SPECTRAL PROPERTIES OF INTEGRAL OPERATORS IN PROBLEMS OF INTERFACE DYNAMICS AND METASTABILITY

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Abstract. In this paper we study some integral operators that are obtained by linearizations of a non local evolution equation for a non conserved order parameter which describes the phase of a fluid. We prove a Perron-Frobenius theorem by showing that there is an isolated, simple, maximal eigenvalue larger than 1 with a positive eigenvector and that the rest of the spectrum is strictly inside the unit ball. Such properties are responsible for the existence of invariant, attractive unstable one dimensional manifolds under the full, non linear evolution. This part of the analysis and the application to interface dynamics and metastability will be carried out in separate papers.

1. Introduction

In this paper we study the eigenvalue problem for an integral operator A on $C^{\text{sym}}(\mathbb{R})$, the space of symmetric, bounded function on $\mathbb{R}$ with sup norm. We suppose that the kernel of A has the form $A(x,y) = p(x)J(x,y)$, with $p(x)$ a symmetric, strictly positive, regular, bounded function and $J(x,y)$ a regular, non negative function of the variable $y-x$ with compact support and integral equal to 1. Further conditions on p and J are specified below, motivated by applications to interface dynamics and metastability. For such p and J we will prove a Perron-Frobenius theorem about the existence of an isolated, positive maximal eigenvalue λ with positive eigenvector. For stability questions it is important to determine the part of the spectrum outside the unit ball. This is not simple in our case because $p(x)$ is both larger and smaller than 1. We will prove that only the maximal eigenvalue λ is > 1, while the rest of the spectrum is strictly inside the unit ball, a property that in the applications is responsible for the existence of an unstable, one dimensional, attractive manifold.

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The problem arises from the analysis of the evolution equation in $C^{\rm sym}(\mathbb{R};[-1,1])$

$$\frac{\partial m_t}{\partial t} = -m_t + \tanh\Big(\beta J\star m_t + \beta h\Big) \tag{1.1}$$

where $\beta > 1$, $h \geq 0$ and

$$(J\star m)(x) := \int_{\mathbb{R}} dy J(x,y)m(y) \tag{1.2}$$

The operators A that we consider are related to the linearization of the right hand side of (1.1). Namely, given $m \in C^{\rm sym}(\mathbb{R};[-1,1])$, we set

$$p_m(x) := \frac{\beta}{\cosh^2\{\beta J\star m(x)\}} \tag{1.3}$$

and define A_m as the operator with kernel

$$A_m(x,y) := p_m(x)J(x,y) \tag{1.4}$$

Then (when $h=0$) $L_m := A_m - 1$ is the linearization around m of the right hand side of (1.1). We will study the operators A_m, we are now adding the subscript m to underline the dependence on m.

Equation (1.1) has been derived from the Glauber dynamics of a one dimensional Ising spin system interacting via a Kac potentials, see [7]. $m_t(x)$ is the spin magnetization density and the condition that $m_t(x)$ is in $[-1,1]$ reflects the fact that the Ising spins have values ± 1. J is the coupling of the spin-spin interaction: $J(x,y)\geq 0$ (ferromagnetic interactions) is an even, C^2 function of $y-x$ (translational invariance) supported by the unit interval (we suppose that $\sup\{x : J(0,x)>0\} = 1$) and normalized to have integral 1. h is an external magnetic field and $\beta = 1/kT$, k the Boltzmann constant and T the absolute temperature.

When $\beta > 1$ there is a phase transition in the underlying spin system, [16], [3], [1]. The pure phases correspond to the stationary, spatially homogeneous solutions of (1.1), thus a pure phase with magnetization $s\in[-1,1]$, is a solution of

$$s = \tanh\Big(\beta s + \beta h\Big) \tag{1.5}$$

Given $\beta > 1$ there is $h^\star > 0$ so that for $0 \leq h < h^\star$ (1.5) has three and only three different roots, denoted by

$$m_\beta^-(h) < m_\beta^0(h) \leq 0 < m_\beta^+(h) \tag{1.6}$$

The two phases $m_\beta^\pm(0)$, $m_\beta^+(0) = -m_\beta^-(0) =: m_\beta > 0$, are thermodynamically stable at $h=0$, while $m_\beta^0(0) = 0$ is unstable. As h increases past 0, $m_\beta^+(h)$ is the only thermodynamically stable phase left, $m_\beta^0(h)$ is still unstable while $m_\beta^-(h)$ becomes metastable. These statements, established in the context of the theory of Equilibrium Statistical Mechanics, see [16], [?], are reflected by the corresponding (obvious) stability properties of the space homogeneous solutions of (1.1).

Interface dynamics concerns the analysis of the Cauchy problem for (1.1) with initial data close to different phases in different regions of space. This problem has been extensively studied in the last years with special attention to the multi-dimensional case where it has been proved that on a suitable space-time scaling limit the evolution is ruled by a motion by mean curvature, see [8], [15] and references therein. In one dimensions when $h > 0$ there are travelling fronts describing the growth of the stable phase at the expenses of the metastable one, see [4] and references therein. When $h = 0$ there are stationary solutions with two coexisting phases: they are all identical, modulo translations and reflection, [10], to "the instanton" $\bar{m}(x)$, which is a C^∞, strictly increasing, antisymmetric function which identically verifies

$$\bar{m}(x) = \tanh\Big(\beta J \star \bar{m}(x)\Big) \tag{1.7}$$

$\bar{m}(x)$ is the stationary pattern that connects the minus and the plus phases, as

$$\lim_{x\to\pm\infty} \bar{m}(x) = \pm m_\beta \tag{1.8}$$

and it has therefore the interpretation of a "diffuse interface". However, since $\bar{m}'(x)$ vanishes exponentially fast as $|x| \to \infty$, [9], then, loosely speaking, the fraction of space not occupied by pure phases is vanishingly small. In this sense, that can be made precise by introducing scalings, the interface is sharp and the transition from one phase to the other one is "instantaneous". That is why $\bar{m}$ (or, more properly $\bar{m}'$) is called the instanton.

As proved in [10] the interface described by the instanton is "stable" and any initial datum "close to an instanton" is attracted and eventually converges to some translate of the instanton. If instead the space occupied by one of the two phases is bounded, for instance a finite interval outside which the other phase is present, then (it is believed that) the latter will prevail and in the end it will be the only one present. This process however may be extremely slow: in the Allen-Cahn equation the velocity of propagation of the majority phase vanishes exponentially fast with the length of the interval where the minority phase is present, [2], [13]. In that case the profile is attracted by an unstable manifold whose points are functions which are close to one phase in an interval, to the other one outside it and each one of the interface patterns is close to an instanton. To prove such a result in the present context we need to study (1.1) in a neighborhood of functions m of the form

$$m_\xi(x) := \bar{m}(\xi - |x|), \qquad \xi > 0 \tag{1.9}$$

which by linearization leads to the operator A_m with either $m = m_\xi$ or m close to it (and ξ large).

This is a first motivation for studying A_m, the second one comes from metastability. When $h > 0$ the metastable phase $m_\beta^-(h)$ is stable under small perturbations, but if the perturbation creates a droplet of the stable phase which is large enough then this will grow and will eventually invade the whole space. In a forthcoming paper we will prove the existence of a critical droplet $m^\star(x)$, which is a stationary solution of (1.1) with $h > 0$. $m^\star(x)$ is close to the stable phase in an interval centered at the origin and to the metastable phase outside it. Moreover we will show that there is a one dimensional unstable manifold through $m^\star$ whose points have a pattern similar to that in $m^\star$, but with a different length of the stable region. In the branch of the manifold where the length is shorter the evolution shrinks it further while it grows if larger. Like before, this manifold attracts all profiles that are in a neighborhood. Thus again we are interested in the analysis of the operator A_m, with $m = m_\xi + h$ or close to such a function.

The existence of these attractive one dimensional unstable manifolds reflects the presence of an isolated, simple, maximal eigenvalue $\lambda_m > 1$ for A_m and of a spectral gap, with the rest of the spectrum strictly in the unit ball. In the present paper we will establish these and other properties of the operators A_m which are the building ingredients for the applications (mentioned above) to interface dynamics and metastability, that will be treated in successive papers. In the next section we state the main results and give an outline of the rest of the paper.

2. Main results

The problem in bounded domains with Neumann conditions

The operator A_m on $C^{\rm sym}(\mathbb{R})$ can be isomorphically regarded as an operator A_m^+ on $C(\mathbb{R}_+)$, by setting $A_m^+ f := A_m f^\star$, where, given $f \in C(\mathbb{R}_+)$, $f^\star \in C^{\rm sym}(\mathbb{R})$ is defined by $f^\star(x) = f(|x|)$. A_m^+ is still an integral operator and its kernel is

$$A_m^+(x,y) = A_m(x,y) + A_m(x,-y), \qquad x, y \in \mathbb{R}_+ \tag{2.1}$$

This can be interpreted as a reflecting boundary condition (hereafter called Neumann) at 0 and the original problem for A_m on $C^{\rm sym}(\mathbb{R})$ is actually the problem on the half line with Neumann conditions at 0. By adding another reflection at $\ell > 1$ we can then define a new operator $A_{m,\ell}$ on $C([0,\ell])$ thus studying problems in bounded domains: setting

$$R_\ell(x) := \begin{cases} |x| & \text{for } -1 \le x \le \ell \\ \ell - (x-\ell) & \text{for } \ell \le x \le \ell+1 \end{cases} \tag{2.2}$$

we define, for x and y in $[0,\ell]$,

$$A_{m,\ell}(x,y) := \sum_{z:R_\ell(z)=y} A_m\big(x,z\big) \tag{2.3}$$

$A_{m,\ell}$ is then the operator on $C([0,\ell])$ with kernel $A_{m,\ell}(x,y)$. The case $\ell = +\infty$ is included by setting $R_{+\infty}(x) := |x|$, then $A_{m,+\infty} = A_m^+$.

We will work in finite volumes and, by proving estimates uniform in ℓ, we will recover the original case in the limit $\ell \to +\infty$. This is not only a technical device, in fact the analysis in the bounded domains has its own interest, see for instance [2] and [13] where, in the context of the Allen-Cahn equation, analogous problems are studied in finite intervals with Neumann boundary conditions. Recalling the discussion in the Introduction where m was taken close to a double instanton, the analysis in $C([0,\ell])$ with Neumann conditions corresponds to two double instantons, one across 0 and the other one across ℓ. The spectral properties in this case reflect the interaction between these two structures, see [2] and [13] for a discussion of these aspects.

When ℓ is finite we have the classical Perron-Frobenius theorem:

2.1 Theorem.

Let $\ell > 1$ and $m \in C([0,\ell],[-1,1])$. Then there are $\lambda_{m,\ell} > 0$, $u_{m,\ell}$ and $v_{m,\ell}$ in $C([0,\ell])$, $u_{m,\ell}$ and $v_{m,\ell}$ strictly positive, so that

$$A_{m,\ell} \star v_{m,\ell} = \lambda_{m,\ell} v_{m,\ell}, \qquad u_{m,\ell} \star A_{m,\ell} = \lambda_{m,\ell} u_{m,\ell} \tag{2.4}$$

($u_{m,\ell}$ and $v_{m,\ell}$ are left and right eigenvectors with eigenvalue $\lambda_{m,\ell}$) and for any $x \in [0,\ell]$

$$v_{m,\ell}(x) = p_m(x) u_{m,\ell}(x) \tag{2.5}$$

Any other point of the spectrum is strictly inside the ball of radius $\lambda_{m,\ell}$.

Being too general the theorem cannot say much about the localization of the spectrum and the dependence on ℓ of $\lambda_{m,\ell}$, $v_{m,\ell}$ and $u_{m,\ell}$, for that we need more assumptions on m. The proof of Theorem 2.1 is classical, we report a version in Section 4 both for completeness and to introduce notions that will be used to study the limit $\ell \to +\infty$. A special role in the proofs is played by a Markov chain whose transition probability is conjugated to $A_{m,\ell}(x,y)$.

Auxiliary Markov chains

By the positivity of $\lambda_{m,\ell}$ and $v_{m,\ell}$,

$$Q_{m,\ell}(x,y) := A_{m,\ell}(x,y) \frac{v_{m,\ell}(y)}{\lambda_{m,\ell} v_{m,\ell}(x)}, \qquad x,y \in [0,\ell] \tag{2.6}$$

is well posed and it defines a transition probability on $[0,\ell]$ conjugated to $A_{m,\ell}$: the spectrum of $A_{m,\ell}$ is obtained from that of $Q_{m,\ell}$ after multiplication by $\lambda_{m,\ell}$. In particular the spectral gap in Theorem 2.1 is related to the mixing properties of the Markov chain with transition probability $Q_{m,\ell}$.

If m is an instanton, $m = \bar{m}$, then $\lambda_{\bar{m}} = 1$ and $v_{\bar{m}} = \bar{m}'$, i.e. $A_{\bar{m}}\bar{m}' = \bar{m}'$, obtained by differentiating the instanton equation (1.7). The analogue of (2.6) defines our basic transition probability:

$$P(x,y) := A_{\bar{m}}(x,y)\frac{\bar{m}'(y)}{\bar{m}'(x)}, \qquad x \text{ and } y \text{ in } \mathbb{R} \tag{2.7}$$

In the problems with a (reflected) instanton at ξ, i.e. $\bar{m}(\xi - x)$, and Neumann conditions, i.e. reflections at 0 and ℓ, $\ell > 2\xi$, $\xi > 1$, an important role will be plaied by the transition probability:

$$Q_{\xi,\ell}(x,y) := \sum_{R_\ell(z)=y} P(\xi - x, \xi - z), \qquad x \text{ and } y \text{ in } [0,\ell] \tag{2.8}$$

The above three Markov chains can be seen as describing similar, discrete time, jump processes of a particle on the line $\mathbb{R}$. The intensity of the jump from x to y is proportional to $J(x,y)$ so that the maximal displacement is one. This is exactly the case for the second one ((2.7)) which has the whole line $\mathbb{R}$ as state space, whereas the first ((2.6)) and the third ((2.8)) ones have the interval $[0,\ell]$ as state space. In these last two cases a reflection rule at 0 and ℓ (see (2.2)) enters into the game when the particle tries to bypass the points 0 and ℓ, respectively. If both x and y are at distance less than 1 from the boundary $\{0,\ell\}$, the intensity of the jumps becomes the sum of two terms: the contribution of the direct jump from x to y and the one of the jump from x to the "mirror" point $z(y)$ of y where $z(y) : R_\ell(z(y)) = y$, $z(y) \in [-1,0] \cup [\ell, \ell+1]$. For instance if $y \in [0,1]$, then $z(y) = |y|$.

For technical reasons we will also introduce in the sequel, another Markov chain with transition probability

$$\tilde{P}(x,y) = P\big(\xi - R_\ell(x), \xi - y\big) \tag{2.9}$$

whose behaviour is almost identical to the one with transition probability (2.8)

The instanton

We will use throughout the paper several properties of $\bar{m}$, some taken from the literature, [5], [9], [10], the others, stated in Theorem 2.2 below and proved in Section 3, are new.

2.2 Theorem.

There are α and a positive, $\alpha_0 > \alpha$ and $c > 0$ so that for $x \geq 0$

$$\left|\bar{m}(x) - (m_\beta - ae^{-\alpha x})\right| + \left|\bar{m}'(x) - a\alpha e^{-\alpha x}\right| + \left|\bar{m}''(x) + a\alpha^2 e^{-\alpha x}\right| \leq ce^{-\alpha_0 x} \tag{2.10}$$

The double instanton

We next specialize to functions close to an instanton, more precisely given $\ell > 2\xi$, $\xi > 1$, we define a finite volume version of m_ξ, see (1.9), that we call the "double instanton", by setting for $x \in [0,\ell]$

$$m^0_{\xi,\ell}(x) := \bar{m}(\xi - x) - ae^{-\alpha(x+\xi)} + ae^{-\alpha(2\ell-\xi-x)} \tag{2.11}$$

When $\ell = +\infty$ we set equal to 0 the last term in (2.11) and write m^0_ξ.

The right hand side of (2.11) should be regarded as (close to) the sum of three instantons: the first one, the basic one, is $\bar{m}(\xi - x)$; the second one is centered at $-\xi$ and the third one at $2\ell - \xi$. These last two are taken in the asymptotic approximation (2.10) which is the dominant term when x is in $[0, \ell]$ and ξ is large. Observe that the instanton at $-\xi$ is obtained by reflecting the basic one, $\bar{m}(\xi - x)$, around the origin while the third one by reflecting around ℓ. Thus the "corrections" to $\bar{m}(\xi - x)$ in (2.11) are due to the Neumann conditions at 0 and ℓ.

We next define the neighborhoods of the double instanton where we will study the spectrum of $A_{m,\ell}$, the choice being dictated by the analysis of the applications mentioned in the Introduction.

2.3 Definition.

Let $\xi > 1$ be fixed.

Given $\ell \in [2\xi, +\infty]$ and $m \in C([0, \ell]; [-1, 1])$, we set

$$\delta^0_\xi m = m - m^0_{\xi,\ell} \tag{2.12}$$

and define $G_{(c,\xi,\ell)}$, $c > 0$, as the set of all m in $C([0, \ell], [-1, 1])$ such that

$$\left|\delta^0_\xi m(x)\right| \le c \begin{cases} e^{-2\alpha\xi} e^{\alpha(\xi - x)} & \text{for } 0 \le x \le \xi \\ e^{-2\alpha\xi} + e^{-2\alpha(\ell-\xi)} e^{\alpha(x-\xi)} & \text{for } \xi < x \le \ell \end{cases} \tag{2.13}$$

We will also consider a subset in $G_{(c,\xi,\ell)}$ indexed by $\delta > 0$ which contains all m such that

$$-\int_{|x-\xi| \le \xi^{1/2}} dx\, \delta^0_\xi m(x) \bar{m}'(\xi - x)^2 \bar{m}(\xi - x) > -c e^{-2(\alpha+\delta)\xi} \tag{2.14}$$

Sharp estimates on $\lambda_{m,\ell}$ and $v_{m,\ell}$

We define $\tilde{m}(x) := \sqrt{C_{\bar{m}}}\bar{m}(x)$, where $C_{\bar{m}}$ is a constant such that

$$\int_{\mathbb{R}} dx \frac{\tilde{m}'(x)^2}{p_{\bar{m}}(x)} = 1 \tag{2.15}$$

and set $\tilde{m}'_\xi(x) = \tilde{m}'(\xi - x)$. We also normalize $u_{m,\ell}(x)$ (and then $v_{m,\ell}$) in such a way that

$$\int_0^\ell \frac{dx}{p_m(x)} v_{m,\ell}(x)^2 \equiv \int_0^\ell dx\, u_{m,\ell}(x) v_{m,\ell}(x) = 1 \tag{2.16}$$

We then have

2.4 Theorem.

For any $c>0$ there are $c_\pm$ and c' all positive so that for all $\ell \ge 2\xi$, $\xi>1$ and all $m \in G_{(c,\xi,\ell)}$

$$1-c_-e^{-2\alpha\xi} \le \lambda_{m,\ell} \le 1+c_+e^{-2\alpha\xi} \tag{2.17}$$

$$u_{m,\ell}(x),\ v_{m,\ell}(x) \ \le\ c_+e^{-\alpha'|\xi-x|}, \qquad \alpha'=\alpha'(\xi):=\alpha-c'e^{-2\alpha\xi} \tag{2.18}$$

$$\Big|v_{m,\ell}(x)-\tilde m'_\xi(x)\Big| \le c_+e^{-2\alpha\xi+\alpha|\xi-x|}\xi^4, \qquad \text{for all } x \text{ such that } |\xi-x|\le \xi/2 \tag{2.19}$$

Moreover for any $c>0$ and $\delta>0$ there is $D>0$ so that if $m\in G_{(c,\xi,\ell)}$ as above and satisfies (2.14)*, then*

$$\lambda_{m,\ell} \ge 1+\frac{D}{2}e^{-2\alpha\xi} \tag{2.20}$$

where D is the parameter defined in (9.40)*.*

(2.18) and the first inequality in (2.17) are proved in Section 5, the other statements in Section 9 together with several other properties of $\lambda_{m,\ell}$, $u_{m,\ell}$ and $v_{m,\ell}$.

Spectral gap, resolvent

Given $\ell>2\xi$, $\xi>1$, $\zeta\in\mathbb{R}$ and $w\in C([0,\ell])$, we set

$$\|w\|_{\zeta,\xi,\ell} := \sup_{x\in[0,\ell]} e^{-\zeta|\xi-x|}|w(x)| \tag{2.21}$$

Given $m\in C^{\rm sym}([0,\ell],[-1,1])$, we then define the linear functional $\pi_{m,\ell}$ on $C([0,\ell])$ as

$$\pi_{m,\ell}(w) := \int_0^\ell dx\, u_{m,\ell}(x)w(x) \tag{2.22}$$

($u_{m,\ell}$ normalized as in (2.16)). We also call $L_{m,\ell}:=A_{m,\ell}-1$.

2.5 Theorem.

Given $c>0$ there are $d_\pm>0$, $\zeta^\star<0$ and $\xi^\star>1$, so that for any $\xi\ge\xi^\star$, $\ell\in[2\xi,+\infty]$, $\zeta^\star<\zeta<0$, $m\in G_{(c,\xi,\ell)}$ and $t\ge0$

$$\|e^{L_{m,\ell}t}\|_{\zeta,\xi,\ell} \le d_+e^{(\lambda_{m,\ell}-1)t} \tag{2.23}$$

and, for any $\tilde w$ such that $\pi_{m,\xi,\ell}(\tilde w)=0$,

$$\|e^{Lt}\tilde w\|_{\zeta,\xi,\ell} \le d_+e^{-d_-t}\|\tilde w\|_{\zeta,\xi,\ell} \tag{2.24}$$

Moreover given $\delta>0$ there is $C>0$ so that if m is in $G_{(c,\xi,\ell)}$ and satisfies (2.14)*, then the inverse $(L_{m,\ell})^{-1}$ exists and*

$$\|(L_{m,\ell})^{-1}\|_{\zeta,\xi,\ell} \le Ce^{2\alpha\xi} \tag{2.25}$$

If $\tilde{w}$ is such that $\pi_{m,\xi,\ell}(\tilde{w}) = 0$ then

$$(L_{m,\ell})^{-1}\tilde{w} = -\int_0^{+\infty} dte^{L_{m,\ell}t}\tilde{w}, \qquad \|(L_{m,\ell})^{-1}\tilde{w}\|_{\zeta,\xi,\ell} \le C\|\tilde{w}\|_{\zeta,\xi,\ell} \tag{2.26}$$

Theorem 2.5 is proved in Section 10. The dependence of the maximal eigenvalues and eigenvectors on ξ is studied in Section 11.

Outline of the paper

As the paper is long we hope that a detailed outline of its content and comments and suggestion on the organization of its reading may be particularly helpful.

Section 3 is devoted to the analysis of the instanton $\bar{m}(x)$ and of its asymptotic behavior as $x \to +\infty$. Its implications on the rest of the paper are through (2.10) and Lemmas 3.1 and 3.3, the rest are proofs that to a first reading may be skipped. Lemma 3.1 is preliminary to the analysis of $\bar{m}(x)$ and concerns properties of a function α_p which are frequently used in the other sections. Lemma 3.3 is about the positivity of the iterates of the kernel $J(x, y)$, it can be read independently of the rest of the section.

Section 4 is about the Perron-Frobenius theorem for $A_{m,\ell}$, here we use that ℓ is finite while m is still quite general; the estimates are not uniform in ℓ. The results, existence of a simple, positive, maximal eigenvalue and positivity of the correspondong eigenvector are classical. We do not refer to the literature because the proofs give an idea of our strategy for the limit $\ell \to +\infty$.

In Section 5 we study functions m close to a double instanton and derive basic estimates on the maximal eigenvalue and eigenvector of $A_{m,\ell}$, however they rely on the validity of some properties of the Markov chain with transition probability $Q_{\xi,\ell}(x, y)$ that are only proved in Section 8.

Sections 6, 7 and 8 are devoted to the analysis of the chains with transition probability $Q_{m,\ell}(x, y)$ and $Q_{\xi,\ell}(x, y)$. Those concerning $Q_{m,\ell}(x, y)$ use the estimates proved in Section 5. Therefore from a logical point of view, one should first read Sections 6, 7 and 8 only for the parts that concern $Q_{\xi,\ell}$. In this way one gets at the end of Section 8 all the estimates needed in Section 5, whose results are then in effect. At that point, then, he can go back to Sections 6 and 7 for the parts that concern $Q_{m,\ell}$ (in Section 8 we only consider $Q_{\xi,\ell}$). As some of the proofs for $Q_{m,\ell}$ and $Q_{\xi,\ell}$ are very similar to each other we have somehow unified them and for this reason we have organized the paper the way it is, even though, logically, we should have followed the other way explained above.

In particular in Section 6 we establish bounds on the expectations of exponential weights, which are used to control the tails at infinity of the chain and provide the necessary tools for taking the limit $\ell \to +\infty$. In Section 7 we use this result to extend the analysis of Section 4 to get results that are uniform in ℓ. In terms of the related Markov chains this amounts to a proof of the decay of the time correlations uniformly in ℓ. In Section 8 we improve the analysis in the case of the chain $Q_{\xi,\ell}$ and exploit the results to prove sharp estimates on its invariant measure.

In Section 9 we improve the analysis of Section 5 with the help of the properties of the auxiliary chains studied in the three preceeding sections. In Section 10 we prove the spectral gap properties and in Section 11 we derive estimates on the dependence of eigenvalues and eigenvectors on ξ in the particular case of the double instanton function.

3. Asymptotic behavior of the instanton

In this Section we study the asymptotic behavior of the instanton $\bar{m}(x)$ as $x \to \infty$. We recall that the instanton $\bar{m}(x)$ is an antisymmetric, continuous increasing function of $x \in \mathbb{R}$ that solves (1.7). In Proposition 2.2 of [9] it is proved that there are c and η positive so that for all $x > 0$

$$|\bar{m}(x) - m_\beta| \le ce^{-\eta x} \tag{3.1}$$

and in Proposition 2.1 of [DOPT2] that $\bar{m}' > 0$. $\bar{m}'$ is an eigenvector of $A_{\bar{m}}$ with eigenvalue 1, namely

$$\bar{m}' = p\, J \star \bar{m}' \tag{3.2}$$

where we use (throughout this subsection) the shorthand notation

$$p(x) \equiv p_{\bar{m}}(x) = \beta[1 - \bar{m}(x)^2] \tag{3.3}$$

(3.2) is obtained by differentiating with respect to x the instanton equation (1.7). After integrating by parts the convolution on the right hand side of (3.2) we deduce that $\bar{m}'$ is a bounded continuous function and by further differentiations that all the derivatives of $\bar{m}$ share such a property.

Since

$$\lim_{|x|\to\infty} p(x) = p_\infty := \beta[1 - m_\beta^2] < 1 \tag{3.4}$$

the obvious conjecture is that the asymptotic behavior of $\bar{m}'(x)$ as $x \to +\infty$ is ruled by the equation

$$v = p_\infty\, J \star v \tag{3.5}$$

Looking for a solution of (3.5) of the form

$$v(x) = e^{-\alpha x} \tag{3.6}$$

we find that α must solve

$$p_\infty \int dy J(0,y) e^{-\alpha y} = 1 \tag{3.7}$$

We are using the convention that when the domain of an integral is not specified then it coincides with the whole $\mathbb{R}$.

3.1 Lemma.

There is a strictly positive, decreasing C^1 function α_p, $p \in (0,1)$, so that $\pm\alpha_p$ are the only solutions of the equation

$$p \int dy\, J(0,y) e^{-\lambda y} = 1, \qquad \lambda \in \mathbb{R} \tag{3.8}$$

Proof.

Given $p \in (0,1)$ let

$$f(\lambda) \equiv p \int dy J(0,y) e^{-\lambda y}$$

Then $f(0) < 1$, $f'(\lambda) > 0$ for all $\lambda > 0$ and $f(\lambda) \to +\infty$ as $\lambda \to +\infty$, hence there is a unique value $\alpha_p > 0$ for which (3.8) holds. By the symmetry of $f(\lambda)$, $-\alpha_p$ is also a solution of (3.8). α_p is obviously a decreasing function of p. Lemma 3.1 is proved. $\square$

We shorthand $\alpha := \alpha_{p_\infty}$, $p_\infty := \beta(1 - m_\beta^2)$, and we introduce the following kernel which will be consider later on:

$$K(x,y) := p_\infty(x,y) e^{-\alpha(y-x)} \tag{3.9}$$

This kernel can be considered as the asymptotic expression for x and y large of the transition probability $P(x,y)$ given by (2.7).

3.2 Theorem.

There are $M > 0$ and $\delta \in (0,\alpha)$ positive so that

$$\lim_{x\to+\infty} e^{\alpha x} \bar{m}'(x) = M, \qquad \lim_{x\to+\infty} e^{\delta x} \Big(e^{\alpha x} \bar{m}'(x) - M \Big) = 0 \tag{3.10}$$

Theorem 3.2 will be proved later. Observe that an analogous statement holds for $x \to -\infty$, as $\bar{m}(x)$ is antisymmetric (and $\bar{m}'(x)$ symmetric). Without loss of generality we thus restrict to $x \geq 0$.

Given a positive integer s we consider the following equation in $C^0(\mathbb{R})$

$$\begin{cases} v(x) = p(x)\, J \star v(x) & \text{for } \ x \geq s \\ v(x) = \bar{m}'(x) & \text{for } \ x < s \end{cases} \tag{3.11}$$

$\bar{m}'$ obviously solves (3.11) and it is its only solution, as we will see. We will also prove that for s large enough there is a Green function $G_s(x,y)$ for (3.11). We will then obtain an expression for $\bar{m}'(x)$, $x > s$, in terms of $G_s(\cdot,\cdot)$ and of $\bar{m}'(y)$, $s-1 \leq y < s$, and that will eventually lead us to the proof of Theorem 3.2.

The Green function $G_s(x,y)$

We are going to prove an identity satisfied by $\bar{m}'(\cdot)$, namely for any $x \geq s$

$$\bar{m}'(x) = \int_{s-1}^{s} dy\, G_s(x,y)\bar{m}'(y) \tag{3.12}$$

where

$$G_s(x,y) = \sum_{n=1}^{\infty} R_s^{(n)}(x,y) \tag{3.13}$$

ans, setting $x = y_0$ and $y = y_n$, $n > 1$,

$$R_s^{(n)}(y_0,y_n) = \int_s^{\infty} dy_1 \cdots \int_s^{\infty} dy_{n-1} \prod_{i=1}^{n} p(y_{i-1})J(y_{i-1},y_i), \tag{3.14}$$

The series in (3.13) converges exponentially fast.

If v solves (3.11), for all $x \geq s$

$$v(x) = p(x)\int_s^{\infty} dy\, J(x,y)v(y) + p(x)\int_{s-1}^{s} dy\, J(x,y)\bar{m}'(y)$$

After N iterations, we get

$$v(x) = \int_{s-1}^{s} dy\, G_s^{(N)}(x,y)\bar{m}'(y) + \int_s^{\infty} dy\, R_s^{(N)}(x,y)v(y) \tag{3.15}$$

where for $x \geq s$ and $y \in \mathbb{R}$

$$G_s^{(N)}(x,y) := \sum_{n=1}^{N} R_s^{(n)}(x,y), \qquad R_s^{(1)}(x,y) := p(x)J(x,y) \tag{3.16}$$

The above quantities have an interpretation in terms of functional integrals over the space of trajectories of the jump process with intensity $p(x)J(x,y)$. $G_s(x,y)$contains trajectories from x to

y with an arbitrary number of jumps, remaining on the right of s. $R_s^{(n)}(x,y)$ contains the same type of trajectories taking place in a time n (n= number of jumps).

Since $p(x)$ decreases (because $\bar{m}(x)$ increases), for all $y_0 > s$

$$R_s^{(n)}(y_0,y) \le p(s)^n J^n(y_0,y) \tag{3.17}$$

¿From (3.1) we have

$$p(s) \equiv \beta[1-\bar{m}(s)^2] \le \beta\big[1-(m_\beta - ce^{-\eta s})^2\big] < 1 \tag{3.18}$$

for all s large enough to which we hereafter restrict.

By (3.17) the second term on the right hand side of (3.15) is bounded by $p(s)^N\|v\|_\infty$ (because $J^n(y_0,y)$ is a probability density). $\|v\|_\infty < \infty$ because by definition any solution v of (3.11) is bounded. The second term in (3.15) thus vanishes as $N \to +\infty$ and

$$v(x) = \int_{s-1}^{s} dy\, G_s(x,y)\bar{m}'(y) \tag{3.19}$$

where $G_s(x,y)$ is defined in (3.13) and, by (3.17), the series in (3.13) converges exponentially fast. (3.19) proves that there is a unique solution to (3.11), which is therefore $\bar{m}'(x)$, hence (3.12).

Reduction to a probability kernel

The purpose is now to bound from above and below the Green function G_s. Since $p(x)$ is a strictly decreasing function of x for $x > 0$,

$$p(x) > p_\infty = \inf_x p(x) \tag{3.20}$$

By (3.1) there is c' so that for $x > 0$

$$p(x) \le p_\infty + c' e^{-\eta x} \tag{3.21}$$

Let $\alpha = \alpha_{p_\infty}$,

$$K(x,y) := p_\infty J(x,y) e^{-\alpha(y-x)}, \qquad \int dy K(x,y) = 1 \tag{3.22}$$

and for $y < s \le x$

$$g_s(x,y) := \sum_{n=1}^{\infty} \int_s^{\infty} dy_1 \cdots \int_s^{\infty} dy_{n-1} \prod_{i=1}^{n} K(y_{i-1},y_i) \tag{3.23}$$

In (3.23) we have written $x = y_0$ and $y = y_n$.

$g_s(x,y)$ is the analogue of the previously defined G_s, (see (3.13)) with the transition probability $K(x,y)$ (see(3.9)) in place of the unnormalized weights $p(x)J(x,y)$. Now the functional integrals assume a probabilistic meaning.

For any $y_0 \geq s$ the series converges uniformly (in y) because its n-th term is bounded by

$$e^{-\alpha(y-y_0)} \int_s^\infty dy_1 \cdots \int_s^\infty dy_{n-1}\, p_\infty^n J(y_{n-1}, y) \prod_{i=1}^{n-1} J(y_{i-1}, y_i) \leq e^{-\alpha(y-y_0)} p_\infty^n \|J\|_\infty \tag{3.24}$$

having bounded $J(y_{n-1}, y) \leq \|J\|_\infty$ and then used that $J^{n-1}(y_0, y_{n-1})$, $n > 1$, is a probability density.

We will prove that for any $x \geq s$ and $y \in [s-1, s)$

$$e^{-\alpha(x-y)} g_s(x,y) \leq G_s(x,y) \leq e^{-\alpha(x-y)}[1+\epsilon(s)] g_s(x,y) \tag{3.25}$$

where

$$\epsilon(s) := c e^{-\eta s} \tag{3.26}$$

with c a suitable, positive constant.

The lower bound in (3.25) follows from (3.20). We will prove the upper bound by studying auxiliary problems of the type (3.11) with kernels obtained by replacing $p(x)$ by a constant p, where $p_\infty \leq p \leq p(s)$. We call $G_s(x,y;p)$ the corresponding Green function and since p is a constant the kernel $pJ(x,y)$ is translationally invariant so that

$$G_s(x,y;p) = G_0(x-s, y-s; p) \tag{3.27}$$

where for $x \geq 0$ and $y < 0$

$$G_0(x,y;p) = \sum_{n=1}^\infty p^n \int_0^\infty dx_1 .. \int_0^\infty dx_{n-1} J(x, x_1)..J(x_{n-1}, y) \tag{3.28}$$

Let $s - 1 \leq y < s \leq N \leq x < N+1$, then by the monotonicity of $p(x)$

$$G_s(x,y) \leq \int_{N-1}^N dx_1\, G_N(x, x_1; p(N)) \int_{N-2}^{N-1} dx_2 G_{N-1}(x_1, x_2; p(N-1)) \ldots$$
$$\cdots \int_s^{s+1} dx_m\, G_s(x_m, y; p(s)) \tag{3.29}$$

with $m := N - s$.

In (3.29) we have decomposed the space of trajectories by specifying the points reached at the first hitting times to the left of $N-1$, $N-2$,...$s+1$ respectively; we have introduced, moreover, the corresponding estimates for $p(x)$.

We will prove later that for any $p^\star \in [p_\infty, 1)$ there is $c^\star$ so that for any $p \in [p_\infty, p^\star]$, all $y \in (-1, 0]$ and $x \in (0, 1]$

$$G_0(x,y;p) \le G_0(x,y;p_\infty)[1 + c^\star(p - p_\infty)] \tag{3.30}$$

Then by (3.21), (3.29) and (3.30)

$$\begin{aligned} G_s(x,y) &\le \int_{N-1}^{N} dx_1 .. \int_s^{s+1} dx_m G_N(x,x_1;p_\infty)..G_s(x_m,y;p_\infty)[1 + c^\star c' e^{-\eta N}] \cdots [1 + c^\star c' e^{-\eta s}] \\ &= G_s(x,y;p_\infty)[1 + c^\star c' e^{-\eta N}] \cdots [1 + c^\star c' e^{-\eta s}] \end{aligned} \tag{3.31}$$

Since $G_s(x,y;p_\infty) = e^{\alpha(y-x)} g_s(x,y)$, (3.31) proves the upper bound in (3.25) with $\epsilon(s)$ as in (3.26) and c determined by $c^\star$, c' and η.

Proof of (3.30).

By (3.28) $d^2 G_0(x,y;p)/dp^2 \ge 0$, hence, since $p > p_\infty$,

$$\begin{aligned} G_0(x,y;p) \le G_0(x,y;p_\infty) + (p - p_\infty) \sum_{n \ge 1} n p^{n-1} \int_0^\infty dx_1 .. \\ .. \int_0^\infty dx_{n-1} J(x,x_1)..J(x_{n-1},y) \end{aligned} \tag{3.32}$$

Since $y < 0$ and $J(x,y) = 0$ if $|x - y| \ge 1$, we have $x_{n-1} \le 1$ and $x_{n-2} \le 2$. We then bound $J(x_{n-2}, x_{n-1})$ by $\|J\|_\infty \mathbf{1}_{x_{n-2} \le 2}$ and get

$$G_0(x,y;p) \le G_0(x,y;p_\infty) + (p - p_\infty) S(x,y) \tag{3.33}$$

where

$$S(x,y) := J(x,y) + C \int_0^\infty dz J(z,y) \tag{3.34}$$

and

$$C = \sum_{n \ge 2} (p^\star)^{n-1} n \int_0^\infty dx_1 .. \int_0^2 dx_{n-2} J(x,x_1)..J(x_{n-3},x_{n-2}) \|J\|_\infty \tag{3.35}$$

Supposing, without loss of generality, that $C \ge 1$, we have

$$S(x,y) \le C\Big(J(x,y) + \int dz J(z,y)\Big) \tag{3.36}$$

The aim is now to bound $S(x,y)$ in terms of $G_0(x,y;p)$. We will use a lemma that appears frequently in this paper. For such a reason it is stated in a slightly more general form than required by the present context.

3.3. Lemma.

For any integer $n \geq 2$ there is a positive integer k_n (with $k_n < n^2$ for all n large enough) and for any $k \geq k_n$ there is $\zeta > 0$ so that for any $x \in [0,n]$ and $y \in [0,n]$

$$\int_0^n dx_1 \cdots \int_0^n dx_{k-1}\, J(x,x_1)\cdots J(x_{k-1},y) > \zeta \tag{3.37}$$

Proof.

We first prove that for any x and y in $[0,n]$ there is a positive integer $k > 1$ and $x_1..x_{k-1}$ in $[0,n]$ so that $J(x,x_1)\cdots J(x_{k-1},y) > 0$.

By the assumption that $\sup\{x : J(0,x) > 0\} = 1$, it follows that there is $\epsilon > 0$ so that

$$J(0,x) > 0 \quad \text{for } 1 - 4\epsilon < x < 1 \tag{3.38}$$

Let a be the midpoint of this interval, i.e.

$$a := \frac{1}{2}\Big(1 + (1 - 4\epsilon)\Big) \tag{3.39}$$

and N the smallest integer such that $\epsilon N \geq n - 1$.

For $0 \leq i \leq N-1$ we set

$$\Lambda_i := J\big(\epsilon i, \epsilon i + a\big) J\big(\epsilon i + a, \epsilon(i+1)\big) > 0 \tag{3.40}$$

observing that the points ϵi, $\epsilon i + a$ are in $[0,n]$. Let $x \leq y \leq n-1$, i and j the integers such that

$$\epsilon i \leq x < \epsilon(i+1), \qquad \epsilon j \leq y < \epsilon(j+1)$$

and

$$\Gamma_+(x,y) := J\big(x, \epsilon i + a\big) J\big(\epsilon i + a, \epsilon(i+1)\big)\Lambda_{i+1}\cdots\Lambda_{j-1} J\big(\epsilon j, \epsilon j + a\big) J\big(\epsilon j + a, y\big) > 0 \tag{3.41}$$

We define $\Gamma_-(x,y)$, $1 \leq x \leq y \leq n$, exactly as we did for Γ_+, but going from the right to the left.

We next define $\Gamma(x,y)$, x and y in $[0,n]$ by setting $\Gamma(x,y) := \Gamma_+(x,y)$ for $x \leq y \leq n-1$; $\Gamma(x,y) := \Gamma_-(x,y)$ for $1 \leq x \leq y$ and $y > n-1$. $\Gamma(x,y) := \Gamma_+(x,n/2)\Gamma_-(n/2,y)$ for $x < 1$ and $y > n-1$ and $\Gamma(x,y) := \Gamma(y,x)$ for $0 \leq y < x \leq n$.

We then set for x and y in $[0,n]$, $\Delta(x,y) := \Gamma(x,n/2)\Gamma(n/2,y)$. By definition $\Delta(x,y)$ is strictly positive and

$$\Delta(x,y) = J(x,x_1)\cdots J(x_{k-1},y)$$

where k is an even integer that depends on x and y, all the points x_i are in $[0,n]$ and one of them is equal to $n/2$. Moreover the range of values of k when x and y vary in $[0,n]$ has a maximum denoted by k_n which is bounded proportionally to n.

Since $n/2+a \in [0,n]$ and

$$J(n/2, n/2+a)J(n/2+a, n/2) > 0 \tag{3.42}$$

we conclude that for any x and y in $[0,n]$ there are points $x_1 \dots x_{k_n-1}$ in $[0,n]$ so that

$$J(x,x_1)\cdots J(x_{k_n-1},y) > 0 \tag{3.43}$$

and the same property holds as well for all the even integers larger than k_n. It also extends to the odd integers larger than k_n: in fact by (3.42) it is enough to prove it for $k = k_n+1$. Suppose first $y-a \in [0,n]$, set $y' := y-a$, then by (3.43)

$$J(x,x_1)\cdots J(x_{k_n-1},y')J(y',y'+a) > 0$$

and $y'+a = y$. If $y-a \notin [0,n]$, $y+a \in [0,n]$. We then set $y' := y+a$ and repeat the previous argument. We have thus shown that given x and y in $[0,n]$ and $k \ge k_n$ there are $x_1,..,x_{k-1}$ in $[0,n]$ so that

$$J(x,x_1)\cdots J(x_{k-1},y) > 0$$

Let $J_n(x,y)$ be the restriction of $J(x,y)$ to $[0,n]^2$. By the continuity of J_n and the above inequality we deduce that also $J_n^k(x,y) > 0$ which, being a continuous function in $[0,n]^2$, is strictly positive. Lemma 3.3 is proved. □

By Lemma 3.3 and (3.28) there are an integer k and $\zeta > 0$ so that for any $x \in [0,1]$ and $y \in [-1,0)$

$$G_0(x,y;p_\infty) \ge p_\infty J(x,y) + p_\infty^{k+1} \int_0^1 dz\,\zeta J(z,y) \tag{3.44}$$

Then there is $C' > 0$ so that

$$G_0(x,y;p_\infty) \ge C'\Big(J(x,y) + \int dz J(z,y)\Big) = \frac{C'}{C} S(x,y) \tag{3.45}$$

(the last equality is (3.34)). (3.30) follows from (3.33) and (3.45). Thus (3.25) is proved.

Estimates on $g_s(x,y)$

By the translation invariance of $K(x,y)$, see (3.22), by (3.23) for $x \ge s$ and $y < s$

$$g_s(x,y) = g_0(x-s, y-s) \tag{3.46}$$

We will first prove that for all $x \ge 0$

$$\int_{-1}^0 dy\, g_0(x,y) = 1 \tag{3.47}$$

This is a consequence of the law of large numbers for independent variables. Indeed by the definition of $K(x,y)$, see (3.22),

$$\int dy\, K(x,y)(y-x) =: b < 0 \tag{3.48}$$

Calling P the probability on $\mathbb{R}^{\mathbb{N}}$ product of identical copies of $K(0,z)dz$, and denoting by z_i the i-th coordinate in $\mathbb{R}^{\mathbb{N}}$, we have, for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\Big(\{\Big|\sum_{i=1}^{n}(z_i - b)\Big| > \epsilon n\}\Big) = 0 \tag{3.49}$$

Given $x \equiv y_0 > 0$, by (3.24) the series (3.23) converges uniformly in y so that

$$\begin{aligned}\int_{-1}^{0} dy\, g_0(x,y) &= \lim_{N\to\infty}\sum_{n=1}^{N}\int_{-1}^{0} dy_n \int_0^\infty dy_1 \cdots \int_0^\infty dy_{n-1}\prod_{i=1}^{n} K(y_{i-1},y_i)\\ &= \lim_{N\to\infty}\Big\{1 - P\Big(\{x + \sum_{i=1}^{n} z_i > 0;\, \forall n \le N\}\Big)\Big\}\\ &\ge \lim_{N\to\infty}\Big\{1 - P\Big(\{x + \sum_{i=1}^{N} z_i > 0\}\Big)\Big\} = 1\end{aligned} \tag{3.50}$$

(3.47) is proved.

3.4 Proposition.

There are $\delta_1 > 0$ and a probability density $\rho(y)$, $y \in [-1,0]$, so that

$$\lim_{x\to\infty} g_0(x,y) = \rho(y), \qquad \lim_{x\to\infty} e^{\delta_1 x}\int_{-1}^{0} dy\, \big|g_0(x,y) - \rho(y)\big| = 0 \tag{3.51}$$

Proof.

We are going to show that for any y, $g(x,y)$ is a Cauchy sequence as $x \to \infty$. Let s be a positive integer and

$$b_s(y) := \sup_{\substack{s\le x<s+1\\ s\le x'<s+1}} \big|g_0(x,y) - g_0(x',y)\big| \tag{3.52}$$

We will prove later that there is $\gamma \in (0,1)$ such that for any $y \in (0,1)$

$$b_s(y) \le \gamma b_{s-1}(y) \tag{3.53}$$

Let then $x \in [s, s+1)$ and $x' \in [n, n+1)$, $n \geq s$. If $n = s$

$$\big|g_0(x,y) - g_0(x',y)\big| \leq b_s(y) \leq \gamma^s b_0(y) \tag{3.54}$$

If instead $n > s$ we write

$$g_0(x',y) = \int_s^{s+1} dz\, g_{s+1}\big(x',z\big) g_0(z,y)$$

Then by (3.47) the function $\lambda(z) := g_{s+1}\big(x',z\big)$ is a probability density so that

$$\big|g_0(x,y) - g_0(x',y)\big| = \Big|\int_s^{s+1} dz\, \lambda(z)[g_0(x,y) - g_0(z,y)]\Big| \leq b_s(y) \leq \gamma^s b_0(y) \tag{3.55}$$

$\{g_0(x,y)\}_{x\geq 0}$ is thus a Cauchy sequence, hence the first limit in (3.51) exists and it defines $\rho(y)$. By letting $x' \to +\infty$ in (3.55) we get

$$\big|\rho(y) - g_0(x,y)\big| \leq \gamma^s b_0(y)$$

so that using (3.24)

$$\begin{aligned}\int_{-1}^0 dy\, \big|\rho(y) - g_0(x,y)\big| &\leq 2\gamma^s \int_{-1}^0 dy \sup_{0\leq x<1} g_0(x,y)\\ &\leq 2\gamma^s \int_{-1}^0 dy \sum_{n\geq 0} \sup_{0\leq x<1} e^{-\alpha(y-x)} p_\infty^n \|J\|_\infty \leq 2\gamma^s \|J\|_\infty \frac{e^{2\alpha}}{1-p_\infty}\end{aligned}$$

This proves the second limit in (3.51) with $\delta_1 < \log \gamma^{-1}$ and together with (3.47) that ρ is a probability density.

Proof of (3.53)

We first show that there are $c \in (0, 1/2)$ and $\zeta > 0$ so that for all $x \in [0,1]$ and y such that $|y + 1/2| \leq c$

$$g_0(x,y) > \zeta \tag{3.56}$$

By the assumption on J, there are $x' \in [0,1]$, $c \in (0,1/2)$ and $\zeta' > 0$ so that

$$\inf_{|x-x'|\leq c} \inf_{|y+1/2|\leq c} J(x,y) \geq \zeta'$$

By Lemma 3.3 there are $h > 0$ and $\epsilon > 0$ so that for all $x \in [0,1]$ and $z \in [0,1]$

$$\int_0^\infty dx_1 \cdots \int_0^\infty dx_{h-1} J(x,x_1)\cdots J(x_{h-1},z) > \epsilon$$

Then for $x \in [0,1]$ and $|y+1/2| \le c$

$$
\begin{aligned}
g_0(x,y) &\ge p_\infty^{h+1} e^{\alpha(x-y)} \int_0^\infty dx_1 \cdots \int_0^\infty dx_{h-1} \int_{x'-c}^{x'+c} J(x,x_1)\cdots J(x_{h-1},z)J(z,y) \\
&> p_\infty^{h+1} \epsilon 2c\zeta'
\end{aligned}
$$

which proves (3.56).

We can suppose that ζ in (3.56) is less that $(2c)^{-1}$. We set

$$
\tilde g_s(x,z) := g_s(x,z) - \zeta \mathbf{1}(|z-(s-1/2)| \le c) \tag{3.57}
$$

which by (3.56) is non negative. For x and x' both in $(s,s+1]$ we have

$$
\Big| g_0(x,y) - g_0(x',y) \Big| = \Big| \int_{s-1}^{s} dz\, [\tilde g_s(x,z) - \tilde g_s(x',z)] g_0(z,y) \Big| \tag{3.58}
$$

Letting $\gamma := 1 - 2\zeta c$ we get

$$
\begin{aligned}
\Big| g_0(x,y) - g_0(x',y) \Big| &= \gamma^{-1} \Big| \int_{s-1}^{s} dz \int_{s-1}^{s} dz'\, \tilde g_s(x,z) \tilde g_s(x',z')][g_0(z,y) - g_0(z',y)] \Big| \\
&\le b_{s-1}\gamma^{-1} \int_{s-1}^{s} dz\, \tilde g_s(x,z) \int_{s-1}^{s} dz'\, \tilde g_s(x',z') = \gamma b_{s-1}
\end{aligned}
$$

which proves (3.53).

Proposition 3.4 is proved. □

Proof of Theorem 3.2

Let $x \ge s$ and

$$
I_s(x) := \int_{-1}^{0} dy\; g_0(x-s,y)[e^{\alpha(s+y)} \bar m'(s+y)] \tag{3.59}
$$

By (3.12) and (3.25)

$$
I_s(x) \le e^{\alpha x} \bar m'(x) \le I_s(x)\big(1+\epsilon(s)\big) \tag{3.60}
$$

Let

$$
I_s^\star := \int_{-1}^{0} dy\; \rho(y)[e^{\alpha(s+y)} \bar m'(s+y)] \tag{3.61}
$$

By (3.51) there are c and δ_1 positive so that

$$
\Big| I_s(x) - I_s^\star \Big| \le c e^{-\delta_1(x-s)} \sup_{|z|\le s} e^{\alpha z} \bar m'(z) \tag{3.62}
$$

Calling $\underline{M}$ and $\bar M$ the liminf and limsup of $e^{\alpha x}\bar m'(x)$ as $x \to +\infty$, we get from (3.60)

$$
I_s^\star \le \underline{M} \le \bar M \le I_s^\star\big(1+\epsilon(s)\big) \tag{3.63}
$$

Therefore for $s > s_0$, s_0 large enough,

$$\bar{M} - \underline{M} \le \epsilon(s) I^\star_s \le \epsilon(s) \int_{-1}^{0} dy\, \rho(y)\big(1+\epsilon(s_0)\big) I_{s_0}(s+y) \tag{3.64}$$

having used (3.60) with s_0 to derive the last inequality.

By (3.60) and (3.63)

$$\limsup_{s\to+\infty} I_{s_0}(s+y) \le \limsup_{s\to+\infty} e^{\alpha(s+y)} \bar{m}'(s+y) \le I^\star_{s_0}[1+\epsilon(s_0)]$$

Then by (3.26) the right hand side of (3.64) vanishes as $s \to +\infty$ which proves the first limit in (3.10). The proof of the second one is similar. Indeed both M and $\bar{m}'(x)e^{\alpha x}$ are in the interval with extremes $\min\{I_s(x), I^\star_s\}$ and $[1+\epsilon(s)]\max\{I_s(x), I^\star_s\}$. Then by (3.62) there is $c > 0$ so that for all s large enough and all $x \ge s$

$$\begin{aligned}\big|M - \bar{m}'(x)e^{\alpha x}\big| &\le |I_s(x) - I^\star_s| + \epsilon(s)[I_s(x) + I^\star_s] \le c[e^{-\delta_1(x-s)} + e^{-\eta s}]\\ &\le 2ce^{-\delta' x}\end{aligned}$$

having chosen s such that $(x-s)\delta_1 = \eta s$ and set

$$\delta' := \frac{\eta\delta_1}{\eta+\delta_1}$$

Theorem 3.2 is proved. $\square$

Theorem 2.2 is a consequence of Theorem 3.2 and of the following corollaries of Theorem 3.2.

3.5. Theorem.

Let $a := M\alpha^{-1}$ and $0 < \delta' < \delta$ with δ as in Theorem 3.2. Then

$$\lim_{x\to+\infty} e^{(\alpha+\delta')x}\big[\bar{m}(x) - \big(m_\beta - ae^{-\alpha x}\big)\big] = 0 \tag{3.65}$$

Proof.

Since $\bar{m}(x) \to m_\beta$ as $x \to +\infty$

$$m_\beta - \bar{m}(x) = \int_x^\infty dy\, \bar{m}'(y)$$

Then, recalling that $M = \alpha a$,

$$\Big|e^{(\alpha+\delta')x}\big[\bar m(x) - \big(m_\beta - ae^{-\alpha x}\big)\big]\Big| = \Big|e^{(\alpha+\delta')x}\int_x^\infty dy\,[a\alpha e^{-\alpha y} - \bar m'(y)]\Big|$$
$$\le \int_x^\infty dy\, e^{(\alpha+\delta')y}\Big|Me^{-\alpha y} - \bar m'(y)\Big|$$
$$= \int_x^\infty dy\, e^{-(\delta-\delta')y}e^{\delta y}\big|M - e^{\alpha y}\bar m'(y)\big|$$

which by Theorem 3.2 vanishes as $x \to +\infty$. Theorem 3.5 is proved. □

3.6. Theorem.

Let $\delta > 0$ be as in Theorem 3.2. Then

$$\lim_{x\to+\infty} e^{\delta x}\big|e^{\alpha x}\bar m''(x) + \alpha M\big| = 0 \tag{3.66}$$

Proof.

By differentiating (3.2) we get

$$\bar m''(x) = -2\beta\bar m(x)\bar m'(x)J\star\bar m'(x) + p(x)J'\star\bar m'(x) \tag{3.67}$$

where $J'(x,y) := \partial J(x,y)\partial x$. By Theorem 3.2

$$\lim_{x\to+\infty} e^{(\alpha+\delta)x}[-2\beta\bar m(x)\bar m'(x)J\star\bar m'(x)] = 0 \tag{3.68}$$

so that

$$\lim_{x\to+\infty} e^{\delta x}\Big|e^{\alpha x}\bar m''(x) - p(x)\int dy\, J'(x,y)e^{-\alpha(y-x)}[e^{\alpha y}\bar m'(y)]\Big| = 0 \tag{3.69}$$

We have

$$\Big|e^{\delta x}p(x)\int dy\, J'(x,y)e^{-\alpha(y-x)}[e^{\alpha y}\bar m'(y) - M]\Big|$$
$$\le p(x)\int_{x-1}^{x+1} dy\,|J'(x,y)|e^{-\alpha(y-x)}e^{\delta}\Big|e^{\delta y}[e^{\alpha y}\bar m'(y) - M]\Big|$$

which by Theorem 3.2 vanishes as $x \to +\infty$. Then by (3.69)

$$\lim_{x\to+\infty} e^{\delta x}\Big|e^{\alpha x}\bar m''(x) - p(x)\int dy\, J'(x,y)e^{-\alpha(y-x)}M\Big| = 0 \tag{3.70}$$

By (3.7) we have

$$\int dy\, J'(x,y)e^{-\alpha(y-x)} = -\alpha \int dy\, J(x,y)e^{-\alpha(y-x)} = -\frac{\alpha}{p_\infty} \tag{3.71}$$

(recall $\alpha = \alpha_{p_\infty}$). (3.70) then becomes

$$\lim_{x\to+\infty} e^{\delta x}\left| e^{\alpha x}\bar{m}''(x) + \frac{p(x)}{p_\infty}\alpha M \right| = 0 \tag{3.72}$$

We have

$$\frac{p(x)}{p_\infty} - 1 = \frac{1}{1-m_\beta^2}[m_\beta + \bar{m}(x)][m_\beta - \bar{m}(x)]$$

Since $0 < \delta < \alpha$, by Theorem 3.5

$$\lim_{x\to+\infty} e^{\delta x}[m_\beta - \bar{m}(x)] = 0$$

Then by (3.72)

$$\lim_{x\to+\infty} e^{\delta x}\left| e^{\alpha x}\bar{m}''(x) + \alpha M \right| = 0 \tag{3.73}$$

Theorem 3.6 is proved. □

Theorem 3.2, 3.5 and 3.6 prove Theorem 2.2.

4. A Perron Frobenius theorem in finite intervals

In this Section we will prove Theorem 2.1 with several other properties of the maximal eigenvalue $\lambda_{m,\ell}$ and the corresponding left and right eigenvectors $u_{m,\ell}$ and $v_{m,\ell}$. We are not yet supposing that m is close to a double instanton, thus our statements refer to general $m \in C^{\rm sym}([0,\ell])$, but the results are not uniform in ℓ.

4.1 Lemma.

The operator $A_{m,\ell}$ is selfadjoint in $L^2([0,\ell], p_m(x)^{-1}dx)$, i.e. for any x and y in $[0,\ell]$

$$\frac{1}{p_m(x)}A_{m,\ell}(x,y) = \frac{1}{p_m(y)}A_{m,\ell}(y,x) \tag{4.1}$$

Moreover for any $f \in C([0,\ell])$

$$\int_0^\ell dy\, A_{m,\ell}(x,y)f(y) = \int_{-1}^{\ell+1} dy\, A_m(x,y)f\big(R_\ell(y)\big) \tag{4.2}$$

Proof.

Recalling the definition of $A_{m,\ell}$, (4.1) becomes $J_\ell(x,y) = J_\ell(y,x)$, where

$$J_\ell(x,y) := \sum_{z: R_\ell(z)=y} J(x,z)$$

Since $J(x,y) = J(y,x)$, we need only consider the case when there is $y' \notin [0,\ell]$ and $J(x,y') > 0$. Then there is $x' \notin [0,\ell]$ with $R_\ell(x') = x$ and by the symmetry of J, $J(x,y') = J(y,x')$ and consequently $J_\ell(x,y) = J_\ell(y,x)$. (4.1) is therefore proved.

To prove (4.2) we write its left hand side as

$$\int_0^\ell dy\, [A_m(x,y) + \mathbf{1}_{R_\ell(y')=y, y'\notin[0,\ell]} A_m(x,y')] f(y) = \int_0^\ell dy\, A_m(x,y) f(y) + \int_{y'\notin[0,\ell]} dy'\, A_m(x,y') f\big(R_\ell(y')\big)$$

Lemma 4.1 is proved. $\square$

Since $A_{m,\ell}$ is selfadjoint in $L^2([0,\ell],dx)$, (2.5) follows from the observation that by multiplying a left eigenvector by p_m we obtain a right eigenvector with the same eigenvalue.

To prove Theorem 2.1 we follow a strategy usual in equilibrium statistical mechanics, [17], [14]. We start from the left eigenvalue problem. Let

$$\mathcal{X}_\ell := \Big\{ \rho \in C([0,\ell]; \mathbb{R}_+) : \int_0^\ell dx\, \rho(x) = 1 \Big\} \tag{4.3}$$

and $N_{m,\ell}$ the [non linear] map on $\mathcal{X}_\ell$ defined by setting

$$[N_{m,\ell}(\rho)](y) := \int_0^\ell dx\, \rho(x) A_{m,\ell}(x,y) \Big\{ \int_0^\ell dy \int_0^\ell dx\, \rho(x) A_{m,\ell}(x,y) \Big\}^{-1} \tag{4.4}$$

Observe that if $N_{m,\ell}$ has a fixed point u:

$$N_{m,\ell}(u) = u \tag{4.5}$$

then u is a left eigenvector

$$u \star A_{m,\ell} = \lambda u \tag{4.6}$$

with

$$\lambda = \int_0^\ell dx \int_0^\ell dy\, u(x) A_{m,\ell}(x,y) \tag{4.7}$$

4.2 Lemma.

There exists a fixed point $u_{m,\ell}$ *solution of* (4.5) *and there is a constant* $c_1 > 0$ *(independent of* m *and* ℓ*) so that for all* ℓ *large enough*

$$0 < u_{m,\ell}(x) \le \sqrt{2c_1}, \quad |u'_{m,\ell}(x)| \le c_1, \qquad \text{for all } x \in [0,\ell] \tag{4.8}$$

Proof.

$\mathcal{R}$, the sup-norm closure of $N_{m,\ell}(\mathcal{X}_\ell)$, is a closed convex subset of $\mathcal{X}_\ell$. We are going to show that $\mathcal{R}$ is compact. To simplify notation we drop (in the course of the proof) the indices m and ℓ. If $\rho \in \mathcal{X}$ then $u = N(\rho)$ is differentiable with derivative

$$u'(y) := \frac{\int dx\, \rho(x) A'(x,y)}{\int dy \int dx\, \rho(x) A(x,y)}$$

($A'(x,y)$ denoting the derivative with respect to y). There are positive constants c' and c'' so that $|A'(x,y)| \le c'$ and for any $x \in [0,\ell]$

$$\int dy\, A(x,y) \ge c'' \int dy\, J(x,y) = c''$$

We then conclude that there is c (that we take ≥ 1) independent of ℓ so that $|u'(y)| \le c$ for all $y \in [0,\ell]$. To prove that $u \le \sqrt{2c}$ we argue by contradiction. Suppose that the maximum of u is attained at $x^\star \ge 0$ and that $u(x^\star) > u^\star := \sqrt{2c}$. Then

$$u(x) \ge [-c(x^\star - x) + u^\star] \mathbf{1}_{x^\star - u^\star c^{-1} \le x \le x^\star}$$

(the right hand side being a function in $[0,\ell]$ for ℓ large enough). Since the inequality is strict for some x, by integrating over x and recalling that the integral of u is 1, we reach a contradiction:

$$1 > \frac{[u^\star]^2}{2c} = 1$$

which implies that the second and the third inequalities in (4.8) are verified in the whole $\mathcal{R}$. By the Ascoli Arzelà theorem, Theorem IV.6.7 in [12], $\mathcal{R}$ is compact and the existence of a solution $u_{m,\ell}$ to (4.5) follows from the Schauder-Tychonoff fixed point theorem, Theorem V.10.5 in [12].

It only remains to prove the first inequality in (4.8) which is again proved by contradiction. If $u(y) = 0$ for some y then $u(x) = 0$ for all x such that $A(x,y) > 0$ (recall that $u \ge 0$, by definition). By iteration $u \equiv 0$ which is in contradiction with u being a probability density. Lemma 4.2 is proved. □

By (4.7)

$$\beta \ge \sup p_m(x) \ge \lambda_{m,\ell} \ge \inf p_m(x) \ge \beta \cosh^{-2}(\beta) \tag{4.9}$$

In fact, since for all $x \in [0, \ell]$

$$\int_{-1}^{\ell+1} dy\, J(x,y) = 1$$

we have using (4.2)

$$\lambda_{m,\ell} = \int_0^\ell dx\, u_{m,\ell}(x) \int_0^\ell dy\, A_{m,\ell}(x,y) = \int_0^\ell dx\, u_{m,\ell}(x) \int_{-1}^{\ell+1} dy\, p_m(x) J(x,y)$$
$$= \int_0^\ell dx\, u_{m,\ell}(x) p_m(x)$$

which is then bounded from above and below respectively by $\sup p_m(x)$ and $\inf p_m(x)$, hence (4.9) after recalling the definition of $p_m(x)$ and that $\|m\|_\infty \le 1$.

4.3 Lemma.

For any $\ell > 1$ there are $c > 0$ and $r < 1$ so that the following holds. Let $m \in C([0,\ell],[-1,1])$, $\lambda_{m,\ell} > 0$, $u_{m,\ell}$ and $v_{m,\ell}$ normalized so that

$$\int_0^\ell dx\, v_{m,\ell}(x) u_{m,\ell}(x) = 1$$

and, given $w \in C([0,\ell]$, let

$$\pi^-_{m,\ell}(w) := \int_0^\ell dx\, w(x) u_{m,\ell}(x), \quad \tilde{w}(x) := w - \pi^-_{m,\ell}(w) v_{m,\ell}(x)$$

Then, for any $n \ge 1$

$$\|A^n_{m,\ell} \star \tilde{w}\|_\infty \le c[\lambda r]^n \|\tilde{w}\|_\infty$$

Proof.

Let $Q_{m,\ell}$ be as in (2.6). By Lemma 3.3 there are $\zeta > 0$ and $k \in \mathbb{N}$ so that for any $|x| \le \ell$ and $|y| \le \ell$, $Q^k_{m,\ell}(x,y) > \zeta$. By classical arguments in the theory of Markov chains (that we recall below) there are then $c > 0$ and $r < 1$ so that for all $n \in \mathbb{N}$ and $y \in [0,\ell]$

$$\sup_{x,x'} \Big| Q^n_{m,\ell}(x,y) - Q^n_{m,\ell}(x',y) \Big| \le c r^n \tag{4.10}$$

In particular this shows that the chain $Q_{m,\ell}$ is ergodic.

Proof of (4.10)

Call $p(x,y) := Q^k_{m,\ell}(x,y) > \zeta$ and, given y,

$$\delta_n(x,x') = \Big| p^n(x,y) - p^n(x',y) \Big|, \quad \delta_n = \sup_{x \ne x'} \delta_n(x,x')$$

We are going to show that

$$\delta_n \le (1-\zeta)^n$$

which implies (4.10). We have

$$\delta_n(x,x') = \Big| \int_0^\ell dz[p(x,z)-\zeta]p^{n-1}(z,y) - \int_0^\ell dz[p(x',z)-\zeta]p^{n-1}(z,y)\Big|$$

Calling $\mu(z) = p(x,z)-\zeta$ and $\mu'(z) = p(x',z)-\zeta$, we notice that they are both non negative and have the same integral. Then, since $p^{n-1}(z,y) \ge 0$,

$$\Big| \int_0^\ell dz[\mu(z)-\mu'(z)]p^{n-1}(z,y)\Big| \le \delta_{n-1} \int_0^\ell dz\, \mu(z) = (1-\zeta)\delta_{n-1}$$

which completes the proof of (4.10).

The probability density

$$\rho_{m,\ell}(x) := u_{m,\ell}(x) v_{m,\ell}(x)$$

is invariant (i.e. a left eigenvector)

$$\rho_{m,\ell} \star Q_{m,\ell} = \rho_{m,\ell}$$

Then

$$\Big| Q^n_{m,\ell}(x,y) - \rho_{m,\ell}(y)\Big| \le \int_0^\ell dx'\, \rho_{m,\ell}(x') \Big| Q^n_{m,\ell}(x,y) - Q^n_{m,\ell}(x',y)\Big| \le cr^n \tag{4.11}$$

Rewriting

$$\pi^-_{m,\ell}(w) = \int_0^\ell dy\, \rho_{m,\ell}(y) \frac{w(y)}{v_{m,\ell}(y)}$$

we have

$$\begin{aligned} \int_0^\ell dy\, A^n_{m,\ell}(x,y)\big[w(y) - \pi^-_{m,\ell}(w)v(y)\big] &= (\lambda_{m,\ell})^n v_{m,\ell}(x) \\ &\times \Big\{ \int_0^\ell dy\, Q^n_{m,\ell}(x,y) \frac{w(y)}{v_{m,\ell}(y)} - \pi^-_{m,\ell}(w) \Big\} \\ &\le (\lambda_{m,\ell})^n v_{m,\ell}(x) \int_0^\ell dy\, \big| Q^n_{m,\ell}(x,y) - \rho_{m,\ell}(y)\big| \frac{w(y)}{v_{m,\ell}(y)} \end{aligned}$$

Then by (4.11) the last term is bounded by

$$c(r\lambda_{m,\ell})^n \|v_{m,\ell}\|_\infty \|v^{-1}_{m,\ell}\|_\infty \|w\|_\infty$$

and since $\|w\|_\infty \le c\|\tilde{w}\|_\infty$, c a constant, Lemma 4.3 is proved. $\square$

Conclusion of the proof of Theorem 2.1

The spectral gap property, the only one still to prove, is a consequence of (i) and (ii) below:

(i) the subspace of the functions $\tilde{w}$ is invariant under $A_{m,\ell}$.

(ii) on such a space the norm of $A_{m,\ell}$ is bounded by $r\lambda_{m,\ell}$.

The proof of (ii) follows directly from Lemma 4.3. To prove (i) we write

$$A_{m,\ell}\tilde{w} = A_{m,\ell}w - \pi^-_{m,\ell}(w)\lambda_{m,\ell}v_{m,\ell} = A_{m,\ell}w - \pi^-_{m,\ell}\big(\lambda_{m,\ell}w\big)v_{m,\ell}$$

and since $u_{m,\ell}$ is a left eigenvector,

$$\pi^-_{m,\ell}\big(\lambda_{m,\ell}w\big) = \int_0^\ell dx\, w(x) \int_0^\ell dy\, u_{m,\ell}(y)A_{m,\ell}(y,x) = \pi^-_{m,\ell}\big(A_{m,\ell}w\big)$$

Thus $A_{m,\ell}\tilde{w} = w' - \pi^-_{m,\ell}(w')$, $w' = A_{m,\ell}w$, which is (i).

4.4. Lemma.

There is $b > 1$ so that for any $|x-y| \le 1$

$$b^{-1} \le \frac{v_{m,\ell}(x)}{v_{m,\ell}(y)} \le b \tag{4.12}$$

Proof.

Dropping the subfix m and ℓ in the course of the proof, by Lemma 3.3 and by the translational invariance of $J(\cdot,\cdot)$, there are k and $\zeta > 0$ so that for any x, y, x' such that $|x-y| \le 1$ and $|x'-y| \le 1$, $J^k(x,x')| \ge \zeta$. Then

$$\begin{aligned} v(x) &= \lambda^{-k} \int dx_1 ... \int dx_k p_m(x_1)\cdots p_m(x_k)J(x,x_1)\cdots J(x_{k-1},x_k)v(x_k) \\ &\ge \cosh^{-2k}\beta \int_{|x'-y|\le 1} dx' \zeta v(x') \end{aligned}$$

because, by (4.9), $\lambda^{-1} > \beta^{-1}$ and $p_m \ge \beta\cosh^{-2}\beta$. On the other hand, using again (4.9), we have

$$\begin{aligned} v(y) &\le \lambda^{-1}\beta \int_{|x'-y|\le 1} dx'\, J(y,x')v(x') \\ &\le \|J\|_\infty \cosh^2\beta \int_{|x'-y|\le 1} dx'\, v(x') \end{aligned}$$

We have thus obtained the first inequality in (4.12) with

$$b^{-1} = \zeta\|J\|^{-1}_\infty \cosh^{-2k-2}(\beta)$$

The second inequality in (4.12) is equivalent to the first one. Lemma 4.4 is proved. □

5. Preliminary bounds

In this section we ourselves restrict to functions m close to a double instanton and derive a lower bound on the maximal eigenvalue $\lambda_{m,\ell}$ of $A_{m,\ell}$ and upper and lower bounds on $u_{m,\ell}$ and $v_{m,\ell}$.

The lower bound on $\lambda_{m,\ell}$

Given $\ell > 2\xi$, $\xi > 1$, and $m \in C([0,\ell],[-1,1])$, we set

$$\phi(x,\xi,m) := \frac{p_m(x)}{p_{\bar m}(\xi - x)}, \qquad 0 \le x \le \ell \tag{5.1}$$

$$\psi(x,\xi) := \frac{\bar m'\big(\xi - R_\ell(x))\big)}{\bar m'\big(\xi - x\big)}, \qquad -1 \le x \le \ell + 1 \tag{5.2}$$

We recall the definition

$$\tilde P(x,y) := P\big(\xi - R_\ell(x), \xi - y\big), \qquad x, y \ \in [-1, \ell+1] \tag{5.3}$$

and denote by $\tilde{\mathbb{E}}_x$, $x \in [-1,\ell+1]$, the expectation of the corresponding Markov chain $\{x_n\}_{n\ge 0}$ starting from x, i.e. $x_0 = x$, and by $\tilde{\mathbb{E}}$ the expectation starting from $u_{m,\ell}(x)dx$, when $u_{m,\ell}$ is normalized to have integral 1.

5.1 Lemma.

For all ξ large enough

$$\psi(x,\xi) \ge 1, \ \ \textit{for all } x \in [-1,\ell+1] \qquad \psi(x,\xi) = 1, \ \ \textit{for all } x \in [1,\ell-1] \tag{5.4}$$

For any $n \ge 1$

$$\lambda_{m,\ell}^n = \tilde{\mathbb{E}}\left(\frac{\bar m'\big(\xi - R_\ell(x_0)\big)}{\bar m'\big(\xi - R_\ell(x_n)\big)} \Big\{ \prod_{i=0}^{n-1} \phi\big(R_\ell(x_i), \xi, m\Big\} \Big\{ \prod_{i=1}^{n} \psi(x_i,\xi) \Big\} \right) \tag{5.5}$$

and for any and $x \in [0,\ell]$

$$\lambda_{m,\ell}^n v_{m,\ell}(x) = \tilde{\mathbb{E}}_x \left(\frac{\bar m'\big(\xi - R_\ell(x)\big) v_{m,\ell}(R_\ell(x_n))}{\bar m'(\xi - R_\ell(x_n))} \Big\{ \prod_{i=0}^{n-1} \phi\big(R_\ell(x_i), \xi, m\Big\} \Big\{ \prod_{i=1}^{n} \psi(x_i,\xi) \Big\} \right) \tag{5.6}$$

Proof.

The second statement in (5.4) follows directly from the definition of R_ℓ. By (2.10) $\bar m'(x)$ is strictly decreasing when x is large, hence there is $\xi' > 1$ so that for all $\xi \ge \xi'$ and all x, $\psi(x,\xi,m) \ge 1$. (5.4) is proved.

We simplify the notation writing in the course of the proof, $u \equiv u_{m,\ell}$, $v \equiv v_{m,\ell}$, $\lambda \equiv \lambda_{m,\ell}$ and $\phi(x)$, $\psi(x)$ dropping ξ and m from the arguments. We write $\lambda^n u(x) = [u \star A^n_{m,\ell}](x)$ and integrate over x, recalling that u is normalized to have integral 1. We then get

$$\lambda^n = \int_0^\ell dx_0 \int_0^\ell dx_1 .. \int_0^\ell dx_n \, u(x_0) A_{m,\ell}(x_0, x_1) A_{m,\ell}(x_1, x_2) .. A_{m,\ell}(x_{n-1}, x_n)$$

and by applying repeatedly (4.2)

$$\lambda^n = \int_0^\ell dx_0 \int_{-1}^{\ell+1} dx_1 .. \int_{-1}^{\ell+1} dx_n \, u(x_0) A_{m,\ell}(x_0, x_1) A_m(R_\ell(x_1), x_2) ... A_m(R_\ell(x_{n-1}), x_n) \tag{5.7}$$

Then

$$\begin{aligned} \prod_{i=0}^{n-1} A_m\big(R_\ell(x_i), x_{i+1}\big) &= \Big\{ \prod_{i=0}^{n-1} \tilde{P}(x_i, x_{i+1}) \Big\} \prod_{i=0}^{n-1} \Big\{ \frac{p_m\big(R_\ell(x_i)\big)}{p_{\bar{m}}\big(\xi - R_\ell(x_i)\big)} \frac{\bar{m}'\big(\xi - R_\ell(x_i)\big)}{\bar{m}'\big(\xi - x_{i+1}\big)} \Big\} \\ &= \Big\{ \prod_{i=0}^{n-1} \tilde{P}(x_i, x_{i+1}) \Big\} \Big\{ \prod_{i=0}^{n-1} \phi\big(R_\ell(x_i)\big) \Big\} \frac{\bar{m}'\big(\xi - R_\ell(x_0)\big)}{\bar{m}'\big(\xi - R_\ell(x_n)\big)} \Big\{ \prod_{i=1}^{n} \psi(x_i) \Big\} \end{aligned} \tag{5.8}$$

which inserted into (5.7) proves (5.5). The proof of (5.6) is completely analogous and omitted. Lemma 5.1 is proved. $\square$

As a corollary of the above result we have:

5.2 Lemma.

There is $\xi_0 > 1$ so that for all $\xi \ge \xi_0$, $\ell \ge 2\xi$ and $m \in C([0,\ell],[-1,1])$

$$\log \lambda_{m,\ell} \ge \int_0^\ell dx \, \chi_{\xi,\ell}(x) \log \phi(x, \xi, m) \tag{5.9}$$

where $\chi_{\xi,\ell}(x)$ is the invariant density of the chain $Q_{\xi,\ell}(x,y)$.

Proof.

By Lemma 5.1 and with the simplified notation used in its proof, for all ξ large enough

$$\lambda^n \ge \tilde{\mathbb{E}} \Big(\frac{\bar{m}'\big(\xi - R_\ell(x_0)\big)}{\bar{m}'(\xi - x_n)} \exp \Big\{ \sum_{i=0}^{n-1} \log \phi\big(R_\ell(x_i)\big) \Big\} \Big)$$

By (2.10) there exists $C = C(\ell, \xi, m) > 0$ so that

$$\inf_{-1 \le x, y \le \ell+1} \frac{\bar{m}'(\xi - x)}{\bar{m}'(\xi - y)} \ge C$$

Then

$$\lambda^n \geq C\tilde{\mathbb{E}}\Big(\exp\{\sum_{i=0}^{n-1} \log\phi\big(R_\ell(x_i)\big)\}\Big)$$

and, by the Jensen's inequality,

$$n\log\lambda \geq \log C + \tilde{\mathbb{E}}\Big(\sum_{i=0}^{n-1} \log\phi(R_\ell(x_i))\}\Big)$$

The expectation involves functions of $\{R_\ell(x_i)\}_{i\geq 0}$ hence it coincides with the expectation E of the chain in $[0,\ell]$ with transition probability $Q_{\xi,\ell}$. Then

$$n\log\lambda \geq \log C + E\Big(\sum_{i=1}^{n-1}\{\log\phi(x_i)\}\Big)$$

We divide by n, let $n\to+\infty$ and get

$$\log\lambda \geq \liminf_{n\to+\infty}\ \mathbb{E}_u\Big(\frac{1}{n}\Big\{\sum_{i=1}^{n-1}\log\phi(x_i)\Big\}\Big)$$

By the ergodicity of the Markov chain, see the proof of Lemma 4.3, we then obtain (5.9). Lemma 5.2 is proved. □

A lower bound on $\lambda_{m,\ell}$ depends on the sign of $\log\phi$. Observe that if $m(x) = \bar{m}(\xi - x)$ for $x\in[0,\ell]$ then $\log\phi = 0$ and $\lambda_{m,\ell}\geq 1$. We restrict hereafter to $m\in G_{(c,\xi,\ell)}$, see Definition 2.3.

Proof of the first inequality in (2.17)

We drop the subfixes m and ℓ from the notation, when no ambiguity may arise. We write

$$\delta_\xi m(x) := m(x) - \bar{m}(\xi - x) \tag{5.10}$$

and observe that there is $c_1 > 0$ so that for $x\in[0,\ell]$

$$\big|\log\phi(x,\xi,m)\big| \leq c_1|\delta_\xi m(x)| \leq c_1 c 2e^{-2\alpha\xi}e^{\alpha|\xi-x|} \tag{5.11}$$

The last inequality is obtained recalling (2.13) and using (8.42) below. Then

$$\begin{aligned}\int_0^\ell dx\,\chi_{\xi,\ell}(x)\log\phi(x) &\geq -\int_0^\ell dx\,\chi_{\xi,\ell}(x)c_1 c2e^{-2\alpha\xi}e^{\alpha|\xi-x|}\\ &\geq -c_1 c2e^{-2\alpha\xi}\int_0^\ell dx\,Ce^{-2\alpha|\xi-x|}e^{\alpha|\xi-x|}\end{aligned} \tag{5.12}$$

which proves the first inequality in (2.17). □

Proof of (2.18)

Since $m \in G_{(c,\xi,\ell)}$ we can use the first inequality in (2.17), hence there are c_1 and c_2 that depend on c only, so that

$$\lambda_{m,\ell}^{-1} p_m(x) \le q(x) := p' + c_2 e^{-\alpha|\xi - x|}, \qquad p' := p_\infty + c_1 e^{-2\alpha\xi} \tag{5.13}$$

Let α_p be as in Lemma 3.1. Recalling that α_p is a regular, decreasing function of p, there is a constant c', determined by c, such that

$$\alpha_{p'} \ge \alpha - c' e^{-2\alpha\xi} =: \alpha', \qquad \alpha \equiv \alpha_{p_\infty} \tag{5.14}$$

We now proceed as in the proof of Theorem 3.2, we just outline the main steps. Let ξ, s and r be such that $\xi_s := \xi - s > 0$ and $q(x) \le r < 1$ for $|\xi - x| \ge s - 1$. Since $v_{m,\ell} = \lambda_{m,\ell}^{-1} A_{m,\ell} \star v_{m,\ell}$ and since, for $0 \le y_0 \le \xi_s$

$$A_{m,\ell}(y_0, y) = A_m(y_0, y)$$

using (2.1) we have

$$v_{m,\ell}(y_0) = \lambda_{m,\ell}^{-1} \int_{-\infty}^{+\infty} A_m(y_0, y) v_{m,\ell}(|y|)$$

so that

$$\begin{aligned} v_{m,\ell}(y_0) = \lambda_{m,\ell}^{-1} &\int_{-\xi_s}^{\xi_s} dy p_m(y_0) J(y_0, y) v_{m,\ell}(|y|) \\ + \lambda_{m,\ell}^{-1} &\int_{\xi_s \le |y| \le \xi_s + 1} dy p_m(y_0) J(y_0, y) v_{m,\ell}(|y|) \end{aligned}$$

Iterating the first integral, similary to the proof of the Theorem 3.2 we get, by (5.13),

$$v_{m,\ell}(y_0) \le \int_{\xi_s \le |y| \le \xi_s + 1} dy\, \bar{G}_s(y_0, y) v_{m,\ell}(|y|) \tag{5.15}$$

where

$$\bar{G}_s(y_0, y) := \sum_{n=1}^{\infty} \int_{-\xi_s}^{\xi_s} dy_1 .. \int_{-\xi_s}^{\xi_s} dy_{n-1}\, q(y_{n-1}) J(y_{n-1}, y) \prod_{i=1}^{n-1} q(y_{i-1}) J(y_{i-1}, y_i) \tag{5.16}$$

We introduce the transition probability kernel

$$T(x, y) := w(x)^{-1} p' J(x, y) w(y), \qquad w(x) := e^{\alpha' x} + e^{-\alpha' x} \tag{5.17}$$

$T(x, y)$ is the analogue of the kernel $K(x, y)$ of Section 3. We want to show that

$$\bar{G}_{\xi_s}(x, y) \le [1 + \epsilon(s)] \bar{g}_{\xi_s}(x, y) \frac{w(x)}{w(y)} \tag{5.18}$$

where

$$\bar{g}_{\xi_s}(y_0, y) := \sum_{n=1}^{\infty} \int_{-\xi_s}^{\xi_s} dy_1 \cdots \int_{-\xi_s}^{\xi_s} dy_{n-1}\, T(y_0, y_1) \cdots T(y_{n-1}, y) \tag{5.19}$$

and $\epsilon(s)$ vanishes as $s \to \infty$. We define the Green functions $G_{\xi_s}(x, y; p)$, $p \in [p', 1)$ by replacing $q(x)$ in (5.16) by p. By the monotonicity of $q(x)$, similarly to (3.29) we have, for $N-1 < |x| \le N$,

$$\bar{G}_{\xi_s}(x, y) \le \int_{N \le |x_1| \le N+1} dx_1 .. \int_{\xi_s - 1 \le |x_m| \le \xi_s} dx_m G_N(x, x_1; q(N)) .. G_{\xi_s}(x_m, y; q(\xi_s)) \tag{5.20}$$

Analogously to (3.30), for any $p^\star \in [p', 1)$ there is $c^\star > 0$ so that for all N and all x, y such that $N-1 < |x| \le N$ and $N < |y| \le N+1$

$$G_N(x, y; p) \le G_N(x, y; p')[1 + c^\star (p - p')] \tag{5.21}$$

We omit the proof of (5.21), very similar to that of (3.30). By (5.20) and (5.21), there are $c > 0$ and $\eta > 0$, so that

$$\bar{G}_{\xi_s}(x, y) \le [1 + c e^{-\eta \xi_s}] \bar{g}_{\xi_s}(x, y) \frac{w(x)}{w(y)} \tag{5.22}$$

We have thus proven (5.18).

Then, by (5.15), for any $x \in [0, \xi_s)$,

$$v(x) \le \int_{\xi_s \le |y| \le \xi_s + 1} dy\, [1 + \epsilon(s)] \bar{g}_{\xi_s}(x, y) \frac{w(x)}{w(y)} v(|y|) \tag{5.23}$$

By Lemma 4.2 v is bounded, moreover there is $c > 0$ so that for $\xi_s < |y| < \xi_s + 1$

$$\frac{w(x)}{w(y)} = \frac{e^{\alpha' x} + e^{-\alpha' x}}{e^{\alpha' y} + e^{-\alpha' y}} \le c e^{\alpha'(x - \xi_s)} \tag{5.24}$$

Since the integral over y of $g_{\xi_s}(x, y)$ is equal to 1, for a suitable constants $c'' > 0$

$$v(x) \le c[1 + \epsilon(s)] e^{\alpha'(x - \xi_s)} \le c'' e^{-\alpha'(\xi - s - x)} \tag{5.25}$$

which proves the first bound in (2.18).

The case $x > \xi$ is completely analogous, reflections around 0 are replaced by reflections around ℓ, we omit the details. (2.18) is proved. □

Lower bounds on $v_{m,\ell}$

The lower bound in (4.8) cannot be strict uniformly in ℓ, because, as we have just seen, $v_{m,\ell}(x)$ is bounded by a decreasing exponential. There is however a strictly positive lower bound on $v_{m,\ell}(x)$ for $|x - \xi|$ in a compact which is uniform in ℓ. This is established in the next Lemma:

5.3. Lemma.

For any $s > 0$ *there is* $\zeta^\star > 0$ *so that if* $\ell > 2\xi$, $\xi > s$ *and* $m \in G_{(c,\xi,\ell)}$, *then*

$$v_{m,\ell}(x) \ge \zeta^\star, \quad u_{m,\ell}(x) \ge \zeta^\star, \qquad \textit{for all } |x-\xi| \le s \tag{5.26}$$

Proof.

Let $u_{m,\ell}$ be normalized to have integral 1. By (2.18) there is $s_1 > 0$, independent of ℓ, so that

$$\int_{\xi-s_1}^{\xi+s_1} dx\, u_{m,\ell}(x) > \frac{1}{2} \tag{5.27}$$

Then there is $z \in (\xi - s_1, \xi + s_1)$ such that

$$u_{m,\ell}(z) \ge \frac{1}{4s_1} \tag{5.28}$$

By the second inequality in (4.8) there is $\epsilon \in (0,1)$ uniquely determined by s_1 and by the constant c_1 in (4.8), so that

$$u_{m,\ell}(x) \ge \frac{1}{8s_1} \qquad \text{for } |x-z| \le \epsilon \tag{5.29}$$

By Lemma 3.3, given $s \ge s_1$ there are an integer k and $\zeta > 0$ so that for any $x \in (\xi - s, \xi + s)$

$$J^k(x,y) \ge \zeta \qquad |y-z| \le 1$$

Then since $p_m(x) \ge \beta \cosh^{-2}(\beta) =: p$

$$\begin{aligned} u_{m,\ell}(x) &\ge \lambda_{m,\ell}^{-k} \int_{\xi-s}^{\xi+s} dx_1 ... \int_{\xi-s}^{\xi+s} dx_k\, u_{m,\ell}(x_1) p_m(x_1) J(x_1,x_2)..p_m(x_k) J(x_k,x) \\ &\ge (p\lambda_{m,\ell}^{-1})^k \int_{z-\epsilon}^{z+\epsilon} dx_k\, J^k(x,x_k) u_{m,\ell}(x_k) \ge (p\lambda_{m,\ell}^{-1})^k \zeta \frac{1}{8s_1} \end{aligned}$$

Lemma 5.3 is proved. □

5.4. Lemma.

There are $s > 0$, $\gamma > 0$ *and* $c > 0$ *so that if* $\ell > 2\xi$, $\xi > s$ *and* $m \in G_{(c,\xi,\ell)}$, *then*

$$v_{m,\ell}(x),\ u_{m,\ell}(x) \ge c e^{-\gamma|\xi - x|}, \qquad \textit{for all } x \in [0,\ell] \tag{5.30}$$

Proof.

By Lemma 5.3 there are $\zeta > 0$ and $s > 0$ for which (5.26) holds. By choosing $c \le \zeta$ we then have (5.30) for $|x-\xi| \le s$. Let us next consider $x > \xi + s$, the other case being analogous is omitted. We write

$$v_{m,\ell}(x) = \lambda_{m,\ell}^{-1} \int dy\, A_{m,\ell}(x,y) v_{m,\ell}(y) \ge C \int_{x-R}^{x-R/2} dy\, J(x,y) v_{m,\ell}(y) \tag{5.31}$$

where $R>0$ is such that $J(0,R)>0$ and $C>0$ such that $\lambda_{m,\ell}^{-1}>C$, see (4.9). Then

$$C_0 := C\int_{x-R}^{x-R/2} dy\, J(x,y) > 0$$

By iterating (5.31) till $y<\xi+s$ we get

$$v_{m,\ell}(x) \ge {C_0}^N \zeta, \qquad N := \left[\frac{2(x-\xi-s)}{R}\right]+1$$

which proves (5.30). Lemma 5.4 is proved. □

6. Markov chains: exponential bounds

In this Section we will prove that the auxiliary Markov chains introduced in Section 2 are exponentially localized uniformly in ℓ. We will exploit the properties of the instanton established in Section 3. We start from the simpler case:

The chain $Q_{\xi,\ell}$

s will hereafter denote a positive integer and

$$N_s := \min_{s-1\le|x|\le s} \bar m'(x) \tag{6.1}$$

Given $\xi>s$ and $|\zeta|<\alpha$, we define the weight function

$$\gamma_{s,\zeta}(x) := \mathbf{1}_{|x-\xi|<s} + \mathbf{1}_{|x-\xi|\ge s} N_s \bar m'(\xi-x)^{-1} e^{\zeta(|x-\xi|-s)} \tag{6.2}$$

6.1 Proposition.

For any $|\zeta|<\alpha$ there are $s^\star>0$ and $r<1$ so that for all $s\ge s^\star$ and all $\ell>\xi>s+1$ the following holds.

For any x: $|x-\xi|>s-1$:

$$\int dy\, Q_{\xi,\ell}(x,y)\gamma_{s,\zeta}(y) \le r\gamma_{s,\zeta}(x) + \mathbf{1}_{|x-\xi|\le s+1} \tag{6.3}$$

For x: $|x-\xi|\le s-1$ the left hand side of (6.3) *is equal to 1.*

Proof.

Writing simply Q for $Q_{\xi,\ell}$, we have for $x \in (\xi+s-1, \ell-1)$

$$\int dy\, Q(x,y)\gamma_{s,\zeta}(y) = \int_{\xi+s}^{\infty} dy\, p_{\bar m}(\xi-x) J(x,y) \frac{\bar m'(\xi-y)}{\bar m'(\xi-x)} N_s\, \bar m'(\xi-y)^{-1} e^{\zeta(y-\xi-s)} + \int_{\xi+s-2}^{\xi+s} dy\, Q(x,y) \tag{6.4}$$

the last term being absent if $x > \xi+s+1$. The first integral on the right hand side is equal to

$$\frac{N_s}{\bar m'(\xi-x)} e^{\zeta(x-\xi-s)} p_{\bar m}(\xi-x) \int_{\xi+s}^{\ell} dy\, J(x,y) e^{\zeta(y-x)} \le \gamma_{s,\zeta}(x) p_{\bar m}(\xi-x) p_\zeta^{-1}$$

where $p_\zeta < 1$ is such that $\alpha_{p_\zeta} = \zeta$, namely

$$p_\zeta \int dy\, J(x,y) e^{\zeta(y-x)} = 1 \tag{6.5}$$

By Lemma 3.1 α_p is a decreasing function of p. Recalling that $\alpha = \alpha_{p_\infty}$, $p_\infty = \lim_{x\to\infty} p_{\bar m}(x)$, and $|\zeta| < \alpha$, we have $p_\zeta > p_\infty$ so that we can choose $s^\star > 0$ and $r < 1$ so that

$$p_{\bar m}(\xi-x) p_\zeta^{-1} < r \qquad \text{for all } |x-\xi| \ge s^\star - 1 \tag{6.6}$$

and (6.3) holds for $x \ge \xi+s$ ($s \ge s^\star$).

Let $\xi+s-1 \le x < \xi+s$ and consider first the case $\zeta \le 0$. By (6.4)

$$\int dy\, Q(x,y)\gamma_{s,\zeta}(y) \le p_{\bar m}(\xi-x) \frac{N_s}{\bar m'(\xi-x)} + 1 = p_{\bar m}(\xi-x)\gamma_{s,\zeta}(x) + 1$$

hence (6.3), because, by (6.6), $p_{\bar m}(\xi-x) \le p_\zeta r < r$.

If $\zeta > 0$, by (6.4)-(6.5)

$$\int dy\, Q(x,y)\gamma_{s,\zeta}(y) \le p_{\bar m}(\xi-x) p_\zeta^{-1} e^{\zeta(x-(\xi+s))} + 1 \le r\gamma_{s,\zeta}(x) + 1$$

because $e^{\zeta(x-\xi-s)} \le 1 = \gamma_{s,\zeta}(x)$ since $x < \xi+s$ and $\zeta > 0$.

The same proof works when $1 < x < \xi-s+1$ and (6.3) obviously holds when $|x-\xi| \le s-1$. We are thus only left with $x \le 1$ and $x \ge \ell-1$, the two cases are similar and we only consider the former. We have

$$\int_0^{\ell} dy\, Q(x,y)\gamma_{s,\zeta}(y) = \int_0^2 dy\, p_{\bar m}(\xi-x) J(x,y) \frac{\bar m'(\xi-y)}{\bar m'(\xi-x)} N_s \bar m'(\xi-y)^{-1} e^{\zeta(\xi-s-y)} + \int_{-1}^{0} dy\, p_{\bar m}(\xi-x) J(x,y) \frac{\bar m'(\xi-y)}{\bar m'(\xi-x)} N_s \bar m'(\xi-|y|)^{-1} e^{\zeta(\xi-s-|y|)} \tag{6.7}$$

We will prove afterwards that given ζ, $|\zeta| < \alpha$, there is $\xi_0 > 0$ so that if $\xi > \xi_0$ for any $\omega \in [0, 1]$

$$\bar{m}'(\xi + \omega)e^{|\zeta|\omega} \le \bar{m}'(\xi - \omega)e^{-|\zeta|\omega} \tag{6.8}$$

When $\zeta < 0$ using (6.8) the second integral in (6.7) is bounded by

$$\int_{-1}^{0} dy\, p_{\bar{m}}(\xi - x)J(x, y)\frac{N_s}{\bar{m}'(\xi - x)}e^{\zeta(\xi - s - y)} \tag{6.9}$$

and we get from (6.7) and (6.6)

$$\int dy\, Q(x, y)\gamma_{s,\zeta}(y) \le \gamma_{s,\zeta}(x)p_{\bar{m}}(\xi - x)\int_{\mathbb{R}} dy\, J(x, y)e^{\zeta(x-y)} \le r\gamma_{s,\zeta}(x) \tag{6.10}$$

because for ξ large enough, $p_{\bar{m}}(\xi - x) \to p_\infty < 1$ as $\xi \to +\infty$.

When $\zeta > 0$ we use that $\bar{m}(\xi - y) \le \bar{m}(\xi - |y|)$ and that $e^{\zeta(\xi - s - |y|)} \le e^{\zeta(\xi - s - y)}$. Using these bounds in the second integral in (6.7) we reduce to the previous case. It thus only remains to prove (6.8).

Proof of (6.8).

By (2.10) for ξ large enough

$$\bar{m}'(\xi + \omega) \le \bar{m}'(\xi - \omega) - 2\omega\alpha^2 a e^{-\alpha(\xi+\omega)}[1 - \epsilon_1(\xi)]$$

where $\epsilon_1(\xi) > 0$ vanishes as $\xi \to +\infty$. To prove (6.8) it is then enough to show that

$$\bar{m}'(\xi - \omega) - 2\omega\alpha^2 a e^{-\alpha(\xi+\omega)}[1 - \epsilon_1(\xi)] \le \bar{m}'(\xi - \omega)e^{-2|\zeta|\omega}$$

hence that

$$1 - 2\omega\alpha\Big(\frac{a\alpha e^{-\alpha(\xi+\omega)}}{\bar{m}'(\xi - \omega)}[1 - \epsilon_1(\xi)]\Big) \;\le\; e^{-2|\zeta|\omega} \tag{6.11}$$

By (2.10)

$$1 - 2\omega\alpha\Big(\frac{a\alpha e^{-\alpha(\xi+\omega)}}{\bar{m}'(\xi - \omega)}[1 - \epsilon_1(\xi)]\Big) \;\le\; 1 - 2\omega\alpha e^{-2\alpha\omega}[1 - \epsilon_2(\xi)] \tag{6.12}$$

where $\epsilon_2(\xi) > 0$ vanishes as $\xi \to +\infty$. Since $|\zeta| < \alpha$ there is $q > 1$ so that if ξ is large enough the right hand side (6.12) is bounded by $1 - 2\omega|\zeta|qe^{-2\alpha\omega}$. There is $\omega_0 > 0$ so that

$$1 - 2\omega|\zeta|qe^{-2\alpha\omega} \le e^{-2|\zeta|\omega} \qquad \text{for all } \omega \le \omega_0 \tag{6.13}$$

We have thus proved that there are ξ_0' and ω_0 positive such that (6.8) holds for any $\xi \ge \xi_0'$ and $\omega \le \omega_0$. We have

$$\bar{m}'(\xi + \omega) \le a\alpha e^{-\alpha(\xi+\omega)}[1 + \epsilon_3(\xi)]$$
$$\bar{m}'(\xi - \omega) \ge e^{-\alpha(\xi-\omega)}[1 - \epsilon_3(\xi)]$$

where $\epsilon_3(\xi) > 0$ vanishes as $\xi \to +\infty$.

(6.8) is then implied by the inequality

$$e^{-2(\alpha-|\zeta|)\omega} \leq \frac{1-\epsilon_3(\xi)}{1+\epsilon_3(\xi)}$$

Recalling that $\alpha > |\zeta|$ this is verified for all $\omega \geq \omega_0$ provided that

$$\frac{1-\epsilon_3(\xi)}{1+\epsilon_3(\xi)} \geq e^{-2(\alpha-|\zeta|)\omega_0}$$

that is verified for all ξ large enough. Proposition 6.1 is proved. □

The existence of an invariant measure for $Q_{\xi,\ell}$, including the case $\ell = +\infty$, follows directly from Proposition 6.1:

6.2. Theorem.

Given any $|\zeta| < \alpha$, let $s^\star$ and r be as in Proposition 6.1 and set $\eta = \alpha + \zeta \in (0, 2\alpha)$. Then there is $c > 0$ so that for any $\xi > s^\star$, $\ell \in [2\xi, +\infty]$, $x \geq 0$ and $n \in \mathbb{N}$

$$\int_0^\ell dy\, Q^n_{\xi,\ell}(x,y) e^{\eta|\xi-y|} \leq c\Big(r^n e^{\eta|\xi-x|} + \frac{e^{\eta s^\star}}{1-r}\Big) \tag{6.14}$$

Moreover the chain $Q_{\xi,\ell}$ has an invariant measure $\nu_{\xi,\ell}(dx) = \chi_{\xi,\ell} dx$ and

$$\int_0^\ell dx\, \chi_{\xi,\ell}(x) e^{\eta|\xi-x|} \ \leq\ c\frac{e^{\eta s^\star}}{1-r} \tag{6.15}$$

Proof.

Writing Q for $Q_{\xi,\ell}$ we have

$$\begin{aligned}\int_0^\ell dy\, Q^n(x,y) e^{\eta|\xi-y|} \leq e^{\eta(s^\star+1)} + \int_{|\xi-y_1|>s^\star+1} dy_1 Q(x,y_1) ... \int_{|\xi-y|>s^\star+1} dy\, Q(y_{n-1},y) e^{\eta|\xi-y|} \\ + \sum_{i=1}^{n-1} \int_{|\xi-z|<s^\star+1} Q^{n-i}(x,z) \int_{|\xi-y_1|>s^\star+1} dy_1 Q(z,y_1) ... \int_{|\xi-y|>s^\star+1} dy\, Q(y_{i-1},y) e^{\eta|\xi-y|}\end{aligned} \tag{6.16}$$

By (6.3)

$$\int_{|\xi-y_1|>s^\star+1} dy_1 Q(x,y_1) ... \int_{|\xi-y|>s^\star+1} dy\, Q(y_{n-1},y) \gamma_{s,\zeta}(y) \leq r^n \gamma_{s,\zeta}(x)$$

By (2.10) there is $c > 1$ so that for $|\xi - y| > s^\star$

$$c^{-1} e^{(\alpha+\zeta)|\xi-y|} e^{-(\alpha+\zeta)s^\star} \le \gamma_{s^\star,\zeta}(y) \le c e^{(\alpha+\zeta)|\xi-y|} e^{-(\alpha+\zeta)s^\star} \tag{6.17}$$

Then the second term on the right hand side of (6.16) is bounded by

$$\int_{|\xi-y_1|>s^\star+1} dy_1 Q(x,y_1) ... \int_{|\xi-y|>s^\star+1} dy\, Q(y_{n-1},y) c e^{(\alpha+\zeta)s^\star} \gamma_{s^\star,\zeta}(y) \le c r^n e^{(\alpha+\zeta)s^\star} \gamma_{s^\star,\zeta}(x)$$
$$\le c^2 r^n e^{(\alpha+\zeta)|\xi-x|}$$

By the same argument we bound the third term on the right hand side of (6.16) by

$$\sum_{i=1}^{n-1} c^2 r^i e^{(\alpha+\zeta)(s^\star+1)} \le \frac{1}{1-r} c^2 e^{(\alpha+\zeta)(s^\star+1)}$$

which proves (6.14).

Let $n \ge 1$ and

$$\nu^{(n)}(dy) := \frac{1}{n} \sum_{i=0}^{n-1} Q^i(x,y) dy \tag{6.18}$$

Then there is c so that for all n

$$\int \nu^{(n)}(dy) e^{\eta|\xi-y|} \le c \tag{6.19}$$

By the Prokhorov theorem the sequence of probability measures $\{\nu_n\}$ is relatively compact and converges weakly by subsequences to a measure ν_ξ that satisfies (6.19). It is easily seen from (6.18) that ν_ξ invariant and that (6.15) holds. Theorem 6.2 is proved. $\square$

A statement analogous to that in Proposition 6.1 holds also for the chain with transition probability $P(x,y)$, see (2.7). The proof is the same without the complication which for $Q_{\xi,\ell}$ was due to $x \in [0,1]$ and $[\ell-1,\ell]$: here we have only one instanton and that problem is absent. The density $\bar{m}'(x)^2/p_{\bar{m}}(x)$ is invariant, as it can be seen by direct inspection, so that the analogue of (6.15) holds trivially.

The chain $Q_{m,\ell}$

Strictly speaking, this subsection should be read after the parts in Sections 7 and 8 which refer to the chain $Q_{\xi,\ell}$. In fact here we will use the first inequality in (2.17) which was proved in the previous Section using (8.42). In fact we suppose $\ell \ge 2\xi$, $\xi > 1$ and $m \in G_{(c,\xi,\ell)}$, so that $\lambda_{m,\ell} \ge 1 - c_- e^{-2\alpha\xi}$, by the first inequality in (2.17), moreover we will use (2.18) and that $\lambda_{m,\ell}^{-1} p_m(x)$ is bounded as in (5.13), i.e. $\lambda_{m,\ell}^{-1} p_m(x) \le p_\infty + c_1 e^{-2\alpha\xi} + c_2 e^{-\alpha|\xi-x|}$.

Let

$$C_s = [1+\epsilon(s)]cb$$

where b is as in Lemma 4.4, $\epsilon(s)$ and c as in (5.23) and (5.24). Recall that $\epsilon(s) \to 0$ as $s \to \infty$, so that C_s is a bounded function of s.

After setting

$$N^+_{s,m} := C_s \min_{\xi+s-1\le x\le \xi+s} v_{m,\ell}(x) \qquad N^-_{s,m} := C_s \min_{\xi-s\le x\le \xi-s+1} v_{m,\ell}(x) \tag{6.20}$$

we define the new weight function

$$\gamma_{s,\zeta,m}(x) := \mathbf{1}_{|x-\xi|\le s} + \mathbf{1}_{x>\xi+s} N^+_{s,m} v_{m,\ell}(x)^{-1} e^{\zeta(x-\xi-s)} + \mathbf{1}_{x<\xi-s} N^-_{s,m} v_{m,\ell}(x)^{-1} e^{\zeta(\xi-s-x)} \tag{6.21}$$

The weight function dangerously depends on m, but the bounds that will be proved in the next Proposition, analogous to Proposition 6.1, do not depend on the particular choice of m in $G_{(c,\xi,\ell)}$.

We now prove that there is ζ so that $\gamma_{s,\zeta,m} \ge 1$. To this end we consider the case $x > \xi + s$ and we write

$$1 = \frac{v_m(x) e^{-\zeta(x-\xi-s)}}{N^+_{s,m}} \gamma_{s,\zeta,m}(x) \tag{6.22}$$

By (5.23) and (5.24)

$$v_m(x) \le \int_{\xi+s-1\le y\le \xi+s} dy\, [1+\epsilon(s)] \bar g_{\xi_s}(x,y) c e^{-\alpha'(x-\xi-s)} v_m(|y|) \tag{6.23}$$

Since $v_m(y)[N^+_{s,m}]^{-1} \le bC_s^{-1}$ and $\bar g_{\xi_s}$ is a probability density, from (6.22) and (6.23) we get

$$1 \le c[1+\epsilon(s)] bC_s^{-1} e^{-(\alpha'-\zeta)(x-\xi-s)} \gamma_{s,\zeta,m}(x) = e^{(-\alpha'-|\zeta|)(x-\xi-s)} \gamma_{s,\zeta,m}(x) \tag{6.24}$$

Therefore for any s and for any $\zeta < \alpha'$, $\gamma_{s,\zeta,m}(x) \ge 1$ for $x > \xi + s$. The proof for $x < \xi - s$ is analogous and it is omitted.

6.3 Proposition.

Let $\xi > 1$ and $c > 0$ be fixed.

There are $\zeta^\star = \zeta^\star(c) > 0$, (see (2.18)), $s^\star = s^\star(c,\zeta)$ and $r = r(c,\zeta) < 1$, so that for any $s \ge s^\star$, $\zeta \in (-\zeta^\star, \alpha')$ and $m \in G_{(c,\xi,\ell)}$ the following holds.

For any x: $|x-\xi| \ge s-1$,

$$\int dy\, Q_{m,\ell}(x,y)\gamma_{s,\zeta,m}(y) \le r\gamma_{s,\zeta,m}(x) + [1+bC_s]\mathbf{1}_{|x-\xi|\le s+1} \tag{6.25}$$

For x: $|x-\xi| < s-1$ the left hand side of (6.25) is equal to one.

Proof.

For $x \in (\xi+s-1, \xi+s)$, using that $\lambda_{m,\ell} > 1$ we have

$$\int dy\, Q_{m,\ell}(x,y)\gamma_{s,\zeta,m}(y) \le 1 + \frac{N^+_{s,m}}{v_{m,\ell}(x)} p_{m,\ell}(x) \int_{\xi+s}^{\xi+s+1} dy\, J(x,y) e^{\zeta(y-\xi-s)}$$

If $\zeta < 0$ the right hand side is obviously bounded by $1 + bC_s$. If $\zeta \geq 0$, using that there is $s^\star$ so that

$$\sup_{|x-\xi|\geq s^\star -1} p_{m,\ell}(x) p_\zeta^{-1} := r < 1 \tag{6.26}$$

we get

$$\int dy\, Q_m(x,y)\gamma_{s,\zeta,m}(y) \leq 1 + bC_s p_{m,\ell}(x) e^{\zeta(x-\xi-s)} \int_{\xi+s}^{\xi+s+1} dy\, J(x,y) e^{\zeta(y-x)} \leq 1 + bC_s$$

For $x \in (\xi+s, \xi+s+1)$

$$\int dy\, Q_m(x,y)\gamma_{s,\zeta,m}(y) \leq 1 + \frac{N^+_{s,m}}{v_{m,\ell}(x)} p_{m,\ell}(x) p_\zeta^{-1} e^{\zeta(x-\xi-s)} \leq 1 + r\gamma_{s,\zeta,m}(x)$$

For $x \in (\xi+s+1, \ell-1)$ we get the above bound with only $r\gamma_{s,\zeta,m}(x)$. The proof is completely analogous for $x \in (1, \xi-s+1)$. The last statement in Proposition 6.3 trivially holds so that we are left with $x \in (0,1)$ and $x \in (\ell-1, \ell)$. We only consider the former case as the other is completely analogous.

We have

$$\int dy\, Q_{m,\ell}(x,y)\gamma_{s,\zeta}(y) \leq \{\frac{N^-_{s,m}}{v_{m,\ell}(x)} e^{\zeta(\xi-s-x)}\}\{\lambda_{m,\ell}^{-1} p_{m,\ell}(x) \int_{-1}^{2} dy\, J(x,y) e^{\zeta(x-|y|)}\}$$

If $\zeta > 0$ we bound the right hand side by replacing $|y| \to y$ so that the second curly bracket is bounded by $r < 1$. If $\zeta < 0$, by (6.26) the second curly brackets is strictly smaller than 1 for $\zeta^\star$ small enough, hence (6.25). Proposition 6.3 is proved. $\square$

6.4. Theorem.

In the same context as in Proposition 6.3, *there is* $c > 0$ *so that for any* $\xi > s^\star$, $\ell \in [2\xi, +\infty]$, $x \geq 0$ *and* $n \in \mathbb{N}$

$$\int_0^\ell dy\, Q^n_{m,\ell}(x,y)\gamma_{s,\zeta,m}(y) \leq r^n \gamma_{s,\zeta,m}(x) + c \tag{6.27}$$

Proof.

We write the analogue of (6.16) with $Q = Q_{m,\ell}$ and $\gamma_{s,\zeta,m}(y)$ instead of $e^{\eta|\xi-y|}$:

$$\int_0^\ell dy\, Q^n(x,y)\gamma_{s,\zeta,m}(y) \leq \sup_{|y-\xi|\leq s+1} \gamma_{s,\zeta,m}(y)$$
$$+ \int_{|\xi-y_1|>s+1} dy_1 Q(x,y_1) ... \int_{|\xi-y|>s^\star+1} dy\, Q(y_{n-1},y)\gamma_{s,\zeta,m}(y)$$
$$+ \sum_{i=1}^{n-1} \int_{|\xi-z|<s+1} Q^{n-i}(x,z) \int_{|\xi-y_1|>s+1} dy_1 Q(z,y_1) ... \int_{|\xi-y|>s+1} dy\, Q(y_{i-1},y)\gamma_{s,\zeta,m}(y)$$

By Lemma 4.4 the first term on the right hand side is bounded by $be^{|\zeta|}$. The second term by $r^n\gamma_{s,\zeta,m}(x)$ and the third term by

$$\sum_{i=1}^{n-1} be^{|\zeta|}r^i \le \frac{be^{|\zeta|}}{1-r}$$

Theorem 6.4 is proved. $\square$

Remark. The invariant measure for the chain $Q_{m,\ell}$ is $\chi_{m,\ell}dx := u_{m,\ell}(x)v_{m,\ell}(x)dx$ as it is easy to see by direct inspection. From (6.27) and (2.18) it follows that the analogue of (6.15) holds also for $\chi_{m,\ell}$.

We conclude this subsection by proving some bounds on $A^n_{m,\ell}(x,y)$ that are corollaries of Proposition 6.3.

6.5. Proposition.

In the context of Proposition 6.3 *and given* $\zeta \in (0,\alpha')$, *there is* $c^\star > 0$ *so that for all* $n > 1$ *and for all* (x,y) *such that either* x, y *are both* $\le \xi + 1$ *or both* $\ge \xi - 1$

$$A^n_{m,\ell}(x,y) \le c^\star e^{-\zeta(|x-y|)}\lambda^n_m \tag{6.28}$$

If either $x > \xi > y$ *or* $y > \xi > x$ *then*

$$A^n_{m,\ell}(x,y) \le c^\star n e^{-\zeta(|x-y|)}\lambda^n_m \tag{6.29}$$

Proof.

Let us consider the case $y > x > \xi$. We define s by setting $s := s^\star$ if $x \le \xi + s^\star$. Otherwise s is the smallest integer such that $\xi + s \ge x$. Given $y > \xi + s$ we set

$$f(z) := \lambda_m^{-1} A_{m,\ell}(z,y)$$

Calling n in (6.28) as $n+1$, we have

$$\begin{aligned}\lambda_m^{-(n+1)} A^{n+1}_{m,\ell}(x,y) &= \lambda_m^{-n} \int dz\, A^n_{m,\ell}(x,z) f(z)\\ &\le v_m(x)[N^+_{s,m}]^{-1} \int dz\, Q^n_{m,\ell}(x,z)\gamma_{s,\zeta}(z)e^{-\zeta(z-\xi-s)} f(z)\\ &\le ce^{-\zeta(y-x)} \int dz\, Q^n_{m,\ell}(x,z)\gamma_{s,\zeta}(z)\end{aligned}$$

because f has support on $|z-y| \le 1$, $f(z) < c$ and $v_m(x) N_{s,m}^{-1}$ is bounded using (4.12). (6.28) then follows from (6.27).

By observing that

$$A_m^n(x,y) = A_m^n(y,x) \frac{p_m(y)}{p_m(x)}$$

we prove (6.28) for $x > y > \xi$. By an analogous argument we prove (6.28) when both x and y are $< \xi$. To prove (6.29) for instance in the case $y > \xi + 1 > \xi > x$ we write

$$A_m^n(x,y) \le \sum_k \int_\xi^{\xi+1} dz\, A_m^k(x,z) A_m^{n-k}(z,y)$$

and apply the previous results. We omit the details.

Proposition 6.4 is proved. □

7. Markov chains: decay of correlations

For each finite ℓ the Markov chains $Q_{m,\ell}$ and $Q_{\xi,\ell}$ satisfy the Döblin condition and there is a spectral gap, as proved in Lemma 4.3. The estimates however are not uniform in ℓ. The idea is then to use the bounds in Propositions 6.1 and 6.3 to control the tails, i.e. the behavior for $|\xi - x|$ large. Notice however that the distance travelled by the chain in one step is at most 1, so that chains that start at distance D from each other are singular with respect to each other at all "times" $n \le D$ steps: the decay therefore is not uniform in ℓ. To overcome this difficulty we will use, as in the previous Section, weighted norms and in such a context we will prove a uniform Döblin theorem. The proof follows by showing that for a suitable n the n-th iterate of the transition probability satisfies a "Dobrushin uniqueness condition" which implies the above statements.

We will in the sequel suppose s, ξ, ℓ and m as in Proposition 6.3 and s, ξ, ℓ as in Proposition 6.1, when referring to $P_{\xi,\ell}$.

Definitions

Joint representations. Let μ and ν be two probability measures on $[0,\ell]$. σ is a joint representation of μ and ν if it is a probability on $[0,\ell]^2$, with marginals μ and ν, i.e. if for any continuous, bounded function f on $[0,\ell]$

$$\int\int \sigma(dx,dx') f(x) = \int \mu(dx) f(x); \qquad \int\int \sigma(dx,dx') f(x') = \int \nu(dx) f(x)$$

The Vaserstein distance. For any x and x' in $[0,\ell]$ we set

$$d_{s,\zeta,m}(x,x') := \begin{cases} 0 & \text{if } x = x' \\ \gamma_{s,\zeta,m}(x) + \gamma_{s,\zeta,m}(x') & \text{if } x \neq x' \end{cases} \tag{7.1}$$

and then define the Vaserstein distance $D_{s,\zeta,m}(\mu,\nu)$ of μ and ν (relative to $d_{s,\zeta,m}$) as

$$D_{s,\zeta,m}(\mu,\nu) := \inf_{\{\sigma\}} \int\int \sigma(dx,dx') d_{s,\zeta,m}(x,x') \tag{7.2}$$

where $\{\sigma\}$ denotes the family of all the joint representations of μ and ν.

7.1. Theorem.
Let $|\mu(dx) - \nu(dx)|$ be the total variation of $\mu - \nu$. Then

$$D_{s,\zeta,m}(\mu,\nu) = \int |\mu(dx) - \nu(dx)| \gamma_{s,\zeta,m}(x) \tag{7.3}$$

Theorem 7.1 is Proposition 4.7 in [6] to which we refer for a proof. Analogous results appear in the original work of Dobrushin, [11], and in earlier papers.

The strategy is now to derive good upper bounds on the $D_{s,\zeta,m}$ distance of $Q^n_{m,\ell}(x,y)dy$ and $Q^n_{m,\ell}(x',y)dy$ by constructing suitable joint representations of these measures, analogous properties hold for $Q_{\xi,\ell}$, that we state without proofs at the end of the Section.

We postpone the proof of the following statements:

(1) *There is an integer $s_0 \geq s^\star$ and for any integer $s \geq s_0$ there is $b_s \in (0,1)$ so that for all $|x-\xi| \leq s$ and y such that $|y-\xi| \leq 1/2$*

$$Q^{s^2}_{m,\ell}(x,y) \geq b_s \tag{7.4}$$

(2) *There are integers $s_1 \geq s_0$ and $s_2 > s_1$ so that for all the integers $s \geq s_2$ and all $|x-\xi| \leq 2s$*

$$\int_{|y-\xi|\leq s_1} dy\, Q^{s^2-s_1^2}_{m,\ell}(x,y) \geq \frac{1}{2} \tag{7.5}$$

(3) *For any $\epsilon > 0$ there is an integer $s_3 \geq s_2$ so that for any integer $s \geq s_3$ and any x such that $|x-\xi| \leq 2s$*

$$\int dy\, Q^{s^2}_{m,\ell}(x,y) \gamma_{s,\zeta,m}(y) \leq 1 + \epsilon \tag{7.6}$$

(4) *There are $r^\star < 1$ and an integer $s_4 \ge s_3$ so that for all the integers $s \ge s_4$ and all x such that $|x-\xi| \ge 2s$*

$$\int dy\, Q^{s^2}_{m,\ell}(x,y)\gamma_{s,\zeta,m}(y) \le r^\star \gamma_{s,\zeta,m}(x) \tag{7.7}$$

(5) *There is an integer $s_5 \ge s_4$ so that for all the integers $s \ge s_5$*

$$\gamma_{s,\zeta,m}(x) \ge 2 \qquad \text{all } x \text{ such that } |x-\xi| \ge 2s \tag{7.8}$$

We choose $s_6 \ge s_5$ in such a way that (7.6) holds with

$$\epsilon = \min\{\frac{1-r^\star}{2}\,,\,\frac{b_{s_1}}{4}\} \tag{7.9}$$

and in the sequel we shorthand $s := s_6$.

7.2. Definition.

For any x and x' in $\mathbb{R}_+$, $x \ne x'$, we define $\sigma_{x,x'}(dydy')$ as the following joint representation of $Q^{s^2}_{m,\ell}(x,y)dy$ and $Q^{s^2}_{m,\ell}(x',y)dy$. If either $|x-\xi| \ge 2s$ or $|x'-\xi| \ge 2s$ we set

$$\sigma_{x,x'}(dydy') := Q^{s^2}_{m,\ell}(x,y)dyQ^{s^2}_{m,\ell}(x',y')dy' \tag{7.10}$$

In the remaining case we set

$$\begin{aligned}\sigma_{x,x'}(dydy') :=& \frac{b_{s_1}}{2}\mathbf{1}_{|y-\xi|\le 1/2}\delta(y-y')dydy'\\ &+(1-\frac{b_{s_1}}{2})^{-1}\Big\{Q^{s^2}_{m,\ell}(x,y)dy - \frac{b_{s_1}}{2}\mathbf{1}_{|y-\xi|\le 1/2}dy\Big\}\Big\{\ y \to y'\ \Big\}\end{aligned} \tag{7.11}$$

The definition is well posed because for $|y-\xi| \le 1/2$ and $|x-\xi| \le 2s$,

$$Q^{s^2}_{m,\ell}(x,y) \ge \int_{|z-\xi|\le s_1} dz Q^{s^2-s_1^2}_{m,\ell}(x,z)Q^{s_1^2}_{m,\ell}(z,y) \ge \frac{b_{s_1}}{2} \tag{7.12}$$

having used (7.4) and (7.5).

7.3. Proposition.

For any x and x'

$$\int \sigma_{x,x'}(dydy')d_{s,\zeta}(y,y') \le r' d_{s,\zeta}(x,x') \tag{7.13}$$

$$r' := \max\left\{\frac{1+r^\star}{2}\ ,\ 1-\frac{b_{s_1}}{4}\right\}\ <\ 1 \tag{7.14}$$

Proof.

Let $x \neq x'$. Suppose first that $|x-\xi| \ge 2s$, $|x'-\xi| \ge 2s$. Then by (7.7) and (7.10)

$$\int \sigma_{x,x'}(dydy')d_{s,\zeta}(y,y') \le r^\star d_{s,\zeta}(x,x'), \qquad r^\star < \frac{1+r^\star}{2} \le r'$$

Suppose now $|x-\xi| \ge 2s$ and $|x'-\xi| < 2s$. Then by (7.10), (7.6), (7.7) and (7.9)

$$\begin{aligned}\int \sigma_{x,x'}(dydy')d_{s,\zeta}(y,y') &\le r^\star\gamma_{s,\zeta,m}(x)+1+\epsilon \le r^\star\gamma_{s,\zeta,m}(x)+1+\frac{1-r^\star}{2}\\ &= \frac{1+r^\star}{2}\gamma_{s,\zeta,m}(x)+\frac{1+r^\star}{2}+\Big([r^\star-\frac{1+r^\star}{2}]\gamma_{s,\zeta,m}(x)+(1-r^\star)\Big)\\ &\le \frac{1+r^\star}{2}d_{s,\zeta}(x,x') \le r' d_{s,\zeta}(x,x')\end{aligned}$$

because $\gamma_{s,\zeta,m}(x') \ge 1$ and, by (5), $\gamma_{s,\zeta,m}(x) \ge 2$ so that

$$[r^\star - \frac{1+r^\star}{2}]\gamma_{s,\zeta,m}(x) + (1-r^\star) \le 0$$

When $|x-\xi| < 2s$, $|x'-\xi| < 2s$ we use (7.11). Recalling that $d_{s,\zeta}(x,x)=0$, by (7.6)

$$\int \sigma_{x,x'}(dydy')d_{s,\zeta}(y,y') \le 2[1+\epsilon-\frac{b_{s_1}}{2}] \le 2[1-\frac{b_{s_1}}{4}] \le r' d_{s,\zeta}(x,x')$$

having used the second inequality in (7.9) and (7.14).

Proposition 7.3 is proved. □

As a corollary of Proposition 7.3 and Theorem 7.1 we have:

7.4. Theorem.

Let $c > 0$, $\delta > 0$, $-\zeta^\star < \zeta < \alpha$, $s \equiv s_6$. Then there are $c' > 0$ and $r < 1$ so that for all $\xi \geq s+1$, $\ell \in [2\xi, +\infty]$ and $m \in G_{(c,\delta,\xi,\ell)}$

$$\int dy \left| Q^n_{m,\ell}(x,y) - Q^n_{m,\ell}(x',y) \right| \gamma_{s,\zeta,m}(y) \leq c' r^n [\gamma_{s,\zeta,m}(x) + \gamma_{s,\zeta,m}(x')] \tag{7.15}$$

for all $x \neq x'$ and all positive integers n.

Proof.

Dropping ℓ from the subfix, we write the positive integers n as

$$n =: s^2 n_0 + n_1, \quad n_1 < s^2$$

with n_0 and n_1 non negative integers. We postpone the proof that

$$\int dy \left| Q^n_{m,\ell}(x,y) - Q^n_{m,\ell}(x',y) \right| \gamma_{s,\zeta,m}(y) \leq r'^{n_0} \left[2c + r_1^{n_1} \big(\gamma_{s,\zeta,m}(x) + \gamma_{s,\zeta,m}(x') \big) \right] \tag{7.16}$$

with r' as in Proposition 7.3 and r_1 equal to the parameter r of Proposition 6.3.

Then (7.15) follows from (7.16) with

$$r := (r')^{1/s^2}$$

recalling that $\gamma_{s,\zeta,m}(x) + \gamma_{s,\zeta,m}(x') \geq 2$.

Proof of (7.16)

We consider the joint representation $\hat\sigma_{x,x'}(dydy')$ of $Q^n_{m,\ell}(x,y)dy$ and $Q^n_{m,\ell}(x',y)dy$ defined as

$$\int\int dy_0 dy'_0 \, Q^{n_1}_{m,\ell}(x,y_0) Q^{n_1}_{m,\ell}(x',y'_0) \int \sigma_{y_0,y'_0}(dy_1 dy'_1) \cdots$$
$$\cdots \int \sigma_{y_{n_0-2},y'_{n_0-2}}(dy_{n_0-1} dy'_{n_0-1}) \sigma_{y_{n_0-1},y'_{n_0-1}}(dydy')$$

where $\sigma_{x,x'}(dydy')$ is defined in Definition 7.2.

By Theorem 7.1 the left hand side of (7.16) is bounded by

$$\int \hat\sigma_{x,x'}(dydy') d_{s,\zeta}(y,y') \leq \int\int dy_0 dy'_0 \, Q^{n_1}_{m,\ell}(x,y_0) Q^{n_1}_{m,\ell}(x',y'_0) r'^{n_0} [\gamma_{s,\zeta,m}(y_0) + \gamma_{s,\zeta,m}(y'_0)]$$

having applied n_0-times Proposition 7.3. (7.16) is then a consequence of Theorem 6.4.

Theorem 7.4 is proved. $\square$

A corollary of Theorem 7.4 is that the measure

$$\nu_{m,\ell}(dx) := \chi_{m,\ell}(x)dx = u_{m,\ell}(x)v_{m,\ell}(x)dx$$

is the unique invariant measure for the chain $Q_{m,\ell}$, with $\ell = +\infty$ included in the statement.

7.5. Theorem.

In the context of Theorem 7.4 there is c'' (independent of the particular choice of ξ, m and ℓ compatible with the requests of Theorem 7.4) so that

$$\int dy \left| Q^n_{m,\ell}(x,y) - \chi_{m,\ell}(y) \right| \gamma_{s,\zeta,m}(y) \le c' r^n \Big(\gamma_{s,\zeta,m}(x) + c'' \Big) \tag{7.17}$$

and if $\rho(y)$ is any invariant density, then $\rho(y) = \chi_{m,\ell}(y)$, Lebesgue almost everywhere.

Proof.

Since $\chi_{m,\ell}$ is invariant

$$\begin{aligned}\int dy \left| Q^n_{m,\ell}(x,y) - \chi_{m,\ell}(y) \right| \gamma_{s,\zeta,m}(y) &= \int dy\, \gamma_{s,\zeta,m}(y) \left| \int dx'\, \chi_{m,\ell}(x') \Big(Q^n_{m,\ell}(x,y) - Q^n_{m,\ell}(x',y') \Big) \right| \\ &\le \int dx'\, \chi_{m,\ell}(x') \int dy \left| Q^n_{m,\ell}(x,y) - Q^n_{m,\ell}(x,y') \right| \gamma_{s,\zeta,m}(y)\end{aligned}$$

(7.17) then follows from (7.16) with c'' not smaller than

$$\int dx'\, \chi_{m,\ell}(x') \gamma_{s,\zeta,m}(x')$$

which is finite recalling that $\chi_m := u_m v_m$ and using the analogue of (6.15) valid for χ_m as discussed at the end of the proof of Theorem 6.4.

It thus remain to prove the uniqueness of the invariant measure, it can be easily seen that any invariant measure is absolutely continuous and it thus have a density ρ. If ρ is such that $\gamma_{s,\zeta,m} \in L^1(\rho(y)dy)$ the statement $\rho = \chi_m$ is a direct consequence of (7.17). Without this assumption we observe that if ρ is invariant and I, I' are compact intervals, then

$$\begin{aligned}\int_I dy\, \rho(y) &= \int_I dy \int_{\mathbb{R}_+} dx\, Q^n_{m,\ell}(x,y)\rho(x) \\ &= \int_{I'} dx\, \rho(x) \int_I dy\, Q^n_{m,\ell}(x,y) + \int_{\mathbb{R}_+ \setminus I'} dx\, \rho(x) \int_I dy\, Q^n_{m,\ell}(x,y)\end{aligned}$$

For any $\epsilon > 0$ let I' be so large that

$$\int_{I'} dx\, \rho(x) \ge 1 - \epsilon$$

By (7.17) there is n so that for any $x \in I$

$$\int_I dy \left| Q^n_{m,\ell}(x,y) - \chi_{m,\ell}(y) \right| < \epsilon$$

Then

$$\left| \int_I dy\, \rho(y) - \int_I dy\, \chi_{m,\ell}(y) \right| < 3\epsilon$$

By the arbitrarity of ϵ and I, $\rho(y)dy = \chi_{m,\ell}(y)dy$.

Theorem 7.5 is proved. $\square$

We conclude this subsection with the analogues of Theorems 7.4 and 7.5 for the chains $Q_{\xi,\ell}$.

7.6. Theorem.

For any $\eta \in (0, 2\alpha)$ there are $c > 0$ and $r < 1$ so that for any $\ell \in (0, +\infty]$, any $n \geq 1$ and any $x \neq x'$ in $[0, \ell]$

$$\int dy \left| Q^n_{\xi,\ell}(x,y) - Q^n_{\xi,\ell}(x',y) \right| e^{\eta|y-\xi|} \leq c r^n \left(e^{\eta|\xi-x|} + e^{\eta|\xi-x'|} \right) \tag{7.18}$$

Moreover denoting by $\chi_{\xi,\ell}(x)$ the invariant density of the chain $Q_{\xi,\ell}$

$$\int dy \left| Q^n_{\xi,\ell}(x,y) - \chi_{\xi,\ell}(y) \right| e^{\eta|y-\xi|} \leq c r^n \left(e^{\eta|\xi-x|} + 1 \right) \tag{7.19}$$

Proof.

By proceeding as in the proof of Theorem 7.4 we prove that for any $|\zeta| < \alpha$ there are $c' > 0$ and $r_0 < 1$ so that for s large enough

$$\int dy \left| Q^n_{\xi,\ell}(x,y) - Q^n_{\xi,\ell}(x',y) \right| \gamma_{s,\zeta,m}(y) \leq c' {r_0}^n \left(\gamma_{s,\zeta,m}(x) + \gamma_{s,\zeta,m}(x')) \right)$$

After recalling the definition of $\gamma_{s,\zeta,m}(x)$, setting $\eta := \alpha + \zeta$ and using (2.10), we conclude that there is $C > 1$ so that

$$C^{-1} e^{\eta|\xi-x|} \leq \gamma_{s,\zeta,m}(x) \leq C e^{\eta|\xi-x|}$$

hence (7.18).

We omit the proof of the analogue of (7.17) and proceeding as above we derive (7.19). Theorem 7.6 is proved. $\square$

For future reference we state the analogue of Theorem 7.6 for P, the proof being completely analogous to the previous one is omitted.

7.7. Theorem.

For any $\eta \in (0, 2\alpha)$ there are $c > 0$ and $r < 1$ so that for any $n \geq 1$ and any $x \neq x'$ in $\mathbb{R}$

$$\int dy \left| P^n(x,y) - P^n(x',y) \right| e^{\eta|y|} \leq c r^n \left(e^{\eta|x|} + e^{\eta|x'|} \right) \tag{7.20}$$

Recall that

$$P(x,y) \equiv P^0(x,y) = \frac{\beta[1-\bar{m}(x)^2]}{\bar{m}'(x)}\, J(x,y)\, \bar{m}'(y)$$

Moreover

$$\int dy \left| P^n(x,y) - \rho(y) \right| e^{\eta|y|} \leq c r^n \left(e^{\eta|x|} + 1 \right) \tag{7.21}$$

where

$$\rho(x) := C_{\bar{m}} \frac{\bar{m}'(x)^2}{\beta[1-\bar{m}(x)^2]}$$

Decay of correlations (CONTINUED)

Proof of (1).

There is $c > 0$ so that for all x and y

$$Q_{m,\ell}(x,y) \geq p_m(x) \lambda_{m,\ell}^{-1} J(x,y) \frac{v_{m,\ell}(y)}{v_{m,\ell}(x)} \geq c J(x,y)$$

having used (4.9) to bound $\lambda_{m,\ell}^{-1}$ and $p_m(x)$ and Lemma 4.4 to bound $v_{m,\ell}(y) v_m(x)^{-1}$, (the first inequality is needed for $x \in (0,1)$ and $x \in (\ell-1, \ell)$).

(1) is then a consequence of Lemma 3.3.

Proof of (2).

Let $s^\star$ be as in Proposition 6.3, $s'' > s' > 2s^\star$, $|x - \xi| \leq 2s''$ and $n = {s''}^2 - {s'}^2$. Then, dropping ℓ from the subfix and denoting by $N_{s^\star,m}^{-1}$ the largest between those with $\pm$,

$$\begin{aligned} \int_{|y-\xi|>s'} dy\, Q_m^n(x,y) &= \int_{|y-\xi|>s'} dy\, Q_m^n(x,y) \gamma_{s^\star,0,m}(y) N_{s^\star,m}^{-1} v_{m,\ell}(y) \\ &\leq c e^{-\alpha^\star s'} N_{s^\star,m}^{-1} \int dy\, Q_m^n(x,y) \gamma_{s^\star,0,m}(y) \end{aligned}$$

where we have used (2.18) with $\alpha^\star = \alpha'(s^\star)$ to bound $v_{m,\ell}(y)$, c being a suitable constant. By Theorem 6.4

$$\int dy\, Q_m^n(x,y)\gamma_{s^\star,0,m}(y) \le r^n \gamma_{s^\star,0,m}(x) + c$$

so that

$$\int_{|y-\xi|>s'} dy\, Q_m^n(x,y) \le c e^{-\alpha^\star s'}\left(N_{s^\star,m}^{-1} + r^n e^{\gamma|\xi - x|}\right) \tag{7.22}$$

having used Lemma 5.4 to write for suitable $C > 0$ and $\gamma > 0$

$$v_{m,\ell}(x)^{-1} \le C^{-1} e^{\gamma|\xi-x|} \tag{7.23}$$

Let s_1 be the first integer such that

$$c e^{-\alpha^\star s_1} e^{\gamma s^\star} \le \frac{1}{4}$$

and s'' the largest for which

$$c e^{-\alpha^\star s_1} r^{s''^2 - s_1^2} e^{\gamma 2 s''} \ge \frac{1}{4}$$

(2) is then proved with such a value of s_1 and $s_2 = s'' + 1$.

Proof of (3).

Let $s \ge s_2$ and $|x - \xi| \le 2s$. Then

$$\int dy\, Q_m^{s^2}(x,y)\gamma_{s,\zeta,m}(y) \le 1 + \int_{|y-\xi|>s} dy\, Q_m^{s^2}(x,y)\gamma_{s,\zeta,m}(y) \tag{7.24}$$

We consider in the last integral the contribution of $\{y - \xi > s\}$ (the argument for the other one is similar and omitted) which can be rewritten as

$$\begin{aligned}
&\int_{y-\xi>s} dy\, Q_m^{s^2}(x,y)\gamma_{s^\star,\zeta,m}(y) \frac{N_{s,m}^+}{N_{s^\star,m}^+} e^{-\zeta(s-s^\star)} \\
&\qquad \le \frac{N_{s,m}^+}{N_{s^\star,m}^+} e^{-\zeta(s-s^\star)}\left(r^{s^2}\gamma_{s^\star,\zeta,m}(x) + c\right) \\
&\qquad \le \frac{e^{\zeta s^\star}}{N_{s^\star,m}^+} c e^{-(\alpha^\star+\zeta)s}\left(r^{s^2} N_{s^\star,m}^{\pm} c' e^{(\gamma+|\zeta|)2s} + c\right)
\end{aligned} \tag{7.25}$$

having used Theorem 6.4 and (7.23); $N_{s^\star,m}^{\pm}$ is either $N_{s^\star,m}^{+}$ or $N_{s^\star,m}^{-}$ according to the value of x. Since $|\zeta| < \alpha^\star$, by choosing s_3 large enough and $s \ge s_3$ we make the last term smaller than ϵ.

(3) is therefore proved.

Proof of (4).

Let $s \ge s_3$ and $|x - \xi| \ge 2s$. From Theorem 6.4 it follows that

$$\int dy\, Q_m^{s^2}(x,y)\gamma_{s,\zeta,m}(y) \le r^{s^2}\gamma_{s^\star,\zeta}(x) + c \tag{7.26}$$

¿From (6.24) it follows that for $|x-\xi| > 2s$

$$1 \le c' e^{-(\alpha' - |\zeta|)s} \gamma_{s,\zeta,m}(x) \tag{7.27}$$

Therefore

$$\int dy\, Q_m^{s^2}(x,y) \gamma_{s,\zeta,m}(y) \le \gamma_{s,\zeta,m}(x) \Big(r^{s^2} + cc' e^{-(\alpha^\star - |\zeta|)s} \Big)$$

Thus given any $r^\star < 1$ we choose $s_4 \ge s_3$ as the first integer for which the last bracket is smaller than $r^\star$. (4) is therefore proved.

Proof of (5).

By (7.27) when $|x-\xi| \ge 2s$

$$\gamma_{s,\zeta,m}(x) \ge (c')^{-1} e^{(\alpha' - |\zeta|)s}$$

Then s_5 is the first integer $\ge s_4$ for which the right hand side is larger than 2. (5) is proved.

8. Markov chains: invariant measure

In this Section we complete the analysis of the Markov chain with transition probability $Q_{\xi,\ell}$ by improving the estimates on its asymptotic behavior and its invariant measure.

We preliminary observe that from (3.2) it follows that

$$\chi^0_\xi(x) = \frac{\tilde{m}'(\xi - x)^2}{p_{\bar{m}}(\xi - x)} \tag{8.1}$$

is the invariant density of the chain $P(\xi - x, \xi - y)$. We also immediately get from (3.2) and the symmetry of $J(x, y)$ the following reversibility condition:

$$\chi^0_\xi(x)P(\xi - x, \xi - y) = \chi^0_\xi(y)P(\xi - y, \xi - x) \tag{8.2}$$

In the next theorem we write

$$d(y, A) := \inf_{x\in A} |y - x|, \qquad y \geq 0, \ A \subset \mathbb{R}_+ \tag{8.3}$$

8.1 Theorem.

There is $C > 0$ such that for any $n \in \mathbb{N}$, $\ell > \xi > s^\star$ ($s^\star$ as in Proposition 6.1)

$$Q^n_{\xi,\ell}(x, y) \leq Ce^{-2\alpha d(y,\{x,\xi\})} \tag{8.4}$$

whenever either $y < \xi$ and $x \geq y$ or $y > \xi$ and $x \leq y$.

Proof.

We write Q for $Q_{\xi,\ell}$ and, for y and z positive, we set

$$f_y(z) := Q(z, y), \qquad \sup_y \|f_y(z)\|_\infty =: c' < \infty \tag{8.5}$$

Hence for all x and y positive and all $n \in \mathbb{N}$

$$Q^{n+1}(x, y) = \int dz\, Q^n(x, z)f_y(z) \leq c' \tag{8.6}$$

Let $s^\star > 0$ and $r < 1$ be the parameters in Proposition 6.1 corresponding to $\zeta = 0$ and set $s := s^\star + 2$. Let $y \leq \xi - s - 1$. For $i \leq n$ we shorthand ($x = x_0$, $y = x_{n+1}$)

$$\mu_i(dx_i \ldots dx_n) := Q(x_i, x_{i+1}) \cdots Q(x_{n-1}, x_n) \prod_{j=i}^{n} [\mathbf{1}_{x_j<\xi-s-1} dx_j] \tag{8.7}$$

Then

$$Q^{n+1}(x,y) = \mathbf{1}_{x<\xi-s} \int \mu_1(dx_1 \dots dx_n) Q(x,x_1) f_y(x_n)$$
$$+ \sum_{i=2}^{n} \int_{\xi-s-1}^{\xi-s} dz\, Q^{i-1}(x,z) \int \mu_i(dx_i \dots dx_n) Q(z,x_i) f_y(x_n) \tag{8.8}$$

For $i < k \le n$ we define

$$\mu^0_{i,k}(dx_i \dots dx_k) := Q(x_i,x_{i+1}) \cdots Q(x_{k-1},x_k) \prod_{j=i}^{k} [\mathbf{1}_{1<x_j<\xi-s-1} dx_j] \tag{8.9}$$

$$\mu_{k+1,n}(dx_{k+1} \dots dx_n) := \mathbf{1}_{x_{k+1}\le 1} \mu_{k+1}(dx_{k+1} \dots dx_n) \tag{8.10}$$

We can then rewrite (8.8) as

$$Q^{n+1}(x,y) = \sum_{i=1}^{3} (\Gamma_i + \Gamma'_i) \tag{8.11}$$

where the Γ_i's refer to the first term and the Γ'_i's to the second term in (8.8):

$$\Gamma_1 := \mathbf{1}_{x<\xi-s} \int \mu^0_{1,n}(dx_1 \dots dx_n) Q(x,x_1) f_y(x_n) \tag{8.12}$$

$$\Gamma'_1 := \sum_{i=2}^{n} \int_{\xi-s-1}^{\xi-s} dz\, Q^{i-1}(x,z) \int \mu^0_{i,n}(dx_i \dots dx_n) Q(z,x_i) f_y(x_n) \tag{8.13}$$

To define the other Γ's let $N = N(y)$ be the first integer such that

$$r^N e^{\alpha y} \le 1 \tag{8.14}$$

and $b > 0$ (independent of y) such that

$$N \le b(y+1) \tag{8.15}$$

We then set (with the understanding that if $n < N$ then $n - N$ should be replaced by 0)

$$\Gamma_2 := \mathbf{1}_{x<\xi-s} \sum_{k=n-N}^{n} \int \mu^0_{1,k}(dx_1 \dots dx_k) \int \mu_{k+1,n}(dx_{k+1} \dots dx_n)$$
$$\times Q(x,x_1) Q(x_k,x_{k+1}) f_y(x_n) \tag{8.16}$$

$$\Gamma'_2 := \sum_{i=2}^{n} \sum_{k=n-N}^{n} \int_{\xi-s-1}^{\xi-s} dz\, Q^{i-1}(x,z) \int \mu^0_{i,k}(dx_i \dots dx_k)$$
$$\times \int \mu_{k+1,n}(dx_{k+1} \dots dx_n) Q(z,x_i) Q(x_k,x_{k+1}) f_y(x_n) \tag{8.17}$$

We are tacitly supposing in the above sum (and in the sequel) that $k > i$. Finally

$$\Gamma_3 := \mathbf{1}_{x<\xi-s} \sum_{k<n-N} \int \mu^0_{1,k}(dx_1 \dots dx_k)\mu_{k+1,n}(dx_{k+1} \dots dx_n) \\ \times \int Q(x, x_1)Q(x_k, x_{k+1})f_y(x_n) \tag{8.18}$$

$$\Gamma'_3 := \sum_{i=2}^{n} \sum_{k<n-N} \int_{\xi-s-1}^{\xi-s} dz\, Q^{i-1}(x, z) \int \mu^0_{i,k}(dx_i \dots dx_k) \\ \times \int \mu_{k+1,n}(dx_{k+1} \dots dx_n)Q(z, x_i)Q(x_k, x_{k+1})f_y(x_n) \tag{8.19}$$

Let $i < k$, $x_i \dots x_k$ all in $(1, \xi - s)$, $P(x, y)$ as in (2.7). Then, from (8.2) for $z \in (\xi - s - 1, \xi - s)$ we get

$$Q(z, x_i)Q(x_i, x_{i+1}) \cdots Q(x_k, x_{k+1}) = P(\xi - x_{k+1}, \xi - x_k) \cdots \\ \cdots P(\xi - x_i, \xi - z) \frac{\bar{m}'(\xi - x_{k+1})^2 p_{\bar{m}}(\xi - z)}{\bar{m}'(\xi - z)^2 p_{\bar{m}}(\xi - x_{k+1})} \tag{8.20}$$

By (8.6) $Q^{i-1}(x, z) \le c'$ so that we have from (8.13) using (8.20)

$$\Gamma'_1 \le \sum_{i=2}^{n} c' \int_{\xi-s-1}^{\xi-s} dz \int \prod_{j=i}^{n} [\mathbf{1}_{1<x_j<\xi-s-1} dx_j] P(\xi - x_n, \xi - x_{n-1}) \cdots \\ \cdots P(\xi - x_i, \xi - z) c e^{-2\alpha(\xi-y)} e^{2\alpha(s+1)} f_y(x_n) \tag{8.21}$$

having used (2.10) and that $|x_n - y| \le 1$ as in the complement $f_y(x_n)$ vanishes. c is a constant, determined by (2.10).

Let $u < \xi - s - 1$ and

$$\pi(u, k) := \int \prod_{j=1}^{k} [\mathbf{1}_{1<x_j<\xi-s-1} dx_j] \int_{\xi-s-1}^{\xi-s} dz\, P(\xi - x, \xi - x_k) \cdot \dots \cdot P(\xi - x_1, \xi - z) \tag{8.22}$$

the probability that the Markov chain with transition probability $P(\xi - u, \xi - w)$ on $\mathbb{R}$ starting from x is for the first time in $(\xi - s - 1, \xi - s)$ at time $k + 1$ without being ever before in $\{x \le 1\}$. Then

$$\Gamma'_1 \le [c' c e^{-2\alpha(\xi-y)} e^{2\alpha(s+1)}] \int_{1}^{\xi-s-1} dx_n \sum_{i=2}^{n} \pi(x_n, n - i) f_y(x_n)$$

Then since $|x_n - y| \le 1$ and $\sum_{i=2}^{n} \pi(x_n, n - i) \le 1$,

$$\Gamma'_1 \le [c' c e^{2\alpha(s+1)} 2c'] e^{-2\alpha(\xi-y)} \tag{8.23}$$

In an analogous way we prove that there is a constant $c > 0$ so that

$$\Gamma_1 \le \mathbf{1}_{x<\xi-s} \int \prod_{j=1}^{n} [\mathbf{1}_{1<x_j<\xi-s} dx_j] f_y(x_n) P(\xi - x_n, \xi - x_{n-1}) \cdots$$
$$\cdots P(\xi - x_2, \xi - x_1) c e^{-2\alpha(\xi-y)} e^{2\alpha(\xi-x)} \tag{8.24}$$

Similarly to (8.6) there is $c'' > 0$ so that for all x, y and $n \ge 0$, $P^n(\xi - x, \xi - y) \le c''$. Then

$$\Gamma_1 \le \mathbf{1}_{x<\xi-s} \int_{y-1}^{y+1} dx_n c' P^{n-1}(\xi - x_n, \xi - x) c e^{-2\alpha(x-y)}$$
$$\le \mathbf{1}_{x<\xi-s} 2cc'c'' e^{-2\alpha(x-y)} \tag{8.25}$$

To bound Γ_i and Γ'_i when $i > 1$ we recall (6.2) and write, using that $|x_n - \xi| > s$, $|x_i - \xi| \ge s + 1$, for all $i = k + 1, \dots, n - 1$, and using again (2.10) and Proposition 6.1,

$$\int \mu_{k+1,n}(dx_{k+1} \dots dx_n) f_y(x_n) = \int \mu_{k+1,n}(dx_{k+1} \dots dx_n) f_y(x_n) \gamma_{s^\star,0}(x_n) \Big(N_{s^\star} \bar{m}'(\xi - x_n)^{-1} \Big)^{-1}$$
$$\le c r^{n-k} \tag{8.26}$$

with c a suitable constant. We use (8.26) when $n - k > N$, while, for $n - k \le N$, we bound the left hand side of (8.26) by c'. Thus by (8.14)

$$\int \mu_{k,n}(dx_{k+1} \dots dx_n) f_y(x_n) \le \begin{cases} c' & \text{if } n - k \le N \\ c r^{n-k-N} & \text{if } n - k > N \end{cases} \tag{8.27}$$

We then have for a suitable constant $\hat{c}$,

$$\Gamma'_2 \le \sum_{i=2}^{n} \sum_{k=n-N}^{n} \int_{\xi-s-1}^{\xi-s} dz\, Q^{i-1}(x, z) \int \mu^0_{i,k}(dx_i \dots dx_k) Q(z, x_i) \hat{c} \mathbf{1}_{x_k \le 2} \tag{8.28}$$

which, for each k, is a term like Γ'_1 with $y \le 2$. Hence

$$\Gamma'_2 \le \hat{c} e^{-2\alpha\xi} N \tag{8.29}$$

with $\hat{c} > 0$ a suitable constant. Analogously

$$\Gamma_2 \le \mathbf{1}_{x<\xi-s} c e^{-2\alpha\xi} N \tag{8.30}$$

For Γ'_3 we use the second alternative in (8.27) and get instead of (8.28)

$$\Gamma'_3 \le \sum_{i=2}^{n} \sum_{k<n-N} \int_{\xi-s-1}^{\xi-s} dz\, Q^{i-1}(x, z) \int \mu^0_{i,k}(dx_i \dots dx_k) \mathbf{1}_{x_k \le 2} c r^{n-k-N} \tag{8.31}$$

The factor r^{n-k-N} allows to control the sum over k so that we can avoid the term N in (8.31) obtaining

$$\Gamma_3' \le ce^{-2\alpha\xi} \tag{8.32}$$

Analogously

$$\Gamma_3 \le \mathbf{1}_{x<\xi-s} ce^{-2\alpha\xi} \tag{8.33}$$

We get from (8.11) using (8.15) with the bounds on Γ_i and Γ_i' obtained so far that there is $c>0$ so that

$$Q^{n+1}(x,y) \le c\Big\{\mathbf{1}_{x<\xi-s}[e^{-2\alpha(x-y)} + e^{-2\alpha\xi}(y+1) + e^{-2\alpha\xi}] + [e^{-2\alpha(\xi-y)} + e^{-2\alpha\xi}(y+1) + e^{-2\alpha\xi}]\Big\} \tag{8.34}$$

Hence

$$Q^{n+1}(x,y) \le c\Big\{\mathbf{1}_{x<\xi-s} e^{-2\alpha(x-y)} + e^{-2\alpha\xi}(y+1)\Big\} \tag{8.35}$$

thus proving (8.4) for $y \le \xi - s - 1$.

The same proof works for $y \ge \xi+s+1$ with ℓ playing the role of the origin. For ease of reference we state the final bound

$$Q^{n+1}(x,y) \le c\Big\{\mathbf{1}_{x>\xi+s} e^{-2\alpha(y-x)} + e^{-2\alpha(\ell-\xi)}(\ell-y+1)\Big\} \tag{8.36}$$

When $|y-\xi| \le s$ we use (8.6) and obtain (8.4) if C is suitably large. Theorem 8.1 is proved. □

8.2 Definition.

Given $\eta \in (0, 2\alpha)$ and $r<1$, let q be the first integer such that

$$r^q e^{\eta} \le 1 \tag{8.37}$$

and for x, y in $\mathbb{R}_+$, $n \in \mathbb{N}$,

$$\tau_{x,y,n} := \beta \mathbf{1}_{\beta\ge 0}, \qquad \beta := [n - \big(|\xi - x| + |\xi - y|\big)q] \tag{8.38}$$

8.3 Proposition.

Let η and r be as in Definition 8.2. Then there is $c>0$ so that for any $x \ne x'$, y and n such that $\tau_{x,y,n} > 0$,

$$\Big|Q^n_{\xi,\ell}(x,y) - Q^n_{\xi,\ell}(x',y)\Big| \le cr^{\tau_{x,x',n}} e^{-\eta|\xi-y|}, \qquad \textit{if } \tau_{x,x',n} > 0 \tag{8.39}$$

$$\Big|Q^n_{\xi,\ell}(x,y) - \chi_{\xi,\ell}(y)\Big| \le cr^{\tau_{x,\xi,n}} e^{-\eta|\xi-y|}, \qquad \textit{if } \tau_{x,\xi,n} > 0 \tag{8.40}$$

Proof.

Renaming $n \to n+1$ and recalling (8.5), the left hand side of (8.39) is bounded by

$$A := \int dz \left| Q^n_{\xi,\ell}(x,z) - Q^n_{\xi,\ell}(x',z) \right| f_y(z) \tag{8.41}$$

By using first (7.18) and then (8.37), having supposed that $\tau \equiv \tau_{x,x',n} > 0$,

$$\begin{aligned} A &\le e^{-\eta(|\xi-y|-1)} c c' r^n \left(e^{\eta|x-\xi|} + e^{\eta|x'-\xi|} \right) \\ &\le c'(ce^\eta) e^{-\eta|\xi-y|} r^\tau r^{q(|\xi-x|+|\xi-x'|)} \left(e^{\eta|x-\xi|} + e^{\eta|x'-\xi|} \right) \\ &\le 2c'(ce^\eta) e^{-\eta|\xi-y|} r^\tau \end{aligned}$$

In an analogous way we prove (8.40) starting from (7.19).

Proposition 8.3 is proved. □

8.4 Proposition.

There is $C > 0$ so that for all $y \in [0,\ell]$

$$\chi_{\xi,\ell}(y) \le C e^{-2\alpha|\xi-y|} \tag{8.42}$$

Proof.

By (8.40) for each y

$$\lim_{n\to+\infty} Q^n_{\xi,\ell}(\xi,y) = \chi_{\xi,\ell}(y) \tag{8.43}$$

(8.42) then follows from (8.4). Proposition 8.4 is proved. □

8.5 Proposition.

Let either $x > y > \xi$ or $x < y < \xi$. Then

$$Q^n_{\xi,\ell}(x,y) \le \begin{cases} c' & \text{if } \tau_{x,\xi,n} = 0 \\ c r^{\tau_{x,\xi,n}} e^{-\eta|\xi-y|} + c e^{-2\alpha|\xi-y|} & \text{otherwise} \end{cases} \tag{8.44}$$

with c' as in (8.6), c as in (8.39) and r as in Definition 8.2.

Proof.

(8.44) is (8.6) when $\tau_{x,\xi,n} = 0$. For $\tau_{x,\xi,n} > 0$ we write

$$Q^n_{\xi,\ell}(x,y) \le \left| Q^n_{\xi,\ell}(x,y) - Q^n_{\xi,\ell}(\xi,y) \right| + Q^n_{\xi,\ell}(\xi,y)$$

(8.44) is then a corollary of Theorem 8.1 and (8.39).

Proposition 8.5 is proved. □

The following theorem provides an approximation of $\chi_{\xi,\ell}$ with χ^0_ξ, see (8.1).

8.6 Theorem.

For any $\delta \in (0,1]$ there is $c > 0$ so that for all $y \in [0,\ell]$

$$\Big|\chi_{\xi,\ell}(y) - \chi^0_\xi(y)\Big| \le c\Big(\big[e^{-2\alpha\xi} e^{-2\alpha|\xi-y|}\xi^{1+\delta} + e^{-2\alpha\xi}(y+1)\big]\mathbf{1}_{y\le\xi} + \big[e^{-2\alpha(\ell-\xi)} e^{-2\alpha|\xi-y|}(\ell-\xi)^{1+\delta} + e^{-2\alpha(\ell-\xi)}(\ell-y+1)\big]\mathbf{1}_{y>\xi}\Big) \tag{8.45}$$

Proof.

Let first $y < \xi$. We write Q for $Q_{\xi,\ell}$ and let n be the smallest integer such that

$$n \ge \xi^{1+\delta} \tag{8.46}$$

By (8.40) there is $c' > 0$ so that

$$\Big|Q^n(\xi,y) - \chi_{\xi,\ell}(y)\Big| \le c' r^n e^{-\eta|y-\xi|} \tag{8.47}$$

whose right hand side is bounded by that in (8.45) for a suitable choice of c.

Let

$$T(x,y) := Q(x,y)\mathbf{1}_{1\le x\le \ell-1}\mathbf{1}_{1\le y\le\ell-1} \tag{8.48}$$

We first prove that for $z \in [\xi-s, \xi-s+1]$

$$\Big|T^k(\xi,z) - Q^k(\xi,z)\Big| \le cke^{-2\alpha\xi} \tag{8.49}$$

In fact the difference on the left hand side is bounded by the probability that at some time $j < k$ the chain with transition probability Q is either in $[0,1]$ or $[\ell-1,\ell]$. This probability is bounded using the argument of proof of Theorem 8.1 and recalling that $\ell > 2\xi$. (8.49) is proved.

Renaming $n \to n+1$ we use (8.8) to rewrite $Q^{n+1}(\xi,y)$. When $x = \xi$ the first term in (8.8) is equal to 0. Thus

$$\begin{aligned}\Big|Q^{n+1}(\xi,y) - \sum_{i=2}^{n}\int_{\xi-s}^{\xi-s+1} dz\, T^{i-1}(\xi,z)\int \mu_i(dx_i\dots dx_n)Q(z,x_i)f_y(x_n)\Big| \\ \le ce^{-2\alpha\xi}\sum_{i=2}^{n} i\int_{\xi-s}^{\xi-s+1} dz \int \mu_i(dx_i\dots dx_n)Q(z,x_i)f_y(x_n) \\ \le ce^{-2\alpha\xi} nCe^{-2\alpha|\xi-y|}\end{aligned} \tag{8.50}$$

The first inequality follows from (8.49). The second one is obtained bounding $i \le n$ and observing that the resulting sum has already been analysed in the proof of Theorem 8.1 (estimates of terms like Γ'_1, Γ'_2, Γ'_3).

The leading term in the sum on the left hand side of (8.50) is obtained by replacing μ_i by $\mu^0_{i,n}$ that has the same structure as the term Γ'_1, see (8.13). The remaining terms are again like Γ'_2 and Γ'_3 and are bounded by the second and the third terms in the second square bracket in (8.34). We then have for a suitable $\hat{c}$,

$$\Big| Q^{n+1}(\xi, y) - \sum_{i=2}^{n} \int_{\xi-s}^{\xi-s+1} dz\, T^{i-1}(\xi, z) \int \mu^0_{i,n}(dx_i \dots dx_n) Q(z, x_i) f_y(x_n) \Big| \le \hat{c}\Big[Cne^{-2\alpha\xi} e^{-2\alpha|\xi-y|} + e^{-2\alpha\xi}(y+1) \Big] \tag{8.51}$$

The right hand side of (8.51) is again bounded by that in (8.45). We omit the proof of the analogous bound for $y > \xi$.

Since Theorem 8.1 and all the subsequences results hold as well for the chain $P(\xi - x, \xi - y)$ on $\mathbb{R}$ with invariant density χ^0_ξ, we get the same expressions (8.51) and (8.47) with Q replaced by P and $\chi_{\xi,\ell}$ by χ^0_ξ. We finally observe that the second term on the left hand side of (8.51) is the same both for Q and P, by the definitions of $\mu^0_{i,n}$ and T, and we obtain (8.45). Theorem 8.6 is proved. $\square$

Observe that by letting ξ and ℓ to $+\infty$ (with $\ell > 2\xi$) we get, for any fixed x and y in $\mathbb{R}_+$,

$$\lim_{\ell,\xi\to+\infty} Q_{\xi,\ell}(x, y) =: Q_\infty(x, y)$$

where

$$Q_\infty(x, y) = p_\infty \Big\{ J(x, y) e^{\alpha(y-x)} + J(x, -y) e^{\alpha(-y-x)} \Big\}, \qquad p_\infty = \beta(1 - m_\beta^2) \tag{8.52}$$

By (2.10) there are $c > 0$ and $\alpha_0 > \alpha$ so that for $x < \xi$

$$Q_\infty(x, y)\Big(1 - ce^{-(\alpha_0-\alpha)(\xi-x)}\Big) \le Q_{\xi,\ell}(x, y) \le Q_\infty(x, y)\Big(1 + ce^{-(\alpha_0-\alpha)(\xi-x)}\Big) \tag{8.53}$$

$Q_\infty(x, y)$ is a transition probability on $\mathbb{R}_+$ and we denote by $\mathbb{P}^{(\infty)}_x$, $x \in \mathbb{R}_+$, the law of the Markov chain $\{x_i\}_{i\ge0}$ with transition probability $Q_\infty(x, y)$ that starts from $x_0 = x$. Let

$$\gamma(x) := \mathbb{P}^{(\infty)}_x\Big(\{x_i > 1,\ i \ge 1\}\Big), \qquad x > 0 \tag{8.54}$$

$$\kappa(x) := \frac{1}{2} \sum_{n=0}^{+\infty} \int_0^1 dy\, [C_{\bar{m}} \frac{e^{2\alpha y}(a\alpha)^2}{p_\infty}] \gamma(y) Q^i_\infty(y, x) \tag{8.55}$$

8.7 Theorem.

Both $\gamma(x)$ and $\kappa(x)$ are strictly positive and finite. Moreover for any $X>0$ there are $\delta>0$ and $c>0$ so that for all $x\in(0,X)$ and all $\ell>2\xi$, $\xi>8X$,

$$\Big|\chi_{\xi,\ell}(x)-e^{-2\alpha\xi}\kappa(x)\Big|\le ce^{-(2\alpha+\delta)\xi} \tag{8.56}$$

Proof.

Proof that $\gamma(x)>0$. Let $\eta\in(0,2\alpha)$, $x\ge 1$, then

$$\begin{aligned}\int_{\mathbb{R}_+} dy\, Q_\infty(x,y)e^{-\eta y} &= p_\infty\int_{\mathbb{R}} dy\, J(x,y)e^{(\alpha-\eta)y}e^{-\alpha x}\\ &=\frac{p_\infty}{p_{\alpha-\eta}}e^{-\eta x}=: re^{-\eta x}\end{aligned} \tag{8.57}$$

where p_ζ is such that $\alpha_{p_\zeta}=\zeta$ and $r<1$ because $|\alpha-\eta|<\alpha$ and if $|\zeta|<\alpha$, $p_\zeta>p_\alpha=p_\infty$.

Analogously if $x\in(0,1)$

$$\begin{aligned}\int_{\mathbb{R}_+} dy\, Q_\infty(x,y)e^{-\eta y} &= p_\infty\int_0^2 dy\, J(x,y)e^{(\alpha-\eta)y}e^{-\alpha x}\\ &+p_\infty\int_{-1}^0 dy\, J(x,y)e^{(\alpha+\eta)y}e^{-\alpha x}\;\le p_\infty\int_{\mathbb{R}} dy\, J(x,y)e^{(\alpha-\eta)y}e^{-\alpha x}\end{aligned} \tag{8.58}$$

having bounded the second integral by changing $\eta\to-\eta$. Thus the bound (8.57) holds for all $x\ge 0$. Then for $z\ge s$ and $X>0$

$$\mathbb{P}_z^{(\infty)}\big(\{x_i\le X\}\big)\le\int_0^X dy\, Q^i_\infty(z,y)e^{-\eta y}e^{\eta X}\le r^ie^{\eta X}e^{-\eta s} \tag{8.59}$$

For any $s>0$ and $x\in(0,1)$ there are k and $\epsilon>0$ so that

$$\mathbb{P}_x^{(\infty)}\Big(\{x_i>1,\ i=1,..,k\}\cap\{x_k>s\}\Big)\ge\epsilon \tag{8.60}$$

Then

$$\begin{aligned}\gamma(x)&\ge\mathbb{P}_x^{(\infty)}\Big(\{x_i>1,i=1,..,k\}\cap\{x_k>s\}\cap\{x_n>1,n>k\}\Big)\\ &\ge\epsilon\Big(1-\sup_{z\ge s}\mathbb{P}_z^{(\infty)}\Big(\cup_{n\le 1}\{x_n>1\}\Big)\end{aligned}$$

and, by (8.59),

$$\gamma(x)\ge\epsilon\Big(1-e^{-\eta(s-1)}\frac{1}{1-r}\Big) \tag{8.61}$$

We choose s so large that the right hand side is positive, thus completing the proof that $\gamma(x)>0$. By an analogous argument we are going to show that $\kappa(x)$, $x\le X$, is bounded. Since $Q_\infty(\cdot,\cdot)$ is bounded, we call c its sup, for $z\ge s$ and $x\le X$:

$$Q^n_\infty(z,x)=\int_0^{X+1} dy Q^{n-1}_\infty(z,y)c\le cr^{n-1}e^{\eta(X+1-s)}$$

Then, calling $n_i=ik$, we have for $y\in(0,1)$m

$$Q^{n_i}_\infty(y,x)\le\sum_{j<i}(1-\epsilon)^j\sup_{z\ge s}Q^{n_i-n_j}_\infty(z,x)\le\sum_{j<i}(1-\epsilon)^j r^{n_i-n_j-1}e^{\eta(X+1-s)}$$

This shows that the sum in (8.55) converges exponentially and $\kappa(x)$ is therefore finite; since it is obviously strictly positive, we are only left with the proof of (8.56).

Proof of (8.56). Let

$$Q^\star_{\xi,\ell}(x,y):=Q_{\xi,\ell}(y,x),\quad Q^\star_\infty(x,y):=Q_\infty(y,x),\qquad \chi\equiv\chi_{\xi,\ell} \tag{8.62}$$

$s_0:=\xi/2$ and $x\equiv x_0<s_0$. We have

$$\chi(x)=\sum_{n\ge0}\int_0^{s_0}dx_1..\int_0^{s_0}dx_n\int_{s_0}^{s_0+1}dx_{n+1}\{\prod_{j=0}^n Q^\star_{\xi,\ell}(x_j,x_{j+1})\}\chi(x_{n+1}) \tag{8.63}$$

Recalling that $\chi^0_\xi(x)=\tilde m'(\xi-x)^2p_{\bar m}(\xi-x)^{-1}$ and (2.15), using (2.10) we have for any $x>0$

$$\lim_{\xi\to+\infty}e^{2\alpha\xi}\chi^0_\xi(x)=:\lambda(x)=C_{\bar m}(a\alpha)^2\frac{e^{2\alpha x}}{p_\infty} \tag{8.64}$$

With this in mind, by analogy with (8.63), we define

$$\rho(x):=e^{-2\alpha\xi}\sum_{n\ge0}\int_0^{s_0}dx_1..\int_0^{s_0}dx_n\int_{s_0}^{s_0+1}dx_{n+1}\{\prod_{j=0}^n Q^\star_\infty(x_j,x_{j+1})\}\lambda(x_{n+1}) \tag{8.65}$$

We will show, see (8.71) below, that $\rho(x)$ and $\chi(x)$ are close to each other.

By (8.45) and (2.10) there are $c>0$ and $\alpha_0>\alpha$ so that for any $x\in[s_0,s_0+1]$

$$\begin{aligned}\big|\chi(x)-e^{-2\alpha\xi}\lambda(x)\big|&\le\big|\chi(x)-\chi^0(x)\big|+\big|\chi^0(x)-e^{-2\alpha\xi}\lambda(x)\big|\\&\le c\Big\{e^{-2\alpha\xi}(x+1)+e^{-(\alpha_0+\alpha)(\xi-x)}\Big\}\le c'e^{-(2\alpha+\delta)\xi}\lambda(x)\end{aligned} \tag{8.66}$$

where δ and c are positive constants, we take $\delta<(\alpha_0-\alpha)\xi/2$. By using (8.53) and (8.66) we get

$$\chi(x)\le e^{-2\alpha\xi}\sum_{n\ge1}\big(1+ce^{-\delta\xi/2}\big)^{n+1}\int_0^{s_0}dx_1..\int_0^{s_0}dx_n\int_{s_0}^{s_0+1}dx_{n+1}\{\prod_{j=0}^n Q^\star_\infty(x_j,x_{j+1})\}\lambda(x_{n+1}) \tag{8.67}$$

The lower bounds holds as well with $c \to -c$. Letting $x \le X$ and recalling that $Q_\infty(\cdot,\cdot)$ is bounded,

$$\int_0^{s_0} dx_1 .. \int_0^{s_0} dx_n \int_{s_0}^{s_0+1} dx_{n+1} \{\prod_{j=0}^{n} Q^\star_\infty(x_j, x_{j+1})\} \lambda(x_{n+1}) \le c e^{2\alpha s_0} \sup_{z\in[s_0,s_0+1]} \mathbb{P}_z^{(\infty)}\Big(x_n \le X+1\Big)$$
$$\le c' e^{2\alpha s_0} e^{-\eta s} e^{\eta X} r^n \tag{8.68}$$

having used (8.59); c and c' are constants, $\eta \in (0, 2\alpha)$ and $r < 1$. For ξ large enough there exists $b > 0$ such that

$$\sum_{n \ge b\xi} c' e^{2\alpha s_0} e^{-\eta s} e^{\eta X} \big(1 + c e^{-\delta\xi/2}\big)^{n+1} r^n \le e^{-2\alpha\xi} \tag{8.69}$$

On the other hand for $n < b\xi$

$$\big(1 + c e^{-\delta\xi/2}\big)^{n+1} - 1 \le 2c e^{-\delta\xi/2} b\xi \tag{8.70}$$

so that there is a new constant c such that

$$\big|\chi(x) - \rho(x)\big| \le c\big(e^{-\delta\xi/2}\xi\rho(x) + e^{-4\alpha\xi}\big) \tag{8.71}$$

We will complete the proof of (8.56) by relating $\rho(x)$ to $\kappa(x)$. Letting $x \equiv x_0$,

$$\rho(x) = e^{-2\alpha\xi} \sum_{i\ge0}\sum_{n\ge0} \int_0^{s_0} dx_1 .. \int_0^{s_0} dx_{i+n} \int_{s_0}^{s_0+1} dy \mathbf{1}_{x_i<1} \prod_{j=i+1}^{n} \mathbf{1}_{x_j \ge 1}$$
$$\times \{\prod_{j=0}^{n+i} Q^\star_\infty(x_j, x_{j+1})\} Q^\star_\infty(x_{i+n}, y)\lambda(y) \tag{8.72}$$

Since $x_j \ge 1$ for $j > i$:

$$Q^\star_\infty(x_i, x_{i+1}) \cdots Q^\star_\infty(x_{i+n}, y)\lambda(y) = \lambda(x_i) Q_\infty(x_i, x_{i+1}) \cdots Q_\infty(x_{i+n}, y) \tag{8.73}$$

and for $x_i < 1$ using (8.59) we get

$$\sum_{n\ge1} \int_1^{s_0} dx_{i+1} .. \int_1^{s_0} dx_{i+n} \int_{s_0}^{s_0+1} dy \{\prod_{j=i}^{n+i} Q^\star_\infty(x_j, x_{j+1})\} Q^\star_\infty(x_{i+n}, y)\lambda(y) - \gamma(x_i)\lambda(x_i) = 0 \tag{8.74}$$

The above identity (8.73) can be interpreted as the reversibility condition for the Markov chain Q_∞ with respect to the weigth $\lambda(x)$

We write

$$\sum_{i>0} \int_0^{s_0} dx_1 .. \int_0^{s_0} dx_{i-1} \prod_{j=0}^{i-1} Q^\star_\infty(x_j, x_{j+1}) = \{ \sum_{i<\xi/4} + \sum_{i\ge\xi/4} \} \int_0^{s_0} dx_1 .. \int_0^{s_0} dx_{i-1} \prod_{j=0}^{i-1} Q^\star_\infty(x_j, x_{j+1})$$

In the first sum we can extend the integral to the whole $\mathbb{R}_+$ because, at each step, the chain moves by one, at most. Thus

$$\Big|\sum_{i>0}\int_0^{s_0} dx_1..\int_0^{s_0} dx_{i-1}\prod_{j=0}^{i-1} Q^\star_\infty(x_j,x_{j+1})-\sum_{i>0} Q^i_\infty(x_i,x)\Big| \le 2\sum_{n\ge \xi/4} \mathbb{P}^{(\infty)}_{x_i}\Big(x_{n-1}\le X+1\Big) \quad (8.75)$$

The last probability is then bounded using (8.59) and by (8.74) and (8.75) we conclude that, given X, there are $c>0$ and $\delta>0$ so that $|\rho(x)-e^{-2\alpha\xi}\kappa(x)|\le ce^{-(2\alpha+\delta)\xi}$ for all $x\le X$. Together with (8.71) this proves (8.56). Theorem 8.7 is proved. $\square$

By an analogous argument with the interval between ℓ and ξ playing the role of the previous one, between 0 and ξ, we have for $x\in[0,X]$

$$\Big|\chi_{\xi,\ell}(\ell-x)-e^{-2\alpha(\ell-\xi)}\kappa(x)\Big|\le ce^{-(2\alpha+\delta)(\ell-\xi)} \quad (8.76)$$

9. Eigenvalues and eigenvectors: sharp estimates

In this section we will prove Theorem 2.4 and several other properties of $u_{m,\ell}$ and $v_{m,\ell}$. We suppose $m\in G_{(c,\xi,\ell)}$, see Definition 2.3 with $\ell>2\xi$ and ξ large enough and recall that in Section 5 we have proved (2.18) and the first inequality in (2.17). At some point we will also suppose that m satisfies (2.14). To simplify notation we write λ, u and v for the maximal eigenvalue and eigenvectors and use other shorthand notation that will be specified as they occur. We proceed as in Section 5, writing identities which are consequence of the eigenvalue equation and manipulating them to evidentiate the terms that will be then recognized as the leading ones.

The first identity

Let $n=[\xi^4]$ (the integer part of ξ^4) and denote by $\tilde m'_\xi(x):=\tilde m'(\xi-x)$, $\tilde m'$ being proportional to $\bar m'$ and satisfying the normalization condition (2.15). We will in this first subsection prove the following identity valid for all $x\in[0,\ell]$:

$$\lambda^n v(x)=\tilde m'_\xi(x)\Big\{\langle v\tilde m'_\xi\rangle \mathbf{1}_{x\le 4\xi}+e^{-2\alpha\xi}n\lambda^n B(\xi)c_\lambda\mathbf{1}_{x\le 4\xi}+U(x,\xi,\lambda)\Big\} \quad (9.1)$$

Before explaining the notation we should keep in mind that the first term on the right hand side is the leading one, the second one represents its main correction and the last one is a negligible remainder term.

For any function $f \in C([0,\ell])$ we are writing

$$\langle f \rangle := \int_0^\ell dx \frac{f(x)}{p_{\bar{m}}(\xi - x)} \tag{9.2}$$

The condition $x \le 4\xi$ in (9.1) actually means x not larger than the min between 4ξ and ℓ, analogous meaning is tacitly given to other analogous conditions. The first term on the right hand side of (9.1) is thus completely defined.

The coefficient c_λ is:

$$c_\lambda := \frac{1}{n} \sum_{i=[\xi^2]}^{n} \lambda^{-i} \le [1 - c_- e^{-2\alpha\xi}]^{-n} \le 2 \tag{9.3}$$

by the first inequality in (2.17) and supposing ξ large enough.

The parameter $B(\xi)$ has fundamental importance in the computation of λ. Its expression is

$$B(\xi) := \sum_{j=1}^{3} B_j(\xi), \qquad B_j(\xi) := e^{2\alpha\xi} \int_0^\ell dx\, \chi_{\xi,\ell}(x) f_j(x) \tag{9.4}$$

with $\chi_{\xi,\ell}(x)$ the invariant density of the Markov chain on $[0,\ell]$ with transition probability $Q_{\xi,\ell}(x,y)$, its existence is proved in Theorem 6.2, other of its properties are established in Section 8. The functions $f_j(x)$ are defined in (9.11), (9.12) and (9.13) below. We will see that the $B_j(\xi)$ and λ^n are bounded functions of ξ so that the second term on the right hand side of (9.1) will be of the order of $\xi^4 e^{-2\alpha\xi}$, recall $n = [\xi^4]$. The term $U(x,\xi,\lambda)$ has a very complicated expression, we will prove that it goes like $\xi^3 e^{-2\alpha\xi}$ and it will thus be negligible.

Proof of (9.1)

We start from (5.6) that we rewrite as

$$\lambda^n \frac{v(x)}{\tilde{m}'_\xi(x)} = \tilde{\mathbb{E}}_x\Big(\prod_{i=0}^{n-1} [g(x_i)+1] \psi(x_n) \frac{v(R_\ell(x_n))}{\tilde{m}'_\xi\big(R_\ell(x_n)\big)} \Big) \tag{9.5}$$

where $x = x_0 \in [0,\ell]$, $R_\ell(x_0) = x_0$; the expectation refers to the Markov chain on $[-1,\ell+1]$ with transition probability (5.3),

$$g(x) := \psi(x)\phi\big(R_\ell(x)\big) - 1 \tag{9.6}$$

where $\phi(x) \equiv \phi(x,\xi,m)$ is defined in (5.1) and $\psi(x) \equiv \psi(x,\xi)$ in (5.2).

By expanding the product of the $[g(x_i)+1]$'s we get

$$\begin{aligned} \lambda^n \frac{v(x)}{\tilde{m}'_\xi(x)} =& \tilde{\mathbb{E}}_x\Big(\psi(x_n) \frac{v(R_\ell(x_n))}{\tilde{m}'_\xi\big(R_\ell(x_n)\big)} \Big) \\ &+ \sum_{i=0}^{n-1} \tilde{\mathbb{E}}_x\Big(g(x_i) \prod_{j=i+1}^{n-1} [g(x_j)+1] \psi(x_n) \frac{v(R_\ell(x_n))}{\tilde{m}'_\xi\big(R_\ell(x_n)\big)} \Big) \end{aligned} \tag{9.7}$$

By conditioning on x_{i+1} and using (9.5) we get

$$\lambda^n \frac{v(x)}{\tilde m'_\xi(x)} = \tilde{\mathbb{E}}_x\Big(\psi(x_n)\frac{v(R_\ell(x_n))}{\tilde m'_\xi\big(R_\ell(x_n)\big)}\Big) + \sum_{i=0}^{n-1} \lambda^{n-(i+1)}\tilde{\mathbb{E}}_x\Big(g(x_i)\frac{v(R_\ell(x_{i+1}))}{\tilde m'_\xi\big(R_\ell(x_{i+1})\big)}\Big) \tag{9.8}$$

Since $\psi(x_0) = 1$ (because $x_0 = x \in [0,\ell]$), the term with $i = 0$ in the last sum is equal to

$$\lambda^{n-1}[\phi(x) - 1]\int dy\, \tilde P(x,y)\frac{v(R_\ell(y))}{\tilde m'_\xi\big(R_\ell(y)\big)} \tag{9.9}$$

We write

$$g = g_1 + g_2 + g_3, \quad g_1 := \phi - 1, \;\; g_2 := \psi - 1, \;\; g_3 := (\phi-1)(\psi-1) \tag{9.10}$$

$$f = f_1 + f_2 + f_3 \tag{9.11}$$

$$f_j(x) = f_j\big(R_\ell(x)\big) := \int\int dy\, dz\, \tilde P\big(x,y\big)\tilde P\big(y,z\big)g_j(y)\frac{v\big(R_\ell(z)\big)}{\bar m'_\xi\big(R_\ell(z)\big)} \tag{9.12}$$

$$\tilde f_2(x) = \tilde f_2\big(R_\ell(x)\big) := \int dy\, \tilde P\big(x,y\big)g_2(y)\frac{v\big(R_\ell(y)\big)}{\bar m'_\xi\big(R_\ell(y)\big)} \tag{9.13}$$

Then

$$\begin{aligned}\lambda^n \frac{v(x)}{\tilde m'_\xi(x)} =& \tilde{\mathbb{E}}_x\Big(\frac{v(x_n)}{\tilde m'_\xi\big(R_\ell(x_n)\big)}\Big) + \tilde{\mathbb{E}}_x\big(\tilde f_2(x_{n-1})\big) + \sum_{i=1}^{n-1}\lambda^{n-(i+1)}\tilde{\mathbb{E}}_x\Big(f\big(R_\ell(x_{i-1})\big)\Big)\\ &+ \lambda^{n-1}[\phi(x)-1]\int dy \tilde P(x,y)\frac{v(y)}{\tilde m'_\xi\big(R_\ell(y)\big)}\end{aligned} \tag{9.14}$$

Since for the first three terms on the right hand side, we have only expectations of functions of $R_\ell(x)$, they are equal to the expectations relative to the Markov process on $[0,\ell]$ with transition probability $Q_{\xi,\ell}(x,y)$, denoted by $\mathbb{E}_x$, $x \in [0,\ell]$. We thus have for $x \in [0,\ell]$

$$\begin{aligned}\lambda^n \frac{v(x)}{\tilde m'_\xi(x)} &= \mathbb{E}_x\Big(\frac{v\big(R_\ell(x_n)\big)}{\tilde m'_\xi(x_n)}\Big) + \mathbb{E}_x\big(\tilde f_2(x_{n-1})\big) + \sum_{i=2}^{n}\lambda^{n-i}\mathbb{E}_x\big(f(x_{i-2})\big)\\ &+ \lambda^{n-1}[\phi(x)-1]\int_{-1}^{\ell+1} dy\, \tilde P(x,y)\frac{v(R_\ell(y))}{\tilde m'_\xi\big(R_\ell(y)\big)}\Big)\end{aligned} \tag{9.15}$$

U is defined so that (9.15) is equal to (9.1). We next write $U(\cdot)$ to make explicit the leading dependence on ξ, the coefficients a_i and b_i below still depend on ξ (and on $x \in (0,\ell)$) but they are bounded.

$$\begin{aligned}U(x,\xi,\lambda) = e^{-2\alpha\xi}\Big\{&\xi^3 a_0(x,\xi) + e^{2\alpha\xi}e^{(\alpha-\alpha')|\xi-x|}\mathbf{1}_{x>4\xi}a_1(x,\xi)\\ &+ n\lambda^n\big[e^{-2\alpha\xi}b_1(x,\xi,\lambda) + e^{(\alpha+(\alpha-\alpha'))|\xi-x|}b_2(x,\xi,\lambda)\mathbf{1}_{x\ge 4\xi}\big]\\ &+ \lambda^n\xi^2 e^{(\alpha+(\alpha-\alpha'))|\xi-x|}b_3(x,\xi,\lambda) + \lambda^n\xi^2\big[e^{2\alpha|\xi-x|}\mathbf{1}_{x\le 4\xi} + e^{\alpha|\xi-x|}\mathbf{1}_{x>4\xi}\big]b_4(x,\xi,\lambda)\\ &+ b_5(x,\xi) + \lambda^{n-1}e^{(\alpha+(\alpha-\alpha'))|\xi-x|}b_6(x,\xi)\Big\}\end{aligned} \tag{9.16}$$

There is $c>0$ so that

$$|a_0(x,\xi)|+\cdots+|b_6(x,\xi,\lambda)|\le c \tag{9.17}$$

The explicit expression of the coefficients a_i and b_i is:

$$a_0(x,\xi):=e^{2\alpha\xi}\xi^{-3}\mathbf{1}_{x\le 4\xi}\Big\{\mathbb{E}_x\Big(\frac{v(R_\ell(x_n))}{\tilde m'_\xi(x_n)}\Big)-\langle v\tilde m'_\xi\rangle\Big\}$$

$$a_1(x,\xi):=\mathbf{1}_{x>4\xi}e^{-(\alpha-\alpha')|\xi-x|}\mathbb{E}_x\Big(\frac{v(R_\ell(x_n))}{\tilde m'_\xi(x_n)}\Big)$$

$$b_1(x,\xi,\lambda):=\mathbf{1}_{0\le x\le 4\xi}e^{4\alpha\xi}\frac{1}{n}\sum_{i=[\xi^2]}^{n}\lambda^{-i}\Big\{\mathbb{E}_x\big(f(x_{i-2})\big)-e^{-2\alpha\xi}B(\xi)\Big\}$$

$$b_2(x,\xi,\lambda):=\mathbf{1}_{4\xi\le x}e^{2\alpha\xi}e^{-(\alpha+(\alpha-\alpha'))|\xi-x|}\frac{1}{n}\sum_{i=[\xi^2]}^{n}\lambda^{-i}\mathbb{E}_x\big(f(x_{i-2})\big)$$

$$b_3(x,\xi,\lambda):=e^{2\alpha\xi}e^{-(\alpha+(\alpha-\alpha'))|\xi-x|}\xi^{-2}\sum_{i=2}^{[\xi^2]-1}\lambda^{-i}\sum_{j=1,3}\mathbb{E}_x\big(f_j(x_{i-2})\big)$$

$$b_4(x,\xi,\lambda):=e^{2\alpha\xi}[e^{-2\alpha|\xi-x|}\mathbf{1}_{x\le 4\xi}+e^{-\alpha|\xi-x|}\mathbf{1}_{x>4\xi}]\xi^{-2}\sum_{i=2}^{[\xi^2]-1}\lambda^{-i}\mathbb{E}_x\big(f_2(x_{i-2})\big)$$

$$b_5(x,\xi):=e^{2\alpha\xi}\mathbb{E}_x\big(\tilde f_2(x_{n-1})\big)$$

$$b_6(x,\xi):=e^{2\alpha\xi}e^{-(\alpha+(\alpha-\alpha'))|\xi-x|}[\phi(x)-1]\int dy\,\tilde P(x,y)\frac{v\big(R_\ell(y)\big)}{\tilde m'_\xi\big(R_\ell(y)\big)}$$

Proof that $|B(\xi)|$ is bounded.

There is $C>0$ so that, using the notation (5.10),

$$\big|\phi(x)-1\big|=\Big|\frac{p_m(x)-p_{\bar m}(\xi-x)}{p_{\bar m}(\xi-x)}\Big|\le C\big|\delta_\xi m\big|\le Cc2e^{-2\alpha\xi}e^{\alpha|\xi-x|} \tag{9.18}$$

The second inequality is proved in (5.11). $\psi(x)$ is bounded, as seen using (2.10), and the ratio $v/\tilde m'$ can be bounded using (2.18) and (2.10). Then, for $j=1,3$,

$$|f_j(x)|\le ce^{-2\alpha\xi}e^{(\alpha+(\alpha-\alpha'))|\xi-x|} \tag{9.19}$$

and by (8.42) and for ξ large enough (so that $\alpha-\alpha'$ is small) we conclude that $B_1(\xi)$ and $B_3(\xi)$ are bounded. Since $\psi(x)=1$ for $1\le x\le \ell-1$, see (5.4), $f_2(x)$ is supported by the two intervals $[0,1]$ and $[\ell-1,\ell]$. By (8.42) and (2.18)-(2.10)

$$\int_0^1 dy\,\chi_{\xi,\ell}(y)|f_2(y)|\le ce^{-2\alpha\xi}e^{(\alpha-\alpha')\xi}\le c'e^{-2\alpha\xi}$$

Analogously

$$\int_{\ell-1}^{\ell} dy\, \chi_{\xi,\ell}(y)|f_2(y)| \le ce^{-2\alpha(\ell-\xi)}e^{(\alpha-\alpha')(\ell-\xi)} \le c'e^{-(2\alpha-c'e^{-2\alpha\xi})(\ell-\xi)}$$

The last term is bounded by $c''e^{-2\alpha\xi}$ uniformly in $\ell > 2\xi$. With this we conclude that also $|B_2(\xi)|$ is bounded. $\square$

Proof that a_i and b_i are bounded

Proof that a_0 is bounded. Let $x < 4\xi$, $r < 1$ and $\eta \in (\alpha - \alpha', 2\alpha)$. By (2.18)-(2.10) (to bound the ratio $v/\tilde{m}'_\xi$) and by (8.40) there are constants c' and c'' so that

$$\begin{aligned}\Big|\mathbb{E}_x\Big(\frac{v(x_n)}{\tilde{m}'_\xi(x_n)} - \int_0^\ell dx\, \chi_{\xi,\ell}(x)\frac{v(x_n)}{\tilde{m}'_\xi(x_n)}\Big| &\le \int dy\, \Big|Q^n_{\xi,\ell}(x,y) - \chi_{\xi,\ell}(y)\Big|ce^{(\alpha-\alpha')|\xi-y|} \\ &\le c'r^{\tau_{x,\xi,n}}\int_0^\ell dy\, e^{(\alpha-\alpha'-\eta)|\xi-y|} \le c''r^{n/2} \end{aligned} \tag{9.20}$$

because, see (8.38), $\tau_{x,\xi,n} = n - |\xi - x|q \ge n - 3\xi q \ge n/2$ for ξ large enough (recall $n = [\xi^4]$).

By Theorem 8.6 with $\delta = 1/2$ and using again (2.18)-(2.10) for the ratio $v/\tilde{m}'_\xi$, there are constants c' and c'' so that

$$\begin{aligned}\Big|\int_0^\ell dx\, \chi_\xi(x)\frac{v(x)}{\tilde{m}'_\xi(x)} - \int_0^\ell dx\, \chi^0_\xi(x)\frac{v(x)}{\tilde{m}'_\xi(x)}\Big| \\ \le c'\int_0^\ell dx\, e^{(\alpha-\alpha')|\xi-x|}e^{-2\alpha\xi}\Big(e^{-2\alpha|\xi-x|}\xi^{3/2} + (x+1)\mathbf{1}_{x<\xi}\Big) \\ \le c''e^{-2\alpha\xi}\xi^{3/2} \end{aligned} \tag{9.21}$$

with c' and c'' suitable constants. By (8.1) and (9.2)

$$\int_0^\ell dx\, \chi^0_\xi(x)\frac{v(x)}{\tilde{m}'_\xi(x)} = \langle v\tilde{m}'_\xi\rangle \tag{9.22}$$

By (9.20), (9.21) and (9.22) we conclude that a_0 is bounded.

Proof that a_1 is bounded. The proof follows from (2.18) and (6.14) with $\eta := \alpha - \alpha'$ (that for ξ large enough is smaller than 2α).

Proof that b_1 is bounded. The contribution of f_1 and f_3 is bounded using (9.19) and (8.40) with $\eta > \alpha + (\alpha - \alpha')$. The sum over i is bounded as in (9.3). The contribution of f_2 is bounded using again (8.40) and recalling that f_2 is bounded and supported in $[0, 1]$.

Proof that b_2 is bounded. Let $x \geq 4\xi$, $\xi^2 - 3 \leq k \leq n$ and $j = 1, 3$. By (9.19)

$$\mathbb{E}_x\big(f_j(x_k)\big) \leq c e^{-2\alpha\xi} \int_0^\ell dy\, Q^k_{\xi,\ell}(x,y) e^{(\alpha+(\alpha-\alpha'))|\xi-y|}$$

We distinguish the values of $y > x$ from the others and use, respectively, (8.4) and (8.6), obtaining:

$$\begin{aligned}\mathbb{E}_x\big(f_j(x_k)\big) &\leq c e^{-2\alpha\xi} e^{(\alpha+(\alpha-\alpha'))|\xi-x|} C \int_x^\ell dy\, e^{-2\alpha|y-x|} e^{(\alpha+(\alpha-\alpha'))|x-y|} \\ &\quad + c e^{-2\alpha\xi} c' \int_0^x dy\, e^{(\alpha+(\alpha-\alpha'))|\xi-y|} \\ &\leq c'' e^{-2\alpha\xi} e^{(\alpha+(\alpha-\alpha'))|\xi-x|} \end{aligned} \tag{9.23}$$

for a suitable constant c''.

By (8.4) and (2.10)-(2.18), there is $c > 0$ so that

$$\Big|\mathbb{E}_x\big(f_2(x_k)\mathbf{1}_{x_k\in[0,1]}\big)\Big| \leq c e^{-2\alpha\xi} e^{(\alpha-\alpha')\xi} \tag{9.24}$$

which is therefore bounded by $c' e^{-2\alpha\xi}$. The remaining contribution of f_2 is bounded by

$$\Big|\mathbb{E}_x\big(f_2(x_k)\mathbf{1}_{x_k\in[\ell-1,\ell]}\big)\Big| \leq \Big|\mathbb{E}_x\big(e^{\eta|\xi-x_k|}\big)\Big| e^{(\alpha-\alpha'-\eta)(\ell-\xi)}$$

having used again (2.10)-(2.18). We take $\eta = \alpha$ in Theorem 6.2, then there are $r < 1$ and c' so that for all ξ large enough the right hand side is bounded by

$$c' e^{(\alpha-\alpha'-\eta)(\ell-\xi)} \Big\{ r^k e^{\eta|\xi-x|} + 1 \Big\}$$

since $\alpha'(\ell-\xi) \geq 2\alpha\xi$ for all ξ large enough and $\ell \geq 4\xi$, the only case when b_2 is relevant. Hence

$$\Big|\mathbb{E}_x\big(f_2(x_k)\mathbf{1}_{x_k\in[\ell-1,\ell]}\big)\Big| \leq c e^{-2\alpha\xi}[e^{\alpha|\xi-x|} + 1] \tag{9.25}$$

By (9.23), (9.24), (9.25) we conclude the proof that b_2 is bounded.

Proof that b_3, b_4 and b_5 are bounded. The proof for b_3 is like in (9.23), for b_4 like in (9.24) and (9.25) as well as for b_5 and are omitted.

Proof that b_6 is bounded. It follows from (2.10)-(2.18) and (9.18), after observing that, by the definition of P, the integral in b_6 is over $|y-x| \leq 1$.

9.1 Lemma.

There are $\tilde B(\xi)$, $\bar B(\xi)$ and a constant $c>0$ so that

$$\lambda^n = \langle v\tilde m'_\xi\rangle^2 + \langle v\tilde m'_\xi\rangle e^{-2\alpha\xi} n\lambda^n \tilde B(\xi), \qquad |B(\xi)c_\lambda - \tilde B(\xi)| \le \frac{c}{\xi} \tag{9.26}$$

$$\lambda^n\langle v\tilde m'_\xi\rangle = \langle v\tilde m'_\xi\rangle + e^{-2\alpha\xi} n\lambda^n \bar B(\xi), \qquad |B(\xi)c_\lambda - \bar B(\xi)| \le \frac{c}{\xi} \tag{9.27}$$

Proof

We multiply both sides of (9.1) by $v(x)/p_m(x)$ and integrate. We use below the convention that if the upper limit of an integral is larger than ℓ, then the integral is understood to be extended only up to ℓ.

$$\begin{aligned}\lambda^n =&\big(\langle v\tilde m'_\xi\rangle + D_1 e^{-2\alpha\xi}\big)\Big\{\langle v\tilde m'_\xi\rangle + e^{-2\alpha\xi} n\lambda^n B(\xi)c_\lambda\Big\}\\ &+ e^{-2\alpha\xi}\Big\{\xi^3 C_0 + C_{0,1} + e^{-2\alpha\xi} n\lambda^n (C_1+C_2) + \lambda^n\xi^2 C_3 + \lambda^n\xi^3 C_4 + C_5 + \lambda^{n-1}C_6\Big\}\end{aligned} \tag{9.28}$$

where the coefficients D and C are defined as follows:

$$\begin{aligned}
D_1 &:= e^{2\alpha\xi}\Big(\int_0^{4\xi}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x) - \langle v\tilde m'_\xi\rangle\Big)\\
C_0 &:= \int_{\mathbb{R}_+}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)a_0(x,\xi)\\
C_{0,1} &:= e^{2\alpha\xi}\int_{4\xi}^{\infty}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)e^{(\alpha-\alpha')|\xi-x|}a_1(x,\xi)\\
C_1 &:= \int_{\mathbb{R}_+}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)b_1(x,\xi,\lambda)\\
C_2 &:= e^{2\alpha\xi}\int_{4\xi}^{+\infty}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)e^{(\alpha+(\alpha-\alpha'))|\xi-x|}b_2(x,\xi,\lambda)\\
C_3 &:= \int_{\mathbb{R}_+}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)e^{(\alpha+(\alpha-\alpha'))|\xi-x|}b_3(x,\xi,\lambda)
\end{aligned}$$

$$C_4 := \xi^{-1}\int_{\mathbb{R}_+}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)[e^{2\alpha|\xi-x|}\mathbf{1}_{x\le 4\xi} + e^{\alpha|\xi-x|}\mathbf{1}_{x>4\xi}]b_4(x,\xi,\lambda)$$

$$C_5 := \int_{\mathbb{R}_+}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)b_5(x,\xi,\lambda)$$

$$C_6 := \int_{\mathbb{R}_+}\frac{dx}{p_m(x)}v(x)\tilde m'_\xi(x)e^{(\alpha+(\alpha-\alpha'))|\xi-x|}b_6(x,\xi)$$

By multiplying both sides of (9.1) by $\tilde m'_\xi(x)/p_{\bar m}(\xi-x)$ and integrating we get

$$\lambda^n\langle v\tilde m'_\xi\rangle = \big(1-D'_1e^{-2\alpha\xi}\big)\Big\{\langle v\tilde m'_\xi\rangle + e^{-2\alpha\xi}n\lambda^nB(\xi)c_\lambda\Big\} + e^{-2\alpha\xi}\Big\{\xi^3C'_0+C'_{0,1}+e^{-2\alpha\xi}n\lambda^n(C'_1+C'_2)+\lambda^n\xi^2C'_3+\lambda^n\xi^3C'_4+C'_5+\lambda^{n-1}C'_6\Big\} \tag{9.29}$$

where

$$D'_1 := e^{2\alpha\xi}\int_{4\xi}^{\infty}dx\,\frac{\bar m'(\xi-x)^2}{p_{\bar m}(\xi-x)} \tag{9.30}$$

and the coefficients C'_i are obtained from the C_i by replacing $v(x)/p_{\bar m}(\xi-x)$ by $\tilde m'_\xi(x)/p_{\bar m}(\xi-x)$. All the coefficients D_i, C_i, D'_i and C'_i are bounded as it follows recalling that the a_i and b_i are bounded and using (2.18) and (2.10). The right hand side of (9.28) can be written in the form

$$<v\tilde m'_\xi>^2 + a<v\tilde m'_\xi> + b = <v\tilde m'_\xi>^2 + <v\tilde m'_\xi>[a+\frac{b}{<v\tilde m'_\xi>}]$$

which identify $\tilde B(\xi)$ in (9.26) in terms of a, b and $<v\tilde m'_\xi>$. To prove the inequality in (9.26) we need a lower bound on $<v\tilde m'_\xi>$ which is derived next.

For ξ large enough $\langle v\tilde m'_\xi\rangle^2\ge 1/2$: in fact either $\langle v\tilde m'_\xi\rangle^2\ge 1$ and then the statement is obviously true or $\langle v\tilde m'_\xi\rangle\le 1$. In such a case by (9.28), since $\lambda\ge 1-c_-e^{-2\alpha\xi}$, there is $c>0$ so that

$$\langle v\tilde m'_\xi\rangle^2 \ge 1-ce^{-2\alpha\xi}n\ge\frac12$$

for ξ large enough. Lemma 9.1 is proved. □

Proof of the second inequality in (2.17)

We can solve explicitly (9.26) and (9.27) in the unknowns $y:=\lambda^n$ and $x:=\langle v\tilde m'_\xi\rangle$ with $\tilde B=\tilde B(\xi)$ and $\bar B=\bar B(\xi)$ thought of as known. Calling $\epsilon':=e^{-2\alpha\xi}n\tilde B$ and $\epsilon'':=e^{-2\alpha\xi}n\bar B$ we get

$$y=\frac{x}{x-\epsilon''},\qquad x=\frac12\Big(\epsilon''-\epsilon'+\sqrt{4+(\epsilon''-\epsilon')^2}\Big) \tag{9.31}$$

Hence

$$\Big|\langle v\tilde m'_\xi\rangle-1\Big|\le c\frac{n}{\xi}e^{-2\alpha\xi} \tag{9.32}$$

$$\Big|\lambda^n-(1+e^{-2\alpha\xi}n\bar B)\Big|\le ce^{-2\alpha\xi}\frac{n}{\xi} \tag{9.33}$$

It follows from (9.33) and (9.3) that there is $c>0$ so that

$$\lambda<1+ce^{-2\alpha\xi} \tag{9.34}$$

We have thus proved (2.17) with c_+ a suitable constant. □

¿From (9.34) there is $c' > 0$ so that

$$c_\lambda > 1 - c' e^{-2\alpha\xi} \tag{9.35}$$

¿From (9.35) and from the first inequality in (2.17) we get for some $\bar c > 0$ that

$$|B(\xi) - \tilde B(\xi)| \le \frac{\bar c}{\xi}, \qquad |B(\xi) - \bar B(\xi)| \le \frac{\bar c}{\xi} \tag{9.36}$$

Proof of (2.19)

We go back to (9.1) that for $|\xi - x| \le 3\xi$ becomes

$$v(x) = \tilde m'_\xi(x)\Big\{\big[1 - \big(1 - \lambda^{-n}\langle v \tilde m'_\xi\rangle\big)\big] + e^{-2\alpha\xi} n B(\xi) c_\lambda + \lambda^{-n} U(x,\xi,\lambda)\Big\} \tag{9.37}$$

We have by (9.32) and (9.33)

$$\Big|1 - \lambda^{-n}\langle v \tilde m'_\xi\rangle\Big| \le c n e^{-2\alpha\xi} \le c' e^{-2\alpha\xi}\xi^4$$

Analogously $e^{-2\alpha\xi} n |B(\xi)| c_\lambda \le c' e^{-2\alpha\xi}\xi^4$. To bound the term with U in (9.37) we use (9.16). The terms with a_1 and b_2 drop out because $|\xi - x| \le \xi/2$, in the others we use that the functions a_i and b_i are bounded and that U in (9.37) is multiplied by $\tilde m'_\xi$. Then the term with b_4 is bounded by $c\xi^2 e^{\alpha|\xi - x|}$ and all the others by $c\xi^3$, $c > 0$ a suitable constant. It is thus the term with b_4 in U which is responsible for the factor $e^{\alpha|\xi-x|}$ that appears on the right hand side of (2.19) which is thus proved.

9.2 Proposition.

Let $B_i(\xi)$, $i = 1, 2, 3$, be as in (9.4). *Then*

$$\lim_{\xi\to+\infty} B_3(\xi) = 0 \tag{9.38}$$

Moreover, letting $\kappa(x)$ as in (8.55) *and supposing that*

$$\lim_{\ell,\xi\to+\infty} e^{2\alpha\xi - 2\alpha(\ell-\xi)} = e^{-2\alpha\omega}$$

with $\omega \in [0, +\infty]$, we have

$$\begin{aligned}\lim_{\ell,\xi\to+\infty} B_2(\xi) = [1 + e^{-2\alpha\omega}][\beta(1 - m_\beta^2)]^2 \int_0^1 dx\, \kappa(x) \int_0^1 dy \int_{-1}^2 dz\, J(x,y) \\ \times J(y,z) e^{-\alpha x}(1 - e^{-2\alpha y})(1 + e^{-2\alpha z}) e^{-\alpha|z|}\end{aligned} \tag{9.39}$$

If m satisfies (2.14), *$B_1 > D/2$, for all ξ large enough, where*

$$D := 2\beta \int_{\mathbb R} dx\, \frac{\tilde m'(x)^2}{p_{\bar m}(x)} \bar m(x) a \sinh(\alpha x)\Big\{1 + e^{-2\alpha\omega}\Big\} \tag{9.40}$$

Moreover, if $\delta^0_\xi m = \delta^{0,1}_\xi m + \delta^{0,2}_\xi m$ *and, for all* x,

$$\lim_{\xi\to+\infty} -\frac{e^{2\alpha\xi}}{2}[\delta^{0,1}_\xi m(\xi - x) - \delta^{0,1}_\xi m(\xi + x)] =: \Delta^0(x) \geq 0 \tag{9.41}$$

while for some c *and* δ *positive*

$$\Big| \int_{|x-\xi|\leq\xi^{1/2}} dx\, \delta^{0,2}_\xi m(x)\bar{m}'(\xi - x)^2 \bar{m}(\xi - x)\Big| \leq ce^{-2(\alpha+\delta)\xi}$$

then

$$\lim_{\xi\to+\infty} B_1(\xi) = D + \int_{\mathbb{R}} 2\beta\tilde{m}'(x)^2\bar{m}(x)\Delta^0(x) > 0 \tag{9.42}$$

Proof.

(9.38) follows from (8.42) using (9.19) and that $f_3(x)$ is supported by $x \leq 1$ and $x \geq \ell - 1$. The explicit expression of B_1 is obtained from (9.12) using the invariance of $\chi_{\xi,\ell}$:

$$B_1(\xi) = e^{2\alpha\xi} \int_0^\ell dx\, \chi_{\xi,\ell}(x) \frac{p_m(x) - p_{\bar{m}}(\xi - x)}{p_{\bar{m}}(\xi - x)} \int_0^\ell dy\, Q_{\xi,\ell}(x, y) \frac{v(y)}{\tilde{m}'_\xi(y)}$$

We use (9.18) and (2.19)-(2.10) when $|\xi - x| \leq \xi^{1/2}$ and (2.18)-(2.10) otherwise. We get

$$\begin{aligned}\Big|B_1(\xi) - e^{2\alpha\xi} \int_{|\xi-x|\leq\xi^{1/2}} dx\, \chi_{\xi,\ell}(x) \frac{p_m(x) - p_{\bar{m}}(\xi - x)}{p_{\bar{m}}(\xi - x)}\Big| \\ \leq ce^{2\alpha\xi} \int_{|\xi-x|\leq\xi^{1/2}} dx\, \chi_{\xi,\ell}(x)[e^{-2\alpha\xi}e^{\alpha|\xi-x|}][e^{-2\alpha\xi}e^{\alpha|\xi-x|}\xi^4 e^{\alpha|\xi-x|}] \\ + ce^{2\alpha\xi} \int_{|\xi-x|\geq\xi^{1/2}} dx\, \chi_{\xi,\ell}(x)[e^{-2\alpha\xi}e^{\alpha|\xi-x|}][e^{(\alpha-\alpha')|x-\xi|}]\end{aligned}$$

where c is a suitable constant.

In the two integrals on the right hand side we bound $\chi_{\xi,\ell}(x)$ by using (8.42). Thus there are c and c' so that

$$\begin{aligned}\Big|B_1(\xi) - e^{2\alpha\xi} \int_{|\xi-x|\leq\xi^{1/2}} dx\, \chi_{\xi,\ell}(x) \frac{p_m(x) - p_{\bar{m}}(\xi - x)}{p_{\bar{m}}(\xi - x)}\Big| \\ \leq c[e^{\alpha\xi^{1/2}}\xi^4 e^{-2\alpha\xi} + e^{-\alpha'\xi^{1/2}}] \leq c'e^{-\alpha'\xi^{1/2}}\end{aligned}$$

We next estimate the integral on the left hand side. There is $c > 0$ so that

$$\Big|\frac{p_m(x) - p_{\bar{m}}(\xi - x)}{p_{\bar{m}}(\xi - x)} + \frac{2\beta^2\bar{m}_\xi(x)[1 - \bar{m}_\xi(x)^2]}{p_{\bar{m}}(\xi - x)}\delta_\xi m(x)\Big| \leq c\delta_\xi m(x)^2$$

We use (5.11) and (8.42) to bound the term with $c\delta_\xi m(x)^2$, giving rise to the second term on the left hand side of the equation below.

By (8.45) and recalling the definition of $p_{\bar m_\xi}$, there is $c>0$ so that

$$\Big|B_1(\xi)-e^{2\alpha\xi}\int_{|\xi-x|\le\xi^{1/2}} dx\,\frac{\tilde m_\xi'(x)^2}{p_{\bar m_\xi}(x)}\Big(-2\beta\bar m_\xi(x)\delta_\xi m(x)\Big)\Big| \le c\Big(e^{-\alpha'\xi^{1/2}}+e^{-2\alpha\xi}\xi^{1/2}+e^{-2\alpha\xi+2\alpha\xi^{1/2}}\Big)$$

We then get

$$\Big|B_1(\xi)+e^{2\alpha\xi}\int_{|\xi-x|\le\xi^{1/2}} dx\,\tilde m_\xi'(x)^2 2\beta\bar m_\xi(x)\delta_\xi m(x)\Big|\le ce^{-\alpha'\xi^{1/2}}$$

with $c>0$ a suitable constant.

We write $\delta_\xi m=\delta_\xi^0 m+m^0-\bar m$, the term $m^0_{\xi,\ell}(x)-\bar m(\xi-x)$ produces in the limit $\xi\to+\infty$ the term D on the right hand side of (9.42). By (2.14) with a suitable δ,

$$\liminf_{\xi\to+\infty} -e^{2\alpha\xi}\int_{|\xi-x|\le\xi^{1/2}} dx\,\tilde m_\xi'(x)^2 2\beta\bar m_\xi(x)\delta_\xi^0 m(x)\ge-\frac{D}{2} \tag{9.43}$$

so that $B_1(\xi)\ge D/2$ for all ξ large enough. For the same reason the contribution of $\delta_\xi^{0,2}m$ vanishes as $\xi\to+\infty$. Since $\tilde m_\xi'(x)^2\bar m_\xi(x)$ is an odd function of $\xi-x$, the contribution of the even part of $\delta_\xi^{0,1}m$ in (9.43) also vanishes. The odd part of $\delta_\xi^{0,1}m$ converges to Δ^0 by (9.41) and since

$$\Big|e^{2\alpha\xi}\delta_\xi^0 m(x)\tilde m_\xi'(x)^2\bar m_\xi(x)\Big|\le ce^{\alpha|\xi-x|}e^{-2\alpha|\xi-x|}$$

by the Lebesgue dominated convergence theorem we obtain (9.42).

The proof of (9.39) is more delicate as it involves the estimate of $v(x)/\tilde m_\xi'(x)$ for x close to 0 and ℓ. While this ratio was proved to be close to 1 when $|\xi-x|\le\xi/2$, this is no longer true when x is close to 0 and ℓ:

9.3 Proposition.

There are $c>0$ and $\delta>0$ so that for $x\in[0,2]$

$$\Big|v_{m,\ell}(x)-C_{\bar m}^{1/2}a\alpha e^{-\alpha\xi}(e^{\alpha x}+e^{-\alpha x})\Big|\le ce^{-(\alpha+\delta)\xi} \tag{9.44}$$

and

$$\Big|\frac{v_{m,\ell}(x)}{\tilde m_\xi'(x)}-\frac{e^{\alpha x}+e^{-\alpha x}}{e^{\alpha x}}\Big|\le ce^{-\delta\xi} \tag{9.45}$$

Proof.

Since (9.45) follows from (2.10) and (9.44) we only need to prove the latter. We proceed as in the proof of (8.56). Let $s_0 := [\xi/2]$, $x_0 \equiv x$, $\lambda \equiv \lambda_{m,\ell}$. Then

$$v_{m,\ell}(x) = \sum_{n\ge1} \int_0^{s_0} dx_1 .. \int_0^{s_0} dx_n \int_{s_0}^{s_0+1} dx_{n+1} \{\prod_{j=0}^{n} \lambda_{m,\ell}^{-1} A_{m,\ell}(x_j, x_{j+1})\} v_{m,\ell}(x_{n+1}) \tag{9.46}$$

Call

$$A_\infty(x,y) := p_\infty \big(J(x,y) + J(x,-y) \big) \tag{9.47}$$

By (2.10) and (2.17) there is $c > 0$ so that for all $x \le s_0 + 1$

$$A_\infty(x,y)[1 - ce^{-\alpha s_0}] \le \lambda_{m,\ell}^{-1} A_{m,\ell}(x,y) \le A_\infty(x,y)[1 + ce^{-\alpha s_0}] \tag{9.48}$$

We set

$$V(x) := C_{\bar m}^{1/2} a\alpha e^{\alpha x} \tag{9.49}$$

and, in analogy with (9.46),

$$\sigma(x) := \sum_{n\ge1} \int_0^{s_0} dx_1 .. \int_0^{s_0} dx_n \int_{s_0}^{s_0+1} dx_{n+1} \{\prod_{j=0}^{n} A_\infty(x_j, x_{j+1})\} V(x_{n+1}) \tag{9.50}$$

By (2.19) and (2.10), there are constants c and c' so that for $x \in [s_0, s_0+1]$

$$\begin{aligned} \big|v_{m,\ell}(x) - e^{-\alpha\xi} V(x)\big| &\le \big|v_{m,\ell}(x) - \tilde m'_\xi(x)\big| + \big|\tilde m'_\xi(x) - e^{-\alpha\xi} V(x)\big| \\ &\le c\big(e^{-2\alpha\xi} e^{\alpha(\xi - x)} \xi^4 + e^{-\alpha_0(\xi - x)}\big) \le c' e^{-\delta\xi} V(x) \end{aligned}$$

with $0 < \delta \le (\alpha_0 - \alpha)/2$. Then

$$v_{m,\ell}(x) \le e^{-\alpha\xi} \sum_{n\ge1} [1 + ce^{-\delta\xi}]^{n+1} \int_0^{s_0} dx_1 .. \int_0^{s_0} dx_n \int_{s_0}^{s_0+1} dx_{n+1} \{\prod_{j=0}^{n} A_\infty(x_j, x_{j+1})\} V(x_{n+1}) \tag{9.51}$$

The lower bound holds as well with $c \to -c$.

Since $p_\infty < 1$ there is $b > 0$ so that for all ξ large enough

$$\sum_{n\ge b\xi} [1 + ce^{-\delta\xi}]^{n+1} \int_0^{s_0} dx_1 .. \int_0^{s_0} dx_n \int_{s_0}^{s_0+1} dx_{n+1} \{\prod_{j=0}^{n} A_\infty(x_j, x_{j+1})\} V(x_{n+1}) \le e^{-\alpha\xi}$$

On the other hand for $n < b\xi$

$$[1 + ce^{-\delta\xi}]^{n+1} \le c' b\xi e^{-\delta\xi}$$

so that there is $c > 0$ and for $x \in [0, 2]$

$$\big|v_{m,\ell}(x) - e^{-\alpha\xi}\sigma(x)\big| \le 2e^{-\alpha\xi} + c\xi e^{-\delta\xi} V(x) e^{-\alpha\xi} \tag{9.52}$$

To estimate $V(x)$ we introduce

$$T(x,y) := w(x)^{-1} A_\infty(x,y) w(y) \tag{9.53}$$

where

$$w(x) := e^{\alpha x} + e^{-\alpha x}$$

so that

$$A_\infty \star w = w$$

Therefore $T(x,y)$ is a transition probability. Denoting by $\mathbb{E}_x$ the expectation for the Markov chain $\{x_n\}_{n\ge 0}$ that starts from $x_0 = x$, we then have

$$\sigma(x) = e^{-\alpha\xi} w(x) \sum_{n\ge 1} \mathbb{E}_x \Big(\{\prod_{i=1}^n \mathbf{1}_{x_i < s_0}\} \frac{V(x_{n+1})}{w(x_{n+1})} \Big)$$

Since

$$\Big| \frac{V(x_{n+1})}{w(x_{n+1})} - C_{\bar m}^{1/2} a\alpha \Big| \le c e^{-2\alpha s_0}$$

For $x \le 2$

$$\Big| \sigma(x) - w(x) C_{\bar m}^{1/2} a\alpha \Big| \le c e^{-2\alpha s_0}$$

which together with (9.52) proves (9.44). Proposition 9.3 is proved. □

The same estimates of Proposition 9.3 hold for $x \in [\ell - 2, \ell]$, we omit the details.

Conclusion of the proof of Proposition 9.2.

It remains to prove (9.39). We rewrite explicitly

$$B_2(\xi) = e^{2\alpha\xi} \int_{\mathbb{R}_+} dx \chi_\xi(x) \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz \tilde{P}(x,y) \tilde{P}(|y|,z) \times \Big(\frac{\tilde{m}'_\xi(|y|)}{\tilde{m}'_\xi(y)} - 1 \Big) \frac{v(|z|)}{\tilde{m}'_\xi(|z|)}$$

By (8.56) we can replace in the above expression $e^{2\alpha\xi}\chi_\xi(x)$ by $\kappa(x)$, the error vanishing exponentially as $\xi \to +\infty$. Moreover by (8.53) we can replace

$$P_\xi(a,b) \longrightarrow p_\infty J(a,b) e^{-\alpha(a-b)}$$

with an error that again vanishes exponentially as $\xi \to +\infty$. Then using (9.45) we get

$$\lim_{\xi\to+\infty} B_2(\xi) = \int_0^1 dx\, \kappa(x) \int_{-1}^0 dy \int_{-1}^2 dz\, p_\infty^2 J(x,y) J(|y|,z) e^{-\alpha(x-y)} e^{-\alpha(|y|-z)} \times (e^{-\alpha(y-|y|)} - 1)(e^{\alpha z} + e^{-\alpha z}) e^{-\alpha|z|}$$

which proves (9.39). Theorem 9.2 is proved. □

Proof of (2.20). It follows immediately from (9.1) and Proposition 9.2. □

10. Spectral gap

In this Section we will prove Theorem 2.5.

We write

$$\|e^{L_{m,\ell}t}w\|_{\zeta,\xi,\ell} \le e^{-t}\sum_{n\ge 0}\frac{t^n}{n!}\|A^n_{m,\ell}w\|_{\zeta,\xi,\ell}$$

A bound as in (2.23) is then obvious when $w = v_{m,\ell}$. By writing

$$w = \tilde w + \pi_{m,\ell}(w)v_{m,\ell}(x), \qquad \pi_{m,\ell}(\tilde w) = 0$$

the proof of (2.23) is reduced to that of (2.24), that we prove next.

Proof of (2.24). We drop m, ξ and ℓ when no ambiguity arises. We have

$$A^n\tilde w(x) = \lambda^n v(x)\int_0^\ell dy\, Q^n(x,y)v(y)^{-1}\tilde w(y)$$

$Q = Q_{m,\ell}$. Letting $\nu(dx) := u(x)v(x)dx$,

$$\int_0^\ell \nu(dy)\tilde w(y)v(y)^{-1} = \pi_{m,\ell}(\tilde w) = 0$$

and since ν is invariant for Q, i.e.

$$\int_0^\ell \nu(dx')\, Q^n(x',y)v(y)^{-1}\tilde w(y) = 0$$

we have

$$A^n\tilde w(x) = \lambda^n v(x)\Big\{\int_0^\ell dy\, Q^n(x,y)v(y)^{-1}\tilde w(y) - \int_0^\ell \nu(dx')\, Q^n(x',y)v(y)^{-1}\tilde w(y)\Big\}$$

hence

$$A^n\tilde w(x) = \lambda^n v(x)\int_0^\ell \nu(dx')\int_0^\ell dy\{Q^n(x,y) - Q^n(x',y)\}\tilde w(y)v(y)^{-1}$$

Then, letting $\|\cdot\| \equiv \|\cdot\|_{\zeta,\xi,\ell}$,

$$|A^n\tilde{w}(x)| \le \lambda^n v(x) \int_0^\ell \nu(dx') \int_0^\ell dy \big|Q^n(x,y) - Q^n(x',y)\big| v(y)^{-1} \|w\| e^{\zeta|\xi-y|}$$

Let s be as in Theorem 7.4, let $N_{s,m} = \min(N^+_{s,m}, N^-_{s,m})$, see (6.20), then

$$v(y)^{-1} e^{\zeta|\xi-y|} \le N^{-1}_{s,m} \gamma_{s,\zeta,m}(y) \mathbf{1}_{|\xi-y|\ge s} + \mathbf{1}_{|\xi-y|\le s} v(y)^{-1} e^{|\zeta|s}$$

By Lemmas 5.3 and 5.4 there is c so that

$$v(y)^{-1} e^{\zeta|\xi-y|} \le c\gamma_{s,\zeta,m}(y)$$

hence, by Theorem 7.4 there are c and $r < 1$ so that

$$|A^n\tilde{w}(x)| \le c\|w\|\lambda^n v(x) \int_0^\ell \nu(dx') \int_0^\ell dy \big|Q^n(x,y) - Q^n(x',y)\big| \gamma_{s,\zeta,m}(y)$$

$$\le c\|w\|\lambda^n v(x) c' r^n \Big(\gamma_{s,\zeta,m}(x) + \int_0^\ell \nu(dx') \gamma_{s,\zeta,m}(x')$$

The last integral is bounded by a constant, see the remark after the proof of Theorem 6.4. Hence recalling that $\gamma_{s,\zeta,m}(x) \ge 1$, there is c so that (if ξ is large enough)

$$|A^n\tilde{w}(x)| \le c\|w\|[r\lambda]^n v(x)\gamma_{s,\zeta,m}(x)$$

and

$$\|A^n\tilde{w}\| \le c\|w\|[r\lambda]^n \sup_{0\le x\le\ell} \{e^{-\zeta|\xi-x|} v(x)\gamma_{s,\zeta,m}(x)\}$$

Recalling that $\gamma_{s,\zeta,m}(x) = 1$ if $|x-\xi| \le s$, by (2.18),

$$\sup_{|x-\xi|\le s} \{e^{-\zeta|\xi-x|} v(x)\gamma_{s,\zeta,m}(x)\} \le e^{|\zeta|s} \sup_{|x-\xi|\le s} v(x) \le c$$

Moreover there is a new constant c so that

$$\sup_{|x-\xi|\ge s} \{e^{-\zeta|\xi-x|} v(x)\gamma_{s,\zeta,m}(x)\} \le c$$

In conclusion if ξ is large enough, there is $c > 0$ so that

$$\|A^n\tilde{w}\| \le c\|w\|[r\lambda]^n, \qquad \|e^{L_{m,\ell}t} w\| \le c\|w\| e^{-t} \sum_{n\ge 0} \frac{[r\lambda t]^n}{n!}$$

hence (2.24).

Proof of (2.26) *and* (2.25). The integral in (2.26) converges in the $\|\cdot\|$ norm, then (2.26) follows from standard arguments. (2.25) follows from the eigenvalue equation $L^{-1}v = (\lambda_m - 1)^{-1} v$ and the inequality (2.17). Theorem 2.5 is proved. □

11. Eigenvalues and eigenvectors: dependence on ξ

To simplify notation we will write in this subsection A_ξ, L_ξ v_ξ and λ_ξ respectively for $A_{m^0_{\xi,\ell}}$, $L_{m^0_{\xi,\ell}}$, $v_{m^0_{\xi,\ell}}$ and $\lambda_{m^0_{\xi,\ell}}$, v_ξ normalized as in (2.16).

We denote by v'_ξ and λ'_ξ the derivative of v_ξ and λ_ξ with respect to ξ, whose existence follows from classical perturbation theory, see [18]. By differentiating $A_\xi v_\xi = \lambda_\xi v_\xi$ and then taking the scalar product in $\mathbb{R}_+$ with u_ξ (the left eigenvector) we get

$$\lambda'_\xi = (u_\xi, A'_\xi v_\xi) \tag{11.1}$$

and

$$(A_\xi - \lambda_\xi)v'_\xi = (A'_\xi - \lambda'_\xi)v_\xi \tag{11.2}$$

By shorthanding $\pi := \pi_{m_\xi}$ we have from (11.1)

$$\pi\Big([A'_\xi - \lambda'_\xi]v_\xi\Big) = 0 \tag{11.3}$$

We can then use (2.26) to solve (11.2) as an equation in v'_ξ obtaining:

11.1 Theorem.

For any $\eta \in (0, 3\alpha/2)$ there are ξ^0 and $c^0 > 0$ so that for all $\xi > \xi^0$

$$|\lambda'_\xi| \le c^0 e^{-\eta\xi} \tag{11.4}$$

$$\begin{aligned} &\Big|v'_\xi(x) - \tilde{m}''(\xi - x)\Big| \le c^0 e^{-\alpha\xi}\xi^4 \quad \textit{for } |\xi - x| \le \xi/2 \\ &|v'_\xi(x)| \le c^0\xi^2 v_\xi(x) \quad \textit{for } |\xi - x| \ge \xi/2 \end{aligned} \tag{11.5}$$

Proof.

To simplify exposition we give an explicit proof only for the case $\ell = \infty$. It will be clear that all the arguments extend with minor changes to the case $\ell > 2\xi$.

Let $x \ge 0$. By definition

$$A'_\xi(x,y) = \Big(\frac{d}{d\xi} p_{m^0_\xi}(x)\Big) J(x,y) \tag{11.6}$$

By (2.10) there is $c > 0$ so that

$$\big|A'_\xi(x,y)\big| \le ce^{-\alpha|\xi - x|} J(x,y) \tag{11.7}$$

Then, recalling that $u_\xi(x) := v_\xi(x)/p_{m^0_\xi}(x)$, there is $c > 0$ so that

$$\Big|\lambda'_\xi - \int_{|\xi-x|\le\xi/2} dx \frac{\tilde m'(\xi-x)}{p_{\bar m}(\xi-x)} \int dy A'_\xi(x,y)\tilde m'(\xi-y)\Big| \le c[e^{-2\alpha\xi}\xi^4 + e^{-(\alpha+2\alpha')\xi/2}] \tag{11.8}$$

The first term on the right hand side bounds the terms that arise when we replace $v_\xi(x)$ by $\tilde m'_\xi(x)$ and $p_{m^0_\xi}(x)$ by $p_{\bar m}(\xi - x)$ for $|\xi - x| \le \xi/2$. When $|\xi - x| > \xi/2$ we bound v_ξ using (2.18). We have also used (11.7).

We set

$$B_\xi(x,y) := p_{\bar m}(\xi-x)J(x,y), \qquad B'_\xi(x,y) := \Big(\frac{d}{d\xi}p_{\bar m}(\xi-x)\Big)J(x,y) \tag{11.9}$$

By (2.10) there is $c > 0$ so that for any $x > 0$

$$\big|B'_\xi(x,y)\big| \le ce^{-\alpha|\xi-x|}J(x,y); \qquad \Big|A'_\xi(x,y) - B'_\xi(x,y)\Big| \le ce^{-2\alpha\xi}e^{\alpha|\xi-x|}J(x,y) \tag{11.10}$$

We then obtain from (11.8)

$$\Big|\lambda'_\xi - \int_{|\xi-x|\le\xi/2} dx \frac{\tilde m'(\xi-x)}{p_{\bar m}(\xi-x)} \int dy B'_\xi(x,y)\tilde m'(\xi-y)\Big| \le ce^{-3\alpha'\xi/2} \tag{11.11}$$

with $c > 0$ a suitable constant. Then there is $c' > 0$ so that

$$\Big|\lambda'_\xi - \int_{\mathbb{R}} dx \frac{\tilde m'(\xi-x)}{p_{\bar m}(\xi-x)} \int dy B'_\xi(x,y)\tilde m'(\xi-y)\Big| \le ce^{-3\alpha\xi/2} \tag{11.12}$$

The integral on the left hand side is equal to 0 because it is the derivative with respect to ξ of the maximal eigenvalue of $B_\xi := A_{\bar m_\xi}$ which is identically 1, $B_\xi \bar m'_\xi = \bar m'_\xi$ as obtained by differentiating (1.7) with respect to x. We have thus proved (11.4).

Recalling (11.3) and shorthanding $w := (A'_\xi - \lambda'_\xi)v_\xi$, by (2.26) and (11.2) we have, for all ξ large enough,

$$v'_\xi = -\int_0^{+\infty} dt\, e^{[L_\xi - (\lambda_\xi - 1)]t} w \tag{11.13}$$

Let $\tau := \xi^2$ and $|\zeta| \le \alpha'$, then by (2.24) and since $0 < \lambda_\xi - 1 \le c_1 e^{-2\alpha\xi}$ there is $c > 0$ so that

$$|v'_\xi + \int_0^\tau dt\, e^{L_\xi t} w\big|_{\zeta,\xi} \le ce^{-2\alpha\xi}\xi^2 \tag{11.14}$$

because $|w|_{\zeta,\xi} < +\infty$. We have

$$\int_0^\tau dt\, e^{L_\xi t}\lambda'_\xi v_\xi = \lambda'_\xi \int_0^\tau dt\, e^{(\lambda_\xi - 1)t} v_\xi$$

hence by (11.4) and (2.17) there is $c > 0$ so that

$$|v'_\xi + \int_0^\tau dt\, e^{L_\xi t} A'_\xi v_\xi\big|_{\zeta,\xi} \le ce^{-\eta\xi}\xi^2 \tag{11.15}$$

Since

$$A'_\xi v_\xi(z) = [\frac{d}{d\xi} p_{m^0_\xi}(z)] \frac{1}{p_{m^0_\xi}(z)} A_\xi v_\xi(z) = [\frac{d}{d\xi} p_{m^0_\xi}(z)] \frac{\lambda_\xi}{p_{m^0_\xi}(z)} v_\xi(z) \tag{11.16}$$

and $\|dp_{m^0_\xi}/d\xi\|_\infty < \infty$ there is $c > 0$ so that for $|\xi - x| > \xi/2$

$$\Big| \int_0^\tau dt\, e^{L^0_\xi t} A'_\xi v_\xi(x) \Big| \le c \int_0^\tau dt\, e^{L^0_\xi t} v_\xi(x) \le c\xi^2 e^{(\lambda_\xi - 1)\tau} v_\xi(x) \le c'\xi^2 v_\xi(x) \tag{11.17}$$

which proves the second inequality in (11.5).

We shorthand

$$\gamma(x) := \mathbf{1}_{|\xi - x| \ge \xi/2} \big(A'_\xi v_\xi \big)(x) \tag{11.18}$$

and write

$$e^{A_\xi t}\gamma = \sum_{n\ge 0} \frac{t^n}{n!} \int dy\, A^n_\xi(x,y)\gamma(y) = \sum_{n \ge 0} \frac{(\lambda_\xi t)^n}{n!} \frac{1}{u_\xi(x)} \int dy\, \gamma(y) u_\xi(y) T^n(y,x) \tag{11.19}$$

where

$$T(y,x) := u_\xi(x) A_\xi(x,y) \frac{1}{\lambda_\xi u_\xi(y)} \tag{11.20}$$

is the transition probability associated to A_ξ via the left eigenfuction u_ξ.

We have

$$T^n(y,x) := \int dz\, T^{n-1}(y,z) T(z,x)$$

$$T(z,x) \le J(z,x) \sup_{|z-x|\le 1} \frac{u_\xi(x)}{u_\xi(z)} \le c$$

having used (4.12), after recalling that $u_\xi(x) = v_\xi(x) p_{m_\xi}(x)^{-1}$. Then

$$\big| e^{A_\xi t}\gamma \big| \le c \sum_{n\ge 0} \frac{(\lambda_\xi t)^n}{n!} \frac{1}{u_\xi(x)} \int dy\, |\gamma(y) u_\xi(y)| \tag{11.21}$$

By (11.7) and (2.18) there is $c > 0$ so that

$$\big|\gamma(y) u_\xi(y)\big| \le c e^{-(2\alpha' + \alpha)|\xi - y|} \mathbf{1}_{|\xi - y| \ge \xi/2} \tag{11.22}$$

By (2.19) and (2.10) for $|\xi - x| \le \xi/2$

$$u_\xi(x)^{-1} \le c e^{\alpha|\xi - x|} \tag{11.23}$$

By (11.21) there are c and c' so that for $|\xi - x| \le \xi/2$

$$\big| e^{A_\xi t}\gamma(x) \big| \le c e^{\lambda_\xi t} e^{-2\alpha'\xi/2} \le c'' e^{-\alpha\xi} e^t \xi \tag{11.24}$$

because, by (2.17), $\lambda_\xi t \le t + (\lambda_\xi - 1)t \le t + ce^{-2\alpha\xi}\xi^2 \le t + c'$. Moreover by (2.18) $\alpha'\xi > \alpha\xi + c$, with c a suitable constant. Therefore for $|\xi - x| \le \xi/2$ we have $e^{L_\xi t}\gamma(x)| \le c'e^{-\alpha\xi}\xi$, then, by (11.15) for any $\eta \in (\alpha, 3\alpha/2)$, there is $c > 0$ so that for all $|\xi - x| \le \xi/2$

$$\left| v'_\xi(x) + \int_0^\tau dt\, [e^{L_\xi t} \mathbf{1}_{|\xi - y| \le \xi/2} A'_\xi v_\xi](x) \right| \le ce^{-\alpha\xi}\xi^3 \tag{11.25}$$

By (11.10) we have for $|y - \xi| \le \xi/2$

$$\left| A'_\xi v_\xi(y) - B'_\xi v_\xi(y) \right| \le ce^{-2\alpha\xi} e^{\alpha|\xi - y|} \sup_{|z-y|\le 1} v_\xi(z) \le c'e^{-2\alpha\xi}$$

Then using (2.19) and (11.25) there is a constant $c > 0$ so that

$$\left| v'_\xi(x) + \int_0^\tau dt\, [e^{L_\xi t} \mathbf{1}_{|\xi - y| \le \xi/2} B'_\xi \tilde{m}'_\xi](x) \right| \le ce^{-\alpha\xi}\xi^3 \tag{11.26}$$

We write

$$\Gamma(x) := \int dy\, \mathbf{1}_{|\xi - x| \le \xi/2} B'_\xi(x, y) \tilde{m}'_\xi(y) \tag{11.27}$$

$$\mu_k(dx_1 ... dx_n) := \mathbf{1}_{x_1 \ge 1} \cdots \mathbf{1}_{x_{k-1} \ge 1} \mathbf{1}_{x_k \in [0,1)} dx_1 \cdots dx_n \tag{11.28}$$

so that

$$\begin{aligned} A^n_\xi \Gamma(x) = & \int dx_1 \cdots dx_n \mathbf{1}_{x_1 \ge 1} \cdots \mathbf{1}_{x_n \ge 1} A_\xi(x, x_1) \cdots A_\xi(x_{n-1}, x_n) \Gamma(x_n) \\ & + \sum_{k=1}^n \int \mu_k(dx_1 ... dx_n) A_\xi(x, x_1) \cdots A_\xi(x_{n-1}, x_n) \Gamma(x_n) \end{aligned} \tag{11.29}$$

Recalling (11.20) and setting

$$S(x, y) := \frac{1}{\lambda_\xi v_\xi(x)} A_\xi(x, y) v_\xi(y) \tag{11.30}$$

we rewrite the k-th term of the sum in (11.29) as

$$\begin{aligned} \lambda^n_\xi \int & \mu_k(dx_1 ... dx_n) T(x_k, x_{k-1}) \cdots T(x_1, x) S(x_k, x_{k+1}) \cdots \\ & \cdots S(x_{n-1}, x_n) \Gamma(x_n) \frac{u_\xi(x_k) v_\xi(x_k)}{u_\xi(x) v_\xi(x_n)} \le c\lambda^n_\xi e^{-2\alpha\xi} e^{\alpha\xi/2} \end{aligned} \tag{11.31}$$

To prove the last inequality we have used (2.18) to bound $u_\xi(x_k)v_\xi(x_k)$; (11.23) to bound $u_\xi(x)^{-1}$. Moreover by (2.19) $\Gamma(x)v_\xi(x)^{-1}$ is bounded and, as we have already seen, $T^n(x, y)$ is bounded for all n, x and y. It is clear that we can treat in a similar way the quantity $B^n_\xi \Gamma(x)$ obtaining the same estimate as (11.31) for the analogous quantity.

We write in the first term on the right hand side of (11.29)

$$A_\xi(x_i,x_{i+1}) = B_\xi(x_i,x_{i+1})\Big([\frac{A_\xi(x_i,x_{i+1})}{B_\xi(x_i,x_{i+1})}-1]+1\Big)$$

Thus the product of the $A_\xi(x_i,x_{i+1})$'s is equal to the product of the $B_\xi(x_i,x_{i+1})$'s times the product of the bracket terms. We then get (writing $x=x_0$)

$$\big|A_\xi^n\Gamma(x)-B_\xi^n\Gamma(x)\big| \le c\lambda_\xi^n e^{-2\alpha\xi+\alpha\xi/2}n + \int_{x_1\ge 1}dx_1\cdots\int_{x_n\ge 1}dx_n B_\xi(x,x_1)\cdots$$
$$\cdots B_\xi(x_{n-1},x_n)\Big|\prod_{i=0}^{n-1}\Big\{1+\Big(\frac{A_\xi(x_i,x_{i+1})}{B_\xi(x_i,x_{i+1})}-1\Big)\Big\}-1\Big||\Gamma(x_n)|$$

There is $\tilde c>0$ so that for all positive x and y

$$\Big|\frac{A_\xi(x,y)}{B_\xi(x,y)}-1\Big|\le \tilde c e^{-\alpha\xi}$$

Then

$$\Big|\prod_{i=0}^{n-1}\Big\{1+\Big(\frac{A_\xi(x_i,x_{i+1})}{B_\xi(x_i,x_{i+1})}-1\Big)\Big\}-1\Big|\le (1+\tilde c e^{-\alpha\xi})^n-1$$

and

$$\int_{x_1\ge 1}dx_1\ldots\int_{x_n\ge 1}dx_n B_\xi(x,x_1)\cdots\cdots B_\xi(x_{n-1},x_n)|\Gamma(x_n)| = \tilde m'(\xi-x)\int_{x_1\ge 1}dx_1\ldots$$
$$\cdots\int_{x_n\ge 1}dx_n P(\xi-x,\xi-x_1)\cdots P(\xi-x_{n-1},\xi-x_n)\frac{|\Gamma(x_n)|}{\tilde m'(\xi-x_n)}\le c$$

for all x and n.

$$\Big|A_\xi^n\Gamma(x)-B_\xi^n\Gamma(x)\Big|\le c\lambda_\xi^n e^{-2\alpha\xi}e^{\alpha\xi/2}n + c\Big\{[1+\tilde c e^{-\alpha\xi}]^n-1\Big\} \tag{11.32}$$

Calling $\epsilon := \tilde c e^{-\alpha\xi}$ we use the inequality

$$(1+\epsilon)^n-1\le \epsilon n(1+\epsilon)^{n-1}$$

Then going back to (11.26)

$$\begin{aligned}\Big|v'_\xi(x)+&\int_0^\tau dt\,[e^{(B_\xi-1)t}\mathbf{1}_{|\xi-y|\le\xi/2}B'_\xi\tilde m'_\xi](x)\Big|\le ce^{-\alpha\xi}\xi^2\\ &+\int_0^\tau dt\,e^{-t}c\sum_{n\ge 0}\frac{t^n}{n!}\Big\{\lambda_\xi^n e^{-3\alpha\xi/2}n+\epsilon n(1+\epsilon)^{n-1}\Big\}\\ &\le ce^{-\alpha\xi}\xi^2 + c\int_0^\tau dt\,e^{-t}\Big\{e^{-3\alpha\xi/2}t\lambda_\xi e^{\lambda_\xi t}+t\epsilon e^{t(1+\epsilon)}\Big\}\\ &\le ce^{-\alpha\xi}\xi^2+c\xi^4e^{-\alpha\xi}\end{aligned} \tag{11.33}$$

By using the same arguments starting from

$$B_\xi \tilde{m}'(\xi - x) = \tilde{m}'(\xi - x)$$

we obtain

$$\Big|\tilde{m}''(\xi - x) + \int_0^\tau dt\, e^{(B_\xi - 1)t} B'_\xi \tilde{m}'(\xi - x)\Big| \le c' e^{-\alpha\xi}\xi^4 \tag{11.34}$$

Theorem 11.1 is proved. □

We conclude the subsection with a corollary of Theorem 11.1.

11.2 Theorem.

There are ξ^0 and $c > 0$ so that for all $\xi \ge \xi^0$ and all $|\xi - x| \le \xi/2$

$$\Big|\frac{d}{d\xi} v_\xi(x) + \frac{d}{dx} v_\xi(x)\Big| \le c e^{-\alpha\xi}\xi^4 \tag{11.35}$$

Proof.

By (11.5)

$$\Big|\frac{d}{d\xi} v_\xi(x) + \frac{d}{dx}\tilde{m}'(\xi - x)\Big| \le c^0 e^{-\alpha\xi}\xi^4 \tag{11.36}$$

For $|\xi - x| \le \xi/2$

$$\frac{d}{dx}\tilde{m}'(\xi - x) = \int dy \Big(\frac{d}{dx}[p_{\bar{m}_\xi}(x) J(x,y)]\Big)\tilde{m}'(\xi - y) \tag{11.37}$$

Recalling the definition of m^0_ξ, there is a constant $c' > 0$ so that for $|\xi - x| \le \xi/2$

$$\Big|\frac{d}{dx} p_{\bar{m}_\xi}(x) - \frac{d}{dx} p_{m^0_\xi}(x)\Big| \le c' e^{-\alpha(\xi + x)}, \qquad \Big|p_{\bar{m}_\xi}(x) - p_{m^0_\xi}(x)\Big| \le c' e^{-\alpha(\xi+x)} \tag{11.38}$$

Moreover by (2.19) (whose validity can be extended to $|\xi - x| \le \xi/2 + 1$) we conclude that for a suitable constant $c'' > 0$

$$\Big|\frac{d}{dx}\tilde{m}'(\xi - x) - \int dy \Big(\frac{d}{dx}[p_{m^0_\xi}(x) J(x,y)]\Big) v_\xi(y)\Big| \le c''[e^{-\alpha 3\xi/2} + e^{-2\alpha\xi + \alpha\xi/2|}\xi^4] \tag{11.39}$$

Then

$$\Big|\frac{d}{dx}\tilde{m}'(\xi - x) - \lambda_\xi \frac{d}{dx} v_\xi(x)\Big| \le 2c'' e^{-\alpha 3\xi/2}\xi^4 \tag{11.40}$$

(11.35) then follows from (11.36), (11.40) and (2.20). Theorem 11.2 is proved. □

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