, \underline{\Gamma}''|\bar{\xi}''_{\Delta^c}) \end{aligned}$$

By Theorem 7.1 the second case above occurs if and only if all cubes of $\delta^{\ell+,\gamma}_{\text{out}}[\Delta]$ are good, while in the third case there are bad cubes in $\delta^{\ell+,\gamma}_{\text{out}}[\Delta]$ so that Σ_{n+1} is non empty. The proof that cubes are good with large probability will be based on Theorem 3.3 and the following lemma:

Lemma 7.3. *Suppose $\Lambda_n(\omega) = \Delta$ and that the third case above is verified, namely ω_{Δ^c} is such that $\mathrm{sp}(\underline{\Gamma}') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] = \mathrm{sp}(\underline{\Gamma}'') \cap \delta^{\ell+,\gamma}_{\text{out}}[\Delta] = \emptyset$ and $q' \cap \delta^{\gamma^{-1}}_{\text{out}}[\Delta] \neq q'' \cap \delta^{\gamma^{-1}}_{\text{out}}[\Delta]$. Suppose also that $\underline{\Gamma}' \cap (T \cup \delta^{\ell+,\gamma}_{\text{out}}[T]) = \underline{\Gamma}'' \cap (T \cup \delta^{\ell+,\gamma}_{\text{out}}[T]) = \emptyset$. Let C in Σ_{n+1}, then C is good if $\omega_\Delta \in \Theta_T(x)$ for all $x \in C$, Θ_T as in* (3.24).

Proof. The proof follows from the definitions of good cubes and $\Theta_T(x)$ because for all $x \in C$, $\Theta_T(x) = \Theta_\Delta(x)$. □

8 Probability estimates.

Recall from the beginning of Part III that μ' and μ'' are obtained by conditioning to the configurations outside Λ the measures ν' and ν'' which are either DLR measures or Gibbs measures $dG_{\Lambda'}(q_{\Lambda'}, \underline{\Gamma}|\bar{q}_{(\Lambda')^c})$ with $\Lambda' \supseteq \Lambda$. Thus if Q^{N^*} is the coupling of μ' and μ'' defined in Subsection 7.4, we obtain a coupling P of ν', ν'' by writing

$$dP(\omega) = d\nu'(\bar{\xi}'_{\Lambda^c})d\nu''(\bar{\xi}''_{\Lambda^c})dQ^{N^*}(\omega_\Lambda|\bar{\omega}_{\Lambda^c}), \quad \omega = (\omega_\Lambda, \bar{\omega}_{\Lambda^c}),\ \bar{\omega}_{\Lambda^c} = (\bar{\xi}'_{\Lambda^c}, \bar{\xi}''_{\Lambda^c}) \tag{8.1}$$

We will prove here that there is a constant c such that for all γ small enough, for any $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable subset Δ of Λ:

$$P\Big(\{\omega : \Lambda_{N^*}(\omega) \supset \Delta\}\Big) \geq 1 - c_1 e^{-c_2 \frac{\mathrm{dist}(\Delta, \Lambda^c)}{\ell_{+,\gamma}}} \tag{8.2}$$

This proves that $(q'_\Lambda, \underline{\Gamma}')$ and $(q''_{\Lambda'}, \underline{\Gamma}'')$ agree in Δ, in the sense of (3.20), with probability $\geq 1 - c_1 e^{-c_2 \frac{\mathrm{dist}(\Delta, \Lambda^c)}{\ell_{+,\gamma}}}$ from which Theorem 3.1 follows. Indeed if ν' and ν'' are two DLR measures, by the arbitrariness of Δ and Λ, (8.2) shows that $\nu' = \nu''$, hence that there is a unique DLR measure. If instead ν' and ν'' are two Gibbs measures $dG_{\Lambda'}(q_{\Lambda'}, \underline{\Gamma}|\bar{q}_{\Lambda'^c})$ and $dG_{\Lambda''}(q_{\Lambda''}, \underline{\Gamma}|\bar{q}_{\Lambda''^c})$, $\Lambda \subset \Lambda'$, $\Lambda \subset \Lambda''$ then (8.2) yields (3.21).

8.1 Reduction to a percolation event

Denote by $\mathcal{A} = \mathcal{A}(\omega)$ the union of all bad cubes contained in Λ and of the cubes in $\delta_{\rm out}^{\ell_{+,\gamma}}[\Lambda]$ with a polymer, namely those cubes C such that $C \subseteq \mathrm{sp}(\Gamma)$, Γ in $\underline{\Gamma}' \cup \underline{\Gamma}''$. Since by its definition any screening set is connected to a bad cube and since any bad cube in Λ is necessarily contained in a screening set, it follows that if $\mathcal{A} \neq \emptyset$ then it is connected to Λ^c. Since the event in (8.2) is bounded by

$$\{\omega : \Lambda_{N^*}(\omega) \supset \Delta\}^c \subset \{\mathcal{A}(\omega) \cap \Delta \neq \emptyset\} \tag{8.3}$$

, it is therefore also bounded by the event that the bad cubes percolate from Δ to Λ^c. Hence, denoting in the sequel by A a connected, $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable subset of $\Lambda \cup \delta_{\rm out}^{\ell_{+,\gamma}}[\Lambda]$,

$$P\Big(\{\Lambda_{N^*} \supset \Delta\}^c\Big) \leq \sum_{x \in \ell_{+,\gamma}\mathbb{Z}^d \cap \Delta} \ \sum_{A: A \ni x, A \cap \delta_{\rm out}^{\ell_{+,\gamma}}[\Lambda] \neq \emptyset} P\big(\{\mathcal{A} = A\}\big) \tag{8.4}$$

We write $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, $\mathcal{A}_i$ the union of cubes of "type i". Cubes of type 1 are those with a polymer, namely C is type 1 if there is Γ in $\underline{\Gamma}' \cup \underline{\Gamma}''$ such that $C \subseteq \mathrm{sp}(\Gamma)$. C is type 2 (also called unsuccessful) if C, say in Σ_{n+1}, is bad and all cubes of $\delta_{\rm out}^{\ell_{+,\gamma}}[\Lambda_n]$ are without polymers (in the above sense). Cubes of type 3 are the remaining ones, they are therefore in the union of all Σ_{n+1} with Σ_{n+1} connected to a type 1 bad cube. Then calling $N_A = |A|/\ell^d_{+,\gamma}$,

$$\text{l.h.s. of (8.4)} \leq \sum_{x \in \ell_{+,\gamma}\mathbb{Z}^d \cap \Delta} \ \sum_{A: A \ni x, A \cap \delta_{\rm out}^{\ell_{+,\gamma}}[\Lambda] \neq \emptyset} 3^{N_A} \max_{A_1 \cup A_2 \cup A_3 = A} P\Big(\bigcap_{i=1}^{3} \{\mathcal{A}_i = A_i\}\Big) \tag{8.5}$$

Since $\mathcal{A}_3 \subset \bigcup_{C \in \mathcal{A}_1} \delta_{\rm out}^{\ell_+,\gamma}[C]$,

$$N_{\mathcal{A}_3} \le 3^d N_{\mathcal{A}_1} \tag{8.6}$$

Therefore $N_{\mathcal{A}_1} + N_{\mathcal{A}_2} + 3^d N_{\mathcal{A}_1} \ge N_{\mathcal{A}}$ and

$$\bigcap_{i=1}^{3} \Big\{ \mathcal{A}_i = A_i \Big\} \;\; \subset \;\; \Big\{ \mathcal{A}_2 = A_2; N_{\mathcal{A}_2} \ge \frac{N_A}{2} \Big\} \cup \Big\{ \mathcal{A}_1 = A_1; N_{\mathcal{A}_1} \ge \frac{N_A}{2(1+3^d)} \Big\} \tag{8.7}$$

We are thus reduced to estimate for any (A_1, A_2, A_3),

$$P\big(\{\mathcal{A}_2 = A_2\}\big), \text{ if } N_{A_2} \ge \frac{N_A}{2}; \quad P\big(\{\mathcal{A}_1 = A_1\}\big), \text{ if } N_{A_1} \ge \frac{N_A}{2(1+3^d)} \tag{8.8}$$

8.2 Peierls estimates

We bound here $P\big(\{\mathcal{A}_1 = A_1\}\big)$ where A_1 is some given set in $\Lambda \cup \delta_{\rm out}^{\ell_+,\gamma}[\Lambda]$. Thus each cube $C \subset A_1$ is either contained in ${\rm sp}(\Gamma)$, $\Gamma \in \underline{\Gamma}'$ or in ${\rm sp}(\Gamma)$, $\Gamma \in \underline{\Gamma}''$ (or both). Thus

$$P\big(\{\mathcal{A}_1 = A_1\}\big) \le 2^{N_{A_1}} \max_{B \subset A_1, N_B \ge N_{A_1}/2} \max\{\nu'({\rm sp}(\underline{\Gamma}) \supset B); \nu''({\rm sp}(\underline{\Gamma}) \supset B)\} \tag{8.9}$$

where ${\rm sp}(\underline{\Gamma}) = \bigcup_{\Gamma \in \underline{\Gamma}} {\rm sp}(\Gamma)$. Let $B = C_1 \cup \dots \cup C_n$, C_i disjoint cubes of $\mathcal{D}^{(\ell_+,\gamma)}$, then, since ν' and ν'' satisfy the Peierls estimates,

$$\begin{aligned} \nu'({\rm sp}(\underline{\Gamma}) \supset B) &\le \sum_{\Gamma_1,\dots,\Gamma_n, {\rm sp}(\Gamma_i) \supset C_i} \nu'\big(\underline{\Gamma} \ni \Gamma_1, ..., \Gamma_n\big) \le \sum_{\Gamma_1,\dots,\Gamma_n, {\rm sp}(\Gamma_i) \supset C_i} e^{-c_{\rm pol}\zeta^2 \ell_{-,\gamma}^d (N_{\Gamma_1} + \dots + N_{\Gamma_n})} \\ &\le e^{-c_{\rm pol}\zeta^2 \ell_{-,\gamma}^d N_B/2} \Big(\sum_{\Gamma: {\rm sp}(\Gamma) \ni C} e^{-c_{\rm pol}\zeta^2 \ell_{-,\gamma}^d N_\Gamma/2} \Big)^{N_B} \le 2^{N_B} e^{-c_{\rm pol}\zeta^2 \ell_{-,\gamma}^d N_B/2} \end{aligned} \tag{8.10}$$

for all γ small enough. Thus

$$P\big(\{\mathcal{A}_1 = A_1\}\big) \le 2^{2N_{A_1}} e^{-c_{\rm pol}\zeta^2 \ell_{-,\gamma}^d N_{A_1}/4} \tag{8.11}$$

8.3 Probability of unsuccessful cubes

We will bound here $P\big(\{\mathcal{A}_2 = A_2\}\big)$. Given any $n > 0$ we define

$$\mathcal{A}_{2,n}(\omega) = \mathcal{A}_2(\omega) \cap \Lambda_n(\omega)^c, \quad O_n(\omega) = N_{\Lambda_n(\omega) \cap A_2} \tag{8.12}$$

$$g_n(\omega) = \chi_n(\omega) \cdot \epsilon^{O_n(\omega)}, \quad \chi_n(\omega) := \mathbf{1}_{\mathcal{A}_{2,n}(\omega)=A_2\cap\Lambda_n(\omega)^c} \tag{8.13}$$

$\epsilon > 0$ will be specified later. We are going to prove that for all n,

$$\mathcal{E}(g_{n+1}) \le \mathcal{E}(g_n) \le \ldots \le \mathcal{E}(g_0) \tag{8.14}$$

where $\mathcal{E}$ is the expectation with respect to P. Since $A_2 \subset \Lambda = \Lambda_0$ and $\mathcal{A}_{2,N^*}(\omega) = \mathcal{A}_2(\omega)$, we then get from (8.14),

$$P\big(\{\mathcal{A}_2 = A_2\}\big) \le \epsilon^{N_{A_2}} \tag{8.15}$$

Recalling (8.1), we set $P^{N^*} = P$ and for $n < N^*$,

$$dP^n(\omega) = d\nu'(\bar\xi'_{\Lambda^c})d\nu'(\bar\xi''_{\Lambda^c})dQ^n(\omega_\Lambda|\bar\omega_{\Lambda^c}), \quad \omega = (\omega_\Lambda, \bar\omega_{\Lambda^c}),\ \bar\omega_{\Lambda^c} = (\bar\xi'_{\Lambda^c}, \bar\xi''_{\Lambda^c}) \tag{8.16}$$

calling $\mathcal{E}^n$ the expectation w.r.t. P^n. We have $\mathcal{E}(g_{n+1}) = \mathcal{E}^{n+1}(g_{n+1})$, hence by (7.9),

$$\begin{aligned}\mathcal{E}(g_{n+1}) = \sum_{\Delta}\sum_{\Delta'\subset\Delta} \epsilon^{N_{A_2\cap\Delta'}} \int P^n(d\,\omega_{\Delta^c})\Big[\mathbf{1}_{\{\Lambda_n=\Delta,\Lambda_{n+1}=\Delta'\}}\chi_n(\omega) \\ \times \int_{\mathcal{A}_2(\omega_\Delta)\supset A_2\cap\Sigma_{n+1}} \pi_\Delta(d\omega_\Delta\,|\,\xi_{\Delta^c})\Big]\end{aligned} \tag{8.17}$$

where $\Sigma_{n+1} = \Delta\setminus\Delta'$. The last integral is equal to 1 if $A_2\cap\{\Delta\setminus\Delta'\} = \emptyset$, while, if this is not the case, by (3.25)

$$\int_{\mathcal{A}_2(\omega_\Delta)\supset A_2\cap\{\Delta\setminus\Delta'\}} \pi_\Delta(d\omega_\Delta\,|\,\xi_{\Delta^c}) \le c(\epsilon_g + e^{-c_{\rm pol}\zeta^2\ell^d_{-,\gamma}/2}), \quad A_2\cap\{\Delta\setminus\Delta'\}\ne\emptyset \tag{8.18}$$

We then get from (8.17),

$$\mathcal{E}(g_{n+1}) \le \mathcal{E}^n(g_n)\max\{1, \frac{c(\epsilon_g + e^{-c_{\rm pol}\zeta^2\ell^d_{-,\gamma}/2})}{\epsilon^{3^d}}\} \tag{8.19}$$

We choose

$$\epsilon = \frac{1}{2}\Big(c(\epsilon_g + e^{-c_{\rm pol}\zeta^2\ell^d_{-,\gamma}/2})\Big)^{3^{-d}} \tag{8.20}$$

so that the max on the r.h.s. of (8.19) is 1 which thus proves (8.14) and (8.15).

8.4 Proof of Theorem 3.1

As we have shown at the beginning of this Section, Theorem 3.1 follows from (8.2) that we prove here.

Given ϵ as in (8.20), for γ small enough we bound the r.h.s of (8.11) as

$$P\big(\{\mathcal{A}_1 = A_1\}\big) \le 2^{2N_{A_1}} e^{-c_{\rm pol}\zeta^2\ell^d_{-,\gamma}N_{A_1}/4} \le \epsilon^{N_{A_1}} \tag{8.21}$$

From (8.4), (8.5), (8.8), (8.21) and (8.15) we then get

$$\begin{aligned}P\Big(\{\Lambda_{N^*}\supset\Delta\}^c\Big) &\le \sum_{x\in\ell_{+,\gamma}\mathbb{Z}^d\cap\Delta}\ \sum_{A:A\ni x, A\cap\delta_{\rm out}^{\ell_{+,\gamma}}[\Lambda]\ne\emptyset} 3^{N_A}2\epsilon^{N_A} \\ &\le 2|\Delta| \sum_{n\ge\frac{\text{dist}(\Delta,\Lambda^c)}{\ell_{+,\gamma}}} (3\epsilon)^n\end{aligned}$$

that implies (8.2). □

Part IV
Appendices

A Operators on Euclidean spaces

For the sake of completeness we recall here some elementary properties of operators on finite dimensional Hilbert spaces used in the previous sections. We call $\mathcal{H}$ the real Hilbert space of vectors $u = \{u(i)\}$ with scalar product

$$(u, v) = \sum_i u(i)v(i) \tag{A.1}$$

where i above ranges in a finite index set on which a distance $|i - j|$ is defined (in our applications i stands for a pair (x, s), with $x \in \ell\mathbb{Z}^d \cap \Lambda$, $s \in \{1, .., S\}$, and either $\ell = \ell_{-,\gamma}$ or $\ell = \gamma^{-1/2}$, Λ being a fixed $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable bounded subset of $\mathbb{R}^d$. Operators on $\mathcal{H}$ are identified to matrices $B = B(i, j)$ by setting $Bu(i) = \sum_j B(i, j)u(j)$. We write $|u|_\infty = \max_i |u(i)|$,

$$\|B\|^2 = \sup_{u \neq 0} \frac{(Bu, Bu)}{(u, u)}, \quad \|B\|_\infty = \sup_{u \neq 0} \frac{|Bu|_\infty}{|u|_\infty} \tag{A.2}$$

Recall that

$$\|B\|_\infty \le \max_i \sum_j |B(i, j)|, \quad \|B\| \le \max_i \{\sum_j |B(i, j)|, \sum_j |B(j, i)|\} =: |B| \tag{A.3}$$

The first inequality in (A.3) is obvious. To prove the second one we write

$$\begin{aligned} \sum_i \Big(\sum_j B(i, j)u(j)\Big)^2 &\le \sum_{i,j_1,j_2} |B(i, j_1)||B(i, j_2)| \frac{1}{2}\big(u(j_1)^2 + u(j_2)^2\big) \\ &\le \sum_{i,j_1,j_2} |B(i, j_1)||B(i, j_2)|u(j_1)^2 \le |B|^2 \sum_i u(i)^2 \end{aligned}$$

In Theorem A.1 below we consider matrices of the form $B = C'A^{-1}C''$, thus including $(QAQ)^{-1}$ (after restricting to $Q\mathcal{H}$) and $PA(QAQ)^{-1}QA$, the matrix considered in (5.37). With in mind these two applications we will suppose the diagonal elements of A strictly positive and large.

Theorem A.1. *Let $B = C'A^{-1}C''$ with $A = D + R$, D a diagonal matrix, and suppose there are $c > 0$, $c' > 0$ and $b > 0$ such that the following holds (recall the definition of the norm $|C|$ given in* (A.3)*).*

$$|C'| + |C''| + |R| \le c \tag{A.4}$$

The diagonal elements $D(i,i)$ of D are such that $D(i,i)\ge b$ for every i. Finally $C'(i,j)=C''(i,j)=R(i,j)=0$ whenever $|i-j|\ge c'\gamma^{-1}$. Then if b is large enough,

$$\|B\|\le \frac{2c^2}{b},\quad \|B\|_\infty\le \max_i\sum_j |B(i,j)|e^{\gamma|i-j|}\le \frac{2c^2e^{2c'}}{b} \tag{A.5}$$

Proof. By (A.3), $\|R\|\le c$. On the other hand $\|D\|^{-1}\le b^{-1}$ and for b so large that $b^{-1}c<1$ the sum on the r.h.s. of (A.6) below converges and

$$A^{-1}=D^{-1}-D^{-1}RD^{-1}+D^{-1}RD^{-1}RD^{-1}-\cdots=\sum_{n=0}^\infty\Big(-D^{-1}R\Big)^nD^{-1} \tag{A.6}$$

as seen by multiplying the r.h.s. of (A.6) from the left by A: we then get $AD^{-1}\big(1-RD^{-1}+\cdots\big)$ which is equal to 1 after writing $AD^{-1}=1+RD^{-1}$ and after telescopic cancellations. Thus (A.6) holds and

$$\|A^{-1}\|\le\sum_{n=0}^\infty b^{-n-1}\|R\|^n\le\frac{1}{b(1-c/b)} \tag{A.7}$$

hence, recalling (A.3), we get the first inequality in (A.5). We write

$$\begin{aligned}\sum_j |B(i,j)|e^{\gamma|i-j|}&\le\sum_{i_1}|C'(i,i_1)|e^{\gamma|i-i_1|}\sum_{i_2}|A^{-1}(i_1,i_2)|e^{\gamma|i_1-i_2|}\sum_j|C''(i_2,j)|e^{\gamma|i_2-j|}\\&\le c^2e^{2c'}\max_{i_1}\sum_{i_2}|A^{-1}(i_1,i_2)|e^{\gamma|i_1-i_2|}\end{aligned}$$

Since $\sum_{i_2}|R(i_1,i_2)|e^{\gamma|i_1-i_2|}\le e^{c'}c$, by (A.6)

$$\sum_{i_2}|A^{-1}(i_1,i_2)|e^{\gamma|i_1-i_2|}\le\sum_{n=0}^\infty b^{-n-1}[e^{c'}c]^n$$

hence the second inequality in (A.5).

□

In the next two theorems we consider a matrix R_1 with small norm, it represents in our applications the matrix $PA(QAQ)^{-1}QA$ which by Theorem A.1 has indeed a small norm (if b is large).

Theorem A.2. *Let $B=A+R_1$; suppose A symmetric, $(u,Au)\ge\kappa(u,u)$ for all u; $\|R_1\|\le\epsilon$ and $\kappa>\epsilon>0$. Then B is invertible and*

$$\|B^{-1}\|\le\frac{1}{\kappa'},\quad \kappa'=\kappa-\epsilon \tag{A.8}$$

Suppose further that

$$\sup_i \sum_j |B(i,j)| e^{\gamma|i-j|} \le a < \infty \tag{A.9}$$

then

$$|B^{-1}(i,j)| \le (\frac{1}{a} + \frac{1}{\kappa'}) \exp\Big\{ - \frac{\kappa'\gamma|i-j|}{a+\kappa'} \Big\} \tag{A.10}$$

Proof. By the integration by parts formula,

$$e^{-Bt} = e^{-At} - \int_0^t e^{-Bs} R_1 e^{-A(t-s)} \tag{A.11}$$

Since $\|e^{-At}\| \le e^{-\kappa t}$,

$$\|e^{-Bt}\| \le e^{-\kappa t} + e^{-\kappa t} \sum_{n=1}^{\infty} \frac{(\epsilon t)^n}{n!} \le e^{-(\kappa-\epsilon)t} \tag{A.12}$$

Then $\int_0^\infty e^{-Bt}$ is well defined and equal to B^{-1}; (A.8) also follows.
Calling e_i the vector with components $e_i(j) = \mathbf{1}_{i=j}$,

$$B^{-1}(i,j) = \int_0^\tau \big(e_i, e^{-Bt} e_j\big) + \int_\tau^\infty \big(e_i, e^{-Bt} e_j\big) \tag{A.13}$$

By (A.12),

$$|\int_\tau^\infty \big(e_i, e^{-Bt} e_j\big)| \le \frac{e^{-\kappa'\tau}}{\kappa'}, \quad \kappa' = \kappa - \epsilon \tag{A.14}$$

By a Taylor expansion:

$$|\big(e_i, e^{-Bt} e_j\big)| \le \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-\gamma|i-j|} \sum_{i_1,..,i_{n-1}} |B(i,i_1)| e^{\gamma|i-i_1|} \cdots |B(i_{n-1},j)| e^{\gamma|j-i_{n-1}|} \tag{A.15}$$

hence using (A.9),

$$|\int_0^\tau \big(e_i, e^{-Bt} e_j\big)| \le \frac{e^{a\tau - \gamma|i-j|}}{a} \tag{A.16}$$

By choosing $\tau = \dfrac{\gamma|i-j|}{a+\kappa'}$ we then get (A.10) from (A.14) and (A.16).

□

Theorem A.3. *Let $B = A + R_1$ as in Theorem A.2; call D the diagonal part of A, $R_0 := A - D$, $R = R_0 + R_1$ and suppose that $\|R\|_\infty < \infty$. Then*

$$\|B^{-1}\|_\infty \le \frac{1}{\kappa} + \frac{\|R\|_\infty}{\kappa^2}\Big(1 + \frac{\|R\|_\infty}{\kappa - \epsilon}\Big) \tag{A.17}$$

Proof. Recalling that $B = D + R$, we use the identity

$$B^{-1} = D^{-1} - D^{-1}RD^{-1} + D^{-1}RB^{-1}RD^{-1}$$

Then

$$B^{-1}(i,j) = (e_i, D^{-1}e_j) - (D^{-1}e_i, RD^{-1}e_j) + \sum_{k,h}(e_i, D^{-1}Re_k)(e_k, B^{-1}e_h)(e_h, RD^{-1}e_j)$$

so that

$$\sum_j |B^{-1}(i,j)| \le \kappa^{-1} + \kappa^{-2}\|R\|_\infty + \|B^{-1}\|\kappa^{-2}\|R\|_\infty^2$$

and (A.17) follows using (A.8).

□

B Proof of Theorem 7.1

In the sequel cubes are always cubes in $\mathcal{D}^{(\ell_{+,\gamma})}$ and a cube C is called "older" than C' if there is n such that $C' \subset \Lambda_n$ and $C \subset \Lambda_n^c$. We will prove the theorem as a consequence of the following property:

Property P. *Let C be a good cube, $x \in \ell_{-,\gamma}\mathbb{Z}^d \cap C$, $\{C_i\}$ the cubes older than C which intersect $B_x(2^d 10^{-10}\ell_{+,\gamma})$. If either $\{C_i\}$ is empty or if all C_i are good, then $q'_\Lambda \cap C_x^{(\ell_{-,\gamma})} = q''_\Lambda \cap C_x^{(\ell_{-,\gamma})}$.*

Before proving Property P, we will use it to prove Theorem 7.1. Suppose that for some N, Λ_N is non empty and that all cubes in $\delta_{\text{out}}^{\ell_{+,\gamma}}[\Lambda_N]$ are good (thus the sequence Λ_n stops at N). Let C be a cube in $\delta_{\text{out}}^{\ell_{+,\gamma}}[\Lambda_N]$, $x \in \ell_{-,\gamma}\mathbb{Z}^d \cap C$ and at distance $\le \gamma^{-1}$ from Λ_N. Then $B_x(2^d 10^{-10}\ell_{+,\gamma}) \cap \Lambda_N^c$ intersects only cubes of $\delta_{\text{out}}^{\ell_{+,\gamma}}[\Lambda_N]$, which are by assumption good; then by Property P, $q'_\Lambda \cap C_x^{(\ell_{-,\gamma})} = q''_\Lambda \cap C_x^{(\ell_{-,\gamma})}$, hence (7.2). (7.3) holds because all cubes of $\delta_{\text{out}}^{\ell_{+,\gamma}}[\Lambda_N]$ are good.

We start the proof of Property P by introducing a new function $M(x)$, $x \in \ell_{-,\gamma}\mathbb{Z}^d$. We set $M(x) = \infty$ outside Λ and at all x which are in bad cubes. The definition of $M(x)$ on the good cubes is given iteratively in Λ_n^c. We thus suppose to have already defined $M(x)$ on all cubes of Λ_n^c and have to define it on $\Sigma_{n+1} = \Lambda_{n+1}^c \setminus \Lambda_n^c$. Let thus $C \subset \Sigma_{n+1}$ and $x \in C$. We set $M(x) = 0$ if $B_x(10^{-10}\ell_{+,\gamma}) \cap \Lambda_n^c = \emptyset$, otherwise

$$M(x) := 1 + \max\Big\{M(y) \big| y \in \ell_{-,\gamma}\mathbb{Z}^d \cap B_x(10^{-10}\ell_{+,\gamma}),\ \ y \text{ such that } C_y^{(\ell_{+,\gamma})} \subset \Lambda_n^c\Big\} \quad \text{(B.1)}$$

To compute the value of $M(x)$, $x \in C$, $C \subset \Sigma_{n+1}$, we need to look at all sequences $y_1, y_2, ...$ such that: $|y_h - y_{h-1}| \le 10^{-10}\ell_{+,\gamma}$, $C^{(\ell_{+,\gamma})}_{y_h}$ is older than $C^{(\ell_{+,\gamma})}_{y_{h-1}}$, $h_0 = x$ and to know whether the cubes $C^{(\ell_{+,\gamma})}_{y_h}$ are good or bad. In principle the sequence may be arbitrarily long but in fact it is not:

Lemma 1. *Let C be a good cube, $x \in C$, then the value of $M(x)$ depends only on whether the cubes $\{C_i\}$ are good or bad, where $\{C_i\}$ is the collection of cubes older than C which intersect $B_x(2^d 10^{-10}\ell_{+,\gamma})$.*

Proof. Since any ball of radius $(2^d 10^{-10} + 1)\ell_{+,\gamma}$ intersects at most 2^d cubes of the partition $\mathcal{D}^{(\ell_{+,\gamma})}$, then any sequence $y_1, y_2, ...$ as above consists at most of 2^d elements. □

Since $\bar{m} = 2^d + 2$, then

$$\text{either } M(x) < \bar{m} - 2 \text{ or } M(x) = +\infty \tag{B.2}$$

We will next prove:

Lemma 2. *Let C be a good cube, $x \in C$, then, if $\bar{m} - M(x) = h > 0$,*

$$q'_\Lambda \cap C^{(\ell_{-,\gamma})}_x = q''_\Lambda \cap C^{(\ell_{-,\gamma})}_x, \qquad \max_{s\in\{1,..,S\}} |\rho^{(\ell_{-,\gamma})}(q'_\Lambda; x, s) - \rho^{(k)}_s| \le \zeta_h \tag{B.3}$$

Proof. The proof is by induction on the "age" of the cubes. We thus suppose that the above statements holds for all cubes of Λ^c_n. Let C be a good cube in Σ_{n+1}, then the above properties hold by the definition of the function K and of good cubes. □

Property P is then an immediate consequence of Lemma 2 and (B.2).

C Mean field

In this appendix we prove Theorems 2.2 and 2.3. Our approach is based on the recent works [12, 10], having in mind that in [12] the total density was set to 1, the temperature being the free parameter, while here we fix the (inverse) temperature $\beta = 1$, the total density x being the free parameter. The two approaches are equivalent, see (2.3).
To achieve our goal, we will need Lemmas C.1, C.2, C.3, C.4 and C.5 below. The first lemma is an essential property relating the total density x to the corresponding constrained minimizer in a one-to-one way. The second and third lemmas respectively deal with the first and second derivatives of the free energy. They show in particular that the sign of the second derivative depends on the roots of some peculiar second degree polynomial. The fourth lemma studies the locations of these roots, while the fifth and last lemma gives a general condition for a piecewise-convex function to have a common tangent at two different points.

The section is organized as follows. We first give some notations and reformulate known results, before stating our auxiliary lemmas. Then we prove Theorems 2.2 and 2.3, while the proofs of lemmas are deferred to the end of the present section.

Notations

For any $x \in (0,+\infty), z \in [0,1]$, we will denote by $\rho^{(z,x)}$ the density vector ρ defined as follows:

$$\rho_i^{(z,x)} = \begin{cases} \frac{1+(S-1)z}{S} x & \text{for } \ i = 1 \\ \frac{1-z}{S} x & \text{for } \ i = 2, \dots, S. \end{cases} \tag{C.1}$$

Notice that $\sum \rho_i^{(z,x)} = x$ and rewrite (2.4) as follows:

$$f^{\mathrm{mf}}(x) = \inf \left\{ F^{\mathrm{mf}}(\rho^{(z,x)}); 0 \le z \le 1 \right\}. \tag{C.2}$$

Now, remarking that $zx = \rho_1^{(z,x)} - \rho_2^{(z,x)}$, we adapt a result from [10, 12]. Namely, recalling Theorem A.1 in [12] or section 3 in [10], and comparing (C.3) and (C.4) below with (A.10) and (A.22) in [12], we know that for any $S > 2$ there exists a threshold

$$x_S := 2 \frac{S-1}{S-2} \ln(S-1) \tag{C.3}$$

such that

- for all $x < x_S$, the function $z \mapsto F^{\mathrm{mf}}(\rho^{(z,x)})$ reaches its minimum at $z = 0$;
- for all $x > x_S$, the function $z \mapsto F^{\mathrm{mf}}(\rho^{(z,x)})$ reaches its minimum at $z = z(x)$, defined as the largest solution of the equation $R(z) = x$ where

$$R(z) := \frac{1}{z} \ln \frac{1+(S-1)z}{1-z}; \tag{C.4}$$

- at $x = x_S$, the function $z \mapsto F^{\mathrm{mf}}(\rho^{(z,x)})$ reaches its minimum at $z = 0$ *and* at $z = z(x_S) = \frac{S-2}{S-1}$.

The statement above means that we have

$$f^{\mathrm{mf}}(x) = \begin{cases} f^{\mathrm{dis}}(x) := F^{\mathrm{mf}}(\rho^{(0,x)}) & \text{if } x \le x_S \\ f^{\mathrm{ord}}(x) := F^{\mathrm{mf}}(\rho^{(z(x),x)}) & \text{if } x \ge x_S. \end{cases} \tag{C.5}$$

First of all, we will see that

Lemma C.1 (Monotony of R and z)**.** *The functions $R : z \to R(z)$ and $z : x \to z(x)$ are both increasing respectively on $[z_S, 1)$ and $[x_S, +\infty)$, where $z_S = \frac{S-2}{S-1}$. They satisfy the relations $R \circ z = Id_{[x_S,+\infty)}$ and $z \circ R = Id_{[z_S,1)}$.*

Moreover

Lemma C.2.

$$\lim_{x\to 0}(f^{\mathrm{mf}})'(x) = -\infty \quad \textit{and} \lim_{x\to+\infty}(f^{\mathrm{mf}})'(x) = +\infty, \tag{C.6}$$

$$\lim_{x\uparrow x_S}(f^{\mathrm{mf}})'(x) - \lim_{x\downarrow x_S}(f^{\mathrm{mf}})'(x) = \left(1-\frac{2}{S}\right)\ln(S-1). \tag{C.7}$$

Lemma C.3.

$$\forall x \le x_S, \quad \frac{d^2 f^{\mathrm{dis}}}{dx^2}(x) = \frac{S-1}{S} + \frac{1}{x} \tag{C.8}$$

$$\forall x \ge x_S, \quad \frac{d^2 f^{\mathrm{ord}}}{dx^2}(x) = \left(\frac{S-1}{S}\right)\frac{z'(x)}{xz(x)}\left[R^+_{z(x)} - x\right]\left[x - R^-_{z(x)}\right], \tag{C.9}$$

where $R^\pm_z$ denotes the roots of the second degree polynomial $P_z(X) := X^2 - b_z X - c_z$ given by

$$\begin{cases} b_z & := & \frac{S(S-2)}{(S-1)[1+(s-1)z]} \\ c_z & := & \frac{S^2}{(S-1)(1-z)[1+(s-1)z]} \end{cases}.$$

According to Lemma C.3, the convexity properties of f^{ord} will follow from the position of the roots of $P_{z(x)}$ with respect to x. We will actually prove the lemma below

Lemma C.4 (Roots of P_z)**.** *The roots of the polynomial P_z are such that $R^-_z < 0 < R^+_z$ and*

- *for any $S \ge 60$, and for all $z \in [z_S, 1)$, $R^+_z > R(z)$;*
- *for any $3 \le S \le 59$, there exists a unique $z^\star_S \in (z_S, 1)$ such that $R^+_{z^\star_S} = R(z^\star_S)$. Moreover, $R^+_z < R(z)$ on $[z_S, z^\star_S)$ and $R^+_z > R(z)$ on $(z^\star_S, 1)$.*

Eventually, the following fact will be helpful to analyze the convex envelope of f^{mf}:

Lemma C.5. *Let $f : (a, b] \to \mathbb{R}$ and $g : [b, c) \to \mathbb{R}$ be convex functions with continuous second derivatives. If $f(b) = g(b)$ and if $\inf_{x<b} f'(x) < g'(b) < f'(b) < \sup_{x>b} g'(x)$, then there exists a common tangent to their respective graphs Γ_f, Γ_g.*

We are now ready to prove our theorems.

Proof of Theorem 2.2. By (C.8), f^{dis} is strictly convex. Let us now study the convexity of f^{ord}. Fixing $x \ge x_S$ we remark that Lemma C.1 implies $z(x) \ge z_S$ and $R(z(x)) = x$ so that Lemma C.4 gives:

- for all S, $x > 0 > R^-_{z(x)}$;
- if $S \ge 60$ then $R^+_{z(x)} > x$;
- if $3 \le S \le 59$, $R^+_{z(x)} < x$ if $x < x^\star_S$ and $R^+_{z(x)} > x$ if $x > x^\star_S$, where $x^\star_S := R(z^\star_S)$.

Therefore, (C.9) shows that if $S \geq 60$ then f^{ord} is strictly convex on $[x_S, \infty)$, while if $S \leq 59$ then f^{ord} is strictly concave on $[x_S, x_S^\star]$ and strictly convex on $[x_S^\star, +\infty)$.

Let us analyze the convex envelope of f^{mf}.

- If $S \geq 60$, (C.5) and Lemma C.2 show that Lemma C.5 applies to $f = f^{\mathrm{dis}}$, $g = f^{\mathrm{ord}}$, $a = 0$, $b = x_S$ and $c = +\infty$.
- If $S \leq 59$ we first have to deal with the concave part of f^{ord}. We introduce the function g defined by

$$g(x) = \begin{cases} f^{\mathrm{ord}}(x_S^\star) + (f^{\mathrm{ord}})'(x_S^\star) \cdot (x - x_S^\star) & \text{if } x \leq x_S^\star \\ f^{\mathrm{ord}}(x) & \text{if } x \geq x_S^\star. \end{cases}$$

  Since $(f^{\mathrm{ord}})''(x_S^\star) = 0$, g is convex and has continuous second derivatives. Moreover, on $[x_S, x_S^\star]$, the graph of g is a line located above the graph of f^{ord} (concavity of f^{ord}); since the latter intersects the (convex) graph of f^{dis}, the graph of g and the graph of f^{dis} intersect at some point with abscisse $b \in (x_S, x_S^\star)$. Besides, the concavity of f^{ord} implies $g'(b) = (f^{\mathrm{ord}})'(x_S^\star) < (f^{\mathrm{ord}})'(x_S)$, while the convexity of f^{dis} implies $(f^{\mathrm{dis}})'(b) > (f^{\mathrm{dis}})'(x_S)$. Thus Lemma C.2 shows that Lemma C.5 applies to $f = f^{\mathrm{dis}}$ and g defined above.

In any case, Lemma C.5 implies that there exists a line T_1 which is simultaneously tangent to the disordered branch of f^{mf} (at some point $x_- < x_S$) and to the ordered branch of f^{mf} (at some other point $x_+ > x_S$). The function $f^{\mathrm{mf}}(x)$ being strictly convex outside $[x_-, x_+]$, the graph of its convex envelope necessarily coincides with T_1 (resp. with the graph of f^{mf}) inside (resp. outside) $[x_-, x_+]$. Denoting by λ_1 the slope of T_1, the convex envelope of $f^{\mathrm{mf}}_{\lambda_1}(x) = f^{\mathrm{mf}}(x) - \lambda_1 x$ is horizontal on $[x_-, x_+]$ and strictly convex outside this segment of minimizers. □

Proof of Theorem 2.3. If ρ is a minimizer of $F = F^{\mathrm{mf}}_{1,\lambda_1}$, then $x = \sum_s \rho_s$ is a minimizer of $f^{\mathrm{mf}}_{1,\lambda_1}$ so that $x \in \{x_-, x_+\}$. If $x = x_- < x_S$, then $\rho = \rho^{(S+1)}$; if $x = x_+ > x_S$, then there exists $k \in \{1, \ldots, S\}$ such that $\rho = \rho^{(k)} := \tau^{1,k} \cdot \rho^{z(x),x}$, where $\tau_{1,k}$ exchanges the first and the k^{th} coordinates. Reciprocally, the above $S+1$ vectors $\rho^{(k)}$ are all minimizers of F. Moreover,

$$\sum_s \rho_s^{(1)} = x^+ > x^- = \sum_s \rho_s^{(S+1)},$$

thus proving (2.5).
We now show the second part of Theorem 2.3 dealing with the Hessian of F. Straightforward computations show:

$$L^{(k)}(s, s') = \frac{\partial^2 F}{\partial \rho_s \partial \rho_{s'}}\Big|_{\rho = \rho^{(k)}} = \frac{1}{\rho_s^{(k)}} \mathbf{1}_{s=s'} + \mathbf{1}_{s \neq s'}$$

Since $\rho^{(k)}$ is a minimizer, $L^k := D^2F(\rho^{(k)})$ is semi-definite positive. Actually, $L^{(k)}$ is definite positive, or else the third order corrections in the Taylor–Lagrange formula would contradict the extremality of ρ^k:

$$\forall s,t,u \quad \frac{\partial^3 F}{\partial\rho_s\partial\rho_t\partial\rho_u} = -\frac{1}{\rho_s^2}\mathbf{1}_{s=t=u}.$$

Taking an orthonormal basis of eigenvectors, the estimate (2.6) holds with $\kappa^* > 0$ the smallest eigenvalue of $L^{(1)}, L^{(S+1)}$. □

This section ends with the proofs of Lemma C.1, Lemma C.2, Lemma C.3, Lemma C.4 and Lemma C.5 which are stated at the beginning of the section and used in the proofs above.

Proof of Lemma C.1. We express $R'(z) = \frac{g(z)}{z^2}$ and show that g is always positive. Recalling (C.4) we have:

$$\begin{aligned} R'(z) &= \frac{1}{z}\cdot\left[\frac{S-1}{1+(S-1)z} + \frac{1}{1-z}\right] - \frac{1}{z^2}\cdot\ln\frac{1+(S-1)z}{1-z}, \\ &= \frac{1}{z^2}\left[\frac{1}{1-z} - \frac{1}{1+(S-1)z} - \ln\frac{1+(S-1)z}{1-z}\right], \\ &= \frac{1}{z^2}g(z). \end{aligned}$$

We now show that g is always positive:

$$\begin{aligned} g(z) &= \frac{1}{1-z} - \frac{1}{1+(S-1)z} - \ln\frac{1+(S-1)z}{1-z}, \\ g'(z) &= \frac{1}{(1-z)^2} + \frac{S-1}{[1+(S-1)z]^2} - \frac{S-1}{1+(S-1)z} + \frac{1}{1-z} \\ &= \frac{Sz\,[2(S-1)z-(S-2)]}{(1-z)^2[1+(S-1)z]^2}. \end{aligned}$$

We see immediately that $g' > 0$ for all $z > \frac{S-2}{2(S-1)}$, so that g increases on $[\frac{S-2}{S-1}, 1)$. On this subinterval, g is thus minimal at $\left(\frac{S-2}{S-1}\right)$ where it takes the value

$$g\left(\frac{S-2}{S-1}\right) = -2\ln(S-1) - \frac{1}{S-1} + (S-1),$$

which increases with S, vanishes at $S=2$, and is strictly positive for all $S \geq 3$. From this it follows that g is strictly positive on $[\frac{S-2}{S-1}, 1)$, which implies that R is strictly increasing with z. Since R goes to $+\infty$ when $z \to 1$, Lemma C.1 is proved. □

Proof of Lemma C.2.
Since $\rho^{(0,x)}$ is the vector $(\frac{x}{S}, \ldots, \frac{x}{S})$, equations (2.1), (C.5), (C.1) give for all $x < x_S$:

$$\begin{aligned} f^{\mathrm{mf}}(x) &= \frac{S(S-1)}{2}\left(\frac{x}{S}\right)^2 + S\frac{x}{S}\left(\ln\frac{x}{S} - 1\right) \\ \left(f^{\mathrm{mf}}\right)'(x) &= \frac{S-1}{S}x + \ln\frac{x}{S}. \end{aligned} \tag{C.10}$$

Recalling (2.1), (C.5) and (C.1), $f^{\mathrm{mf}}(x) = F(x, z(x))$ holds for all $x > x_S$, where

$$F(x,z) = \frac{1}{2}\frac{S-1}{S}x^2(1-z^2) + (S-1)\frac{x(1-z)}{S}\ln\frac{x(1-z)}{S} + \frac{x(1+(S-1)z)}{S}\ln\frac{x(1+(S-1)z)}{S} - x. \tag{C.11}$$

Using (C.11) and recalling that $\left(\frac{\partial F}{\partial z}\right)_{|z(x)} = 0$, we have for all $x > x_S$:

$$\left(f^{\mathrm{mf}}\right)'(x) = \left(\frac{\partial F}{\partial x}\right)_{\big|x,z(x)} + z'(x)\left(\frac{\partial F}{\partial z}\right)_{\big|z(x)}$$
$$= \frac{S-1}{S}x(1-z^2) + (S-1)\frac{1-z}{S}\left[\ln\frac{x(1-z)}{S} + 1\right] + \frac{1+(S-1)z}{S}\left[\ln\frac{x(1+(S-1)z)}{S} + 1\right] - 1 \tag{C.12}$$
$$= \frac{S-1}{S}x + \ln\frac{x}{S} + \ln(1-z) + \frac{xz}{S}. \tag{C.13}$$

From (C.4), we know that $x \geq \frac{1}{z}\log\frac{1}{1-z}$, thus

$$\left(f^{\mathrm{mf}}\right)'(x) \geq \left(\frac{S-1}{Sz} - 1\right)\log\frac{1}{1-z} + \ln\frac{x}{S} + \frac{xz}{S},$$
$$\geq \ln\frac{x}{S} + \frac{xz}{S}. \tag{C.14}$$

From Lemma C.1, $z(x) \to z_S$ as $x \to x_S$ thus (C.7) follows from (C.10) - (C.13). Similarly, $z(x) \to 1$ as $x \to \infty$, thus (C.6) follows by taking limits in (C.10) and (C.14). □

Proof of Lemma C.3.
First notice that (C.8) follows from (C.10). Using (C.12) we get:

$$\left(f^{\mathrm{mf}}\right)''(x) = \frac{1}{x} + \frac{S-1}{S}(1-z^2-2xz'z) + \frac{S-1}{S}z'\ln\frac{1+(S-1)z}{1-z}$$
$$= \frac{1}{x} + \frac{S-1}{S}\left(1-z^2-2xz'z\right) + \frac{S-1}{S}z'zx$$
$$= \frac{z'}{x}\left(R' + \frac{S-1}{S}\left[(1-z^2)xR' - zx^2\right]\right)$$
$$= -\frac{1}{x}\frac{z'}{z}P_{z(x)}(x)$$

where we used $\frac{1}{z'(x)} = R'(z(x))$ ($= R'$ by abusing notations) and $zR' = -R(z) + \frac{S}{(1-z)[1+(S-1)z]}$ (from (C.4)). This achieves the proof of (C.9). □

Proof of Lemma C.4.

- roots of P_z

  We notice that the discriminant of P_z

$$\Delta(P_z) = \frac{S[S+(3S-4)z]}{(1-z)}$$

  is always positive, so that the two distinct roots of P_z are given by

$$R_z^{\pm} = \frac{S}{2(S-1)}\left[\frac{(S-2)\pm\sqrt{\frac{S[S+(3S-4)z]}{1-z}}}{1+(S-1)z}\right]. \tag{C.15}$$

  For all positive z we have $\frac{S+(3S-4)z}{1-z} \geq S$, thus R_z^- is negative while R_z^+ is positive.

- sign of $R_z^+ - R(z)$

  We will actually analyze the sign of $H_S(z) := z[R_z^+ - R(z)]$, showing it is strictly monotone and thus vanishes at most once. Using (C.4) and (C.15) we get

$$H_S(z) = \frac{Sz}{2(S-1)(1+(S-1)z)}\left[S-2+\sqrt{\Delta(P_z)}\right] - \log\frac{1+(S-1)z}{1-z} \tag{C.16}$$

$$H_S'(z) = \frac{S^2\left[S+2(2S-3)z+(S-2)(2S-3)z^2+\left(-1+2(S+2)z+(2S-3)z^2\right)\sqrt{\Delta(P_z)}\right]}{2(S-1)(1-z)^2(1+(S-1)z)^2\sqrt{\frac{S(S+(3S-4)z)}{1-z}}}$$

$$= \frac{S^2\left[A(z)+B(z)\sqrt{\Delta(P_z)}\right]}{2(S-1)(1-z)^2(1+(S-1)z)^2\sqrt{\Delta(P_z)}} \tag{C.17}$$

  In the formula (C.17) above, the denominator as well as the polynomial $A(z)$ in the numerator are clearly positive for all $z > 0$. Since the polynomial $B(z)$ is increasing for $z > 0$ and since $B(z_S) \geq -1 + 2(S-1)z_S = 2S-5 > 0$, we deduce that $H_S'(z)$ is always positive for $z \in [z_S, 1)$.

- H_S vanishes exactly once $\iff S \leq 59$

  We now check for which values of S the function H_S actually vanishes somewhere on $[z_S, 1)$. As $z \to 1$, the leading term in H_S diverges like $(1-z)^{-1/2}$, so that $H_S(z) \to +\infty$. Thus H_S will vanish exactly once if and only if $H_S(z_S) \leq 0$.

$$G(S) = H_S(z_S) = \frac{S(S-2)(S-2+\sqrt{S(8-11S+4S^2)})}{2(S-1)^3} - 2\log(S-1)$$

$$G'(S) = \frac{S\left[2S^4-10S^3+27S^2-40S+24-(4S^2-13S+12)\sqrt{S(8-11S+4S^2)}\right]}{2(S-1)^4\sqrt{S(8-11S+4S^2)}}$$

  and

$$\begin{aligned} G'(S) = 0 &\iff 2S^4-10S^3+27S^2-40S+24 = (4S^2-13S+12)\sqrt{S(8-11S+4S^2)} \\ &\iff (2S^4-10S^3+27S^2-40S+24)^2 = S(8-11S+4S^2)(4S^2-13S+12)^2 \\ &\iff 4(S-2)^2(S-1)^3(S^3-19S^2+48S-36) = 0. \end{aligned}$$

The last bracket reaches a local (negative) maximum at $S = \frac{19-\sqrt{217}}{3} \approx 1.4$ and a local (negative) minimum at $S = \frac{19-\sqrt{217}}{3} \approx 11.2$. Therefore it has exactly one root $S^\star$, is negative before this root and positive after it. Numerical computations give $S^\star \approx 16.2$.

From this, we know that G is decreasing on $[3, S^\star]$ and increasing on $[S^\star, \infty)$. Since $G(3) < 0$ and since $G(S)$ diverges like $+\sqrt{S}$ as $S \to \infty$, we get that G has exactly one root $\bar{S} > S^\star$, is negative before it and positive after it. Numerical computations show $\bar{S} \approx 59.1$. $\square$

Proof of Lemma C.5. We will use the notation

$$K := \{\alpha \in [b, c); \alpha \geq b \text{ and } T_g(\alpha) \cap \Gamma_f \neq \emptyset\},$$

where $T_g(\alpha)$ denotes the tangent to Γ_g at α.
Since $f(b) = g(b)$, we have $b \in K$, and K is non-empty. Besides, by continuity of g', there exists $b_0 \in (b, c)$ such that $g'(b_0) = f'(b)$; since f, g are strictly convex, elements of K are bounded from above by b_0 and $\alpha^\star := \sup K \leq b_0$ is well defined.
Now, let α_n an increasing sequence converging to $\alpha^\star$. By definition, $T_g(\alpha_n)$ intersects Γ_f, and we denote by x_n the abscisse of the intersection point which is the closest to b, so that $f'(x_n) \geq g'(\alpha_n) \geq g'(b)$. We now show that x_n is a bounded decreasing sequence:

- On $\{x \geq x_n\}$, Γ_f is above $T_f(x_n)$ (convexity of f), which in turn is above $T_g(\alpha_n)$ (definition of x_n), and therefore above $T_g(\alpha_{n+1})$ (convexity of g). Thus Γ_f may not intersect $T_g(\alpha_{n+1})$ after abscisse x_n, and $x_{n+1} \leq x_n$.
- By continuity of f', there exists $b_1 \in (a, b)$ such that $f'(b_1) = g'(b)$, thus $f'(x_n) \geq g'(b)$ implies $x_n \geq b_1$ (convexity of f).

Thus $x_n \to x^\star \in [b_1, b] \subset (a, b]$, and by continuity of f, g, g', $T_g(\alpha^\star)$ intersects Γ_f at $(x^\star, f(x^\star))$. In particular, $\alpha^\star \in K$ and $f'(x^\star) \geq g'(\alpha^\star)$.
If we had $f'(x^\star) > g'(\alpha^\star)$ we could apply the implicit function function theorem to $\Psi(\alpha, x) = g(\alpha) + g'(\alpha)(x - \alpha) - f(x)$ to deduce that K contains a neighborhood of $\alpha^\star$, thus contradicting the maximality of $\alpha^\star$. Therefore $f'(x^\star) = g'(\alpha^\star)$ and $T_g(\alpha^\star) = T_f(x^\star)$ is actually tangent to Γ_f. $\square$

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