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Spin Systems with Long Range Interactions

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In these notes I will review some of the results on Ising spin systems with Kac potentials. The basic feature of these models is that they are finite range approximations of mean field interactions, i.e., the interaction between particles is parametrized by their range γ^{-1}, the limit $\gamma \to 0$ corresponding to mean field.

Kac potentials were introduced in the 1960s by Kac, Uhlenbeck and Hemmer and had a great success in establishing the validity of the van der Waals theory. In fact starting with the works of Kac, Uhlenbeck and Hemmer ([43],[44],[45],) and Lebowitz and Penrose, [49] it had been possible to prove in the limit $\gamma \to 0$, phase transitions and metastability.

In Section 1, which serves as an introductory section, the Lebowitz and Penrose limit and its relation with the van der Waals phase transition theory is explained in detail.

These results however refer to the behavior of the system only in the limit $\gamma \to 0$ which does not correspond any longer to a possible interaction with statistical mechanics.

Only recently the theory of Kac potential has been revisited with the idea of proving the validity of the old results without taking the limit $\gamma \to 0$, but working with positive values of γ, maybe with γ very small yet strictly positive. The first results along these lines have been obtained by Cassandro and Presutti, [19] and Bovier and Zahrahdník, [10] about phase transitions for Ising, ferromagnetic systems; see also Bodineau and Presutti. [7] for more general interactions and Cassandro, Marra and Presutti, [20] for the analysis in a neighborhood of the critical temperature.

Maybe the most surprising result in this direction has been obtained by Lebowitz, Mazel and Presutti, [51] who, using Kac potentials, have proved a long standing conjecture on the existence, in the context of Gibbsian statistical mechanics, of a liquid-vapour phase transition for point particles in R^d, $d \geq 2$.

In Section 2 I will discuss the main ideas of the proof of the existence of phase transition given in [10] and [19] for the spin system with Kac potential in $d \geq 2$ dimensions. Since the key point of the proof is a generalization of the Peierls argument, [52], I first recall the well-known proof of phase transition at low temperatures for the 2-d, nearest neighborhood, ferrogmanetic, Ising spin system.

We note in passing, but do not discuss in these lectures, another major success in this topic. In the study of surface tension the van der Waals theory has been proved rigorously by Benois, Bodineau, Buttà and Presutti, [5] (see also references therein) in the limit $\gamma \to 0$, and very recently by Bodineau, [8], for $\gamma > 0$ small and fixed. The analysis has profited by the strict relation between large deviations and the theory of Γ-convergence in variational problems and has led to the validity of the Wulff shape for the equilibrium droplets in Ising models.

In the last section I will review results on the time evolution. Indeed the theory has been applied successfully also in nonequilibrium, where the spinodal decomposition (i.e., escape from an unstable equilibrium) has been characterized for Ising spin systems with reversible Glauber dynamics and Kac interactions, [30]. The successive motion of the interfaces has been also studied; it has been proven in several works,(see [27], [47] and references therein) to be governed by the law of "motion by curvature", thus giving a complete description of the phenomena of development and motion of interfaces.

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Therefore this is not a review paper: the bibliography is incomplete, I refer to the original papers for a more exhaustive list of references and historical remarks. Some of the material of these notes is taken from: E. Presutti, notes of lectures at the Max Plank Institute, Leipzig 1999.

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1. Ising spin systems

The Ising spin configurations are points of the space $\mathcal{X} = \{-1,1\}^{\mathbb{Z}^d}$ denoted by

$$\sigma \in \mathcal{X}, \qquad \sigma = \big(\sigma(x),\, x \in \mathbb{Z}^d\big)\,;$$

$\sigma(x) \in \{-1.1\}$ is the spin at site x in the configuration σ. Analogously, spin configurations in $\Lambda \subset \mathbb{Z}^d$ are elements of $X_\Lambda = \{-1,1\}^\Lambda$,

$$\sigma_\Lambda \in \mathcal{X}_\Lambda, \qquad \sigma_\Lambda = (\sigma_\Lambda(x),\, x \in \Lambda)\,.$$

Definition 1.1. Energies *Let Λ be a bounded region of $\mathbb{Z}^d$. The energy of σ_Λ in Λ is*

$$H(\sigma_\Lambda) = -\frac{1}{2}\sum_{x \neq y \in \Lambda} J(x,y)\sigma_\Lambda(x)\sigma_\Lambda(y) - h\sum_{x\in\Lambda}\sigma_\Lambda(x). \tag{1.1}$$

Given $\sigma_{\Lambda^c} \in X_{\Lambda^c}$, the energy of σ_Λ in Λ with boundary conditions σ_{Λ^c} outside Λ is

$$H(\sigma_\Lambda|\sigma_{\Lambda^c}) = H(\sigma_\Lambda) - \sum_{x\in\Lambda}\sum_{y\ \in\Lambda^c} J(x,y)\sigma_\Lambda(x)\sigma_{\Lambda^c}(y) \tag{1.2}$$

where $h \in \mathbb{R}$ and for the function J we assume