Fields Institute Communications
Volume 6, 1996

Spinodal Decomposition and Interface Dynamics for Glauber Evolution with Kac Potential

Anna De Masi
Dipartimento di Matematica Pura ed Applicata
Università di L'Aquila
Via Vetoio (Coppito), 67100 L'Aquila
Italy

1 Introduction

We derive a mathematically rigorous theory of the spinodal decomposition for an Ising spin system in $\mathbb{Z}^d$ with Glauber dynamics and Kac potential. We also characterize the spatial distribution of the clusters of the pure phases (pattern formation) and their successive dynamics (interface dynamics).

A Kac potential is a function $J_\gamma(x,y)$, $\gamma \in (0,1)$, $x, y \in \mathbb{Z}^d$ of the form

$$J_\gamma(x,y) = \gamma^d J(\gamma|x-y|) \qquad x \neq y \in \mathbb{Z}^d \tag{1.1}$$

where the function $J(|r|)$, $r \in \mathbb{R}^d$ has compact support, i.e. $J(|r|) = 0$, if $|r| > 1$ and it is smooth, i.e. $J \in C^2(\mathbb{R}^d)$. We only consider ferrogmanetic interactions, i.e.,

$$J(|r|) \geq 0 \qquad \text{for all } r \in \mathbb{R}^d \tag{1.2}$$

and we assume, for simplicity, that

$$\int dr J(|r|) = 1. \tag{1.3}$$

The Kac potential has been proposed by M. Kac in the context of Equilibrium Statistical Mechanics to study the phase transition for systems with long range attractive (ferrogmanetic) forces proving that in the limit $\gamma \to 0$ its phase diagram is that of the Van der Waals theory (Kac, Uhlenbeck and Hemmer [1963], Lebowitz and Penrose [1966]).

1991 *Mathematics Subject Classification.* Primary: 82C20, Secondary: 82C22, 82C24, 82C26, 60K35.

Denoting by $\sigma(x) = \pm 1$ the spin variable at $x \in \mathbb{Z}^d$, we define the energy interaction of the spin $\sigma(x)$ with all the others as

$$H_\gamma(\sigma) = -\frac{1}{2}\sum_x h_\gamma(x;\sigma)\sigma(x) \tag{1.4}$$

$h_\gamma(x;\sigma)$, the *effective magnetic field at* x, is defined as

$$h_\gamma(x;\sigma) = \sum_{y \neq x} J_\gamma(x,y)\sigma(y) + h \qquad h \in \mathbb{R} \tag{1.5}$$

h is an external magnetic field.

In the limit $\gamma \to 0$ there is phase transition at $h = 0$ and at $\beta > \beta_c$, where the inverse critical temperature is

$$\beta_c = 1.$$

Lebowitz and Penrose [1966], have in fact proven that the free energy density $F_\gamma(\beta, m)$ at the inverse temperature β and mean magnetization m converges as $\gamma \to 0$ to $F(\beta, m)$ and the latter, as a function of m with β fixed, is strictly convex when $\beta < 1$ while it has a flat part when $\beta > 1$. The flat part (that defines the region where there is phase transition) occurs in the interval $-m_\beta \leq m \leq m_\beta$ with m_β the positive solution of the equation

$$m_\beta = \tanh(\beta m_\beta).$$

2 Glauber evolution

The Glauber dynamics is a spin flip Markov process on $\{-1,1\}^{\mathbb{Z}^d}$: the spin at $x \in \mathbb{Z}^d$ flips at rate

$$c_\gamma(x,\sigma) = \frac{e^{-\beta\sigma(x)h_\gamma(x;\sigma)}}{e^{-\beta h_\gamma(x;\sigma)} + e^{\beta h_\gamma(x;\sigma)}}. \tag{2.1}$$

The generator of the process acts on the cylinder functions f as

$$L_\gamma f(\sigma) = \sum_x c_\gamma(x,\sigma)[f(\sigma^{(x)}) - f(\sigma)] \tag{2.2}$$

where $\sigma^{(x)}$ is obtained from σ by flipping the spin at x and leaving the others unchanged. I refer to Presutti's paper in this volume for more comments on the definition of this dynamics.

In the sequel σ_t denotes the configuration at time t.

The parameter β in (2.1) and (2.2) has the physical meaning of the inverse temperature. The rate function $c_\gamma(x,\sigma)$ has been chosen in such a way to satisfy detailed balance with respect to the Gibbs measure at the inverse temperature β and energy interaction as given by (1.4).

We are interested in the behaviour of the evolution in the limit $\gamma \to 0$. Observe that the energy interaction $J_\gamma(x,y)\sigma(x)\sigma(y)$ between two given spins at $x \neq y \in \mathbb{Z}^d$ vanishes in the limit $\gamma \to 0$, while the effective magnetic field $h_\gamma(x,\sigma)$ stays bounded. This is therefore like a mean field limit and to avoid its pathologies we will look at the system for very small but nonzero γ.

Far from the phase transition region nothing particular is expected: there is only one Gibbs state invariant for the evolution and the system cannot but evolve toward it. If instead $h = 0$ and $\beta > \beta_c$ in the limit $\gamma \to 0$ there are two pure phases with magnetization $\pm m_\beta$. We will assume that initially the magnetization is 0. This is an unstable state and a stochastic fluctuation will make the system escape from its initial state. Such fluctuations in far apart regions are independent hence the phase will separate in a non trivial spatial pattern. The interface is the magnetization pattern in the transition region which separate one phase from the other. The shape and the evolution of the interfaces will be charaterized in the sequel.

3 Time and space scales

The various phenomena we are going to observe in the model appear in different space-time scales. To this purpose we use different scale-dependent coordinates. The *microscopic scale* is the one where the stochastic dynamics lives, hence the space-time coordinates are the original ones, (x, t) where $x \in \mathbb{Z}^d$. In the *mesoscopic scale* the space units is the range of the interaction. We denote the space-time coordinates in this scale by (r, t), t being the same as in the micro-scale, while $r \in \mathbb{R}^d$ is related to the microscopic coordinate $x \in \mathbb{Z}^d$ by

$$r = \gamma x. \tag{3.1}$$

The space and time in the *macroscopic scale* are denoted by (ξ, τ). These are defined as those units in which we observe the motion of the interface after the phase separation. As an outcome of the analysis it turns out that

$$r = \xi\sqrt{\log\gamma^{-1}} \quad \xi \in \mathbb{R}^d, \qquad t = \tau\log\gamma^{-1} \quad \tau > 0. \tag{3.2}$$

The reason why $\log\gamma^{-1}$ is the right scale in (3.2) will be explained later.

4 The order parameter

The relevant quantity is the magnetization which is the (non conserved) order parameter of the model.

We define the "actual" magnetization $M_\gamma(r, \sigma_t)$ at the mesoscopic point r at time t as a block spin variable. That is, given any $\eta \in (0, 1)$, we consider the region $B_\gamma(x)$ in $\mathbb{Z}^d$ defined as

$$B_\gamma(x) := \{y \in \mathbb{Z}^d : |x - y| \le \gamma^{-\eta}\} \qquad 0 < \eta < 1 \tag{4.1}$$

and denoting by

$$|B_\gamma(x)| = \text{ cardinality of } B_\gamma(x) \tag{4.2}$$

we let

$$M_\gamma(r, \sigma_t) = \frac{1}{|B_\gamma(x)|}\sum_{y \in B_\gamma(x)} \sigma_t(y) \qquad x = [\gamma^{-1} r] := \text{integer part of } \gamma^{-1} r. \tag{4.3}$$

Observe that, since $\eta > 0$ the volume $|B_\gamma(x)| \backsim \gamma^{-\eta d}$, diverges as $\gamma \to 0$ depressing the statistical fluctuations. On the other hand, since $\eta < 1$ the block size is much smaller than the patterns to observe which are not smaller than the range of the interaction, i.e. $\backsim \gamma^{-1}$.

5 The initial state

The initial probability distribution of the spins is a measure μ_γ on $\{-1,1\}^{\mathbb{Z}^d}$ such that

$$\mathbb{E}_{\mu_\gamma}\big(\sigma(x)\big) = m_0(\gamma x) \tag{5.1}$$

where $\mathbb{E}_{\mu_\gamma}$ denotes the average with respect to μ_γ and the *mesoscopic profile* $m_0(r)$, $r \in \mathbb{R}^d$ is in $C^3(\mathbb{R}^d)$, and $\|m_0\|_\infty \le 1$.

We also assume that μ_γ is a product measure, i.e. "chaos" at $t = 0$ holds. That is for all n and all distinct sites $x_1, \dots, x_n$ in $\mathbb{Z}^d$,

$$\mathbb{E}_{\mu_\gamma}\big(\prod_{i=1}^n \sigma(x_i)\big) = \prod_{i=1}^n \mathbb{E}_{\mu_\gamma}\big(\sigma(x_i)\big). \tag{5.2}$$

6 The mesoscopic behaviour

The next theorem states that the distribution at any time $t > 0$ approximates a mesoscopic profile $m(\cdot, t)$ solution to a deterministic equation, the mesoscopic equation, with initial condition $m_0(\cdot)$.

Theorem 6.1 *(De Masi et al. [1994]) Let μ_γ be as in Section 5. Then, for all $\beta > 0$ and $h \in \mathbb{R}$ the following holds.*
(i) (Propagation of chaos) For any time $t > 0$ and any $n \ge 1$ there is c so that for any distincts sites $x_1, \dots x_n$ in $\mathbb{Z}^d$, and any $\gamma > 0$,

$$\Big|\mathbb{E}^\gamma_{\mu_\gamma}\big(\prod_{i=1}^n \sigma_t(x_i)\big) - \prod_{i=1}^n \mathbb{E}^\gamma_{\mu_\gamma}\big(\sigma_t(x_i)\big)\Big| \le c\gamma^{d/2}. \tag{6.1}$$

$\mathbb{E}^\gamma_{\mu_\gamma}$ denotes the average with respect to the process with initial distribution μ_γ.
(ii) There are $\delta > 0$ and a smooth function $m(r,t)$ so that for all $t \ge 0$

$$\lim_{\gamma \to 0} \sup_r P_{\mu_\gamma}\Big(|M_\gamma(r, \sigma_t) - m(r,t)| > \gamma^\delta\Big) = 0 \tag{6.2}$$

where $M_\gamma(r, \sigma_t)$ is the order parameter defined in Section 4. Moreover the function $m(r,t)$ is the unique solution of

$$\frac{\partial m}{\partial t} = -m + \tanh\Big\{\beta\big[J \star m + h\big]\Big\} \tag{6.3}$$

$$m(r,0) = m_0(r). \tag{6.4}$$

In (6.3) the $\star$ indicates the convolution, i.e.

$$(J \star m)(r) = \int dr' J(|r - r'|) m(r'). \tag{6.5}$$

6.1 Remarks on the proof of Theorem 6.1 The proof of (i) is based on cluster expansions techniques that give sharp estimates on some sort of truncated correlation, see also De Masi and Presutti [1991] for applications of the same techniques to other models. Weaker statements which are valid for more general spin systems with Kac potentials, are proved in Comets and Eisele [1988].

To understand the origin of the differential equation (6.3), (6.4) and (6.5) notice that

$$\begin{aligned} \frac{d}{dt}\mathbb{E}^{\gamma}_{\mu_\gamma}(\sigma_t(x)) &= \mathbb{E}^{\gamma}_{\mu_\gamma}\big(- 2\sigma_t(x) c(x, \sigma_t)\big) \\ &= \mathbb{E}^{\gamma}_{\mu_\gamma}\Big(- \sigma_t(x) + \tanh\Big\{\beta\big[\sum_y J_\gamma(x,y)\sigma_t(y) + h\big]\Big\}\Big). \end{aligned} \tag{6.6}$$

Then (6.3), (6.4) and (6.5) follows from proving that

$$\lim_{\gamma \to 0} \sup_x \big|\mathbb{E}^{\gamma}_{\mu_\gamma}(\sigma_t(x)) - m(\gamma x, t)\big| = 0.$$

Moreover, using this and (i), there is c so that

$$\lim_{\gamma \to 0} \sup_x \Big|\mathbb{E}^{\gamma}_{\mu_\gamma}\Big(\tanh\Big\{\beta\big[\sum_y J_\gamma(x,y)\sigma_t(y) + h\big]\Big\}\Big) - \tanh\Big\{\beta[J \star m + h]\Big\}\Big| = 0. \tag{6.7}$$

7 The non local evolution equation

To be more specific from now on we assume that the external magnetic field h is non negative. It will turn out that the equation (6.3), (6.4) and (6.5) behaves in the space-time regimes of interest as the reaction diffusion equation

$$\frac{\partial m}{\partial t} = \Delta m - V'(m) \tag{7.1}$$

with $V(m)$ a polynomial function of m.

Adding and subtracting to the right hand side of (6.3) the term $\tanh\{\beta[m+h]\}$ we have that

$$\frac{\partial m}{\partial t} = \mathcal{R}(m) + \mathcal{D}(m) \tag{7.2}$$

where

$$\begin{aligned} \mathcal{R}(m) &= -m + \tanh\{\beta[m+h]\} && (7.3)\\ \mathcal{D}(m) &= \tanh\Big\{\beta\big[(J \star m) + h\big]\Big\} - \tanh\{\beta[m+h]\}. && (7.4) \end{aligned}$$

The function $\mathcal{R}(m)$, $m \in \mathbb{R}$, is strictly local and is of the form $-V'(m)$. It has one or three zeros according to whether $\beta \le \beta_c$ or $\beta > \beta_c$ respectively. Call $m_\beta^- < m_\beta^0 < m_\beta^+$ the three zeros and observe that $m_\beta^- = -m_\beta^+$ if $h = 0$, while

$|m_\beta^-| < m_\beta^+$ if $h > 0$. Furthermore, denoting by $\mathcal{R}'(m)$ the derivative of $\mathcal{R}$ with respect to m,

$$\mathcal{R}'(m_\beta^0) < 0 \ \text{ if } \beta < \beta_c, \qquad \mathcal{R}'(m_\beta^0) = 0 \qquad \text{if } \beta = \beta_c \tag{7.5}$$

$$\mathcal{R}'(m_\beta^0) > 0, \ \mathcal{R}'(m_\beta^\pm) < 0 \qquad \beta > \beta_c. \tag{7.6}$$

8 Phase separation: initial state

From now on we consider $\beta > \beta_c$ and we prepare the system in an unstable state. More precisely we assume that the initial mesoscopic profile is

$$m_0(r) \equiv m_\beta^0 \qquad \text{for all } r \in \mathbb{R}^d \tag{8.1}$$

where m_β^0 is defined in Section 7 and verifies (7.6). We then denote by μ_0 the initial probability distribution as in Section 5.

Notice that if m does not depend on r, i.e. it is a constant profile, then $\mathcal{D}(m) = 0$. Since $\mathcal{R}(m_\beta^0) = 0$, m_β^0 is a stationary solution to (7.2), (7.3) and (7.4). From Theorem 6.1 we then conclude that for all fixed $t \geq 0$, the order parameter verifies

$$\lim_{\gamma\to 0} \sup_r P_{\mu_\gamma}^\gamma \left(|M_\gamma(r,t) - m_\beta^0| > \gamma^\delta \right) = 0. \tag{8.2}$$

But, since by (7.6), m_β^0 is unstable for the mesoscopic evolution (6.3), (6.4) and (6.5) if we keep γ small but fixed and look at the system at longer times, then the instability will have important effects, as we are going to see.

8.1 Phase separation: linear theory Since by (8.2) the difference $M_\gamma(r,t) - m_\beta^0$ is small if γ is small, it is conceivable to approximate it with $u_\gamma(r,t)$, the solution of (7.2) linearized around m_β^0, that is

$$\frac{\partial u_\gamma}{\partial t} = \mathcal{R}'(m_\beta^0) u_\gamma + \mathcal{D}'(m_\beta^0)[J \star u_\gamma - u_\gamma] \tag{8.3}$$

$$u_\gamma(r,0) = \sigma([\gamma^{-1} r]) - m_\beta^0. \tag{8.4}$$

The initial datum $u_\gamma(r,0)$, $r \in \mathbb{R}^d$ is determined by a family of independent, identically distributed, mean zero random variables.

We confine the following discussion to the case $h = 0$ for which $m_\beta^0 = 0$. For the case $h > 0$, the same picture is expected to hold (Gobron (in preparation)).

If $h = 0$ then

$$\mathcal{R}'(0) = \beta - 1 \quad \text{and} \quad \mathcal{D}'(0) = \beta \tag{8.5}$$

so that (8.3) and (8.4) becomes

$$\frac{\partial u_\gamma}{\partial t} = (\beta - 1) u_\gamma + \beta [J \star u_\gamma - u_\gamma] \tag{8.6}$$

$$u_\gamma(r,0) = \sigma([\gamma^{-1} r]). \tag{8.7}$$

It is not difficult to prove that the solution to (8.6) and (8.7) can be written as

$$u_\gamma(r,t) = e^{(\beta-1)t}\mathcal{M}_\gamma(r,t) \tag{8.8}$$

where

$$\mathcal{M}_\gamma(r,t) := \int dr' P_t(r,r')\sigma([\gamma^{-1}r']) \tag{8.9}$$

and $P_t(r,r')$ is the probability density of going from r to r' in a time t with random jumps of intensity βJ.

For P_t the central limit theorem holds, that is for t large enough

$$P_t(r,r') \backsim (\tfrac{1}{D2\pi t})^{d/2} e^{-(r-r')^2/2Dt} \tag{8.10}$$
$$D = \int dr J(|r|) r^2. \tag{8.11}$$

From (8.9) and (8.10) and (8.11) it then follows that $\mathcal{M}_\gamma(r,t)$ is approximatively the average of $\sigma(x)$ for x ranging in a set whose volume is of the order $(\gamma^{-1}\sqrt{t})^d$. Then since the $\sigma(x)$'s are i.i.d. mean zero random variables the central limit theorem implies that $\mathcal{M}_\gamma(r,t)$ goes to zero as the square root of the volume.

From the above discussion we then conclude that for any fixed r

$$u_\gamma(r,t) = e^{(\beta-1)t}\mathcal{M}_\gamma(r,t) \backsim e^{(\beta-1)t}\gamma^{-d/2}t^{-d/4}. \tag{8.12}$$

Define

$$t_c = \tau_c \log\gamma^{-1} \qquad \tau_c = \frac{d}{2(\beta-1)} \tag{8.13}$$

we then have that

$$u_\gamma(r,t_c) \backsim t_c^{-d/4}. \tag{8.14}$$

On the other hand the space dependence of $u_\gamma(r,t_c)$ is ruled by $P_t(r,\cdot)$ therefore for $u_\gamma(r,t_c)$ to change significantly, r must vary by $\sqrt{t_c}$.

All the above can be proven to hold for the spin system only if we neglect an initial time layer which is the period of time when the noise fluctuations are relevant. To explain this we first simplify notation letting

$$\lambda = \frac{1}{\sqrt{\log\gamma^{-1}}}. \tag{8.15}$$

We can prove that our order parameter evolves essentially according to (8.6) but for times $\tau\lambda^{-2}$ with $\tau \in [\tau_0, \tau_c]$, for any $\tau_0 > 0$. Before stating this result we need the following definition.

Definition 8.1 *Given any $\tau_0 > 0$ we define $m_{\gamma,\tau_0}(x,t|\sigma)$, $x \in \mathbb{Z}^d$, $t \geq \lambda^{-2}\tau_0$, as the solution of the following discrete version of (6.3)*

$$\frac{\partial m_{\gamma,\tau_0}}{\partial t} = -m_{\gamma,\tau_0} + \tanh\{\beta J_\gamma \circ m_{\gamma,\tau_0}\} \qquad J_\gamma \circ m(x) = \sum_y J_\gamma(x,y)m(y)$$

$$m_{\gamma,\tau_0}(x, \lambda^{-2}\tau_0|\sigma) = \sigma(x). \tag{8.16}$$

For any $\tau \in [\tau_0, \tau_c]$ and any $\xi \in \mathbb{R}^d$ we define

$$X_\gamma(\xi) = t_c^{d/4} e^{(\beta-1)(\tau_c-\tau)} \int dr P_{(\tau_c-\tau)}(\lambda^{-1}\xi, r) m_{\gamma,\tau_0}([\gamma^{-1}r], \lambda^{-2}\tau|\sigma). \tag{8.17}$$

We then have the following result.

Theorem 8.1 *(De Masi et al. [1992]) The stochastic process $X_\gamma(\xi)$ converges in distribution as $\gamma \to 0$ to the mean zero Gaussian process $X(\xi)$ with covariance $C(\xi,\xi')$ given by*

$$C(\xi,\xi') = e^{-(\xi-\xi')^2/4D\tau_c} \tag{8.18}$$

where D is given in (8.11).

8.2 Phase separation: non linear analysis up to the critical time As predicted by the linear theory the relevant space-time scale for phase separation will be (ξ, τ) as in (3.2).

Theorem 8.2 *(De Masi et al. [1992]) There exists $\delta > 0$ such that for all $R > 0$*

$$\lim_{\gamma\to 0} P_{\mu_\gamma}\Big(\sup_{\tau\leq\tau_c} \sup_{|\xi|\leq R} |M_\gamma(\lambda^{-1}\xi, \sigma_{\lambda^{-2}\tau}) - m_\beta^0| > \gamma^\delta\Big) = 0. \tag{8.19}$$

8.3 Phase separation: development of interfaces After t_c the non linear effects become relevant and the phases develop; for this reason we call t_c the critical time. At time t_c

$$M_\gamma(\lambda^{-1}\xi, \sigma_{t_c}) \backsim (\frac{1}{t_c})^{d/4} X_\gamma(\xi)$$

where $X_\gamma(\xi)$ is defined in (8.17). After t_c, say at t^*, where

$$t^* = t_c + (\log\log\gamma^{-1})^2 \equiv \tau_c\lambda^{-2} + (\log\lambda)^2 \tag{8.20}$$

large clusters of the phases $m_\beta^\pm$ will cover the whole space. More precisely, we will see that at time t^*, for typical configurations σ_{t^*}, $M_\gamma(\lambda^{-1}\xi, \sigma_{t^*}) \backsim m_\beta^{\chi(\xi)}$, where $\chi(\xi) = \text{sign } X_\gamma(\xi)$, except in a neighbourhood of $\{X_\gamma = 0\}$. In this way we prove that the block spin variables have the cluster structure described above. The analysis goes further proving that the region $\{X_\gamma = 0\}$ where the different clusters meet are made of regular connected surfaces, called interfaces as I am going to explain below.

Theorem 8.3 *(De Masi et al. [1992]) Let $h = 0$. For any ϵ and δ there is $\mathcal{G}_\gamma$ such that*

$$P_{\mu_\gamma}(\mathcal{G}_\gamma) > 1 - \epsilon$$

and if $\sigma_{t^*} \in \mathcal{G}_\gamma$ then the following holds.

There is a set Σ union of smooth surfaces Γ called interfaces such that the following holds. Let $d(\xi, \Sigma)$ denote the distance of ξ from Σ.
(i) If $d(\xi, \Sigma) > \delta$ then

$$|M_\gamma(\lambda^{-1}\xi, \sigma_{t^*}) - m_\beta \, sign \, X_\gamma(\xi)| \leq \epsilon.$$

(ii) Let $\xi_0 \in \Sigma$ and ν be the normal to Σ at ξ_0. Then

$$|M_\gamma(\lambda^{-1}\xi_0 + r, \sigma_{t^*}) - \bar{m}(r \cdot \nu)| \leq \epsilon \qquad \text{for all } |r| \leq \delta\lambda^{-1}.$$

The function $\bar{m}(z)$, $z \in \mathbb{R}$ is the stationary solution (instanton) connecting the two pure phases $m_\beta^\pm$. That is

$$0 = -\bar{m} + \tanh\{\beta[\hat{J} \star \bar{m}]\} \tag{8.21}$$

$$\lim_{z\to\pm\infty} \bar{m}(z) = m_\beta^\pm \tag{8.22}$$

where

$$\hat{J}(z, z') = \int_{\mathbb{R}^{d-1}} dy J\big(((z - z')^2 + y^2)^{1/2}\big). \tag{8.23}$$

The set Σ can be taken equal to the set $\{X(\xi) = 0\}$, where $X(\xi)$ is the random variable of Theorem 8.2.

In the case $h > 0$ the same statement is expected to hold, but in this case the function $\bar{m}$ is the travelling front connecting the two pure phases $m_\beta^\pm$. Thus, for $h > 0$ small enough, there is $c = c(h) \leq 0$ such that $\bar{m}(z - ct)$ solves the evolution equation, that is

$$-c\bar{m}' = -\bar{m} + \tanh\{\beta[\hat{J} \star \bar{m} + h]\} \tag{8.24}$$

$$\lim_{z\to\pm\infty} \bar{m}(z) = m_\beta^\pm \tag{8.25}$$

where $\hat{J}$ is defined in (8.22).

The existence and stability of the travelling front $\bar{m}$ has been proven (De Masi, Gobron and Presutti [1993]) for $h > 0$, the case $h = 0$ has been studied in Dal Passo and de Mottoni [1991], De Masi et al. [1993] and De Masi et al. (to appear).

9 Motion of the interfaces

By Theorem 8.3 the configuration σ_{t^*} at time t^* is in the set $\mathcal{G}_\gamma$ with large probablity. Therefore, the order parameter $M_\gamma(r, \sigma_{t^*})$ has values $m_\beta^\pm$ inside and outside regular surfaces. We expect that after t^* the evolution is deterministically governed by the non local evolution equation (6.3) whith initial datum $M_\gamma(r, \sigma_{t^*})$. On the other hand, if we consider the evolution equation (6.3) with such an initial datum, we expect that the interfaces move according to rules that depend on h, as I explain below.

Case h > 0. Since $h > 0$, the phase m_β^+ prevails on the other one and move on macroscopic times which are scaled as

$$t = \tau\lambda^{-1}.$$

The front $\bar{m}$ moves rigidly with velocity pointing toward the m_β^- phase and directed along the normal. Namely if we follow a point ξ of the interface its equation of motion is

$$\frac{d\xi}{d\tau} = c\nu, \qquad \xi(0) = \xi_0 \tag{9.1}$$

where ν is the normal in $\xi(\tau)$ at the interface directed toward the interior and $c = c(h)$ is the speed of the travelling front.

Case h = 0. There is no bias toward any of the two phases, $m_\beta^+ = -m_\beta^-$. The only driving force is the curvature of the interface and the motion is slower than in the case $h > 0$. In fact the time has to be scaled as

$$t = \tau\lambda^{-2} \qquad \tau > \tau_c.$$

The interface moves by mean curvature, namely its points move according to the equation

$$\frac{d\xi}{d\tau} = \theta k(\xi(\tau))\nu \tag{9.2}$$

where ν is the normal as before, $k(\xi)$ is the signed mean curvature at $\xi \in \Gamma$ and θ is a positive constant depending on the potential J. The value of θ is known rather explicitely, see De Masi et al. [1994], and it satisfies an Einstein relation proposed by Spohn [1993], and verified in our case by Buttà [1993].

We have not yet a complete proof of the validity of this picture, but we have solved the following intermediate problem.

Let $\Gamma_0 \subset \mathbb{R}^d$ be a compact domain whose boundary Σ_0 is a connected C^∞ surface. For instance Γ_0 is the sphere of radius 1 centered at the origin. For any $\lambda < 1$, define the function $u_0^\lambda(r)$, $r \in \mathbb{R}^d$ as follows.

$$u_0^\lambda(r) = \begin{cases} m_\beta^+ & \text{if } r \in \lambda^{-1}\Gamma_0 \\ m_\beta^- & \text{otherwise} \end{cases} \tag{9.3}$$

Consider the Glauber dynamics with an initial state as in Section 5 and assume that the initial mesoscopic profile

$$m_0(r) = u_0^\lambda(r)$$

where u_0 is defined in (9.3). Then the following holds.

Theorem 9.1 *There are a and b positive such that the following holds.*
(i) Case $h > 0$ (Gobron (in preparation)). For any $\tau > 0$, let Σ_τ, be the surface Σ_0 evoluted according to (9.1). Let $\tau^\star > 0$ be such that the motion is regular for $\tau < \tau^\star$. Let $d(\xi, \Sigma_\tau)$ be the distance of a point $\xi \in \mathbb{R}^d$ from Σ_τ. If $d(\xi, \Sigma_\tau) \geq \lambda^a$, then

$$\lim_{\gamma\to 0} P^{\gamma}_{\mu_\gamma}\left(|M_\gamma(\lambda^{-1}\xi, \lambda^{-2}\tau_c + \lambda^{-1}\tau) - m^{\pm}_{\beta}| \le \lambda^b\right) = 1 \textit{ for all } \tau < \tau^\star \quad (9.4)$$

for ξ, inside, respectively outside Γ_τ.
(ii) Case $h = 0$ (De Masi et al. [1994a]). There exists $\theta > 0$ and $\tau^\star > 0$ such that the following holds. Let Σ_τ, $\tau < \tau^\star$ be the surface Σ_0 evoluted according to (9.2) and let $d(\xi, \Sigma_\tau)$ be the distance of a point $\xi \in \mathbb{R}^d$ from Σ_τ. If $d(\xi, \Sigma_\tau) \ge \lambda^a$, then

$$\lim_{\gamma\to 0} P^{\gamma}_{\mu_\gamma}\left(|M_\gamma(\lambda^{-1}\xi, \lambda^{-2}\tau) - m^{\pm}_{\beta}| \le \lambda^b\right) = 1 \qquad \textit{for all } \tau \in (\tau_c, \tau^\star) \quad (9.5)$$

for ξ, inside, respectively outside Γ_τ.

To prove this Theorem we show that the order parameter M_γ is close to $m^\lambda(r,t)$, the solution to (6.3) with initial datum u_0^λ. We then use the results proven in De Masi et al. [1993] for $m^\lambda(r,t)$.

Many aspects of the above analysis are common to the model introduced in De Masi, Ferrari and Lebowitz [1985] whose mesoscopic equation is the reaction diffusion equation (7.1). This stochastic dynamics is a continuous time Markov process whose generator is of the form $\gamma^{-2}L_0 + L_G$, $\gamma \in (0,1)$. L_0 is the generator of the symmetric simple exclusion process and L_G generates a Glauber dynamics with finite range interaction between nearest neighbour spins.

The spinodal decomposition for this model has been studied in De Masi et al. [1994b] in dimension one and Giacomin [1992] in dimensions $d \le 3$. The motion by curvature is established in Bonaventura [1994] and Katsoulakis and Souganidis [1992]. The result in Bonaventura is (as in our Theorem 9.1) only up to the time in which the motion by curvature is smooth. In the analysis of Bonaventura the magnetization around the interfaces is fully characterized as the instanton solution $\bar{m}$ defined in (8.21). The result uses the analysis done in Mottoni and Schatzman [1989] for the reaction diffusion equation. Instead the statement in Katsoulakis and Souganidis [1992] concerns only regions that do not include the interface, but it is valid also past the appearence of singularities. The techniques that are used are based on the notion of generalized motion by curvature developed in Evans and Spruck [1991] and applied to the reaction diffusion equations in Evans, Soner and Souganidis (to appear).

References

Bonaventura, L. [1994], *Motion by curvature in an interacting spin system,* in Ann. Inst. H. Poincaré, Nonlinear Analysis Theory Methods and Applications.

Buttà, P. [1993], *On the validity of an Einstein relation in models of interface dynamics,* J. Stat. Phys., **72**, 1401-1406.

Comets, F. and Eisele, Th. [1988], *Asymptotic dynamics, noncritical and critical fluctuations for a geometric long range interacting model,* Commun. Math. Phys., **118**, 531-568.

Dal Passo, R. and de Mottoni, P. [1991], *The heat equation with a nonlocal density dependent advection term,* preprint.

De Masi, A., Ferrari, P.A. and Lebowitz, J.L. [1985], *Rigorous derivation of reaction diffusion equations with fluctuations*, Phys. Rev. Lett., **55**, 1947-1949, *Reaction diffusion equations for interacting particle systems*, J. Stat. Phys., **44**, 589-644, [1986].

De Masi, A., Gobron, T. and Presutti, E. [1993], *Travelling fronts in non local evolution equations*, preprint.

De Masi, A. and Presutti, E. [1991], *Mathematical methods for hydrodynamic limits*, Lectures Notes in Mathematics, **1501**, Springer Verlag, New York, Heidelberg, Berlin.

De Masi, A., Orlandi, E., Presutti, E. and Triolo, L. [1994a], *Glauber evolution for Kac potentials: I. Mesoscopic and macroscopic limits, interface dynamics*, Nonlinearity, **7**, 633-696.

De Masi, A., Orlandi, E., Presutti, E. and Triolo, L. [1992], *Glauber evolution with Kac potentials. II. Spinodal decomposition*, CARR report n. 28/92.

De Masi, A., Orlandi, E., Presutti, E. and Triolo, L. [1993], *Uniqueness of the instanton profile and global stability in non local evolution equations*, CARR report n. 10/93.

De Masi, A., Orlandi, E., Presutti, E. and Triolo, L. [to appear], *Stability of the interface in a model of phase separation*, The Royal Society of Edinburg Proceedings A (Mathematics).

De Masi, A., Orlandi, E., Presutti, E. and Triolo, L. [1993], *Motion by curvature by scaling non local evolution equations*, J. Stat. Phys., **73**, 543-570.

De Masi, A., Pellegrinotti, A., Presutti, E. and Vares, M.E. [to appear 1994b], *Spatial patterns when phases separate in an interacting particle system*, Annals of Probability, **22**, 334-371.

de Mottoni, P. and Schatzman, M. [1989], *Évolution géometrique d'interfaces*, C.R. Acad. Sci. Paris, **309**, 453-458.

Evans, L.C. and Spruck, J. [1991], *Motion of level sets by mean curvature*, J. Differential Geometry, **33**, 635-681; *Motion of level sets by mean curvature. II.*, Trans. Amer. Math. Soc., **330**, (1992), 321-332; *Motion of level sets by mean curvature. III.*, J. Geom. Anal., **2**, (1992), 121-150.

Evans, L.C., Soner, H.M. and Souganidis, P.E. [to appear], *Phase transitions and generalized motion by mean curvature*, Comm. Pure and Appl. Math.

Giacomin, G. [1992], *Phase separation and random domain paterns in a stochastic particle model*, preprint, Rutgers; *Interface formation and global spatial structure in a reaction-diffusion model*, preprint (1992).

Gobron, T., in preparation.

Kac, M., Uhlenbeck, G. and Hemmer, P.C. [1963], *On the Van der Waals theory of vapor-liquid equilibrium*, I. Discussion of a one dimensional model, J. Math. Phys., **4**, 216-228; II. Discussion of the distribution functions, J. Math. Phys., **4**, 229-247, (1963); III. Discussion of the critical region, J. Math. Phys., **5**, 60-74, (1964).

Katsoulakis, M.A. and Souganidis, E. [1992], *Interacting particle systems and generalized motion by mean curvature*, preprint.

Lebowitz, J. and Penrose, O. [1966], *Rigorous treatment of the Van der Waals Maxwell theory of the liquid vapour transition*, J. Math. Phys., **7**, 98.

Presutti, E., *Critical fluctuations in a spin system*, In this volume.

Spohn, H. [1993], *Interface motion in models with stochastic dynamics*, J. Stat. Phys., **71**, 1081-1132.