Critical droplet for a non local mean field equation

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Abstract. In this paper we prove the existence of a stationary, spatially non homogeneous solution of a non local, mean field equation which describes the phases of a fluid.

1. Introduction

The main goal of this paper is to prove the existence of the critical droplet which marks the saddle point in the path from the metastable to the stable state; the work is done in the context of one dimensional Ising spins with Kac interactions and Glauber dynamics. By exploiting the spectral estimates derived in the first paper of this series, [**2**], we will prove here the existence of the critical droplet. The series will continue with papers on the existence of one dimensional manifolds which connect the critical droplet respectively to the stable and to the metastable states. In the final paper we will then prove that the preferred path of the spin system will follow the time reversed motion along the latter manifold and, after "jumping" across the critical droplet it will "land" on the other manifold following the classical motion toward the stable state.

We start with a brief description of the underlying spin model, even though our analysis refers only to a deterministic, mesoscopic equation which describes the limit behavior of the stochastic Glauber dynamics on the spins. The spin model can be described as follows. At each side of the one-dimensional lattice there is a spin variable with two different values ± 1. The value of the spin at x is flipped at rates which depend on the value of the others spins. This interaction is long range, of order γ^{-1}, with γ small. More precisely, the interaction among two given spin at x and y is $\gamma J(\gamma|x-y|)$, where $J(r)$ $r \in R$ is a smooth symmetric, probability density (thus $J \geq 0$) with compact support, the precise assumptions are stated in Definition 2.1 below. These type of interactions have been studied in [**5**], [**6**], in order to derive the van der Waals phase transition theory. Lebowitz and Penrose in [**7**] give a rigorous explanation of metastability in this context.

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The free energy density of the spatially homogeneous spin state with constant magnetization s is given by

$$f(s) = -\frac{1}{2}s^2 - hs - \beta^{-1} i(s), \qquad s \in [-1,1] \tag{1.1}$$

$$i(s) = -\frac{1+s}{2}\log\{\frac{1+s}{2}\} - \frac{1-s}{2}\log\{\frac{1-s}{2}\} \tag{1.2}$$

where $-s^2/2$ represents the internal energy density, $-hs$ is the energy density due to the external field h and $i(s)$ is its entropy (which is the entropy of a $\{\pm 1\}$ Bernoulli process with average s).

The stationary points of the mean field free energy density solve the following mean field equation in $\mathbb{R}$

$$s = \tanh\big(\beta[s+h]\big) \tag{1.3}$$

Given $\beta > 1$ there is $h^\star > 0$ so that for $0 \le h < h^\star$ (1.3) has three and only three different roots, denoted by

$$m_\beta^-(h) < m_\beta^0(h) \le 0 < m_\beta^+(h) \tag{1.4}$$

For $h > 0$, $|m_\beta^-(h)| < m_\beta^+(h)$ and $m_\beta^0(h) < 0$; for $h = 0$, $m_\beta^0(0) = 0$ and $m_\beta^+(0) = -m_\beta^-(0) =: m_\beta$.

The diagram of $f(s)$ versus s at $h = 0$ changes when β increases past 1 from a single to a double well shape with two minimizers, $m_\beta^\pm(0)$. As h increases past 0 at $\beta > 1$ fixed, the two roots $m_\beta^\pm(0)$ move to $m_\beta^\pm(h)$ with $f\big(m_\beta^+(h)\big) < f\big(m_\beta^-(h)\big)$, hence $m_\beta^+(h)$ is the only minimizer while $m_\beta^-(h)$ is just a local minimum. For this reason they are interpreted respectivley as the magnetizations in the stable and in the metastable phases.This interpretation is fully justified by the analysis of Ising spin system with Kac potentials, [**7**].

As mentioned in the beginning, our ultimate purpose is to characterize the tunnelling from the metastable to the stable phase in the above described, $d = 1$ Ising system with Glauber dynamics and Kac interactions. As we will see, the metastable phase is stable under small perturbations and therefore it will persist for a long time and till when, due to a large deviation event, a large enough region becomes occupied by the stable phase. Afterwards this region expands indefinitely and the stable phase grows everywhere. The intuition that instead the system goes to the stable phase passing through the intermediate phase $m_\beta^0(h)$ is definitely wrong. In fact the nucleation of the stable phase from the metastable one is characterized by the formation of "a critical droplet" which breaks the spatial homogeneity, and, in our case, it is given by the bump $m_J^\star$, solution of the following non local, one dimensional equation (in $L^\infty(\mathbb{R};[-1,1])$)

$$m_J^\star(x) = \tanh\Big(\beta[J \star m_J^\star(x) + h]\Big) \tag{1.5}$$

with asymptotic conditions

$$\lim_{|x|\to\infty} m_J^\star(x) = m_\beta^-(h) \tag{1.6}$$

In (1.5) we have used the notation

$$(f \star g)(x) := \int_{\mathbb{R}} dy f(x-y) g(y) \tag{1.7}$$

Observe that (1.5) is a space-dependent version of the mean field equation.

In this paper we will prove the existence of "the bump" $m_J^\star$ for $\beta > 1$ and $h > 0$ small enough using Newton's method as we are going to explain in the next Section.

2. Definitions, results, outline of proofs

We first state the assumptions on the probability density J in (1.5):

Definition 2.1. $J(x)$, $x \in \mathbb{R}$, is a $C^3(\mathbb{R})$, symmetric, non negative, non increasing in $\mathbb{R}$ function such that

$$\int_{\mathbb{R}} dy\, J(y) = 1 \tag{2.1}$$

$$\sup \{x \in \mathbb{R} : J(x) > 0\} \;=\; 1 \tag{2.2}$$

We will study (1.5) perturbatively around $h = 0$. However, when $h = 0$ there is no critical droplet and no metastable phase as well, because $m_\beta^\pm(0) \equiv m_\beta^\pm$ are both minimizers of the free energy (1.1). There are however many spatially non homogeneous solutions of (1.5), the relevant one for our purposes is the instanton $\bar{m}(x)$, whose asymptotes at $\pm\infty$ are $m_\beta^\pm$. Its existence has been proved in [**1**], [**4**], its stability properties have been derived in [**3**], [**4**] and in Section 3 of [**2**]. The main properties of the instanton are:

Theorem 2.2 ([**3**], [**4**], [**2**])**.** There exists a solution $\bar{m}_J(x)$ of (1.5) with $h = 0$ and $\beta > 1$. (To avoid heavy notations we will write $\bar{m}$ instead of $\bar{m}_J$).

$$\bar{m}(x) = \tanh\Big(\beta J \star \bar{m}(x)\Big) \tag{2.3}$$

which is a C^∞, strictly increasing, antisymmetric function with asymptotes

$$\lim_{x\to\pm\infty} \bar{m}(x) = \pm m_\beta \tag{2.4}$$

$\bar{m}(x)$ is, modulo translation, the unique solution of (2.3) with asymptotes (2.4). Moreover, letting $\alpha > 0$ be such that

$$\beta(1 - m_\beta^2) \int dy J(y - x) e^{-\alpha(x-y)} = 1 \tag{2.5}$$

there are, $a > 0$, $\alpha_0 > \alpha$ and $c > 0$ so that for all $x \ge 0$

$$\Big|\bar{m}(x) - (m_\beta - a e^{-\alpha x})\Big| + \Big|\bar{m}'(x) - a\alpha e^{-\alpha x}\Big| + \Big|\bar{m}''(x) + \alpha^2 a e^{-\alpha x}\Big| \le c e^{-\alpha_0 x} \tag{2.6}$$

where $\bar{m}'$ and $\bar{m}''$ are respectively the first and second derivatives of $\bar{m}$.

Our main result is

Theorem 2.3. Given $\beta > 1$, there is $h_0 > 0$ and for any $h \in (0, h_0]$ there is $m_J^\star \in C^{\rm sym}(\mathbb{R}; [-1, 1])$ (i.e. the space of symmetric, continuous functions with range in $[-1, 1]$) which solves (1.5) with asymptotes as in (1.6). (In the sequel we will write $m^\star$ instead of $m_J^\star$).

Moreover there are $\xi^\star = \xi^\star(h)$ and $c > 0$ so that

$$\lim_{h\to 0^+} \sup_{x\le 0} \Big| m^\star(x) - \bar m(x+\xi^\star)\Big| = 0 \tag{2.7}$$

$$\lim_{h\to 0^+} e^{2\alpha\xi^\star(h)} h = c \tag{2.8}$$

where α is as in (2.5).

We may think of (1.5) as a fixed point problem relative to the map on the space of functions $m(x)$ given by the r.s.h. of the equation itself. Such an approach was used by Dal Passo and de Mottoni, [**1**], to prove the existence of the instanton (when $h = 0$). After setting the problem in the class of antisymmetric functions, they have showed that the transformation $m \to \tanh\{\beta J \star m\}$ is a contraction, in a suitable norm and in a suitable neighborhood of $\bar m$.

When $h > 0$ is small it is natural to conjecture that the instanton has still an important role. We then introduce, for any $\xi > 1$, the symmetric function given for non negative x by

$$m^0_\xi(x) := \bar m_\xi(x) - a e^{-\alpha(x+\xi)}, \qquad \bar m_\xi(x) = \bar m(\xi - x) \tag{2.9}$$

Except for the second term (whose origin is technical) m^0_ξ is an instanton on the negative half line shifted by $-\xi$ and its mirror image on the positive half line. A neighborhood of m^0_ξ (with $\xi > 0$ to be chosen suitably as a function of h) is therefore the natural place where to look for the bump $m^\star$. We cannot hope however to have a contraction property like in [**1**]: the analysis in [**2**] shows in fact that the transformation $m \to \tanh\{\beta(J\star m + h)\}$ is expansive (in some direction) in a neighborhood of m^0_ξ.

The natural alternative is to use the Newton method, whose application, however, is not at all straightforward in our case. We need in fact, from one side, h small in order to use the perturbative analysis of [**2**] but, on the other side, if h is small the derivative operator involved in the Newton method has an eigenvalue very close to zero. We are thus confronted with a small divisors problem. Let us briefly recall the Newton method in its more elementary, one real variable setting, underlining the questions concerning the small divisors.

The Newton method

The method is designed for computing the zeros of a smooth function $f(x)$ in terms of the limit points of the orbits of a map T (the Newton map). $T(x)$ is defined on $\{z : f'(z) \ne 0\}$ as the solution of the linear equation

$$f(x) + f'(x)[T(x) - x] = 0 \tag{2.10}$$

The search of a zero is succesful if one has a point x_0 for which the Newton orbit $T^n(x_0)$ is well defined for all $n \ge 1$ and converges to a point $x^\star$ where $f'(x^\star) \ne 0$. $x^\star$ is then the required zero because, by the continuity of T, $T(x^\star) = x^\star$, hence, by (2.10), $f(x^\star) = 0$. All the zeros $x^\star$ such that $f'(x^\star) \ne 0$ can in principle be found with this procedure because any point x_0 sufficiently close to $x^\star$ gives rise to a succesful Newton orbit. This is evidently not too usefull as it requires an a-priori knowledge of the unknown zero $x^\star$. The truly constructive criteria are those which involve x_0 alone, like the following one whose formulation takes care of the small denominator problems that arise in our case when the magnetic field becomes small. In Section 4, Theorem 4.2, we will prove an infinite dimensional version of the following lemma:

Lemma 2.4. There exist universal, positive constants c, c', δ, ϵ' so that if f, x_0 and $\epsilon \in (0, \epsilon']$ satisfy the relations

$$|f(x_0)| \le c\epsilon^{2+\delta}, \quad f'(x_0) \ge c'\epsilon, \quad \|f''\|_\infty \le c'' \tag{2.11}$$

then $\{T^n(x_0)\}_{n\ge 1}$ is a succesful Newton orbit.

In our setup

$$f(m) := -m + \tanh\{\beta J \star m + \beta h\} \tag{2.12}$$

Let L_m be the operator on $C^{\rm sym}(\mathbb{R}; [-1,1])$ defined by,

$$L_m(\psi) := -\psi + p_m J \star \psi; \qquad p_m(x) := \frac{\beta}{\cosh^2\{\beta J \star m(x)\}} \tag{2.13}$$

then L_{m+h} is the derivative $Df|_m$ of $f(m)$ at m, recall in fact that

$$Df\big|_m(\psi) = \frac{d}{dt} f(m + t\psi)\big|_{t=0} \tag{2.14}$$

Similarly to the one dimensional case the Newton map is

$$m \to T(m) = m + \psi, \qquad \psi = -L_{m+h}^{-1} f(m) \tag{2.15}$$

To be well defined we need the invertibility of L_{m+h}, which has been proved in [**2**] in a suitable neighborhood of m^0_ξ. In the next section, Section 3, we will recall this and other properties proved in [**2**] which will be used in the rest of the paper. In Section 4 we will state and prove the analogue of 2.4. The important point in the whole method is then to find the good starting point, a problem that we will solve in the remaining sections.

3. Some spectral properties

In this section we report properties proved in [**2**]. about the spectrum of L_m, when m is in a neighborhood of m^0_ξ, properties that will be crucial for the applicability of the Newton method and for proving the existence of the bump $m^\star$, as explained in the previous section.

Definition 3.1. We fix $\xi > 1$ and, given $m \in C^{\rm sym}(R; [-1,1])$, we set

$$\delta^0_\xi m = m - m^0_\xi \tag{3.1}$$

We then define $G_{(c,\xi)}$, $c>0$, as the set of all m in $C^{\rm sym}(R,[-1,1])$ such that

$$\left|\delta_\xi^0 m(x)\right| \le c \begin{cases} e^{-2\alpha\xi}e^{\alpha(\xi-x)} & \text{for } 0\le x\le \xi \\ e^{-2\alpha\xi} & \text{for } \xi<x \end{cases} \tag{3.2}$$

We will also consider a subset $G_{(c,\xi,\delta)}$ of $G_{(c,\xi)}$, $\delta>0$, made of all m such that

$$-\int_{|x-\xi|\le \xi^{1/2}} dx\, \delta_\xi^0 m(x)\bar m'(\xi-x)^2\bar m(\xi-x) > -ce^{-2(\alpha+\delta)\xi} \tag{3.3}$$

A first result on the spectrum of L_m for general m is

Theorem 3.2 ([**2**])**.** Let $m\in C^{\rm sym}(R,[-1,1])$. Then there are $\lambda_m>-1$, u_m and v_m in $C^{\rm sym}(\mathbb{R})$, u_m and v_m strictly positive, so that

$$L_m v_m = \lambda_m v_m, \quad u_m L_m = \lambda_m u_m \tag{3.4}$$

and for any $x\ge 0$

$$v_m(x) = p_m(x)u_m(x) \tag{3.5}$$

Any other point of the spectrum is strictly inside the ball of radius λ_m.

We define $\tilde m(x) := \sqrt{C_{\bar m}}\bar m(x)$, where $C_{\bar m}$ is a constant such that

$$\int_{\mathbb{R}} dx \frac{\tilde m'(x)^2}{p_{\bar m}(x)} = 1 \tag{3.6}$$

and set $\tilde m'_\xi(x) = \tilde m'(\xi-x)$. We also normalize $u_m(x)$ (and then v_m) in such a way that

$$\int_0^\infty \frac{dx}{p_m(x)} v_m(x)^2 \equiv \int_0^\infty dx\, u_m(x)v_m(x) = 1 \tag{3.7}$$

We then have

Theorem 3.3 ([**2**])**.** For any $c>0$ there are $c_\pm$ and c' all positive so that for all $\xi>1$ and all $m\in G_{(c,\xi)}$

$$-c_- e^{-2\alpha\xi} \le \lambda_m \le c_+ e^{-2\alpha\xi} \tag{3.8}$$

$$u_m(x),\ v_m(x) \ \le\ c_+ e^{-\alpha'|\xi-x|}, \qquad \alpha'=\alpha'(\xi):=\alpha-c'e^{-2\alpha\xi} \tag{3.9}$$

$$\left|v_m(x)-\tilde m'_\xi(x)\right| \le c_+ e^{-2\alpha\xi+\alpha|\xi-x|}\xi^4, \qquad \text{for all } x \text{ such that } |\xi-x|\le \xi/2 \tag{3.10}$$

There is $\hat\xi$ so that for any $c>0$, $\delta>0$ $\xi>\hat\xi$ and $m\in G_{(c,\xi,\delta)}$

$$\lambda_m \ge \frac{D}{2}e^{-2\alpha\xi} \tag{3.11}$$

where

$$D := 2\beta\int_{\mathbb{R}} dx\, \frac{\tilde m'(x)^2}{p_{\bar m}(x)}\bar m(x) a\sinh(\alpha x) \tag{3.12}$$

Moreover

$$\Big|\frac{dv_{m^0_\xi}}{d\xi} - \tilde m''(\xi - x)\Big| \le c_0 e^{-\alpha\xi}\xi^4 \qquad \text{for} |\xi - x| \le \xi/2 \tag{3.13}$$

$$\Big|\frac{dv_{m^0_\xi}}{d\xi}\Big| \le c_0 \xi^2 v_{m^0_\xi} \qquad \text{for } |\xi - x| > \xi/2 \tag{3.14}$$

Finally,

$$\Big|\frac{dv_{m^0_\xi}(x)}{d\xi} + \frac{dv_{m^0_\xi}(x)}{dx}\Big| \le c e^{-\alpha\xi}\xi^4 \tag{3.15}$$

Given $m \in C^{\rm sym}(R, [-1, 1])$, we define the linear functional π_m on $C(R_+)$ as

$$\pi_m(w) := \int_0^\infty dx\, u_m(x) w(x) \tag{3.16}$$

(u_m normalized as in (3.7)).

Theorem 3.4 ([**2**])**.** Given $c > 0$ there are $d_\pm > 0$, and $\xi^\star > 1$, so that for any $\xi \ge \xi^\star$, $m \in G_{(c,\xi)}$ and $t \ge 0$

$$\|e^{L_m t}\|_\infty \le d_+ e^{\lambda_m t} \tag{3.17}$$

and, for any $\tilde w \in C(R_+)$ such that $\pi_m(\tilde w) = 0$,

$$\|e^{L_m t}\tilde w\|_\infty \le d_+ e^{-d_- t}\|\tilde w\|_\infty \tag{3.18}$$

Moreover given $\delta > 0$ there is $C > 0$ so that if m is in $G_{(c,\xi,\delta)}$, then the inverse $(L_m)^{-1}$ exists and

$$\|(L_m)^{-1}\|_\infty \le C e^{2\alpha\xi} \tag{3.19}$$

If $\tilde w$ is such that $\pi_{m,\xi}(\tilde w) = 0$ then

$$(L_m)^{-1}\tilde w = -\int_0^{+\infty} dt e^{L_m t}\tilde w, \qquad \|(L_m)^{-1}\tilde w\|_\infty \le C\|\tilde w\|_\infty \tag{3.20}$$

4. The Newton method and the bump

In this section we will prove that for h small enough there is $m \in C^{\rm sym}(\mathbb{R}; [-1, 1])$ such that (recall the definition (2.15)) the Newton orbit $\{T^n(m)\}_{n\ge 0}$ is succesful, namely it is well defined for all n and it converges as $n \to \infty$ to the bump $m^\star$ (i.e. a solution of (1.5)). The proof is divided into two parts. In the first one, which is the argument of this section, we prove that if m satisfies the criterion stated in Definition 4.1 below, then $T^n(m) \to m^\star$, see Theorem 4.2. Roughly speaking the criterion requires that m should be close to m^0_ξ with

$\xi \approx \log h^{-1}$ and moreover that $\|f(m)\|_\infty \le o(h^2)$. In Section 7 we construct a function which in Section 8 is shown to fulfill the criterion of Definition 4.1.

Definition 4.1. Recalling the Definition 3.1, we let $F(c,\delta,h,\zeta)$, $c>1$, δ, ζ positive, be the set of all $m \in C^{\rm sym}(\mathbb{R};[-1,1])$ such that $m+h \in G_{(c,\xi,\delta)}$ with ξ such that

$$c^{-1}h \le e^{-2\alpha\xi} \le ch \tag{4.1}$$

and furthermore

$$\|f(m)\|_\infty \le ch^{2+\zeta} \tag{4.2}$$

Theorem 4.2. For any $C>1$ and any δ and ζ positive, there are C' and h' positive so that if $h \le h'$ and $m \in F(C,\delta,h,\zeta)$ then for any $n \ge 1$ $T^n(m)$ is well defined,

$$\|T^n(m) - T^{n-1}(m)\|_\infty \le C'h[C'h^\zeta]^{2^n} \tag{4.3}$$

and

$$\lim_{n\to\infty} \|T^n(m) - m^\star\|_\infty = 0 \tag{4.4}$$

where $m^\star \in C^{\rm sym}(\mathbb{R};[-1,1])$ solves (1.5) and (1.6).

Proof.

In the course of the proof we will determine h'. Setting $m_n := T^n(m)$ we will first prove by induction on n that there is $\delta_1 > 0$ so that, for all $h \le h'$, $m_n + h \in G_{(2C,\xi,\delta_1)}$. Suppose that this is true for $1 \le n \le N$, $N \ge 1$. Then by Theorem 3.4 (which requires ξ large enough, a condition which is satisfied by choosing h' small enough and recalling that $e^{-2\alpha\xi} \le Ch$) there is $c>0$ so that by (3.19)

$$\|L^{-1}_{m_n+h}\|_\infty \le ch^{-1} \tag{4.5}$$

Setting $\psi_n := m_n - m_{n-1}$, $m_0 = m$, by definition (of the Newton map T)

$$\psi_n = -L^{-1}_{m_{n-1}+h} f(m_{n-1}), \qquad \|f(m_n)\|_\infty \le c''\|\psi_n\|^2_\infty \tag{4.6}$$

The second inequality is obtained by expanding to second order in ψ_n the function $f(m_n) = f(m_{n-1}+\psi_n)$ and observing that the 0-th and the first order terms vanish by the first relation in (4.6). Since ψ_n is defined in terms of $L_{m_{n-1}}$, (4.6) holds till $n = N+1$. By (4.5) we get from (4.6)

$$\|f(m_n)\|_\infty \le c''[ch^{-1}\|f(m_{n-1})\|_\infty]^2 \tag{4.7}$$

Calling

$$g_n := \|f(m_n)\|_\infty [(c'')^{1/2} ch^{-1}]^2 \tag{4.8}$$

we get

$$g_n \le g^2_{n-1}, \qquad g_n \le g_0^{2^n}, \;\; g_0 := \|f(m)\|_\infty [(c'')^{1/2} ch^{-1}]^2 \tag{4.9}$$

Thus

$$\|f(m_n)\|_\infty \le (c'')^{-1} c^{-2} h^2 [c'' c^2 h^{-2} C h^{2+\zeta}]^{2^n} \le c_0^{2^n} h^{2+2^n\zeta} \tag{4.10}$$

which holds for $n \le N+1$, $h \le h_1$ and with $c_0 > 0$ a suitable constant which depends on C. By (4.5) and (4.6)

$$\|\psi_{n+1}\|_\infty \le c c_0^{2^n} h^{1+2^n\zeta} \tag{4.11}$$

For all h small enough (this is the last condition on h')

$$c\sum_{n\ge 0}(c_0 h^\zeta)^{2^n} \le h^{\zeta/2} \tag{4.12}$$

Then

$$\phi_{N+1} := m_{N+1} - m = \psi_1 + \cdots + \psi_{N+1}, \qquad \|\phi_{N+1}\|_\infty \le h^{1+\zeta/2} \tag{4.13}$$

We need to prove that $m_{N+1} + h \in G_{(2C,\delta_1,\xi)}$. Let

$$\Psi_{N+1} := m_{N+1} + h - m^0_\xi, \qquad \Psi := m + h - m^0_\xi \tag{4.14}$$

Then

$$\Psi_{N+1} = \Psi + \phi_{N+1} \tag{4.15}$$

so that recalling that $h \le Ce^{-2\alpha\xi}$ and that $m + h \in G_{(c,\xi,\delta)}$

$$\begin{aligned}\|\Psi_{N+1}(x)e^{-\alpha|\xi-x|}\|_\infty &\le \|\Psi(x)e^{-\alpha|\xi-x|}\|_\infty + [Ce^{-2\alpha\xi}]^{1+\zeta/2}\\ &\le Ce^{-2\alpha\xi}\Big(1 + C^{\zeta/2}e^{-\alpha\zeta\xi}\Big) \le 2Ce^{-2\alpha\xi}\end{aligned} \tag{4.16}$$

if h' is small enough. We have thus proved that the first inequality in (3.2) is satisfied by $m_{N+1} + h$ with $c = 2C$.

For the second one we have

$$\begin{aligned}\sup_{x\ge\xi}|\Psi_{N+1}(x)| &\le \sup_{x\ge\xi}|\Psi(x)| + \|\phi_{N+1}\|_\infty \le Ce^{-2\alpha\xi} + [Ce^{-2\alpha\xi}]^{1+\zeta/2}\\ &\le 2Ce^{-2\alpha\xi}\end{aligned} \tag{4.17}$$

Thus (3.2) is verified by $m_{N+1} + h$ with $c = 2C$.

We will next prove that also (3.3) is verified. We write

$$\begin{aligned}-\int_{|\xi-x|\le\xi^{1/2}} dx\, \Psi_{N+1}(x)\bar m'(\xi-x)^2\bar m(\xi-x) &\ge -Ce^{-(2\alpha+\delta)\xi} - [Ce^{-2\alpha\xi}]^{1+\zeta/2}c\\ &\ge -2Ce^{-(2\alpha+\delta_1)\xi}\end{aligned}$$

with

$$\delta_1 := \frac{1}{2}\min\{\delta, \frac{\alpha\zeta}{2}\} \tag{4.18}$$

and for h' small enough. We have thus concluded the proof by induction that $m_n + h \in G_{(2C,\xi,\delta_1)}$ for all $n \ge 1$. (4.3) is then proved by (4.11) and (4.12). Moreover

$$m^\star := \lim_{n\to+\infty} m_n = m^0_\xi + \sum_{n\ge 1}\psi_n \tag{4.19}$$

is well defined and since $f(m)$ is a continuous function of m in the topology of the sup norm,

$$f(m^\star) = \lim_{n\to+\infty} f(m_n) = 0 \tag{4.20}$$

by (4.6).

Theorem 4.2 is proved. □

5. Ansatz on the starting point of the Newton orbit

In this section we make an educated ansatz on the starting point m of the Newton orbit. In Sections 6, 7 and 8 we will prove that m is indeed a good choice, as it will be shown to satisfy the conditions of Definition 4.1. Then, according to Theorem 4.2, the Newton orbit $T^n(m)$ converges to a bump, $m^\star$, as $n \to +\infty$ and Theorem 2.2 is proved.

The 0-th order approximation

We use a perturbative method close in spirit to the Chapmann-Enskogg expansion of the Boltzmann equation. The 0-th order approximation will be $m^0_\xi(x)$, which is defined in (2.9). Why $m^0_\xi(x)$ and not $m_\xi(x)$? ($m_\xi(x)$ the symmetric function obtained from $m^0_\xi(x)$ by dropping the somewhat misterious last term in (2.9)). To compare the two, let

$$f_0(m) = -m + \tanh\{\beta J \star m\}$$

(i.e. $f(m)$ after dropping h from the argument of the hyperbolic tangent)

$$b_\xi = f_0(m^0_\xi) \equiv \tanh\{\beta J \star m^0_\xi\} - m^0_\xi, \qquad \tilde b_\xi = f_0(m_\xi) \tag{5.1}$$

$\tilde b_\xi(x) = 0$ for $x \ge 1$, so that in this region m_ξ is definitely the best choice. However $\tilde b_\xi(x) \approx e^{-\alpha\xi}$ for $x \in [0,1]$. Instead $b_\xi(x) \ne 0$ when $x \ge 1$ but $|b_\xi(x)| \le ce^{-\alpha_0\xi}$ and such a better behavior at $x \in [0,1]$ compensates for being worse elsewhere making m^0_ξ overall a better choice than m_ξ.

In Lemma 5.1 below we will not only prove that $|b_\xi(x)|$ is bounded in $[0,1]$ by $ce^{-\alpha_0\xi}$, but also that it is bounded by $ce^{-2\alpha\xi}$ in $[1,\xi]$ and by $ce^{-2\alpha\xi}e^{-2\alpha(x-\xi)}$ in $[\xi,\infty)$.

Let $k_\xi \in C^{\rm sym}(\mathbb{R})$ be defined for $x \ge 0$ as

$$k_\xi(x) := e^{-2\alpha\xi} k^0_\xi(x) \tag{5.2}$$

where $k^0_\xi(x) = k^0(\xi - x)$ and

$$k^0(y) := \frac{a}{1-m_\beta^2}[m_\beta + \bar m(y)]\{e^{\alpha y}[m_\beta - \bar m(y)]\} \tag{5.3}$$

Lemma 5.1. Let $\alpha_0 > \alpha$ be as in Theorem 2.2. Then there is $c > 0$ so that for all ξ and $x \ge 0$

$$\Big|b_\xi(x) + k_\xi(x)\Big| \le c\Big(e^{-2\alpha(\xi+x)} + \mathbf{1}_{0\le x\le 1} e^{-\xi\alpha_0}\Big) \tag{5.4}$$

$$\Big|\frac{d}{d\xi}\big(e^{2\alpha\xi}b_\xi(x)\big) + \frac{d}{dx}\big(e^{2\alpha\xi}b_\xi(x)\big)\Big| \le c\Big(e^{(2\alpha-\alpha_0)\xi}\mathbf{1}_{0\le x\le 1} + e^{-2\alpha\xi}e^{2\alpha(\xi-x)}\Big) \tag{5.5}$$

Proof.
As $m^0_\xi(x)$ is continued symmetrically through the origin, the expression on the r.h.s. of (2.9) does not remain valid for $x < 0$. We denote by $R_\xi(x)$ their difference, setting however $R_\xi(x) = 0$ for $x < -1$, thus $R_\xi(x) = 0$ unless $x \in [-1, 0)$, where

$$R_\xi(x) = m^0_\xi(x) - [\bar m(\xi - x) - ae^{-\alpha(x+\xi)}] = [\bar m(x+\xi) - ae^{-\alpha(\xi-x)}] - [\bar m(\xi - x) - ae^{-\alpha(x+\xi)}]$$

We can then write

$$R_\xi(x) = \mathbf{1}_{x\in[-1,0]}\Big\{[m_\beta - ae^{-\alpha(\xi-x)} - \bar m(\xi-x)] + [\bar m(x+\xi) - m_\beta + ae^{-\alpha(x+\xi)}]\Big\} \tag{5.6}$$

Then, by Theorem 2.2, there are $c>0$ and $\alpha_0>\alpha$ so that

$$|R_\xi(x)| \le ce^{-\alpha_0\xi} \tag{5.7}$$

For $x \ge 0$,

$$\begin{aligned} b_\xi(x) = \tanh\Big(\beta\int dy\, J(y-x)\{\bar m(\xi-y) - ae^{-\alpha(y+\xi)} + R_\xi(y)\}\Big) \\ -\tanh\Big\{\beta\int dy\, J(y-x)\bar m(\xi-y)\Big\} + ae^{-\alpha(x+\xi)} \end{aligned} \tag{5.8}$$

Recalling the definition (2.13) of L_m, after a Taylor expansion we get

$$b_\xi = L_{\bar m_\xi}\big(-ae^{-\alpha(x+\xi)}\big) + G_\xi \tag{5.9}$$

where

$$G_\xi = p_{\bar m_\xi} J\star R_\xi + \frac{1}{2} z_{\bar m_\xi}[J\star e^{-\alpha(y+\xi)}]^2 + H_\xi \tag{5.10}$$

$$z_{\bar m_\xi} := \beta\tanh''\{\beta J\star \bar m_\xi(x)\} \tag{5.11}$$

with $H_\xi(x)$ the remainder, so that

$$|H_\xi(x)| \le ce^{-3\alpha(x+\xi)} + e^{-2\alpha_0\xi}\mathbf{1}_{x\in[0,1]} \tag{5.12}$$

Then

$$|G_\xi(x)| \le c[e^{-2\alpha(x+\xi)} + \mathbf{1}_{x\in[0,1]}e^{-\alpha_0\xi}] \tag{5.13}$$

with $c>0$ a suitable constant.

Recalling that $\bar m_\xi$ is defined in (2.9), we have

$$L_{\bar m_\xi}\big(e^{-\alpha(x+\xi)}\big) = e^{-2\alpha\xi}e^{\alpha(\xi-x)}\frac{m_\beta^2 - \bar m(\xi-x)^2}{1-m_\beta^2} \tag{5.14}$$

because, by (2.5),

$$\beta\int dy\, J(y-x)e^{\alpha(y-x)} = \frac{1}{1-m_\beta^2} \tag{5.15}$$

(5.4) follows from (5.9), (5.14) and (5.13).

By (5.1) and (5.8) we have for $x \ge 0$

$$\begin{aligned} e^{2\alpha\xi}b_\xi(x) = e^{2\alpha\xi}\tanh\Big(\beta\int dy\, J(y-x)\{\bar m(\xi-y) - ae^{-2\alpha\xi}e^{\alpha(\xi-y)} + R_\xi(y)\}\Big) \\ -e^{2\alpha\xi}\bar m(\xi-x) + ae^{\alpha(\xi-x)} \end{aligned}$$

Then

$$e^{-2\alpha\xi}\frac{d}{d\xi}\Big(e^{2\alpha\xi}b_\xi(x)\Big) = I_1 + \cdots + I_5 \tag{5.16}$$

where

$$I_1 := 2\alpha\Big(\tanh\big\{\beta J\star m^0_\xi(x)\big\} - \bar m(\xi-x)\Big) \tag{5.17}$$

$$I_2 := 2\alpha\cosh^{-2}\big\{\beta J\star m^0_\xi(x)\big\}\beta\int dy\, J(y-x)[ae^{-2\alpha\xi}e^{\alpha(\xi-y)}] \tag{5.18}$$

$$I_3 := \cosh^{-2}\{\beta J\star m^0_\xi(x)\}\beta\int dy\, J'(y-x)\big(m^0_\xi(y)-R_\xi(y)\big) \tag{5.19}$$

where $J'(y-x)$ denotes the derivative of $J(y-x)$ with respect to x.

$$I_4 := \frac{d}{dx}m^0_\xi(x) \tag{5.20}$$

$$I_5 := \cosh^{-2}\{\beta J\star m^0_\xi(x)\}\frac{d}{d\xi}\Big(\beta J\star R_\xi(x)\Big) \tag{5.21}$$

We have

$$\Big|I_1+I_2\Big| \le c\Big(\mathbf{1}_{x\in[0,1]}\|R_\xi\|_\infty + [e^{-2\alpha\xi}e^{\alpha(\xi-x)}]^2\Big) \tag{5.22}$$

$$\begin{aligned} I_3+I_4 &= -\frac{d}{dx}\tanh\{\beta J\star m^0_\xi\} + \frac{d}{dx}m^0_\xi(x) + \cosh^{-2}\{\beta J\star m^0_\xi(x)\}\beta J\star R'_\xi(x)\\ &= -\frac{d}{dx}b_\xi(x) + \cosh^{-2}\{\beta J\star m^0_\xi(x)\}\beta J\star R'_\xi(x)\end{aligned}$$

where $R'_\xi(x) = dR_\xi(x)/dx$. By (5.6) we get for $x\in[-1,0]$

$$\frac{d}{dx}R_\xi(x) = \bar m'(x+\xi) - a\alpha e^{-\alpha(\xi-x)} + \bar m'(\xi-x) - a\alpha e^{-\alpha(x+\xi)} \tag{5.23}$$

$$\frac{d}{d\xi}R_\xi(x) = \bar m'(x+\xi) + a\alpha e^{-\alpha(\xi-x)} - \bar m'(\xi-x) - a\alpha e^{-\alpha(x+\xi)} \tag{5.24}$$

By (2.6) the right hand sides are bounded by $ce^{-\alpha_0\xi}$, c a suitable constant. With the first expression we complete the bound of I_3+I_4 and with the second one that of I_5. In this way we prove (5.5).

Lemma 5.1 is proved. □

The next orders of the perturbative expansion.

We write

$$m = m^0_\xi + \psi_1 + \psi_2 \tag{5.25}$$

and want to determine ξ, m^0_ξ, ψ_1 and ψ_2 as 0-th, first and second order approximations to the bump. According to perturbation theory we will determine recursively the various order of approximation. We write

$$f(m) = \tanh\{\beta J\star(m^0_\xi+\psi_1+\psi_2)+\beta h\} - \tanh\{\beta J\star m^0_\xi\} + b_\xi - (\psi_1+\psi_2) \tag{5.26}$$

with b_ξ defined in (5.1).

We consider ψ_1, ψ_2 and h as "infinitesimals" and we Taylor expand to second order. Writing L_ξ and p_ξ for $L_{m^0_\xi}$ and $p_{m^0_\xi}$ we get

$$\begin{aligned} f(m) = L_\xi(\psi_1+\psi_2) + hp_\xi + b_\xi + \frac{1}{2}z_\xi\{J\star(\psi_1+\psi_2)\}^2 + \frac{h^2}{2}z_\xi\\ + hz_\xi\{J\star(\psi_1+\psi_2)\} + \frac{1}{3!}\tilde z_\xi(J\star\psi_1)^3 + 0_3\big(\psi_1,\psi_2,h\big)\end{aligned} \tag{5.27}$$

where setting

$$z(\cdot) = \beta^2\tanh''(\cdot), \qquad \tilde z(\cdot) = \beta^3\tanh'''(\cdot) \tag{5.28}$$

we define $z_\xi(x)$ and $\tilde z_\xi(x)$ as respectively $z(\cdot)$ and $\tilde z(\cdot)$ computed at $\beta J\star m^0_\xi(x)$. 0_3 is the remainder term,

$$\Big|0_3\big(\psi_1,\psi_2,h\big)\Big| \le \;\; c \sum_{i_1+i_2+i_3=3; i_1<3} |J\star\psi_1|^{i_1}|J\star\psi_2|^{i_2}h^{i_3} \tag{5.29}$$

with c a suitable constant.

The first order equation is

$$L_\xi\psi_1 + hp_\xi + b_\xi = 0 \tag{5.30}$$

It looks natural to solve (5.30) by choosing $\xi=\xi_0\equiv\xi_0(h)$ so that the sum of the last two terms on the left hand side has no component along the eigenvector with maximal eigenvalue of L^0_ξ, which is very close to 0 for h small. Using the notation (3.16), we then set

$$\pi_{m^0_{\xi_0}}\Big(hp_{\xi_0}+b_{\xi_0}\Big)=0 \tag{5.31}$$

In Proposition 5.2 below we will see that indeed (5.31) has a solution, at least for h small enough, and that $e^{-\alpha\xi_0}=0(\sqrt{h})$, in agreement with the previous claims on the length of the bump. Even neglecting the difference between b_ξ and $-k_\xi$, ψ_1 is at best of the order of h. We then have from (5.27)

$$\begin{aligned} f(m) &= L_{\xi_0}\psi_2 + \frac12 z_{\xi_0}[J\star\psi_1]^2 + \frac12 z_{\xi_0}\Big([J\star\psi_2]^2+2[J\star\psi_1][J\star\psi_2]^2\Big)+\frac{h^2}{2}z_{\xi_0}\\ &\qquad + hz_{\xi_0}\{J\star(\psi_1+\psi_2)\} + \frac{1}{3!}\tilde z_{\xi_0}(J\star\psi_1)^3 + 0_3\big(\psi_1,\psi_2,h\big)\\ &= \frac12 z_{\xi_0}\Big([J\star\psi_2]^2+2[J\star\psi_1][J\star\psi_2]^2\Big) + hz_{\xi_0}J\star\psi_2 + 0_3\big(\psi_1,\psi_2,h\big) \end{aligned} \tag{5.32}$$

having chosen ψ_2 so that

$$L_{\xi_0}\psi_2 + \frac12 z_{\xi_0}\{J\star\psi_1\}^2 + \frac{h^2}{2}z_{\xi_0} + hz_{\xi_0}J\star\psi_1 + \frac{1}{3!}\tilde z_{\xi_0}(J\star\psi_1)^3 = 0 \tag{5.33}$$

In this way $f(m)$ may only be $0(h^2)$. In fact, if ψ_1 is of the order of h and the sum of the second, the third and the fourth terms in (5.33) has a component along the eigenvector with maximal eigenvalue of L_{ξ_0},then ψ_2 is also of the order of h and the term in (5.32) is of the order of h^2 and not smaller as requested in Definition 4.1. Unless a miracolous but unlikely cancellation, this approach is not going to work.

To avoid the impasse we expand also ξ (with a procedure which may remind of the Chapman-Enskogg expansion). Set $\tau := {\xi_0}^2$ and let $\Gamma\in C^{\rm sym}(\mathbb{R})$ be defined for $x\ge 0$ as

$$\Gamma(x) := \int_0^\tau dt e^{L_{\xi_0}t}\Big(hp_{\xi_0}(x)+b_{\xi_0}(x)\Big) \tag{5.34}$$

By Theorem 3.4 (details are given later) $\Gamma(x)$ is a good approximation to $\psi_1(x)$. The new ansatz which replaces (5.25) has still the form

$$m = m^0_\xi + \psi + \psi^\star \tag{5.35}$$

but ξ, ψ and $\psi^\star$ will be different from ξ_0, ψ_1 and ψ_2. We rewrite (5.27) as

$$f(m) = L_\xi(\psi+\psi^\star) + hp_\xi + b_\xi + \frac{1}{2}z_\xi\Big(J\star[\Gamma+h]\Big)^2 + \frac{1}{3!}\tilde z_\xi(J\star\Gamma)^3$$
$$+\frac{1}{2}z_\xi\Big\{[J\star(\psi+\psi^\star)]^2 - [J\star\Gamma]^2\} + 2h\{J\star(\psi+\psi^\star-\Gamma)\Big\}$$
$$+\frac{1}{3!}\tilde z_\xi\Big\{(J\star\psi)^3-(J\star\Gamma)^3\Big\} + 0_3\big(\psi,\psi^\star,h\big) \tag{5.36}$$

which is obtained after the Taylor expansion of (5.27) by adding and subtracting the term

$$\frac{1}{2}z_\xi\Big(J\star[\Gamma+h]\Big)^2 + \frac{1}{3!}\tilde z_\xi(J\star\Gamma)^3$$

which is the same as the sum of the second to the fifth terms on the l.h.s. of (5.33), if we neglect the difference between Γ and ψ_1.

Choice of parameters.

We choose $\xi=\xi_1$, $\xi_1=\xi_1(h)$, so that

$$\pi_{m^0_{\xi_1}}\bigg(hp_{\xi_1} + b_{\xi_1} + \frac{1}{2}z_{\xi_1}\Big(J\star[\Gamma+h]\Big)^2 + \frac{1}{3!}\tilde z_{\xi_1}(J\star\Gamma)^3\bigg) = 0 \tag{5.37}$$

ψ so that

$$L_{\xi_1}\psi + hp_{\xi_1} + b_{\xi_1} + \frac{1}{2}z_{\xi_1}\Big(J\star[\Gamma+h]\Big)^2 + \frac{1}{3!}\tilde z_{\xi_1}(J\star\Gamma)^3 = 0 \tag{5.38}$$

and finally $\psi^\star$ so that

$$L_{\xi_1}\psi^\star + \frac{1}{2}z_{\xi_1}\Big\{[J\star\psi]^2 - [J\star\Gamma]^2 + 2h[J\star(\psi-\Gamma)]\Big\} = 0 \tag{5.39}$$

Then, setting m as in (5.35),

$$f(m) = \frac{1}{2}z_{\xi_1}\Big\{[J\star\psi^\star]^2 + 2[J\star\psi][J\star\psi^\star] + 2hJ\star\psi^\star\Big\} + \frac{1}{3!}\tilde z_{\xi_1}\Big\{(J\star\psi)^3-(J\star\Gamma)^3\Big\}$$
$$+0_3\big(\psi,\psi^\star,h\big) \tag{5.40}$$

We will prove that ξ_1 is close to ξ_0 as (5.31) differs from (5.37) by terms of order h^2. We will then show that ψ and Γ are so close to each other that $\psi^\star$ has order higher than h and $m = m^0_{\xi_1} + \psi + \psi^\star$ will satisfy the conditions of Definition 4.1.

6. Existence of ξ_0 and of ξ_1

In this section we will prove that for all h small enough there exist solutions ξ_0 and ξ_1 of (5.31) and respectively (5.37). We will also establish a bound on $\Gamma(x)$.

Proposition 6.1. For any $h > 0$ small enough there are ξ_0 and ξ_1 which verify (5.31) and respectively (5.37). Both $he^{2\alpha\xi_0}$ and $he^{2\alpha\xi_1}$ have limits when $h \to 0^+$ and these limits coincide. Moreover for any $\alpha^\star < \alpha$ there is $c > 0$ so that

$$|\xi_1 - \xi_0| \le ce^{-(2\alpha_0-\alpha)\xi_1}, \qquad |\Gamma(x)| \le c\big[e^{\alpha^\star(\xi_0-x)}e^{-(\alpha_0+\alpha^\star)\xi_0} + e^{-2\alpha\xi_0}\big] \tag{6.1}$$

Proof.
Let $\xi^\star \equiv \xi^\star(h)$ be such that $e^{2\alpha\xi^\star(h)}h = 1$ and let

$$\zeta = \xi - \xi^\star(h), \qquad \zeta_0(h) = \xi_0(h) - \xi^\star(h), \;\; \zeta_1(h) = \xi_1(h) - \xi^\star(h) \tag{6.2}$$

In the following we will consider ζ and h as independent variables, with then ξ a function of h and ζ via (6.2), $\xi = \xi(h,\zeta)$. Consequently (5.31)-(5.37) are now thought of as equations in ζ which depend parametrically on h. We multiply (5.31) by h^{-1}, then ζ_0 solves the equation

$$F^0_h(\zeta) := A_h(\zeta) - e^{-2\alpha\zeta}B^0_h(\zeta) = 0 \tag{6.3}$$

where setting $v_\xi := v_{m^0_\xi}$, $u_\xi := u_{m^0_\xi}$, $\lambda_\xi := \lambda_{m^0_\xi}$ and $p_\xi := p_{m^0_\xi}$ and recalling (3.16) with the relation $u_\xi p_\xi = v_\xi$

$$A_h(\zeta) := \int_{\mathbb{R}_+} dx\, v_\xi(x), \qquad \xi = \xi(h,\zeta) \tag{6.4}$$

$$B^0_h(\zeta) := -\int_{\mathbb{R}_+} \frac{dx}{p_\xi} v_\xi e^{2\alpha\xi} b_\xi, \qquad \xi = \xi(h,\zeta) \tag{6.5}$$

Analogously, by (5.37) ζ_1 solves the equation

$$F_h(\zeta) := F^0_h(\zeta) + B^1_h(\zeta) = 0 \tag{6.6}$$

where

$$B^1_h(\zeta) := \int_{\mathbb{R}_+} \frac{dx}{p_\xi} v_\xi \bigg(\frac{1}{2} z_\xi \Big(h^{-1}[J\star\Gamma]^2 + 2J\star\Gamma + h\Big) + h^{-1}\frac{1}{3!}\tilde{z}_\xi (J\star\Gamma)^3\bigg) \tag{6.7}$$

We postpone the proof that $A_h(\zeta)$, $B^0_h(\zeta)$ and $B^1_h(\zeta)$ are for any h continuous and differentiable functions of ζ, that they and their derivatives with respect to ζ are continuous functions of h and that, uniformly on the compacts,

$$\lim_{h\to 0^+} A_h(\zeta) = A > 0; \qquad \lim_{h\to 0^+} B^0_h(\zeta) = B^0 > 0; \qquad \lim_{h\to 0^+} B^1_h(\zeta) = 0 \tag{6.8}$$

$$\lim_{h\to 0^+} \frac{\partial}{\partial\zeta} A_h(\zeta) = \lim_{h\to 0^+} \frac{\partial}{\partial\zeta} B^0_h(\zeta) = \lim_{h\to 0^+} \frac{\partial}{\partial\zeta} B^1_h(\zeta) = 0 \tag{6.9}$$

Existence of $\xi_0(h)$

(6.8) and (6.9) imply the existence of $\xi_0(h)$ for all h small enough. In fact let $\zeta_0(0)$ be such that

$$A - e^{-2\alpha\zeta_0(0)}B^0 = 0, \qquad e^{2\alpha\zeta_0(0)} = \frac{B^0}{A} \tag{6.10}$$

Then there is $h' > 0$ so that for all $h \le h'$ and all $|\zeta - \zeta_0(0)| \le 1$

$$\frac{d}{d\zeta} F^0_h(\zeta) \ge \frac{1}{2}[2\alpha e^{-2\alpha\zeta} B^0] \ge \frac{1}{2}[2\alpha A e^{-2\alpha}] \tag{6.11}$$

Since $F_h^0\big(\zeta_0(0)\big) \to 0$ as $h \to 0^+$ there is $h'' \le h'$ so that for all $h \le h''$

$$\big|F_h^0\big(\zeta_0(0)\big)\big| \le \frac{1}{10}[2\alpha A e^{-2\alpha}] \tag{6.12}$$

Then for any $h \le h''$ there is a unique value $\zeta_0(h) \in [\zeta_0(0)-1, \zeta_0(0)+1]$ where $F_h^0\big(\zeta_0(h)\big) = 0$. We have thus proved the existence of $\xi_0(h) = \xi^\star(h) + \zeta_0(h)$ for all $h \le h''$. Moreover

$$h e^{2\alpha\xi_0(h)} = e^{2\alpha\zeta_0(h)}, \qquad \lim_{h\to 0^+} h e^{2\alpha\xi_0(h)} = e^{2\alpha\zeta_0(0)} \tag{6.13}$$

After proving the first inequality in (6.1) we will conclude that also $\lim h e^{2\alpha\xi_1(h)} = e^{2\alpha\zeta_0(0)}$.

Existence of $\xi_1(h)$

By the same arguments there is $h''' \le h''$ so that for all $h \le h'''$ and all $|\zeta - \zeta_0(0)| \le 1$

$$\frac{d}{d\zeta} F_h(\zeta) \ge \frac{1}{2}[2\alpha A e^{-2\alpha}] \tag{6.14}$$

We also take $h^{iv} \le h'''$ so that for all $h \le h^{iv}$

$$\big|F_h\big(\zeta_0(0)\big)\big| \le \frac{1}{10}[2\alpha A e^{-2\alpha}] \tag{6.15}$$

Then there is a unique solution $\zeta_1 = \zeta_1(h)$ of $F_h(\zeta) = 0$ in $[\zeta_0(0) - 1, \zeta_0(0) + 1]$.

Proof of the first inequality in (6.1)

We have

$$\big|\zeta_1(h) - \zeta_0(h)\big| \le \frac{2}{2\alpha A e^{-2\alpha}} \max_{|\zeta-\zeta_0(0)|\le 1} \big|B_h^1(\zeta)\big| \tag{6.16}$$

We will prove that there are $h^v \le h^{iv}$ and $c > 0$ so that for all $h \le h^v$

$$\big|B_h^1(\zeta)\big| \le c e^{(\alpha - 2\alpha_0)\xi}, \quad \text{for all } |\zeta - \zeta_0(0)| \le 1 \tag{6.17}$$

so that

$$\big|\xi_1(h) - \xi_0(h)\big| = \big|\zeta_1(h) - \zeta_0(h)\big| \le c e^{(\alpha-2\alpha_0)\xi} \tag{6.18}$$

which proves the first inequality in (6.1).

Proof of the first two limits in (6.8)

Since for any c and δ, $m_\xi^0 \in G_{(c,\xi,\delta)}$ the conclusions of Theorem 3.3 apply to $m \equiv m_\xi^0$. Then by (3.9) and (3.10)

$$\lim_{\xi\to+\infty} \int_{\mathbb{R}_+} dx\, v_\xi(x) = \int_{\mathbb{R}} dx\, \tilde m'(x) = A \tag{6.19}$$

which proves the first limit in (6.8). We will next prove that

$$B^0 = \int_{\mathbb{R}} dx \frac{\tilde m'(x) k^0(x)}{\beta(1 - \bar m(x)^2)} \tag{6.20}$$

k^0 being defined in (5.2). In fact by (3.9) and (5.2)-(5.4) the contribution of $x \in [0,1]$ to the integral in (6.5) is bounded by

$$c \int_0^1 dx\, e^{-\alpha|\xi - x|} \{k^0(\xi - x) + c[e^{-2\alpha x} + e^{(2\alpha-\alpha_0)\xi}]\} \tag{6.21}$$

that vanishes as $\xi \to +\infty$ because $\alpha_0 > \alpha$ and $\|k^0\|_\infty < \infty$.

When $x > 1$ the last term is absent. We then use (3.9) when $|\xi - x| \geq \xi/2$ and (3.10) otherwise. We then obtain the second limit in (6.8) with B^0 as in (6.20).

Proof of the first two limits in (6.9)

The first limit in (6.9) is a straight consequence of (3.13) and (3.14). For the second one we write

$$\begin{aligned}\frac{\partial}{\partial \zeta} B_h^0(\zeta) = &- \int_{|\xi-x|\leq \xi/2} dx\, \frac{d}{dx}\Big(v_\xi(x) \frac{e^{2\alpha\xi} b_\xi(x)}{p_\xi(x)}\Big) + \Lambda_1 + \Lambda_2 + \Lambda_3 \\ &+ \int_{|\xi-x|>\xi/2} dx\, \Big\{ \frac{dv_\xi}{d\xi} \frac{e^{2\alpha\xi} b_\xi(x)}{p_\xi(x)} - v_\xi \frac{d}{dx}\Big(\frac{e^{2\alpha\xi} b_\xi(x)}{p_\xi(x)}\Big)\Big\}\end{aligned} \tag{6.22}$$

where

$$\Lambda_1 := \int_{|\xi-x|\leq\xi/2} dx\, \Big\{ \frac{d}{d\xi} v_\xi(x) + \frac{d}{dx} v_\xi(x) \Big\} \frac{e^{2\alpha\xi} b_\xi(x)}{p_\xi(x)} \Big\} \tag{6.23}$$

$$\Lambda_2 := \int_{\mathbb{R}_+} dx\, \frac{v_\xi(x)}{p_\xi(x)} \Big\{ \frac{d}{d\xi} e^{2\alpha\xi} b_\xi(x) + \frac{d}{dx} e^{2\alpha\xi} b_\xi(x) \Big\} \tag{6.24}$$

$$\Lambda_3 := \int_{\mathbb{R}_+} dx\, v_\xi(x) e^{2\alpha\xi} b_\xi(x) \Big\{ \frac{d}{d\xi} \frac{1}{p_\xi(x)} + \frac{d}{dx} \frac{1}{p_\xi(x)} \Big\} \tag{6.25}$$

The first integral on the right hand side of (6.22) vanishes as $\xi \to +\infty$ by (3.10), (2.6) and (5.2)-(5.4).

The first term in the second integral vanishes by (3.13) and (5.2)-(5.4). The other one can be written as

$$\int_{|\xi-x|>\xi/2} dx\, \frac{e^{2\alpha\xi} b_\xi(x)}{p_\xi(x)} \frac{d}{dx} v_\xi(x) - \Big[v_\xi \frac{e^{2\alpha\xi} b_\xi(x)}{p_\xi(x)} \Big]_0^{\xi/2} - \Big[v_\xi \frac{e^{2\alpha\xi} b_\xi(x)}{p_\xi(x)} \Big]_{3\xi/2}^{\infty} \tag{6.26}$$

We have

$$\frac{d}{dx} v_\xi(x) = [1 + \lambda_\xi]^{-1} \frac{d}{dx} \int dy\, p_\xi(x) J(y - x) v_\xi(y)$$

hence by (3.9) and (3.8)

$$\Big| \frac{d}{dx} v_\xi(x) \Big| \leq c e^{-\alpha'|\xi - x|} \tag{6.27}$$

Using again (3.9) and (5.2)-(5.4) we conclude that (6.26) vanishes as $\xi \to +\infty$.

By (3.15) and (5.2)-(5.4) Λ_1 vanishes as $\xi \to +\infty$. Λ_2 vanishes because of (5.5) and (3.9) (recall that $\alpha_0 > \alpha$). Λ_3 vanishes because of (5.2)-(5.4), (3.9) and because there is $c > 0$ so that

$$\Big| \frac{d}{d\xi} \frac{1}{p_\xi(x)} + \frac{d}{dx} \frac{1}{p_\xi(x)} \Big| \leq c e^{-\alpha(\xi + x)} \tag{6.28}$$

(recall that $p_\xi(x) = p_{m_\xi^0(x)}$).

We have therefore proved the first two limits in (6.9).

The existence of $\xi_0(h)$ (and hence of Γ) follows, for all h small enough, from the first two limits in (6.9) and it is thus established.

The bound (6.1) *on* Γ

We set

$$\gamma = \gamma(h) := -\pi_{\xi_0}\Big(hp_{\xi_0} - k_{\xi_0}\Big) \tag{6.29}$$

(k_ξ as in (5.2)) and write

$$\Gamma(x) := \int_0^\tau dt\, e^{L_{\xi_0} t}\Big\{\Big(hp_{\xi_0} - k_{\xi_0} + \gamma v_{\xi_0}\Big) + \Big(b_{\xi_0} + k_{\xi_0} - \gamma v_{\xi_0}\Big)\Big\} \tag{6.30}$$

By the definition of ξ_0

$$\gamma = -\pi_{\xi_0}\Big(k_{\xi_0} + b_{\xi_0}\Big) \tag{6.31}$$

Using (5.4) and (3.9) there is $c' > 0$ so that

$$|\gamma| \le \int_{\mathbb{R}_+} dx u_{\xi_0}(x) c e^{-2\alpha(\xi_0 + x)} + \int_0^1 dx u_{\xi_0}(x) c e^{-\alpha_0 \xi_0} \le c' e^{-(\alpha+\alpha_0)\xi_0} \tag{6.32}$$

By (3.18)

$$\begin{aligned} \|\int_0^\tau dt\, e^{L_{\xi_0} t}\Big(hp_{\xi_0} - k_{\xi_0} + \gamma v_{\xi_0}\Big)\|_\infty \;&\le\; d_+ \int_0^\tau dt\, e^{-d_- t}\Big(\|hp_{\xi_0}\|_\infty \\ &+ \|k_{\xi_0}\|_\infty + \|\gamma v_{\xi_0}\|_\infty\Big) \;\le\; c' e^{-2\alpha\xi_0} \end{aligned} \tag{6.33}$$

having used (6.32) the fact that $h < c e^{-2\alpha\xi_0}$, and that $\|k_{\xi_0}\|_\infty \le c e^{-2\alpha\xi_0}$. By (5.4)

$$\begin{aligned} \Big|\int_0^\tau dt\, e^{L_{\xi_0} t}\Big(b_{\xi_0} + k_{\xi_0} - \gamma v_{\xi_0}\Big)\Big| \;&\le\; c\tau|\gamma| + \int_0^\tau dt\, e^{L_{\xi_0} t}\Big(\mathbf{1}_{y\in[0,1]} e^{-\alpha_0 \xi_0} \\ &+ e^{-4\alpha\xi_0} e^{2\alpha(\xi_0 - y)}\Big) \end{aligned} \tag{6.34}$$

We will prove that for any $\alpha^\star < \alpha$ there is $c > 0$ so that

$$\int_0^\tau dt\, [e^{L_{\xi_0} t} \mathbf{1}_{y\in[0,1]}](x) \le c e^{-\alpha^\star x} \tag{6.35}$$

$$\int_0^\tau dt\, e^{L_{\xi_0} t} e^{2\alpha(\xi_0 - y)} \le c e^{2\alpha\xi_0} \tag{6.36}$$

Then by (6.30)-(6.36), recalling that $\tau = \xi_0^2$

$$\big|\Gamma(x)\big| \le c\Big(e^{-2\alpha\xi_0} + \xi_0^2 e^{-(\alpha+\alpha_0)\xi_0} + e^{-(\alpha_0+\alpha^\star)\xi_0} e^{\alpha^\star(\xi_0 - x)} + e^{-2\alpha\xi_0}\Big) \tag{6.37}$$

which proves the second inequality in (6.1).

Proof of (6.35). By Proposition 6.3, equations (6.28)-(6.29) of [**2**], for any $\zeta \in (0,\alpha)$ there is $c>0$ so that for all $x>0$ and $y>0$

$$
\begin{aligned}
e^{L_{\xi_0}t}(x,y) &\le e^{-t}\sum_n \frac{[(\lambda_{\xi_0}+1)t]^n}{n!}c^\star n e^{-\zeta|x-y|}\\
&\le c^\star t(\lambda_{\xi_0}+1)e^{t\lambda_{\xi_0}}e^{-\zeta|x-y|}
\end{aligned}
\tag{6.38}
$$

If $y\in[0,1]$ by (3.8) and recalling that $t\le\tau=\xi_0^2$ there is $c>0$ so that

$$
e^{L_{\xi_0}t}(x,y)\le c\xi_0^2 e^{-\zeta x}
\tag{6.39}
$$

(6.35) then follows with $\alpha^\star\in(0,\zeta)$.

Proof of (6.36). Let

$$
\gamma' = \pi_{m^0_{\xi_0}}\Big(e^{2\alpha(\xi_0-y)}\Big)
\tag{6.40}
$$

there is $c>0$ so that

$$
|\gamma'| \le \int_{|x-\xi_0|\le\xi/2} dx\,\frac{v_{\xi_0}(x)}{p_{\xi_0}(x)}e^{2\alpha(\xi_0-x)} + \int_{|x-\xi_0|\ge\xi/2} dx\,\frac{v_{\xi_0}(x)}{p_{\xi_0}(x)}e^{2\alpha(\xi_0-x)} \le ce^{\alpha\xi_0}
\tag{6.41}
$$

having used (3.9). Then by (3.18), there are c and c' so that for any $t\le\tau=\xi_0^2$

$$
\begin{aligned}
\Big|[e^{L_{\xi_0}t}e^{2\alpha(\xi_0-y)}](x)\Big| &\le c\Big(e^{t\lambda_{\xi_0}}\|\gamma' v_{\xi_0}\|_\infty + e^{-d_- t}\|e^{2\alpha(\xi_0-y)}-\gamma' v_{\xi_0}\|_\infty\Big)\\
&\le c'\Big(e^{\alpha\xi_0}+e^{2\alpha\xi_0}e^{-d_- t}\Big)
\end{aligned}
$$

hence (6.36).

Proof of (6.17) *and of the third limit in* (6.8)

Let $|\zeta-\zeta_0(0)|\le 1$ and h so small that $|\zeta_0(h)-\zeta_0(0)|\le 1$. Then

$$
|\xi-\xi^\star(h)|\le|\zeta_0(0)|+1
\tag{6.42}
$$

because $\xi=\xi^\star(h)+\zeta=\xi^\star(h)+\zeta_0(0)+[\zeta-\zeta_0(0)]$. Analogously,

$$
|\xi-\xi_0(h)|\le 2
\tag{6.43}
$$

because $\xi-\xi_0(h)=\zeta-\zeta_0(h)+\zeta_0(0)-\zeta_0(0)$. Then by (3.9) and (3.10) there is $c>0$ so that

$$
\begin{aligned}
\Big|B^1_h(\zeta)\Big| &\le c\int_0^\xi dx\,e^{-\alpha(\xi-x)}\Big\{h^{-1}e^{-2(\alpha_0+\alpha^\star)\xi_0}e^{2\alpha^\star(\xi_0-x)}+e^{\alpha^\star(\xi_0-x)}e^{-(\alpha_0+\alpha^\star)\xi_0}+h\Big\}\\
&\quad +c\int_\xi^\infty dx\,e^{-\alpha'(x-\xi)}\Big\{h^{-1}e^{-2(\alpha_0+\alpha^\star)\xi_0}+e^{-(\alpha_0+\alpha^\star)\xi_0}+h\Big\}\\
&\le c\Big\{e^{(2\alpha^\star-\alpha)\xi}e^{-2(\alpha_0+\alpha^\star)\xi}h^{-1}+h+e^{-(\alpha_0+\alpha^\star)\xi}+h^{-1}e^{-2(\alpha_0+\alpha^\star)\xi_0}\\
&\quad +e^{-(\alpha_0+\alpha^\star)\xi_0}+h\Big\} \;\le\; c'e^{(\alpha-2\alpha_0)\xi}
\end{aligned}
$$

Proof of the third limit in (6.9)

We denote by $\{\cdot\}$ the factor that multiplies v_ξ/p_ξ in the integral in (6.7). Then

$$\frac{d}{d\zeta}B^1_h(\zeta) = \int_{\mathbb{R}_+} dx \Big(\Big(\frac{d}{d\xi}v_\xi\Big)\frac{1}{p_\xi}\{\cdot\} + v_\xi\Big(\frac{d}{d\xi}\frac{1}{p_\xi}\Big)\{\cdot\} + \frac{v_\xi}{p_\xi}\frac{d}{d\xi}\{\cdot\}\Big) \tag{6.44}$$

By (3.13) and (3.14), there are c and δ positive so that

$$\Big|\frac{d}{d\xi}v_\xi(x)\Big| \le c\Big(e^{-\alpha|\xi-x|} + e^{-2\alpha\xi}e^{-\delta|\xi-x|}\Big) \tag{6.45}$$

By (6.44) the first term on the right hand side of (6.38) can be bounded proceeding as in the proof of (6.17) and shown to vanish as $h \to 0^+$, we omit the details. Since $dp_\xi^{-1}/d\xi$ is bounded the same argument works for the second term which also vanishes when $h \to 0^+$. By observing that $dz_\xi^{-1}/d\xi$ and $d\tilde z_\xi^{-1}/d\xi$ are bounded we are again in the same situation thus proving that also the third term vanishes as $h \to 0^+$. This completes the proof of the third inequality in (6.9).

Having proved (6.8) and (6.9) the previous analysis is justified and we have the existence of $\xi_0(h)$ and of $\xi(h)$. The first inequality in (6.1) which is the same as (6.18) was proved under the assumption of validity of (6.17) which has been established above, so that (6.18) is also proved. Proposition 6.1 is proved. □

7. Existence of ψ and $\psi^\star$

Since $\xi_1(h) \to +\infty$ as $h \to 0^+$, for all h small enough $\xi_1(h)$ verifies the conditions of Theorem 3.4 that can therefore be applied in the present context and used to solve (5.38), thus finding ψ. Let $\tau := \xi_0^2$ and

$$\tilde\psi(x) := \int_0^\tau dt\, e^{L_{\xi_1}t}\Big\{hp_{\xi_1} + b_{\xi_1} + \frac{1}{2}z_{\xi_1}\Big(J\star[\Gamma+h]\Big)^2 + \frac{1}{3!}\tilde z_{\xi_1}(J\star\Gamma)^3\Big\} \tag{7.1}$$

Recalling that, by the choice of ξ_1, the curly bracket term has no component along v_{ξ_1}, by (3.18) and (3.20) we get

$$\|\tilde\psi - \psi\|_\infty \le ce^{-d_-\xi_0^2} \tag{7.2}$$

for h so small that ξ_1 is as large as required by Theorem 3.4.

Lemma 7.1. There are $\delta > 0$ and $c > 0$ so that for all h small enough

$$\big|\tilde\psi(x) - \Gamma(x)\big| \le ce^{\alpha(\xi_1-x)_+}e^{-(3\alpha+\delta)\xi_1} \tag{7.3}$$

where $(y)_+ = y\mathbf{1}_{y>0}$.

Proof.

We write

$$\tilde\psi(x)-\Gamma(x)=\int_0^\tau dt\,\Big([e^{L_{\xi_1}t}-e^{L_{\xi_0}t}]\{hp_{\xi_0}+b_{\xi_0}\}+\,e^{L_{\xi_1}t}[F_1+F_2]\Big) \tag{7.4}$$

with

$$F_2:=\frac12 z_{\xi_1}\Big(J\star\Gamma\Big)^2+\{p_{\xi_1}J\star R_{\xi_1}-p_{\xi_0}J\star R_{\xi_0}+H_{\xi_1}-H_{\xi_0}\} \tag{7.5}$$

$R_\xi(x)$ being defined in (5.6) and H_ξ in (5.10)

$$F_1:=h(p_{\xi_1}-p_{\xi_0})+\big(\hat b_{\xi_1}-\hat b_{\xi_0}\big)+\frac12 z_{\xi_1}\Big\{\Big(J\star[\Gamma+h]\Big)^2-\Big(J\star\Gamma\Big)^2\Big\}+\frac{1}{3!}\tilde z_{\xi_1}\Big(J\star\Gamma\Big)^3 \tag{7.6}$$

$$\hat b_{\xi_1}(x):=b_{\xi_1}-p_{\xi_1}J\star R_{\xi_1}-H_{\xi_1} \tag{7.7}$$

Analysis of the term containing F_1

By (3.17) and (3.8)

$$\|\int_0^\tau dt\,e^{L_{\xi_1}t}F_1\|_\infty\le d_+\tau e^{c_+\tau e^{-2\alpha\xi}}\|F_1\|_\infty\le c'\xi_0^2\|F_1\|_\infty \tag{7.8}$$

with c' suitable constant. By (6.1) $\|\Gamma\|_\infty\le ce^{-\alpha_0\xi_0}$ and since $h\le ce^{-2\alpha\xi_1}$ we have

$$\|F_1\|_\infty\le h\|p_{\xi_1}-p_{\xi_0}\|_\infty+\|\hat b_{\xi_1}-\hat b_{\xi_0}\|_\infty+c'e^{-(\alpha_0+2\alpha)\xi_1} \tag{7.9}$$

c' a suitable constant. The third term is all right because

$$c'{\xi_1}^2e^{-(\alpha_0+2\alpha)\xi_1}\le ce^{-(3\alpha+\delta)\xi_1} \tag{7.10}$$

for suitable values of the parameters.

We will next show that also the two other terms on the right hand side of (7.9) are bounded by the right hand side of (7.3). By (6.1) and (2.6) there are c and c' so that

$$h|p_{\xi_1}(x)-p_{\xi_0}(x)|\le ch\bar m'(\xi_1-x)|\xi_1-\xi_0|\le c'e^{-2\alpha\xi_1}e^{-\alpha|\xi_1-x|}e^{(\alpha-2\alpha_0)\xi_1} \tag{7.11}$$

By (5.9), (5.10) and (7.7)

$$\hat b_{\xi_1}=L_{\bar m_{\xi_1}}\big(e^{-\alpha(x+\xi_1)}\big)+\hat G_{\xi_1},\qquad \hat G_\xi:=\frac12 z_{\bar m_\xi}[J\star e^{-\alpha(y+\xi)}]^2 \tag{7.12}$$

We then have

$$\|\hat b_{\xi_1}-\hat b_{\xi_0}\|_\infty\le\|\hat G_{\xi_1}-\hat G_{\xi_0}\|_\infty+\|L_{\bar m_{\xi_1}}\big(e^{-\alpha(x+\xi_1)}\big)-L_{\bar m_{\xi_0}}\big(e^{-\alpha(x+\xi_0)}\big)\|_\infty \tag{7.13}$$

By (5.14)

$$\begin{aligned}\frac{d}{d\xi}L_{\bar m_\xi}\big(e^{-\alpha(x+\xi)}\big)\Big|_{\xi=\xi_1}&=-\alpha e^{-2\alpha\xi_1}e^{\alpha(\xi_1-x)}\frac{m_\beta^2-\bar m(\xi_1-x)^2}{1-m_\beta^2}\\&\quad-2\bar m(\xi_1-x)\bar m'(\xi_1-x)e^{-2\alpha\xi_1}e^{\alpha(\xi_1-x)}[1-m_\beta^2]^{-1}\end{aligned}$$

Hence by (2.6) and (6.1)

$$\|L_{\bar m_{\xi_1}}\big(e^{-\alpha(x+\xi_1)}\big)-L_{\bar m_{\xi_0}}\big(e^{-\alpha(x+\xi_0)}\big)\|_\infty\le c|\xi_1-\xi_0|e^{-2\alpha\xi_1}\le c'e^{(\alpha-2\alpha_0-2\alpha)\xi_1} \tag{7.14}$$

Recalling (7.12) there is $c>0$ so that

$$\|\hat G_{\xi_1}-\hat G_{\xi_0}\|_\infty\le\|\frac{d}{d\xi}\frac12 z_{\bar m_\xi}[J\star e^{-\alpha(y+\xi)}]^2\Big|_{\xi=\xi_1}\|_\infty|\xi_1-\xi_0|\le ce^{(\alpha-2\alpha_0)\xi_1}e^{-2\alpha\xi_1} \tag{7.15}$$

In conclusion

$$\|F_1\|_\infty \le c\Big(e^{-(2\alpha_0+\alpha)\xi_1} + e^{-(\alpha_0+2\alpha)\xi_1}\Big) \tag{7.16}$$

which by (7.8) shows that the contribution of F_1 to the integral in (7.4) is bounded by the right hand side of (7.3) for suitable values of the parameters.

Analysis of the term containing F_2

By (6.38) for any $\alpha^\star \in (0,\alpha)$ there is $c>0$ so that (recall $\tau = \xi_0^2$)

$$\Big|\int_0^\tau dt\, e^{L_{\xi_1}t} F_2(x)\Big| \le c\xi_1^4 \int_{\mathbb{R}_+} dy\, e^{-\alpha^\star|x-y|}|F_2(y)| \tag{7.17}$$

By (6.1)

$$e^{-\alpha^\star|x-y|}\frac{1}{2}z_{\xi_1}\Big(J\star\Gamma(y)\Big)^2 \le ce^{-\alpha^\star|x-y|}\Big\{[e^{\alpha^\star(\xi_1-y)}e^{-(\alpha_0+\alpha^\star)\xi_1}]^2 + e^{-4\alpha\xi_1}\Big\} \tag{7.18}$$

The integral over $y\in[0,\xi_1]$ of the the first term on the right hand side is bounded by

$$\begin{aligned} &c\int_0^{\xi_1} dy\, e^{-\alpha^\star(x-y)}[e^{\alpha^\star(\xi_1-y)}e^{-(\alpha_0+\alpha^\star)\xi_1}]^2 \le c' e^{-2(\alpha_0+\alpha^\star)\xi_1} e^{\alpha^\star(\xi_1-x)}e^{\alpha^\star\xi_1}\\ &= c' e^{\alpha^\star(\xi_1-x)}e^{-(2\alpha_0+\alpha^\star)\xi_1} \end{aligned}$$

with c and c' suitable constants. The integral over $y>\xi_1$ is bounded by

$$c\int_{\xi_1}^\infty dy\, e^{-\alpha^\star(x-y)}[e^{-(\alpha_0+\alpha^\star)\xi_1}]^2 \le c' e^{-2(\alpha_0+\alpha^\star)\xi_1} \tag{7.19}$$

Thus

$$\Big|\int_0^\tau dt\, e^{L_{\xi_1}t}\frac{1}{2}z_\xi\Big(J\star\Gamma\Big)^2(x)\Big| \le c\xi_0^4\Big\{e^{-4\alpha\xi} + e^{\alpha^\star(\xi-x)}e^{-(2\alpha_0+\alpha^\star)\xi_1} + e^{-2(\alpha_0+\alpha^\star)\xi_1}\Big\} \tag{7.20}$$

which is bounded by the right hand side of (7.3) (with a suitable choice of the parameters).

By (5.12)

$$\begin{aligned} \int_{\mathbb{R}_+} dy\, e^{-\alpha^\star|x-y|}|H_{\xi_1}(y)| &\le c\int_{\mathbb{R}_+} dy\, e^{-\alpha^\star|x-y|}\{e^{-3\alpha(y+\xi)} + e^{-2\alpha_0\xi_1}\mathbf{1}_{y\in[0,1]}\}\\ &\le c' e^{-\alpha^\star x}[e^{-3\alpha\xi_1}+e^{-2\alpha_0\xi_1}] \le c' e^{\alpha^\star(\xi_1-x)}[e^{-(3\alpha+\alpha^\star)\xi_1}+e^{-(2\alpha_0+\alpha^\star)\xi_1}] \end{aligned}$$

Thus

$$\Big|\int_0^\tau dt\, e^{L_{\xi_1}t}H_{\xi_1}(x)\Big| \le c\xi_0^4 c' e^{\alpha^\star(\xi_1-x)}[e^{-(3\alpha+\alpha^\star)\xi_1}+e^{-(2\alpha_0+\alpha^\star)\xi_1}] \tag{7.21}$$

which is also bounded by the right hand side of (7.3) (with a suitable choice of the parameters).

The last term to estimate is

$$\begin{aligned} \Big|p_{\xi_1}J\star R_{\xi_1}(x) - p_{\xi_0}J\star R_{\xi_0}(x)\Big| &\le \Big|\frac{d}{d\xi}\big(p_\xi J\star R_\xi(x)\big)\Big|_{\xi=\xi_1}\Big||\xi_1-\xi_0|\\ &\le \mathbf{1}_{x\in[0,1]}|\xi_1-\xi_0|c\Big(e^{-\alpha_0\xi_1}e^{-\alpha\xi_1}+e^{-\alpha_0\xi_1}\Big) \end{aligned}$$

We have used that, by (2.6), $|dp_\xi/d\xi| \le ce^{-\alpha\xi}$; we have laso used (5.7) to bound R_ξ and (5.24) to bound $dR_\xi/d\xi$. We thus have

$$\Big|\int_0^\tau dt\, e^{L_{\xi_1}t}\Big(p_{\bar m_{\xi_1}} J\star R_{\xi_1}(x) - p_{\bar m_{\xi_0}} J\star R_{\xi_0}(x)\Big)\Big| \le c\xi_0^4 e^{-\alpha^\star x} e^{(\alpha-3\alpha_0)\xi_1}$$
$$\le c'\xi_0^4 e^{\alpha^\star(\xi_1-x)} e^{(\alpha-3\alpha_0-\alpha^\star)\xi_1}$$

which is again bounded by the right hand side of (7.3). The bound of the term containing F_2 is completed.

Conclusion of the proof of Lemma 7.1

The last term to estimate in (7.4) is

$$A := \int_0^\tau dt\, \Big([e^{L_{\xi_1}t} - e^{L_{\xi_0}t}]\{hp_{\xi_0} + b_{\xi_0}\} \tag{7.22}$$

By the integration by parts formula

$$A = \int_0^\tau dt \int_0^t ds\, e^{L_{\xi_1}(t-s)}[L_{\xi_1} - L_{\xi_0}]e^{L_{\xi_0}s}\Big(hp_{\xi_0} + b_{\xi_0}\Big) \tag{7.23}$$

Using (6.38), by (5.2) and (5.4) there is $c>0$ so that

$$e^{L_{\xi_0}s}\Big(|hp_{\xi_0}| + |b_{\xi_0}|\Big)(x) \le s\int_{\mathbb{R}_+} dy\, e^{-\alpha^\star|x-y|} c\Big(h + e^{-2\alpha\xi_0}\|k^0\|_\infty$$
$$+ e^{-2\alpha(\xi_0+y)} + \mathbf{1}_{y\in[0,1]} e^{-\xi_0\alpha_0}\Big)$$
$$\le c'\xi_0^2\Big(e^{-2\alpha\xi_0} + e^{\alpha^\star(\xi_0-x)}[e^{-(2\alpha+\alpha^\star)\xi_0} + e^{-(\alpha_0+\alpha^\star)\xi_0}\Big)$$

because $s \le \tau = \xi_0^2$. We have

$$[L_{\xi_1} - L_{\xi_0}](x,y) = [p_{\xi_1}(x) - p_{\xi_0}(x)]J(y-x) \tag{7.24}$$

Then by (6.1) and (2.6) there is $c>0$ so that

$$\Big|[L_{\xi_1} - L_{\xi_0}](x,y)\Big| \le ce^{-\alpha|\xi_1-x|}e^{(\alpha-2\alpha_0)\xi_1}J(y-x) \tag{7.25}$$

Thus applying again (6.38)

$$|A| \le c\xi_0^4 \int_{\mathbb{R}_+} dy \int_{\mathbb{R}_+} dz\, e^{-\alpha|x-y|}e^{-\alpha|\xi-y|}e^{(\alpha-2\alpha_0)\xi_1}J(y,z)\Big(e^{-2\alpha\xi_1}$$
$$+ e^{\alpha^\star(\xi-z)}[e^{-(2\alpha+\alpha^\star)\xi_1} + e^{-(\alpha_0+\alpha^\star)\xi_1}]\Big)$$
$$\le c'\xi_0^4 e^{(\alpha-2\alpha_0)\xi_1}\Big(e^{-2\alpha\xi_1} + [e^{-(2\alpha+\alpha^\star)\xi_1} + e^{-(\alpha_0+\alpha^\star)\xi_1}]\Big)$$

Lemma 7.1 is proved. □

We next study $\psi^\star$ which is defined as the solution of (5.39). Recalling Theorem 3.2 we have

Lemma 7.2. There are $\delta > 0$ and $c > 0$ so that

$$|\psi^\star(x)| \le c\Big(e^{-(2\alpha+\delta)\xi_1} v_{\xi_1}(x) + e^{-(3\alpha+\delta)\xi_1}\Big) \tag{7.26}$$

Proof.
We shorthand (5.39) as

$$L_{\xi_1}\psi^\star + K = 0 \tag{7.27}$$

Then

$$\psi^\star = -\lambda_{\xi_1}^{-1}\omega v_{\xi_1} - (L_{\xi_1})^{-1}[K - \omega v_{\xi_1}] \tag{7.28}$$

where

$$\omega := \pi_{m^0_\xi}\big(K\big) = \int_{\mathbb{R}_+} dx\, \frac{v_{\xi_1}}{p_{\xi_1}} K \tag{7.29}$$

We are going to prove that there is $c > 0$ so that

$$|\omega| \le c e^{-(4\alpha+\delta)\xi_1} \tag{7.30}$$

We rewrite K as

$$2K = z_{\xi_1}\Big\{[J\star(\psi-\Gamma)][2h + 2J\star\Gamma + J\star(\psi-\Gamma)\Big\} \tag{7.31}$$

By (6.1), (7.2) and (7.3)

$$|K(x)| \le c\{e^{\alpha(\xi_1-x)_+} e^{-(3\alpha+\delta)\xi_1}\}\{e^{-2\alpha\xi_1} + e^{\alpha^\star(\xi_1-x)} e^{-(\alpha_0+\alpha^\star)\xi_0}\} \tag{7.32}$$

Then by (3.9)and (3.10) we get

$$|\omega| \le c\int_{\mathbb{R}_+} dx\, \frac{v_{\xi_1}}{p_{\xi_1}} |K(x)| \le c'\Big(\xi e^{-(5\alpha+\delta)\xi_1} + e^{-(\alpha_0+\alpha^\star+3\alpha+\delta-\alpha^\star)\xi_1}\Big) \tag{7.33}$$

which proves (7.30).

By (3.20) and (3.8) there is $c > 0$ so that

$$|\psi^\star(x)| \le c\Big\{e^{2\alpha\xi_1}|\omega| v_{\xi_1}(x) + \big(\|K\|_\infty + |\omega|\big)\Big\} \tag{7.34}$$

which proves (7.26). Lemma 7.2 is proved. □

8. $m^0_{\xi_1} + \psi + \psi^\star$ satisfies the conditions in Definition 4.1

To have lighter notation we write in this section ξ for $\xi_1 = \xi_1(h)$.

Proof of (4.2)

We need to show that there are $\zeta > 0$ and $c > 0$ so that

$$\|f\big(m^0_\xi + \psi + \psi^\star\big)\|_\infty \le c e^{-(4\alpha+\zeta)\xi} \tag{8.1}$$

We go back to (5.40) and examine separately all the terms. For the first one we use (7.26) and get

$$\|z_\xi [J\star\psi^\star]^2\|_\infty \le c e^{-2(2\alpha+\delta)\xi} \tag{8.2}$$

After writing $\psi = (\psi - \tilde\psi) + (\tilde\psi - \Gamma) + \Gamma$, see (7.1), we bound the second term by

$$\begin{aligned} \|z_\xi [J\star\psi][J\star\psi^\star]\|_\infty \le c\Big(& [e^{-d-\xi^2} + e^{-(2\alpha+\delta)\xi}] e^{-(2\alpha+\delta)\xi} \\ & + \sup_{x\ge 0} [e^{\alpha^\star(\xi-x)} e^{-(\alpha_0+\alpha^\star)\xi} + e^{-2\alpha\xi}][e^{-(2\alpha+\delta)\xi} e^{-\alpha'|\xi-x|} + e^{-(3\alpha+\delta)\xi}]\Big) \end{aligned} \tag{8.3}$$

We have used (6.1) and (7.2) for $\psi - \tilde\psi$, (7.3) for $\tilde\psi - \Gamma$ and (6.1) for Γ; we have used (7.26) for $\psi^\star$ and (3.9) to bound v_ξ (which appears in (7.26)).

Since for ξ large enough, α' can be chosen as close to α as we need, we have $\alpha^\star < \alpha'$ and $\alpha_0 + \alpha^\star > 2\alpha$ for ξ large enough. In such a case the right hand side of (8.3) is bounded by $ce^{-(4\alpha+\zeta)\xi}$, for suitable values of c and ζ.

For the third term in (5.40) we have by (7.26) and since $h < ce^{-2\alpha\xi}$

$$\|z_\xi h [J\star\psi^\star]\|_\infty \le c e^{-2\alpha\xi} e^{-(2\alpha+\delta)\xi} \tag{8.4}$$

For the fourth term in (5.40) we write

$$\|\tilde z_\xi \Big\{ (J\star\psi)^3 - (J\star\Gamma)^3 \Big\}\|_\infty \le c[e^{-\alpha_0\xi}]^2 e^{-(2\alpha+\delta)\xi} \tag{8.5}$$

which is obtained using (6.1), (7.2), (7.3) and observing that the largest contribution comes from $(J\star\Gamma)^2[J\star(\tilde\psi - \Gamma)]$.

The term with $0_3\big(\psi,\psi^\star,h\big)$ is bounded as in (5.29) with ψ_1 and ψ_2 replaced by ψ and $\psi^\star$. The leading term is $\|J\star\psi\|_\infty^2 \|J\star\psi^\star\|_\infty$. Since $\|\psi\|_\infty \le c\|\Gamma\|_\infty$ we get

$$\|0_3\big(\psi,\psi^\star,h\big)\|_\infty \le c[e^{-\alpha_0\xi}]^2 e^{-(2\alpha+\delta)\xi} \tag{8.6}$$

thus (4.2) is proved.

Proof of (3.3)

Shorthanding $m = m_\xi^0 + \psi + \psi^\star$, we need to prove that the function $m + h$ satisfies (3.3). We have

$$\delta_\xi^0(m+h) = m + h - m_\xi^0 = \psi + \psi^\star + h$$

and split

$$\delta_\xi^0(m+h) = \delta_\xi^{0,1} + \delta_\xi^{0,2} \tag{8.7}$$

We next define $\delta_\xi^{0,1}$ ($\delta_\xi^{0,2}$ is then determined by (8.7)).

$$\delta_\xi^{0,1}(x) := -\int_0^\tau dt \int_{|\xi-y|\le\xi^{1/2}} dy\, G_{\xi,t}(x,y) k_\xi^{\mathrm{odd}}(y) \tag{8.8}$$

where $G_{\xi,t}(x,y)$ is defined in (8.10) below and

$$k_\xi^{\mathrm{odd}}(x) = \frac{1}{2}[k_\xi(x) - k_\xi(2\xi - x)] \tag{8.9}$$

(i.e. it is the odd part of the function k_ξ defined in (5.2), where oddness is w.r.t. the point ξ). The kernel $G_{\xi,t}(x,y)$ is defined by setting $A_\xi(x,y)=p_{m^0_\xi}(x)J(y-x)$ and

$$G_{\xi,t}(x_0,y)=e^{-t}\sum_{n\ge 0}\frac{t^n}{n!}\int dx_1..dx_{n-1}\Big(\prod_{i=0}^{n-1}\mathbf{1}_{x_i\in(1,2\xi-1)}\Big)A_\xi(x,x_1)..A_\xi(x_{n-1},y) \tag{8.10}$$

As we will see $G_{\xi,t}(x,y)$ is a good approximation to $e^{L_\xi t}(x,y)$.

We will prove that there are c and δ positive so that

$$\Big|\int_{|\xi-x|\le\xi^{1/2}} dx\,\delta^{0,2}_\xi m(x)\bar m'_\xi(x)^2\bar m_\xi(x)\Big|\le ce^{-(2\alpha+\delta)\xi} \tag{8.11}$$

We will also prove that $\delta^{0,1}_\xi$ is odd (w.r.t. the point ξ) and that for $x\ge\xi$

$$\delta^{0,1}_\xi(x)\ge 0 \tag{8.12}$$

Since $\bar m'_\xi$ is symmetric, $\bar m_\xi$ antisymmetric and non positive for $x\ge\xi$, (8.11) and (8.12) prove (3.3).

Proof of (8.11)

We write $\delta^{0,2}_\xi$ as

$$\delta^{0,2}_\xi := h+\psi^\star+[\psi-\tilde\psi]+\Big\{\tilde\psi+\int_0^\tau dt\int_{|\xi-y|\le\xi^{1/2}}dy\,G_{\xi,t}(x,y)k^{\rm odd}_\xi(y)\Big) \tag{8.13}$$

and examine separately the contribution to the integral in (8.11) of all the terms on the right hand side of (8.13).

The first term with h vanishes by symmetry. By (2.6) and (7.26) there are $c>0$ and $c'>0$ so that

$$\Big|\int_{|\xi-x|\le\xi^{1/2}}dx\,\psi^\star(x)\bar m'_\xi(x)^2\bar m_\xi(x)\Big|\le c\int dx\,e^{-2\alpha|\xi-x|}e^{-(2\alpha+\delta)\xi}\le c'e^{-(2\alpha+\delta)\xi} \tag{8.14}$$

The contribution of $[\psi-\tilde\psi]$ is also bounded as on the right hand side of (8.11) as it follows using (7.2).

We use the expression (7.1) for $\tilde\psi$. The last two terms can be bounded using (3.17) and (6.1): there are c and c' so that

$$\begin{aligned}\|\int_0^\tau dt\,e^{L_\xi t}\Big(\frac12 z_\xi[J\star(\Gamma+h)]^2+\frac{1}{3!}\tilde z_\xi(J\star\Gamma)^3\|_\infty &\le c\tau e^{\tau d_+ e^{-2\alpha\xi}}\Big(\|\Gamma\|_\infty+h^2\Big)\\ &\le c'\xi^2e^{-2\alpha_0\xi}\end{aligned} \tag{8.15}$$

(recall that $\tau=\xi^2$). Since $\alpha_0>\alpha$ (8.15) is bounded by $c''e^{-(2\alpha+\delta')\xi}$, for suitable values of $c''>0$ and $\delta'>0$ in agreement with (8.11).

The remaining term in $\tilde\psi$ gives rise to

$$M_1:=-\int_{|\xi-x|\le\xi^{1/2}}dx\,\bar m'_\xi(x)^2\bar m_\xi(x)\int_0^\tau dt\,e^{L_\xi t}\big\{hp_\xi+b_\xi\big\} \tag{8.16}$$

We will next show that there are c and δ positive so that

$$|M_1-M_2|\le ce^{-(2\alpha+\delta)\xi} \tag{8.17}$$

where

$$M_2 := -\int_{|\xi-x|\le\xi^{1/2}} dx\,\bar m_\xi'(x)^2\bar m_\xi(x)\int_0^\tau dt\,e^{L_\xi t}\{hp_\xi - k_\xi\} \tag{8.18}$$

In fact by (5.4) there is $c>0$ so that

$$|M_1-M_2|\le c\int_{|\xi-x|\le\xi^{1/2}} dx\,\bar m_\xi'(x)^2\bar m_\xi(x)\int_0^\tau dt\,e^{L_\xi t}(x,y)\{e^{-2\alpha(\xi+y)}+\mathbf{1}_{y\in[0,1]}e^{-\xi\alpha_0}\} \tag{8.19}$$

Using (6.38) and (2.6) we then obtain (8.17), we omit the details.

We next compare M_2 with

$$M_3 := -\int_{|\xi-x|\le\xi^{1/2}} dx\,\bar m_\xi'(x)^2\bar m_\xi(x)\int_0^\tau dt\,G_{\xi,t}(x,y)\{hp_\xi - k_\xi\} \tag{8.20}$$

We write

$$\begin{aligned} e^{L_\xi t}(x,y) = G_{\xi,t}(x,y) + e^{-t}\sum_{n\ge0}\frac{t^n}{n!}\sum_{k=1}^{n-1}\int dx_1..dx_{n-1}\Big(\prod_{i=1}^{k-1}\mathbf{1}_{x_i\in(1,2\xi-1)}\Big)\mathbf{1}_{x_k\notin(1,2\xi-1)} \\ \times A_\xi(x,x_1)\dots A_\xi(x_{k-1},x_k)A_\xi^{n-k}(x_k,y) \end{aligned} \tag{8.21}$$

By Proposition 6.3 of [**2**] for any $0<\zeta<\alpha$ there is c so that for all $k\ge1$

$$A_\xi^k(x,y)\le ck(1+\lambda_\xi)^k\,e^{-\zeta|x-y|}$$

Then there are $c>0$ and $c'>0$ so that

$$\begin{aligned} \Big|e^{L_\xi t}(x,y)-G_{\xi,t}(x,y)\Big| &\le ce^{\lambda_\xi t}[(1+\lambda_\xi)t]^2e^{-\zeta(x+y)}] \\ &\le c'\xi^4e^{-\zeta(x+y)} \end{aligned} \tag{8.22}$$

Using (8.22) and (2.6) we have for suitable $c>0$ and $\delta>0$

$$|M_2-M_3|\le ce^{-(2\alpha+\delta)\xi} \tag{8.23}$$

We next observe that

$$G_{\xi,t}(\xi-x,\xi-y)=G_{\xi,t}(\xi+x,\xi+y) \tag{8.24}$$

while $\bar m_\xi'(x)^2\bar m_\xi(x)$ is antisymmetric. Then the even part of $hp_\xi-k_\xi$ in (8.20) does not contribute to the integral and we are left with the odd one. Since hp_ξ is even this term drops out and we are left with the odd part of k_ξ. This compensates exactly the last term on the right hand side of (8.13) (which has the opposite sign) so that (8.11) is proved.

$\delta_\xi^{0,1}$ is an odd function because such is $k_\xi^{\rm odd}$ and because $G_{\xi,t}$ is even, see (8.24). By (5.2)

$$k_\xi(y)-k_\xi(2\xi-y)=e^{-2\alpha\xi}a\frac{m_\beta^2-\bar m(\xi-y)^2}{1-m_\beta^2}[e^{\alpha(\xi-y)}-e^{-\alpha(\xi-y)}] \tag{8.25}$$

which is non positive for $y\ge\xi$. Thus $-k_\xi^{\rm odd}$ is an odd function which is non negative for $y\ge\xi$. To prove that $\delta_\xi^{0,1}$ has the same property we use for the first time in the paper the assumption that $J(x)$ is not increasing for $x\ge0$.

Lemma 8.1. Let $f_\xi(x)$ be a bounded function antisymmetric around ξ and non negative for $x \ge \xi$. Then for any $x \ge \xi$ and $t \ge 0$

$$G_{\xi,t} f(x) \ge 0 \tag{8.26}$$

Proof.

Let $x \ge \xi$, then

$$\int_{\mathbb{R}} dy\, J(y-x) f_\xi(y) = \int_\xi^\infty dy\, f_\xi(y)[J(y-x) - J(2\xi - y - x)]$$

Let $x \ge \xi$, $y \ge \xi$ and

$$A_\xi^+(x,y) = p_{m_\xi^0}(x)[J(y-x) - J(2\xi - y - x)] \tag{8.27}$$

Since $|2\xi - y - x| \ge |y-x|$, it follows, by the monotonicity of $J(\cdot)$ in $\mathbb{R}_+$, that

$$A_\xi^+(x,y) \ge 0 \tag{8.28}$$

On the other hand

$$\int_{\mathbb{R}} dy\, G_{\xi,t}(x,y) f_\xi(y) = e^{-t} \sum_{n \ge 0} \frac{t^n}{n!} \int_\xi^\infty dx dx_1 .. dx_{n-1} \Big(\prod_{i=1}^{n-1} \mathbf{1}_{x_i \in (\xi, 2\xi - 1)}\Big)$$
$$\times A_\xi^+(x,x_1) .. A_\xi^+(x_{n-1},y) f_\xi(y) \tag{8.29}$$

which by (8.28) is non negative. Lemma 8.1 is proved. □

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