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Glauber evolution with Kac potentials: II. Fluctuations*

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Abstract. In this paper we continue the analysis of the Glauber evolution in Ising systems with Kac interactions. In the first paper, we have proved that in a continuum limit, called the mesoscopic limit, the magnetization density converges to the solution of a non-local deterministic equation. Here we study the fluctuations around the limit proving convergence to a generalized Ornstein–Uhlenbeck process. We also prove asymptotic formulae for the correlation functions that improve those established in the previous paper and that will be used in a successive paper to study phase separation.

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1. Introduction

This is the second paper in a series devoted to the analysis of the Glauber evolution of ± 1 valued spins on the lattice $\mathbb{Z}^d$, interacting via a Kac potential. The purpose of the series is to extend to non-equilibrium the equilibrium theory of Kac interactions developed in the late 1960s, thus giving a full justification to the van der Waals theory in the context of statistical mechanics.

We refer to [5] and references therein for more comments on the model, we only recall here the two main features of a Kac potential, namely that its range diverges as γ^{-1}, where γ is a positive scaling parameter that eventually goes to 0, and that the total interaction energy of any single spin with all the others is bounded uniformly in γ. In [6] it is proved that in the mesoscopic limit $\gamma \to 0$, where space is scaled by the same γ, the limit magnetization density $m(r, t)$ solves the non-local equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta(J \star m + h)\} \tag{1.1}$$

$$(J \star m)(r) = \int \mathrm{d}r' \, J([r - r'])m(r') \,. \tag{1.2}$$

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The deterministic behaviour described by (1.1) is a mean-field effect due to the scaling of the interaction and to the validity of a law of large numbers for the density field. In this paper we study the fluctuations proving in theorem 2.8 below that the small deviations from the deterministic limit converge to a generalized Ornstein–Uhlenbeck process.

Fluctuations have a relevant role in non-equilibrium statistical mechanics, see for instance the fluctuating hydrodynamic theory in [13] where the case of stochastic interacting particle systems is explicitly discussed. Fluctuations are particularly important when the system is in a critical state where they may produce large, macroscopic effects. This will be discussed at length in the next paper [7], where we study the phase separation. We also refer to [2, 8, 9] for the study of the fluctuations of the interface in the Ginzburg–Landau equation with noise and to [1, 7] for the analysis of the one-dimensional fluctuations at the critical point for the same model considered here. The fluctuations in such a case solve, in a suitable limit, the stochastic quantization equation of Euclidean field theory. A more detailed discussion of all these aspects may be found in an earlier version of this paper [5].

All the proofs in this and in the subsequent paper [7], are consequences of sharp estimates on some special functions, the v and the ω functions, whose analysis is reminiscent of the cluster expansion in equilibrium statistical mechanics. The v and the ω functions are a linear combination of the correlation functions. The v functions define 'a distance' between the actual measure at time t and the product measure with averages $m(\gamma x, t)$, $m(r, t)$ solving (1.1). In [5] we have proved that the v functions of 'order' n (i.e. sum of expectations of products of at most n spins) are bounded by $c_n\gamma^{dn/2}$, c_n a positive coefficient that depends only on n. The ω functions are special linear combinations of the v functions and are similar to the truncated correlations functions. They 'measure the distance' of the v functions from the moments of a Gaussian process. The main result in this paper is theorem 2.6 where we show that the ω functions are bounded by $c'_n\gamma^{2dn/3}$, $c'_n > 0$, which vanishes faster than the bound, $c_n\gamma^{dn/2}$, on the v-functions. Thus to first order the spins are independent, while to the next order they are mutually correlated as in a Gaussian distribution. In this way we prove that all the moments of the fluctuation fields converge to the moments of a limit Ornstein Uhlenbeck process. The weak convergence of the whole process is then a simple consequence of the Holley and Stroock theory [9].

The paper is organized as follows. In section 2 we state the main results, in section 3 we study the ω functions proving theorem 2.6, and finally in section 4 we prove the convergence of the fluctuation fields to the generalized Ornstein–Uhlenbeck process.

2. Main definitions and results

For ease of reference we first recall the basic definitions from [5] and state precisely the bounds on the v functions mentioned in the introduction.

The Glauber dynamics is a Markov process with state space $\{-1, 1\}^{\mathbb{Z}^d}$, whose elements, the spin configurations, are denoted by σ, $\sigma = \{\sigma(x),\ x \in \mathbb{Z}^d\}$. σ_t, $t \geq 0$, denotes the spin configuration (i.e. the state of the process) at time $t \geq 0$.

The generator of the process, denoted by L_γ, $\gamma > 0$, is defined by its action on the cylindrical functions f:

$$L_\gamma f(\sigma) = \sum_{x\in\mathbb{Z}^d} c_\gamma(x, \sigma)[f(\sigma^x) - f(\sigma)] \tag{2.1}$$

where

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if} \quad y \neq x \\ -\sigma(x) & \text{if} \quad y = x \end{cases}$$

$$c_\gamma(x,\sigma) = \frac{\mathrm{e}^{-\beta h_\gamma(x)\sigma(x)}}{\mathrm{e}^{-\beta h_\gamma(x)} + \mathrm{e}^{\beta h_\gamma(x)}} \tag{2.2}$$

$$h_\gamma(x) = h + (J_\gamma \circ \sigma)(x) \qquad (J_\gamma \circ \sigma)(x) = \sum_{y\neq x} J_\gamma(x,y)\sigma(y) \tag{2.3}$$

and, finally,

$$J_\gamma(x,y) = \gamma^d J(\gamma[x-y]) \tag{2.4}$$

with $J \in C^3(\mathbb{R}^d)$, spherically symmetric and vanishing when $|r| \geq 1$, see [5].

$\beta > 0$ is the inverse temperature, $h \in \mathbb{R}$ an external magnetic field and $\gamma > 0$ the scaling parameter of the Kac potential, that goes eventually to 0.

Definition 2.1 *(The initial distribution.) In this paper we always consider initial distributions μ that are product measures on $\{-1,1\}^{\mathbb{Z}^d}$. Sometimes we restrict to a particular class μ^γ of product measures determined by a function $m_0 \in C^0(\mathbb{R}^d)$ such that $\|m_0\|_\infty \leq 1$; μ^γ is then the product measure such that*

$$\mathbb{E}_{\mu^\gamma}(\sigma(x)) = m_0(\gamma x) \qquad x \in \mathbb{Z}^d\,. \tag{2.5}$$

Finally for any probability μ, we denote by $\mathbb{P}^\gamma_\mu$ the law of the Markov process generated by L_γ with initial measure μ; $\mathbb{E}^\gamma_\mu$ denotes the corresponding expectation.

In many applications μ will simply be the measure supported by a single configuration σ_0, then $\mathbb{E}_\mu(\sigma(x)) = \sigma_0(x)$. For a while we keep $\gamma > 0$ fixed and for notational simplicity we will often drop the dependence on γ.

The basic quantities that have been extensively studied in [6] are the correlation functions, i.e. the expectation of products of spins all at the same time. We will use the following notation: $\underline{x}$ denotes a finite subset of $\mathbb{Z}^d$, S the collection of all such subsets, $S^{(n)}$, $n \geq 0$, the subset of S of all the sets $\underline{x}$ whose cardinality, $|\underline{x}|$, is n. Given $\underline{x}$ and $\underline{y}$ in S we denote by $\underline{x}+\underline{y}$ the union of $\underline{x}$ and $\underline{y}$ and by $\underline{x}-\underline{y}$, $\underline{y} \subset \underline{x}$, their set theoretical difference. We then write

$$\sigma_t(\underline{x}) = \prod_{x\in\underline{x}} \sigma_t(x)\,. \tag{2.6}$$

We will study the deviation of the expectation of $\sigma_t(\underline{x})$ from the product of the expectations of its factors $\sigma_t(x)$'s and determine the leading term as $\gamma \to 0$.

Rather than centring the spins it is convenient to consider the variables

$$\tilde{\sigma}_t(\underline{x}) = \prod_{x\in\underline{x}} [\sigma_t(x) - m_t(x)] \tag{2.7}$$

where m_t solves

$$\frac{\mathrm{d}m_t(x)}{\mathrm{d}t} = -m_t(x) + \tanh\{\beta[(J_\gamma \circ m_t)(x) + h]\} \tag{2.8}$$

with initial condition

$$m_0(x) = \mathbb{E}_\mu\big(\sigma(x)\big)\,. \tag{2.9}$$

Equation (2.8) is a discretized version of (1.1). Its solution depends in general on γ and, under suitable assumptions on the initial data, converges to the solution of (1.1) [6].

The v functions are defined by

$$v_t(\underline{x}) = \mathbb{E}_\mu\big(\tilde{\sigma}_t(\underline{x})\big)\,. \tag{2.10}$$

The initial measure μ in (2.10) will always be a product measure. μ specifies m_0 by (2.9), m_t by (2.8) and then $\tilde{\sigma}_t$ by (2.7). Everything in (2.10) depends on γ: the law of the process, the variable σ_t and v_t. For ease of reference we report below theorem 2.3.4 of [5].

Theorem 2.2 *There are $a > 0$, c_n, $n \geq 1$, and C so that for any initial product measure, for any $t \leq a \log\gamma^{-1}$ and $\underline{x} \in S$,*

$$|v_t(\underline{x})| \leq c_{|\underline{x}|}\,\mathrm{e}^{C|\underline{x}|t}\gamma^{d|\underline{x}|/2} \tag{2.11}$$

where $|\underline{x}|$ denotes the cardinality of $\underline{x}$.

If $aC < d/2$, the right-hand side of (2.11) vanishes as a power of $\gamma^{|\underline{x}|}$.

As mentioned in the introduction we will prove that the leading term in the v functions is given by 'a Gaussian approximation'. We need several definitions before stating this result in theorem 2.6.

We call V_t the linear space of all the cylindrical functions $f(\sigma_t)$, σ_t the spin configuration at time t. V_t is the linear span of the set $\{\sigma_t(\underline{x}), \underline{x} \in S\}$. The linear span of $\{\sigma_t(\underline{x}), \underline{x} \in S^{(n)}\}$ will be denoted by $V_t^{(n)}$.

Let $M(S)$ be the linear space of functions on S. Then

$$\underline{\sigma}_t := \{\sigma_t(\underline{x}), \underline{x} \in S\} \qquad \underline{\tilde{\sigma}}_t := \{\tilde{\sigma}_t(\underline{x}), \underline{x} \in S\} \tag{2.12}$$

are random variables with values in $M(S)$: namely, every spin configuration σ_t at time t determines the values $\sigma_t(\underline{x})$ and $\tilde{\sigma}_t(\underline{x})$; these collections specify the values of $\underline{\sigma}_t$ and $\underline{\tilde{\sigma}}_t$ at the given configuration σ_t.

We use the convention

$$\sigma_t(\emptyset) = \tilde{\sigma}_t(\emptyset) = 1\,. \tag{2.13}$$

As usual in equilibrium statistical mechanics, [12], we introduce an algebraic structure in $M(S)$ defined by the $*$ product

$$(f * g)(\underline{x}) = \sum_{\underline{y}\subset\underline{x}} f(\underline{y})g(\underline{x}-\underline{y})\,. \tag{2.14}$$

It is then readily seen that, setting

$$\underline{m}_t = \left\{m_t(\underline{x}) = \prod_{x\in\underline{x}} m_t(x)\right\} \qquad m_t(\emptyset) = 1 \tag{2.15}$$

with m_t solution of (2.8);

$$\underline{\sigma}_t = \underline{\tilde{\sigma}}_t * \underline{m}_t \qquad \underline{\tilde{\sigma}}_t = \underline{\sigma}_t * \underline{m}_t^{-1} \tag{2.16}$$

where, $\underline{m}_t^{-1}$, the $*$ inverse of $\underline{m}_t$, is

$$\underline{m}_t^{-1}(\underline{x}) = (-1)^{|\underline{x}|}\underline{m}_t(\underline{x})\,. \tag{2.17}$$

For future reference we observe that the identity for the $*$ product is I, where $I(\underline{x}) = 0$ if $|\underline{x}| > 0$ and $I(\emptyset) = 1$. Moreover $f \in M(S)$ has a $*$ inverse if and only if $f(\emptyset) \neq 0$: in fact we can construct f^{-1} iteratively, setting $f^{-1}(\emptyset) = f(\emptyset)^{-1}$, then solving

$$0 = (f^{-1} * f)(x) \qquad \text{which gives} \quad f^{-1}(x) = -f(\emptyset)^{-1}[f^{-1}(\emptyset)f(x)]$$

and so forth.

We next introduce the linear operator K_t on $M(S)$ as follows:

Definition 2.3 *Let $m \in [-1,1]^{\mathbb{Z}^d}$, m_t, $t \geq 0$, the solution of (2.8) with initial datum m. Then for any $t \geq 0$, x and y in $\mathbb{Z}^d$ we set*

$$k_t(x,y,m) = -\mathbf{1}_{x,y} + f_x^{(1)}(m_t)\beta J_\gamma(x,y) \tag{2.18}$$

where $\mathbf{1}_{x,y}$ is the Kronecker delta, and denoting by $\tanh^{(n)}(\cdot)$ the nth derivative of the hyperbolic tangent,

$$f_x^{(n)}(m) = \tanh^{(n)}\{\beta[J_\gamma \circ m(x) + h]\} \qquad f_x(m) = f_x^{(0)}(m)\,. \tag{2.19}$$

We then define the linear operator K_t on $M(S)$ by setting

$$(K_t g)(\underline{x}) = \sum_{x\in\underline{x}}\sum_{y\notin\underline{x}-x} k_t(x,y,m)g(\underline{x}-x+y) \tag{2.20}$$

and let $K_t^{(n)}$, $n \geq 0$, the restriction of K_t to $M(S^{(n)})$. We finally denote by $K_t(\underline{x},\underline{y})$ the kernel of K_t:

$$(K_t g)(\underline{x}) = \sum_{\underline{y}} K_t(\underline{x},\underline{y})g(\underline{y})\,. \tag{2.21}$$

Observe that $k_t(x,y,m)$ is the kernel of the operator obtained by linearizing (2.8). By the help of K_t we next define the W functions:

Definition 2.4 *Given m as in definition 2.3, we define $\kappa_t(x,y)$, dependent on m and interpreted as the value of the function κ_t on $S^{(2)}$, as*

$$\kappa_t(x,y) = \beta J_\gamma(x,y)\big\{f_y^{(1)}(m_t)[1-m_t(x)^2] + f_x^{(1)}(m_t)[1-m_t(y)^2]\big\} \tag{2.22}$$

$w_t : S^{(2)} \to \mathbb{R}$ denotes the solution of

$$\frac{\mathrm{d}w_t}{\mathrm{d}t} = K_t^{(2)}w_t + \kappa_t \qquad w_0 = 0\,. \tag{2.23}$$

We then define W_t as an element of $M(S)$ by setting $W_t = 0$ on $S^{(2n+1)}$ while, on $S^{(2n)}$,

$$W_t(\underline{x}) = \sum_{\{(i_1,j_1),\ldots,(i_n,j_n)\}}\prod_{\ell=1}^{n} w_t(x_{i_\ell},x_{j_\ell}) \qquad \underline{x} = (x_1,\ldots,x_{2n}) \tag{2.24}$$

where the sum is over all the partitions $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ *of* $\{1, \ldots, 2n\}$ *into* n *disjoint pairs.*

Observe that if ξ_x, $x \in \underline{x}$, are centred Gaussian variables with covariance $w_t(x, y)$, then $W_t(\underline{x})$ is the expectation of their product. We will prove that $W_t(\underline{x})$ is the leading term in the expectation of $\tilde{\sigma}(\underline{x})$. It is easy to see that there is a constant C such that

$$\|K_t^{(2)}\| \le C \tag{2.25}$$

as on operator on the space $M(S^{(2)})$, equipped with sup norm. Moreover there is a constant c such that $\|k_t\|_\infty \le c\gamma^d$. We then have by (2.23)

$$\|w_t\|_\infty \le c\,\mathrm{e}^{Ct}\gamma^d\,. \tag{2.26}$$

Then

$$|W_t(\underline{x})| \le \left(c\mathrm{e}^{Ct}\gamma^d\right)^{|\underline{x}|/2}\,. \tag{2.27}$$

The relation between $\underline{\tilde{\sigma}}_t$ and W_t is conveniently expressed in terms of the $\hat{\omega}$ functions as

$$\underline{\tilde{\sigma}}_t = \hat{\omega}_t * W_t \qquad \hat{\omega}_t = \underline{\tilde{\sigma}}_t * W_t^{-1} \tag{2.28}$$

where the $*$ inverse is

$$W_t^{-1}(\underline{x}) = (-1)^{|\underline{x}|/2} W_t(\underline{x})\,. \tag{2.29}$$

Observe that $\hat{\omega}(x) = \tilde{\sigma}(x)$ for $x \in \mathbb{Z}^d$. The proof of (2.29) is obtained by checking that for $|\underline{x}| > 0$

$$0 = \sum_{\underline{y} \subset \underline{x}} (-1)^{|\underline{y}|/2} W_t(\underline{y}) W_t(\underline{x} - \underline{y})\,. \tag{2.30}$$

We define

$$\omega_t(\underline{x}) = \mathbb{E}_\mu\big(\hat{\omega}_t(\underline{x})\big) \tag{2.31}$$

observing that

$$v_t = \omega_t * W_t \qquad \omega_t = v_t * W_t^{-1} \tag{2.32}$$

As we shall see in theorem 2.6 the ω functions are much smaller than the v functions so that the leading term in the first equality in (2.32) is the one with only W_t, the other terms are corrections and are smaller the larger is the order (i.e. the number of sites) of ω_t. We next specify the assumptions needed in theorem 2.6.

Definition 2.5 *(The assumptions). Given* $t \ge s \ge 0$, *we denote by* $U_{t;s}$ *the flow on* $M(S)$ *defined as follows. Let* $f \in M(S)$, $f_{s'}$, $s \le s' \le t$ *the solution of*

$$\frac{\mathrm{d}f_{s'}}{\mathrm{d}s'} = K_{s'} f_{s'} \qquad f_s = f\,. \tag{2.33}$$

Then $U_{t;s} f = f_t$. We also denote by $U_{t;s}(\underline{x}, \underline{y})$ the matrix elements of $U_{t;s}$, in agreement with definition 2.3, and by $U^{(n)}_{t;s}$ the restriction of $U_{t;s}$ to $M(S^{(n)})$.

We suppose that there are increasing functions a_t, b_t, ($b_t \geq ct$ for some c), both larger than 1, and coefficients c'_n, so that, for all $t \geq s$

$$\|U^{(n)}_{t;s}\|_\infty \leq c'_n a^n_{t-s} \qquad \|w_t\|_\infty \leq c'_2 b_t \gamma^d \tag{2.34}$$

and that

$$b_t^{-1} a_{t-s} b_s \leq c'_0 \,. \tag{2.35}$$

It is readily seen that the above assumptions are always satisfied by the choice

$$a_t = b_t = \mathrm{e}^{Ct} \tag{2.36}$$

for a suitable C. In some cases however, other choices for a_t and b_t are available and more convenient.

Theorem 2.6 *Let a_t, b_t and c'_n as in the Assumptions 2.5; then there are $a > 0$, ℓ_n and c_n so that for all γ, all $t \leq a \log \gamma^{-1}$, all $\underline{x}$ and all initial measures μ (that are product measures)*

$$|\omega_t(\underline{x})| \leq c_{|\underline{x}|} t^{\ell_{|\underline{x}|}} b_t^{|\underline{x}|} \gamma^{2d|\underline{x}|/3} \qquad \text{for all} \quad |\underline{x}| \geq 3 \tag{2.37}$$

$$|\omega_t(\underline{x})| \leq c_2 t^{\ell_2} b_t^5 \gamma^{2d} \qquad \text{for all} \quad |\underline{x}| = 2 \tag{2.38}$$

$$|\omega_t(x)| \leq c_1 t b_t^2 \gamma^d \qquad \text{for all} \quad x \in \mathbb{Z}^d \,. \tag{2.39}$$

By recalling (2.32) and (2.34) the leading contribution to $v_t(\underline{x})$ differs from $W_t(\underline{x})$ by a term that vanishes faster than $[b_t \gamma^{d/2}]^{|\underline{x}|}$. This allows us to go beyond the law of large numbers, as we did in [6], and to study the fluctuation fields. We will see in theorem 2.8 that they converge to a Ornstein–Uhlenbeck process by proving convergence of all the moments and the weak convergence of the process.

Before defining the fluctuation fields we recall from [6] the definition of the density fields. For $\phi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of smooth functions with fast decay to infinity, we define the (magnetic) density field

$$X^\gamma_t(\phi) = \gamma^d \sum_{x \in \mathbb{Z}^d} \phi(\gamma x) \sigma_t(x) \,. \tag{2.40}$$

By using the Chebishev inequality and (2.11), (we do a similar computation in section 4), we can easily see that for any $\delta > 0$

$$\lim_{\gamma \to 0} \sup_{0 \leq t \leq a \log \gamma^{-1}} \mathbb{P}^\gamma_{\mu^\gamma} \left(\left| X^\gamma_t(\phi) - \int \mathrm{d}r\, \phi(r) m_t(r) \right| > \delta \right) = 0 \tag{2.41}$$

where $m_t(r)$ is the solution of (1.1) with initial condition $m_0(r)$, m_0 and μ^γ as in definition 2.1. By using martingale techniques, see for instance [13], it is also possible to show that if $\mathbb{P}^\gamma$ is the law of the fields $\{X^\gamma_t(\phi)\}$ thought of as the canonical variables $\{X_t(\phi)\}$ in $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$, then $\mathbb{P}^\gamma$ converges weakly on the compacts of $\mathbb{R}_+$ to $\mathbb{P}$, the law supported by the trajectory $t \to m_t(r)\,\mathrm{d}r$.

We next consider the fluctuation fields

$$Y_t^\gamma(\phi) = \gamma^{d/2} \sum_{x\in\mathbb{Z}^d} \phi(\gamma x)[\sigma_t(x) - m_t^\gamma(x)] \tag{2.42}$$

where $m_t^\gamma(x)$ is the solution of (2.8) with initial condition $m_0(\gamma x)$, here we make explicit the dependence on γ. The distribution of $\{\sigma_t\}$ in (2.42) is $\mathbb{P}^\gamma_{\mu^\gamma}$ with μ^γ as in definition 2.1.

We then consider the space $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ and its canonical variables

$$Y_t(\phi) = \int \mathrm{d}r\, \phi(r)\xi(r,t)$$

where $\{\xi(r,t),\ r\in\mathbb{R}^d,\ t\ge 0\}$ are the elements of $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$. We denote by $\hat{\mathbb{P}}^\gamma$ the law on $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ whose marginal on the variables $Y_t(\phi)$ is the same as the distribution of the variables $Y_t^\gamma(\phi)$ induced by $\mathbb{P}^\gamma_{\mu^\gamma}$. We finally let $\hat{\mathbb{E}}^\gamma$ denote the expectation with respect to $\hat{\mathbb{P}}^\gamma$.

Definition 2.7 *(The Ornstein–Uhlenbeck process.) Let $\hat{\mathbb{P}}$ be the law of the Gaussian process on $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ with mean zero and covariance*

$$\hat{\mathbb{E}}\big(Y_s(\phi)Y_t(\psi)\big) = C_{s,t}(\phi,\psi) \equiv \int \mathrm{d}r\,\mathrm{d}r'\, C_{s,t}(r,r')\phi(r)\psi(r') \tag{2.43}$$

where, for any $0\le s<t$:

$$C_{s,t}(r_1,r_2) = \int \mathrm{d}r_2'\, \mathrm{e}^{\mathcal{L}(t-s)}(r_2,r_2')C_{s,s}(r_1',r_2') \tag{2.44}$$

and for any $t\ge 0$

$$C_{t,t}(r,r') = (1-m_t(r)^2)\delta(r-r') + C_t^0(r,r') \tag{2.45}$$

where $\mathcal{L}$ is the operator with kernel,

$$\mathcal{L}(r,r',t) = -\delta(r-r') + \mathcal{L}^0(r,r',t) \qquad \mathcal{L}^0(r,r',t) = \ell(r,t)\beta J(|r-r'|) \tag{2.46}$$

$$\ell(r,t) = \frac{1}{\cosh^2\{\beta(J*m_t)(r)+h\}} \tag{2.47}$$

and $C_t^0(r,r')$ solves

$$\begin{aligned}\frac{\mathrm{d}C_t^0(r,r')}{\mathrm{d}t} &= \int \mathrm{d}r''[\mathcal{L}(r,r'',t)C_t^0(r'',r') + \mathcal{L}(r',r'',t)C_t^0(r,r'')]\\ &\quad + \mathcal{L}^0(r,r',t)\,[1-m_t(r')^2] + \mathcal{L}^0(r,r',t)[1-m_t(r)^2] \\ C_0^0(r,r') &\equiv 0\,.\end{aligned} \tag{2.48}$$

Observe that $\mathcal{L}(r,r',t)$ in (2.46) and $\ell(r,t)$ in (2.47) are the continuous version of $k_t(x,y,m)$ and $f_x^{(1)}(m)$ defined in (2.18) and (2.19).

It can be seen that $\hat{\mathbb{P}}$ is supported by the solutions $\xi(r,t)$ of the stochastic equation

$$\frac{\mathrm{d}\xi(r,t)}{\mathrm{d}t} = \int \mathrm{d}r' \, \mathcal{L}(r,r',t)\xi(r',t) + \frac{\mathrm{d}B(r,t)}{\mathrm{d}t} \, . \tag{2.49}$$

The noise $B(r,t)$ is Gaussian with mean zero and covariance

$$\hat{\mathbb{E}}\left(\frac{\mathrm{d}B(r,t)}{\mathrm{d}t}\frac{\mathrm{d}B(r',t')}{\mathrm{d}t'}\right) = \delta(t-t')\delta(r-r')2b(r,t) \tag{2.50}$$

where

$$b(r,t) = 1 - m_t(r)\tanh\{\beta[(J \star m_t)(r) + h]\} \, . \tag{2.51}$$

The initial condition $\xi(r,0)$ of (2.49) is a Gaussian process with mean 0 and covariance

$$C_{0,0}(r,r') = \delta(r,r')\left(1 - m_0(r)^2\right) .$$

The distribution $\hat{\mathbb{P}}$ of the solution (2.49) is the Gaussian process with mean zero and covariances (2.44) and (2.45).

Observe that $\hat{\mathbb{P}}$ is a ‘generalized Ornstein–Uhlenbeck process’ since only the variables $Y_t(\phi)$ have a meaning as real functions, the variables $\xi(r,t)$ being only defined as distributions.

Theorem 2.8 *With the above notation, $\hat{\mathbb{P}}^\gamma$ converges weakly to $\hat{\mathbb{P}}$ on $D([0,T],\mathcal{S}'(\mathbb{R}^d))$, for any given $T > 0$. Also the $\hat{\mathbb{P}}^\gamma$-expectation of the product of any finite number of $Y_t(\phi)$'s converges to the corresponding $\hat{\mathbb{P}}$-expectation, for any fixed t.*

3. Proof of theorem 2.6

The proof starts from the computation of the derivative of $\omega_t(\underline{x})$:

$$\frac{\mathrm{d}\omega_t(\underline{x})}{\mathrm{d}t} = \mathbb{E}_{\mu^\gamma}\left(D_t^\star \hat{\omega}_t(\underline{x})\right) \tag{3.1}$$

where

$$D_t^\star = L_\gamma + \frac{\partial}{\partial t} \tag{3.2}$$

where L_γ is the generator of the process, see equation (2.1), and $\partial/\partial t$ the derivative with respect to time of m_t and W_t.

$D_t^\star$ is a linear operator on V_t and the whole proof follows from a good characterization of its action. We first need some more notation:

Definition 3.1 *Given a linear operator $A^\star$ on V_t we define $A(\underline{x},\underline{y})$ so that*

$$A^\star \hat{\omega}_t(\underline{x}) = \sum_{\underline{y}} A(\underline{x},\underline{y})\hat{\omega}_t(\underline{y}) \tag{3.3}$$

and the operator A *on* $M(S)$ *as*

$$(Af)(\underline{x}) = \sum_{\underline{y}} A(\underline{x}, \underline{y}) f(\underline{y}) . \tag{3.4}$$

We call χ_n *the projection onto* $M(S^{(n)})$*:*

$$(\chi_n f)(\underline{x}) = \begin{cases} f(\underline{x}) & \text{if} \quad |\underline{x}| = n \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

and

$$\chi_{\le n} = \sum_{k \le n} \chi_k . \tag{3.6}$$

We also define the operators $p^{\pm}_{t,\alpha}$ *on* $M(S)$ *as*

$$(p^{\pm}_{t,\alpha} f)(\underline{x}) = p^{\pm}_{t,\alpha}(|\underline{x}|) f(\underline{x}) \qquad p^{\pm}_{t,\alpha}(|\underline{x}|) = [\gamma^{2d/3-\alpha} b_t]^{\pm|\underline{x}|} \tag{3.7}$$

with either $\alpha = d/2$ *or* $\alpha = 2d/9$*;* b_t *is the function introduced in (2.34).*

We finally denote by $\|A\|_\infty$ *the norm of the operator* A *on the space* $M(S)$ *equipped with sup norm and set*

$$|A|_{t,\alpha} = \|p^-_{t,\alpha} A p^+_{t,\alpha}\|_\infty \qquad |A|_{t,\alpha,n} = \|\chi_n p^-_{t,\alpha} A p^+_{t,\alpha}\|_\infty . \tag{3.8}$$

Proposition 3.2 *Given any* $N > 2$ *there are two linear operators on* V_t*,* $A^\star_t$ *and* $R^\star_t$*, and coefficients* $c(n)$ *so that*

$$D^\star_t = K^\star_t + A^\star_t + R^\star_t \tag{3.9}$$

with

$$K^\star_t \hat\omega_t(\underline{x}) = \sum_{\underline{y}} K_t(\underline{x}, \underline{y}) \hat\omega_t(\underline{y})$$

see equation (2.21);

$$|R^\star_t \hat\omega_t(\underline{x})| \le c(|\underline{x}|) \sum_{x \in \underline{x}} \left| J_\gamma \circ \tilde\sigma_t(x) \right|^N . \tag{3.10}$$

Moreover, $A^\star_t : V^{(n)}_t \to V^{(n+N)}_t$ *and, recalling (3.8),*

$$|A_t|_{t,\alpha,n} \le c(n)\gamma^\alpha \qquad \|\chi_n A_t\|_\infty \le c(n) . \tag{3.11}$$

Proof. To have lighter notation we drop t from $\hat\omega_t$, $D^\star_t$, $K^\star_t$, $R^\star_t$, $\ldots$. By the definition of $\hat\omega$, if $x \notin \underline{x}$:

$$\hat\omega(\underline{x} + x) = \tilde\sigma(x)\hat\omega(\underline{x}) - \sum_{y \in \underline{x}} w(x, y)\hat\omega(\underline{x} - y) . \tag{3.12}$$

Recalling (2.19) for notation,

$$D^\star \tilde\sigma(x) = -\tilde\sigma(x) + f_x(\sigma) - f_x(m)\,. \tag{3.13}$$

We then get

$$D^\star \hat\omega(\underline{x}) = \sum_{x\in\underline{x}} \left\{ \hat\omega(\underline{x}-x)[f_x(\sigma) - f_x(m) - \tilde\sigma(x)] - \tfrac{1}{2} \sum_{y\in\underline{x}-x} \hat\omega(\underline{x}-x-y)\dot w(x,y) \right\} \tag{3.14}$$

where $\dot w(x, y)$ is the right-hand side of (2.23). After a Taylor expansion we get

$$f_x(\sigma) - f_x(m) = \sum_{k=1}^{N-1} \frac{1}{k!} f_x^{(k)}(m)[\beta J_\gamma \circ \tilde\sigma(x)]^k + r(x,\sigma,m,N)[\beta J_\gamma \circ \tilde\sigma(x)]^N \tag{3.15}$$

and identify

$$R^\star \hat\omega(\underline{x}) := \sum_{x\in\underline{x}} \hat\omega(\underline{x}-x) r(x,\sigma,m,N)[\beta J_\gamma \circ \tilde\sigma(x)]^N\,. \tag{3.16}$$

Since $|\tilde\sigma(x)| \le 2$, by (2.34)

$$|\hat\omega(\underline{x})| \le c(|\underline{x}|) \qquad \text{where } c(|\underline{x}|) \text{ is a combinatorial factor}$$

hence (3.10).

We thus have

$$D^\star \hat\omega(\underline{x}) = \Omega_1 + \Omega_2 + R^\star \hat\omega(\underline{x}) \tag{3.17}$$

$$\Omega_1 = \sum_{x\in\underline{x}} \left\{ \hat\omega(\underline{x}-x)\big(-\tilde\sigma(x) + f_x^{(1)}(m)\beta J_\gamma \circ \tilde\sigma(x)\big) - \tfrac{1}{2} \sum_{y\in\underline{x}-x} \hat\omega(\underline{x}-x-y)\dot w(x,y) \right\} \tag{3.18}$$

$$\Omega_2 = \sum_{x\in\underline{x}} \hat\omega(\underline{x}-x) \sum_{k=2}^{N-1} \frac{1}{k!} f_x^{(k)}(m)[\beta J_\gamma \circ \tilde\sigma(x)]^k =: \sum_{\underline{y}} \Omega_2(\underline{x},\underline{y})\hat\omega(\underline{y})\,. \tag{3.19}$$

We write

$$\Omega_1 = \Omega_{1,1} + \Omega_{1,2} + \Omega_{1,3} \tag{3.20}$$

with, see equation (2.18),

$$\begin{aligned} \Omega_{1,1} = \sum_{x\in\underline{x}} \Bigg\{ & f_x^{(1)}(m) \sum_{z\notin\underline{x}} \beta J_\gamma(x,z)\tilde\sigma(z)\hat\omega(\underline{x}-x) \\ & - \tfrac{1}{2} \sum_{y\in\underline{x}-x} \hat\omega(\underline{x}-x-y) \sum_{z\notin\underline{x}} \Big(k(x,z,m)w(z,y) + k(y,z,m)w(x,z) \Big) \Bigg\} \end{aligned} \tag{3.21}$$

$$\Omega_{1,2} = \sum_{x\in\underline{x}} \left\{ \sum_{y\in\underline{x}-x} w(x,y)\hat\omega(\underline{x}-x-y) - \tilde\sigma(x)\hat\omega(\underline{x}-x) \right\} \tag{3.22}$$

$$\begin{aligned} \Omega_{1,3} = \sum_{x\in\underline{x}} \sum_{y\in\underline{x}-x} \Bigg\{ & f_x^{(1)}(m)\beta J_\gamma(x,y)\tilde\sigma(y)\hat\omega(\underline{x}-x) - \tfrac{1}{2}\hat\omega(\underline{x}-x-y)\kappa(x,y) \\ & - \tfrac{1}{2} \sum_{z\in\underline{x}-x-y} \hat\omega(\underline{x}-x-y)[k(x,z,m)w(z,y) + k(y,z,m)w(x,z)] \Bigg\}\,. \end{aligned} \tag{3.23}$$

Thus the first term, $\Omega_{1,1}$, takes into account the contribution from $J_\gamma \circ \tilde{\sigma}(x)$ and $\dot{w}(x, y)$ due to the sites not in $\underline{x}$. The contribution to $\dot{w}(x, y)$ from $k(x, x)$ and $k(y, y)$ is in $\Omega_{1,2}$, that also accounts for the first term on the right-hand side of (3.18). The remaining terms are collected in $\Omega_{1,3}$.

We observe that the last term on the right-hand side of (3.21) is equal to

$$-\sum_{x\in\underline{x}}\sum_{y\in\underline{x}-x}\sum_{z\notin\underline{x}}\hat{\omega}(\underline{x}-x-y)k(x,z,m)w(z,y)\,.$$

Thus, using (3.12) and (2.18) we have

$$\Omega_{1,1} = \sum_{x\in\underline{x}}\sum_{z\in\underline{x}-x} k(x,z,m)\hat{\omega}(\underline{x}+z-x)\,.$$

By equation (3.12)

$$\Omega_{1,2} = -\sum_{x\in\underline{x}}\hat{\omega}(\underline{x})$$

hence

$$\Omega_{1,1} + \Omega_{1,2} = K^\star\hat{\omega}(\underline{x})\,. \tag{3.24}$$

We next consider $\Omega_{1,3}$. Let $x \in \underline{x}$ and $y \in \underline{x} - x$ be fixed, then, using (3.12),

$$\hat{\omega}(\underline{x}-x) = \tilde{\sigma}(y)\hat{\omega}(\underline{x}-x-y) - \sum_{z\in\underline{x}-x} w(y,z)\hat{\omega}(\underline{x}-x-y-z)\,. \tag{3.25}$$

Since

$$\tilde{\sigma}(y)^2 = 1 - m(y)^2 - 2m(y)\tilde{\sigma}(y) \tag{3.26}$$

and recalling the definition of $\kappa(x, y)$, see (2.22), the second term on the right-hand side of (3.23) is cancelled by a term contributing to the first one, precisely the one that comes from $1 - m(y)^2$ in (3.25). Thus

$$\begin{aligned}\Omega_{1,3} = -\sum_{x\in\underline{x}}\sum_{y\in\underline{x}-x}\Bigg\{&k(x,y,m)2m(y)\tilde{\sigma}(y)\hat{\omega}(\underline{x}-x-y)\\ &+\sum_{z\in\underline{x}-x-y} w(y,z)(k(x,y,m)\tilde{\sigma}(y)\hat{\omega}(\underline{x}-x-y-z)+k(x,z,m)\hat{\omega}(\underline{x}-x-y))\Bigg\}\\ =:\ &\sum_{\underline{y}}\Omega_{1,3}(\underline{x},\underline{y})\hat{\omega}(\underline{y})\,.\end{aligned} \tag{3.27}$$

We have proven so far that

$$D^\star\hat{\omega} = K^\star\hat{\omega} + \Omega_2 + \Omega_{1,3} + R^\star\hat{\omega} \tag{3.28}$$

with $\Omega_{1,3}$ as in (3.27) and Ω_2 as in (3.19). We have also proved the bound (3.10) for $R^\star\hat{\omega}(\underline{x})$.

All the terms in Ω_2 and $\Omega_{1,3}$ are of the form $\tilde{\sigma}(x_1)\dots\tilde{\sigma}(x_n)\hat{\omega}(\underline{z})$, for some $n \geq 1$, $x_1 \dots x_n$ and $\underline{z}$. Multiplication by $\tilde{\sigma}(x)$ is a linear operator on V_t, identified by the matrix elements $S_x(\underline{x}, \underline{y})$ via the identity

$$\tilde{\sigma}(x)\hat{\omega}(\underline{x}) = \sum_{\underline{y}} S_x(\underline{x}, \underline{y})\hat{\omega}(\underline{y}) \,. \tag{3.29}$$

Using equation (3.12) we obtain that for $x \notin \underline{x}$

$$\sum_{\underline{y}} S_x(\underline{x}, \underline{y})\hat{\omega}(\underline{y}) = \hat{\omega}(\underline{x} + x) + \sum_{y \in \underline{x}} w(x, y)\hat{\omega}(\underline{x} - y) \,. \tag{3.30}$$

The same equation can be used to find also the other ones. Let $x \in \underline{x}$ and denote by $\underline{y} = \underline{x} - x$, then

$$\begin{aligned}\tilde{\sigma}(x)\hat{\omega}(\underline{y} + x) &= \tilde{\sigma}(x)^2\hat{\omega}(\underline{y}) - \tilde{\sigma}(x)\sum_{y \in \underline{y}} w(x, y)\hat{\omega}(\underline{y} - y) \\ &= (1 - m(x)^2)\hat{\omega}(\underline{y}) - 2m(x)\tilde{\sigma}(x)\hat{\omega}(\underline{y}) - \sum_{y \in \underline{y}} w(x, y)\tilde{\sigma}(x)\hat{\omega}(\underline{y} - y) \,.\end{aligned} \tag{3.31}$$

Thus for $x \notin \underline{x}$, $|\underline{x}| = n$ from (3.30) and (2.34) we get,

$$\begin{aligned}p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |S_x(\underline{x}, \underline{y})| p^+_{t,\alpha}(|\underline{y}|) &\leq b_t\gamma^{2d/3-\alpha} + n\gamma^{-(2d/3-\alpha)}b_t^{-1}c_2'b_t\gamma^d \\ &\leq A_{\gamma,n} := c_1 n\gamma^{d/3+\alpha/2}b_t\end{aligned} \tag{3.32}$$

because $\alpha = 0$ or $\alpha = 2d/9$ and

$$\gamma^{2d/3-\alpha} \leq \gamma^{d/3+\alpha/2}$$

for all $t \leq a\log\gamma^{-1}$ and all γ small enough.

For $x \in \underline{x}$ we use (3.31) and we observe that we can apply the same estimates as before for the second and third term on the right-hand side of (3.31). Then from (2.34) we get

$$\begin{aligned}p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |S_x(\underline{x}, \underline{y})| p^+_{t,\alpha}(|\underline{y}|) &\leq \gamma^{-(2d/3-\alpha)}b_t^{-1} + 2\gamma^{-(2d/3-\alpha)}b_t^{-1}A_{\gamma,n} \\ &\quad + n[\gamma^{-(2d/3-\alpha)}b_t^{-1}]^2 c_2' b_t\gamma^d A_{\gamma,n} \leq c_2 n^2\gamma^{-(2d/3-\alpha)} \,.\end{aligned} \tag{3.33}$$

Recalling (3.27) and using (3.32) we have

$$\begin{aligned}p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |\Omega_{1,3}(\underline{x}, \underline{y})| p^+_{t,\alpha}(|\underline{y}|) &\leq c\Big(\big[\gamma^{-(2d/3-\alpha)}b_t^{-1}\big]^2\gamma^{d/3+\alpha/2}\gamma^d b_t \\ &\quad + \big[\gamma^{-(2d/3-\alpha)}b_t^{-1}\big]^3\gamma^{d/3+\alpha/2}\gamma^{2d}b_t + \big[\gamma^{-(2d/3-\alpha)}b_t^{-1}\big]^2\gamma^{2d}b_t\Big)\end{aligned} \tag{3.34}$$

where c is proportional to $|\underline{x}|^3$. We thus have

$$p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |\Omega_{1,3}(\underline{x}, \underline{y})| p^+_{t,\alpha}(|\underline{y}|) \leq c\big\{\gamma^{5\alpha/2}b_t^{-1} + \gamma^{d/3+7\alpha/2}b_t^{-2} + \gamma^{2d/3+2\alpha}b_t^{-1}\big\} \,. \tag{3.35}$$

We next observe that from (3.32) and (3.33), given $x \in \underline{x}$,

$$\sum_{y} |J_\gamma(x,y)| p^-_{t,\alpha}(|\underline{x}|) \sum_{\underline{y}} |S_y(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) c_1' A_{\gamma,n}$$

$$+ \sum_{y\in\underline{x}} |J_\gamma(x,y)| p^-_{t,\alpha}(|\underline{x}|) \sum_{\underline{y}} |S_y(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|)$$

$$\le c_1' A_{\gamma,n} + c_2' n^3 \gamma^d \gamma^{-(2d/3-\alpha)} \le c\gamma^{d/3+\alpha/2} . \tag{3.36}$$

Hence, recalling (3.19):

$$p^-_{t,\alpha}(|\underline{x}|) \sum_{\underline{y}} |\Omega_2(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) \le c \max_{2\le k\le N} [\gamma^{-(2d/3-\alpha)} b_t^{-1}][\gamma^{d/3+\alpha/2}]^k \le c\gamma^{2\alpha} b_t^{-1} . \tag{3.37}$$

We have thus concluded the proof of proposition 3.2. □

Proof of (2.37). Using proposition 3.2 we rewrite (3.1) as follows. Let $\omega_t = \{\omega_t(\underline{x}), \underline{x} \in S\}$ in $M(S)$, $\omega_t(\emptyset) \equiv 1$. Then

$$\frac{\mathrm{d}\omega_t}{\mathrm{d}t} = K_t\omega_t + A_t\omega_t + R_t \tag{3.38}$$

$$R_t(\underline{x}) = \mathbb{E}_{\mu^\gamma}\big(R_t^\star \hat{\omega}_t(\underline{x})\big) . \tag{3.39}$$

K_t is as in definition 2.3 with $K^{(0)} \equiv 0$; A_t is related to $A_t^\star$ by definition 3.1, with $A_t(\emptyset, \underline{y}) \equiv 0$.

Both A_t and R_t depend on N, N will be specified later in terms of the value $n = |\underline{x}|$ for which we wish to prove (2.37). Recalling that $I(\emptyset) = 1$ and $I(\underline{x}) = 0$, $|\underline{x}| > 0$, we have

$$\omega_t = I + \int_0^t \mathrm{d}s\, U_{t,s}\big(A_s\omega_s + R_s\big) \tag{3.40}$$

because $\omega_0(\underline{x}) \equiv 0$, $|\underline{x}| > 0$; $U_{t,s}$ is defined in definition 2.5.

Given $h^\star > n$ we call

$$\chi^\star = \chi_{\le h^\star} \qquad \omega_t^- = \chi^\star\omega_t \qquad \omega_t^+ = (1-\chi^\star)\omega_t . \tag{3.41}$$

We then have

$$\omega_t^- = I + \int_0^t \mathrm{d}s\, U_{t,s}\chi^\star\big(A_s\omega_s^- + A_s\omega_s^+ + R_s\big) . \tag{3.42}$$

After H iterations we get

$$\omega_t^- = I + \int_0^t \mathrm{d}s U_{t,s}\chi^\star[A_s\omega_s^+ + R_s] + \sum_{k=1}^{H-1}\{\Gamma_k(t) + \Lambda_k(t) + \Delta_k(t)\} + \Delta(t)$$

$$= \sum_{k=0}^{H-1}\{\Gamma_k(t) + \Lambda_k(t) + \Delta_k(t)\} + \Delta(t) \tag{3.43}$$

where

$$\Gamma_k(t) = \int_0^t \mathrm{d}s\, s_1 \dots \int_0^{s_{k-1}} \mathrm{d}s\, s_k U_{t,s_1} A^\star_{s_1} \dots U_{s_{k-1},s_k} A^\star_{s_k} I \qquad A^\star_t = \chi^\star A_t \tag{3.44}$$

$$\Lambda_k(t) = \int_0^t \mathrm{d}s\, s_1 \dots \int_0^{s_{k-1}} \mathrm{d}s_k\, U_{t,s_1} A^\star_{s_1} \dots U_{s_{k-1},s_k} A^\star_{s_k} \left[\int_0^{s_k} \mathrm{d}s\, U_{s_k,s} \chi^\star R_s \right] \tag{3.45}$$

$$\Delta_k(t) = \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{k-1}} \mathrm{d}s\, s_k U_{t,s_1} A^\star_{s_1} \dots U_{s_{k-1},s_k} A^\star_{s_k} \left[\int_0^{s_k} \mathrm{d}s\, U_{s_k,s} A^\star_s \omega^+_s \right] \tag{3.46}$$

$$\Delta(t) = \int_0^t \mathrm{d}s\, s_1 \dots \int_0^{s_{H-1}} \mathrm{d}s\, s_H U_{t,s_1} A^\star_{s_1} \dots U_{s_{H-1},s_H} A^\star_{s_H} \omega^-_{s_H}\,. \tag{3.47}$$

We fix n and we need to bound $|\omega_t(\underline{x})|$ with $|\underline{x}| = n$. Observe that

$$\omega_t(\underline{x}) = \omega^-_t(\underline{x}) \qquad \text{for all } \underline{x} \text{ such that } |\underline{x}| = n\,. \tag{3.48}$$

We start by bounding from $\Gamma_k(t)$. We write

$$\chi_n \Gamma_k(t) = \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{k-1}} \mathrm{d}s_k\, \chi_n U_{t,s_1} A^\star_{s_1} \chi_{\le(n+N)} \dots \chi_{\le(n+(k-1)N)} U_{s_{k-1},s_k} A^\star_{s_k} I \tag{3.49}$$

because $A^\star_s : V_s^{(j)} \to V_s^{(j+N)}$, by proposition 3.2. Recalling (3.7), we write for $\alpha = 0$ and $\alpha = 2d/9$:

$$\tilde{U}_{t,s} = p^-_{t,\alpha} U_{t,s} p^+_{s,\alpha} \qquad A^{(\alpha)}_t = p^-_{t,\alpha} A^\star_t p^+_{t,\alpha}\,. \tag{3.50}$$

Observe that $\tilde{U}_{t,s}$ does not depend on α.

We next choose $\alpha = 0$ and, recalling that $p^\pm_t(\emptyset) = 1$, we get

$$\chi_n \Gamma_k(t) = p^+_{t,0}(n) \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{k-1}} \mathrm{d}s_k\, \chi_n \tilde{U}_{t,s_1} A^{(0)}_{s_1} \chi_{\le(n+N)} \dots \chi_{\le(n+(k-1)N)} \tilde{U}_{s_{k-1},s_k} A^{(0)}_{s_k} I\,. \tag{3.51}$$

By equations (2.34), (2.35) and proposition 3.2 we have

$$\|\chi_\ell \tilde{U}_{t,s}\|_\infty \le c'_\ell a^\ell_{t-s} b^{-\ell}_t b^\ell_s \le c'_\ell (c'_0)^\ell \qquad \|\chi_\ell A^{(0)}_t\|_\infty \le c(\ell)\,. \tag{3.52}$$

Hence there is a constant $c_0(n, h^\star, N, H)$ so that for $k \le H - 1$

$$\|\chi_n \Gamma_k(t)\|_\infty \le c_0(n, h^\star, N, H)[b_t \gamma^{2d/3}]^n \frac{t^k}{k!}\,. \tag{3.53}$$

From equation (3.45) we get

$$\|\chi_n \Lambda_k(t)\|_\infty = p^+_{t,\alpha}(n) \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{k-1}} \mathrm{d}s_k\, \chi_n \tilde{U}_{t,s_1} A^{(\alpha)}_{s_1} \chi_{\le(n+N)} \dots$$
$$\dots \chi_{\le(n+(k-1)N)} \tilde{U}_{s_{k-1},s_k} A^{(\alpha)}_{s_k} \int_0^{s_k} \mathrm{d}s\, \tilde{U}_{s_k,s} p^-_{s,\alpha} \chi^\star R_s\,. \tag{3.54}$$

By proposition 3.2 we have

$$\|\chi_\ell A^{(\alpha)}_t\|_\infty \le c(\ell)\gamma^\alpha\,. \tag{3.55}$$

Then there is a constant $c_1 \equiv c_1(n, h^\star, N, H)$ so that

$$\begin{aligned}\|\chi_n \Lambda_k(t)\|_\infty &\le c_1 b_t^n \gamma^{(2d/3-\alpha)n} \gamma^{\alpha k} \frac{t^{k+1}}{(k+1)!} \gamma^{-(2d/3-\alpha)h^\star} \sup_{s\le t} \sup_x \mathbb{E}_{\mu^\gamma}\left(|J_\gamma \circ \tilde\sigma(x)|^N\right)\\ &\le c_1 [b_t \gamma^{(2d/3)}]^n \gamma^{\alpha(h^\star - n)} \gamma^{\alpha k} \frac{t^{k+1}}{(k+1)!} \gamma^{-(2d/3)h^\star} \sup_{s\le t} \sup_x \mathbb{E}_{\mu^\gamma}\left(|J_\gamma \circ \tilde\sigma(x)|^N\right).\end{aligned} \tag{3.56}$$

By equation (2.11), for every even N there is c so that

$$\mathbb{E}_{\mu^\gamma}\left(|J_\gamma \circ \tilde\sigma(x)|^N\right) \le c\left[\mathrm{e}^{Ct}\gamma^{d/2}\right]^N. \tag{3.57}$$

Expression (3.57) is obtained by expanding the product on the left-hand side. Terms with all sites distinct give rise to a v function that is bounded using (2.11). This behaves as the right-hand side of (3.57). All the other terms are smaller, as there is an extra factor J_γ whenever two sites are equal (and possibly two sites less in the v function).

Since $h^\star > n$, $t \le a \log \gamma^{-1}$ and $b_t \le \mathrm{e}^{Ct}$ we have for a small enough and for a suitable constant $c_2 \equiv c_2(n, h^\star, N, H)$,

$$\|\chi_n \Lambda_k(t)\|_\infty \le c_2 [b_t \gamma^{(2d/3)}]^n \gamma^{-(2d/3)h^\star} \gamma^{(d/2-aC)N}. \tag{3.58}$$

Analogously,

$$\begin{aligned}\chi_n \Delta_k(t) = p^+_{t,\alpha}(n) \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{k-1}} \mathrm{d}s_k\ \chi_n \tilde U_{t,s_1} A^{(\alpha)}_{s_1} \chi_{\le(n+N)} \dots \chi_{\le(n+(k-1)N)} \tilde U_{s_{k-1},s_k} A^{(\alpha)}_{s_k}\\ \times \int_0^{s_k} \mathrm{d}s\ \tilde U_{s_k,s} A^{(\alpha)}_s p^-_{s,\alpha} \omega^+_s.\end{aligned} \tag{3.59}$$

By theorem 2.1

$$\sup_{s\le a\log\gamma^{-1}} \sup_{|\underline{x}|=h} |\omega_s(\underline{x})| \le c'_h \gamma^{(d/2-aC)h} \tag{3.60}$$

where c'_h is a suitable constant. Hence there is a constant $c_3 \equiv c_3(n, h^\star, N, H)$, such that for $\alpha = 2d/9$ and $t \le a \log \gamma^{-1}$,

$$\begin{aligned}\|\chi_n \Delta_k(t)\|_\infty &\le c_3 b_t^n \gamma^{(2d/3-\alpha)n} \gamma^{\alpha k} \frac{t^{k+1}}{(k+1)!} \sup_{h^\star \le h \le h^\star + N} c'_h\ \gamma^{[(d/2-aC)-(2d/3-\alpha)]h}\\ &\le c_3 [b_t \gamma^{2d/3}]^n \gamma^{-2dn/9} \gamma^{[(d/18)-aC)]h^\star}.\end{aligned} \tag{3.61}$$

We bound $\Delta(t)$ in an analogous way, getting the existence constants $c_4 \equiv c_4(n, h^\star, N, H)$ and $c'_4 \equiv c'_4(n, h^\star, N, H)$, such that for all $t \le a \log \gamma^{-1}$,

$$\begin{aligned}\|\chi_n \Delta(t)\|_\infty &\le c_4 b_t^n \gamma^{(2d/3-\alpha)n} \gamma^{\alpha H} \frac{t^H}{H!} \gamma^{-(2d/3-\alpha)h^\star}\\ &\le c'_4 [b_t \gamma^{2d/3}]^n \gamma^{\alpha(h^\star-n)} \left(\gamma^\alpha \log \gamma^{-1}\right)^H \gamma^{-(2d/3)h^\star}.\end{aligned} \tag{3.62}$$

We first choose $h^\star > n$ so that (see (3.61))

$$\gamma^{-2dn/9} \gamma^{[(d/18)-aC]h^\star} \le 1. \tag{3.63}$$

Then we chose H and N so that (see (3.62) and (3.58), respectively)

$$\left(\gamma^{\alpha}\log\gamma^{-1}\right)^{H}\gamma^{-(2d/3)h^{\star}} \le 1 \qquad \gamma^{(d/2)-aC)N}\gamma^{-(2d/3)h^{\star}} \le 1\,. \tag{3.64}$$

From equations (3.43), (3.53), (3.58), (3.61), (3.62) and the choice of $h^{\star}$, H and N, we finally get that there is a constant $\bar{c}$ so that

$$\sup_{|\underline{x}|=n} |\omega_t^{-}(\underline{x})| \le \bar{c}\left\{[b_t\gamma^{2d/3}]^n t^H + [b_t\gamma^{2d/3}]^n\right\}. \tag{3.65}$$

The inequality (2.37) is therefore proven.

Proof of (2.38), (2.39). We go back to (3.9) recalling that the action of $A_t^{\star}$ is given by $\Omega_2 + \Omega_{1,3}$, see (3.28). Let $\underline{x} = (x_1, x_2)$ in (3.19), then, for a suitable constant c,

$$|\Omega_2(\underline{x}, \emptyset)| \le c[b_t\gamma^d]^2 \tag{3.66}$$

as it only arises from values of $k \ge 3$ in (3.19) and the largest contribution comes from the product of two w_t's for which we use (2.34). Analogously we have, for a suitable constant c,

$$\sum_{y} |\Omega_2(\underline{x}, y)| \le c[b_t\gamma^d] \qquad \sum_{|\underline{y}|=2} |\Omega_2(\underline{x}, \underline{y})| \le c[b_t\gamma^d] \qquad \sum_{|\underline{y}|\ge 3} |\Omega_2(\underline{x}, \underline{y})| \le c\,. \tag{3.67}$$

When $|\underline{x}| = 1$ we get

$$|\Omega_2(x, \emptyset)| \le c[b_t\gamma^d] \qquad \sum_{y} |\Omega_2(x, y)| \le c[b_t\gamma^d] \tag{3.68}$$

$$\sum_{|\underline{y}|=2} |\Omega_2(x, \underline{y})| \le c \qquad \sum_{|\underline{y}|\ge 3} |\Omega_2(x, \underline{y})| \le c\,. \tag{3.69}$$

From equation (3.27) we get $\Omega_{1,3}(\underline{x}) = 0$ if $|\underline{x}| = 1$, while, if $|\underline{x}| = 2$,

$$|\Omega_{1,3}(\underline{x}, \emptyset)| \le c\gamma^d b_t\gamma^d \tag{3.70}$$

(the first factor γ^d comes from $k(x, z, m)$ the other factor from bounding w_t in the second term on the right-hand side of (3.27)):

$$\sum_{y} |\Omega_{1,3}(\underline{x}, y)| \le c\gamma^d\,. \tag{3.71}$$

We call

$$\psi_t = \sup_{|\underline{x}|=2}\sup_{s\le t} |\omega_s(\underline{x})| \qquad \phi_t = \sup_{|\underline{x}|=1}\sup_{s\le t} |\omega_s(\underline{x})|\,. \tag{3.72}$$

We then get

$$\psi_t \le \int_0^t \mathrm{d}s\, |U_{t,s}\chi_2\big(A_s\omega_s + R_s\big)| \qquad \phi_t \le \int_0^t \mathrm{d}s\, |U_{t,s}\chi_1\big(A_s\omega_s + R_s\big)|\,. \tag{3.73}$$

We use (2.35), (3.16) and (3.57) to conclude that

$$\|U_{t,s}\chi_2\|_\infty \le (b_t c_0')^2 \qquad \|U_{t,s}\chi_1\|_\infty \le b_t c_0' \qquad \sup_{s\le a\log\gamma^{-1}} \|R_s\|_\infty \le \bar{c}\big(\gamma^{(d/2)-a}\big)^N . \tag{3.74}$$

Furthermore, using (2.37) that has been already proven, we have from (3.67) that there is $\bar{c}$ so that for all $s \le a\log\gamma^{-1}$

$$\sum_{|\underline{y}|\ge 3} |\Omega_2(\underline{x},\underline{y})||\omega_s(\underline{y})| \le N \sup_{3\le k\le N} s^{\ell_k} b_s^k \gamma^{2dk/3} \le \bar{c}\; s^{\ell_3} b_s^3 \gamma^{2d} . \tag{3.75}$$

From all this we then get that, for a suitable constant c,

$$\psi_t \le \int_0^t \mathrm{d}s\, (b_t c_0')^2 c\big\{[(b_s\gamma^d)^2 + b_s\gamma^d\phi_s + b_s\gamma^d\psi_s + s^{\ell_3} b_s^3\gamma^{2d}] + [b_s\gamma^{2d} + \gamma^d\phi_s]\big\} . \tag{3.76}$$

Analogously

$$\phi_t \le \int_0^t \mathrm{d}s\, (b_t c_0')c\big\{b_s\gamma^d + b_s\gamma^d\phi_s + \psi_s + s^{\ell_3} b_s^3\gamma^{2d}\big\} . \tag{3.77}$$

Hence, for a suitable c',

$$\psi_t \le c' t b_t^2\big\{(b_t\gamma^d)^2 + b_t\gamma^d\phi_t + t^{\ell_3} b_t^3\gamma^{2d}\big\} \tag{3.78}$$

$$\phi_t \le c' t b_t\big\{b_t\gamma^d + \psi_t + t^{\ell_3} b_t^3\gamma^{2d}\big\} . \tag{3.79}$$

It then follows that, for a suitable c'', and $\ell_3 \ge 1$

$$\psi_t \le c'' t^{\ell_3+1} b_t^5\gamma^{2d} \qquad \phi_t \le c'' t b_t^2\gamma^d . \tag{3.80}$$

Theorem 2.6 is thus proven. □

4. Fluctuations

In this section we prove theorem 2.8. The key ingredients of the proof are the bounds on the ω functions. We start with a lemma which follows quite easily from theorem 2.6. We will now write explicitly the dependence on γ in the v functions and the ω functions.

Lemma 4.1 *There are $a > 0$ and, for any odd integer n, c_n, $n \ge 1$, and $\bar{C}$ so that for all initial measures μ (that are product measures), for all $t \le a\log\gamma^{-1}$ and for all $\underline{x} \in S$, $|\underline{x}| = n$,*

$$|v_t^\gamma(\underline{x})| \le c_{|\underline{x}|}\, \mathrm{e}^{\bar{C}|\underline{x}|t}\gamma^{d(|\underline{x}|+1)/2} \tag{4.1}$$

where $|\underline{x}|$ denotes the cardinality of $\underline{x}$.

Proof. Using the first equality in (2.32) we write

$$v_t^\gamma(\underline{x}) = \sum_{\underline{y}} \omega_t^\gamma(\underline{x}-\underline{y}) W_t^\gamma(\underline{y}) .$$

We recall that $W_t^\gamma(\underline{y}) = 0$ if $|\underline{y}|$ is odd, so that in all terms there is a ω function. Equation (4.1) then follows from (2.27), (2.37) and (2.39). $\square$

Recalling the notation below (2.42) and the definition 2.7 we first prove in the next lemma the convergence of the moments of $Y_t^\gamma(\phi)$, thus proving the last statement of theorem 2.8.

Lemma 4.2 *Let $n \geq 2$ and $\phi_1, \ldots, \phi_n$ in $\mathcal{S}(\mathbb{R}^d)$. Then, for any $t > 0$*

$$\lim_{\gamma\to 0} \hat{\mathbb{E}}^\gamma\left(\prod_{i=1}^n Y_t^\gamma(\phi_i)\right) = \hat{\mathbb{E}}\left(\prod_{i=1}^n Y_t(\phi_i)\right).$$

Proof. Let $n \geq 2$ and let $\underline{x} \in S^{(n)}$, $\underline{x} = (x_1, \ldots, x_n)$. We define a partition of the set $\{1, \ldots, n\}$ by saying that i and j are in the same atom of the partition if and only if $x_i = x_j$. We call $h(\underline{x})$ the number of atoms of the partition so that

$$\{1, \ldots, n\} = \bigcup_{k=1}^{h(\underline{x})} A_k \qquad A_k \subset \{1, \ldots, n\} \qquad A_k \cap A_h = \emptyset \text{ if } k \neq h\,. \tag{4.2}$$

We also denote by $f(\underline{x})$ the number of atoms A_k such that $|A_k| = 1$. We suppose to have ordered the atoms of the partition in such a way that the first A_ℓ, $\ell \leq f(\underline{x})$ are singletons, i.e.

$$A_\ell = \{i_\ell\} \qquad \ell = 1, \ldots, f(\underline{x})\,. \tag{4.3}$$

We then have

$$n \geq h(\underline{x}) \geq f(\underline{x}) \geq 0 \qquad h(\underline{x}) \leq f(\underline{x}) + \frac{n - f(\underline{x})}{2} \tag{4.4}$$

and also

$$\prod_{i=1}^n \tilde{\sigma}_t(x_i) = \prod_{\ell=1}^{f(\underline{x})} \tilde{\sigma}_t(x_{i_\ell}) \prod_{k=f(\underline{x})+1}^{h(\underline{x})} \tilde{\sigma}_t(x_{i_k})^{|A_k|} \qquad i_k := \min\{i : i \in A_k\}\,. \tag{4.5}$$

We now observe that for any $k \geq 1$

$$\begin{aligned}
\tilde{\sigma}_t(x)^k &= \frac{1+\sigma(x)}{2}[1 - m_t^\gamma(x)]^k + \frac{1-\sigma(x)}{2}[-1 - m_t^\gamma(x)]^k \\
&= \tilde{\sigma}_t(x)\left\{\frac{[1 - m_t^\gamma(x)]^k}{2} - \frac{[-1 - m_t^\gamma(x)]^k}{2}\right\} \\
&\quad + \frac{1+m_t^\gamma(x)}{2}[1 - m_t^\gamma(x)]^k + \frac{1-m_t^\gamma(x)}{2}[-1 - m_t^\gamma(x)]^k \\
&= \tilde{\sigma}_t(x) a(k, m_t^\gamma(x)) + b(k, m_t^\gamma(x))
\end{aligned} \tag{4.6}$$

with a and b defined by the last equality. Thus

$$\begin{aligned}
a(1, m_t^\gamma(x)) = 1 \qquad & b(1, m_t^\gamma(x)) = 0 \\
a(2, m_t^\gamma(x)) = -2m_t^\gamma(x) \qquad & b(2, m_t^\gamma(x)) = 1 - m_t^\gamma(x)^2
\end{aligned} \tag{4.7}$$

and there are constants c_k so that

$$|a(k, m_t^\gamma(x))| \le c_k \qquad |b(k, m_t^\gamma(x))| \le c_k\,. \tag{4.8}$$

Since

$$\hat{\mathbb{E}}^\gamma\left(\prod_{i=1}^n Y_t^\gamma(\phi_i)\right) = \gamma^{dn/2} \sum_{\underline{x}\in S^{(n)}} \left(\prod_{i=1}^n \phi_i(\gamma x_i)\right)\mathbb{E}_{\mu^\gamma}^\gamma\left(\prod_{i=1}^n \tilde{\sigma}_t(x_i)\right). \tag{4.9}$$

Denoting by $[(f+1)/2]$ the integer part of $(f+1)/2$, we then have

$$\begin{aligned}
&\left|\gamma^{dn/2} \sum_{\underline{x}\in S^{(n)}} \left(\prod_{i=1}^n \phi_i(\gamma x_i)\right)\mathbb{E}_{\mu^\gamma}^\gamma\left(\prod_{i=1}^n \tilde{\sigma}_t(x_i)\right)\right| \\
&\qquad \le c\gamma^{dn/2} \sum_{h,f}^{\star} \sum_{\underline{x}\in S^{(n)}} \mathbf{1}(h(\underline{x}) = h,\, f(\underline{x}) = f)\left(\prod_{i=1}^n |\phi_i(\gamma x_i)|\right)\gamma^{d[(f+1)/2]} \\
&\qquad \le c'\gamma^{dn/2} \sum_{h,f}^{\star} \gamma^{-dh}\gamma^{d[(f+1)/2]}
\end{aligned} \tag{4.10}$$

where the starred sum is restricted to the non-negative integers h and f which satisfy (4.4). We have applied (2.11) when f is even, observing that the integer part of $(f+1)/2$, denoted by $[(f+1)/2]$, is equal to $f/2$. We instead use (4.1) when f is odd, in which case $[(f+1)/2] = (f+1)/2$. By equation (4.4), the right-hand side of (4.10) is uniformly bounded in γ and it vanishes in the limit $\gamma \to 0$, unless f is even and $h = (f+n)/2$. This is possible only when n is also even, we shall therefore restrict ourselves henceforth to this case.

We thus have

$$\hat{\mathbb{E}}^\gamma\left(\prod_{i=1}^{2n} Y_t^\gamma(\phi_i)\right) = G_\gamma + R_1^\gamma \tag{4.11}$$

where

$$\begin{aligned}
G_\gamma = \gamma^{dn} \sum_{k=0}^n \sum_{\substack{I\subset\{1,\dots 2n\}\\ |I|=2k}} \sum_{\underline{z}\in S_{\neq}^{(2(n-k)}} \left(\prod_{i\notin I} \phi_i(\gamma z_i)\right) v_t^\gamma(\underline{z}) \\
\times \sum_{\{(i_1,j_1)\dots(i_k,j_k)\}} \sum_{\substack{\underline{x}\in S_{\neq}^{(k)}\\ \underline{x}\cap\underline{z}=\emptyset}} \left(\prod_{\ell=1}^k \phi_{i_\ell}(\gamma x_\ell)\phi_{j_\ell}(\gamma x_\ell)[1 - m_t^\gamma(x_\ell)^2]\right)
\end{aligned} \tag{4.12}$$

where $S_{\neq}^{(i)}$ is the subset of $S^{(i)}$ with configurations with sites different from each other and therefore $\underline{z} = \{z_i,\, i \notin I\}$ is a configuration with $2(n-k)$ sites different from each other. The sum over $(i_1, j_1)\dots(i_k, j_k)$ is over all the partitions of I into k elements (atoms), each with two indices. The remainder R_1^γ can be estimated as in (4.10) and vanishes when $\gamma \to 0$. By equation (2.32) we have

$$v_t^\gamma(\underline{y}) = W_t^\gamma(\underline{y}) + \sum_{\emptyset\neq\underline{z}\subset\underline{y}} \omega_t^\gamma(\underline{z}) W_t^\gamma(\underline{y} - \underline{z})\,. \tag{4.13}$$

Then by theorem 2.6, for any positive p,

$$\lim_{\gamma\to 0}\gamma^{-dp}\sup_{\underline{y}\in S_{\neq}^{(2p)}}\left|v_t^\gamma(\underline{y})-W_t^\gamma(\underline{y})\right|=0\,. \tag{4.14}$$

We then substitute in (4.12) the following:

$$v_t^\gamma(\underline{z})=W_t^\gamma(\underline{z})+[v_t^\gamma(\underline{z})-W_t^\gamma(\underline{z})] \tag{4.15}$$

and we obtain

$$G_\gamma=G'_\gamma+R_2^\gamma$$

where

$$\begin{aligned}G'_\gamma=\sum_{k=0}^{n}\sum_{\substack{I\subset\{1,\ldots 2n\}\\|I|=2k}}\gamma^{2d(n-k)}\sum_{\underline{z}\in S_{\neq}^{(2(n-k))}}\left(\prod_{i\notin I}\phi_i(\gamma z_i)\right)\gamma^{-d(n-k)}W_t(\underline{z})\\ \times\sum_{\{(i_1,j_1)\ldots(i_k,j_k)\}}\gamma^{dk}\sum_{\substack{\underline{x}\in S_{\neq}^{(k)}\\ \underline{x}\cap\underline{z}=\emptyset}}\left(\prod_{\ell=1}^{k}\phi_{i_\ell}(\gamma x_\ell)\phi_{j_\ell}(\gamma x_\ell)[1-m_t^\gamma(x_\ell)^2]\right)\end{aligned} \tag{4.16}$$

and R_2^γ vanishes in the limit $\gamma\to 0$. From equation (2.8) and the choice of μ^γ in definition 2.1 we have

$$\lim_{\gamma\to 0}\sup_{x\in\mathbb{Z}^d}|m_t^\gamma(x)-m_t(\gamma x)|=0\qquad\text{for all } t\geq 0 \tag{4.17}$$

where m_t solves (1.1) with initial condition m_0.

We finally observe that from (4.17), (2.23) and (2.48) it follows that

$$\lim_{\gamma\to 0}\sup_{x,y\in\mathbb{Z}^d}|\gamma^{-d}w_t(x,y)-C_t^0(\gamma x,\gamma y)|=0\qquad\text{for all } t\geq 0 \tag{4.18}$$

namely, that $C_t^0(r,r')$ in (2.48) is the continuous version, suitable normalized, of $w_t(x,y)$.

From equations (2.45), (4.17) and (4.18) we then have that

$$\lim_{\gamma\to 0}\left|G'_\gamma-\hat{\mathbb{E}}\left(\prod_{i=1}^{2n}Y_t(\phi_i)\right)\right|=0\,. \tag{4.19}$$

The lemma is thus proven. $\square$

To complete the proof of theorem 2.8 we need to prove the convergence of the fluctuation fields as a process on $D([0,T],\mathcal{S}'(\mathbb{R}^d))$. This is done following criteria which are by now standard in this field. Their applicability is granted in our case by the bounds proven on the v and ω functions. We give below some details and references.

Tightness.

We first prove tightness of $\hat{\mathbb{P}}^\gamma$ on $D([0,T],\mathcal{S}'(\mathbb{R}^d))$. By Mitoma's theorem, [11], it is enough to show that the marginal of $\hat{\mathbb{P}}^\gamma$ on $\{Y_t(\phi),\ t\in[0,T]\}$ is tight. For this we use the criterion stated in theorem 2.6.3 of [4]. We thus need to prove that there is c such that

$$\sup_{0\le t\le T} \hat{\mathbb{E}}_{\mu^\gamma}\big(F_1^\gamma(t)^2+F_2^\gamma(t)^2+Y_t^\gamma(\phi)^2\big)\le c \tag{4.20}$$

where

$$F_1^\gamma(t)=\left(L+\frac{\mathrm{d}}{\mathrm{d}t}\right)Y_t^\gamma(\phi) \tag{4.21}$$

(the time derivative acts only on m_t^γ)

$$F_2^\gamma(t)=LY_t^\gamma(\phi)^2-2Y_t^\gamma(\phi)LY_t^\gamma(\phi)\,. \tag{4.22}$$

For any function $f\in C^3(\mathbb{R})$ we have

$$\begin{aligned}\left(L+\frac{\mathrm{d}}{\mathrm{d}t}\right)f\big(Y_t^\gamma(\phi)\big)&=\sum_x c_\gamma(x,\sigma_t)\left[f\left(Y_t^\gamma(\phi)-2\gamma^{d/2}\phi(\gamma x)\sigma_t(x)\right)-f\big(Y_t^\gamma(\phi)\big)\right]\\&\quad-f'\big(Y_t^\gamma(\phi)\big)\gamma^{d/2}\sum_x\phi(\gamma x)\frac{\mathrm{d}}{\mathrm{d}t}m_t^\gamma x)\\&=f'\big(Y_t^\gamma(\phi)\big)F_1^\gamma(t)+\tfrac12 f''\big(Y_t^\gamma(\phi)\big)\left(4\gamma^d\sum_x c_\gamma(x,\sigma_t)\phi(\gamma x)^2\right)+R_1^\gamma(t)\end{aligned} \tag{4.23}$$

$$F_1^\gamma(t)=\gamma^{d/2}\sum_x\phi(\gamma x)\left(L^x+\frac{\mathrm{d}}{\mathrm{d}t}\right)\tilde{\sigma}_t(x) \tag{4.24}$$

where

$$L^x g(\sigma)=c_\gamma(x,\sigma)[g(\sigma^x)-g(\sigma)] \tag{4.25}$$

and for any $T>0$ there is c so that

$$|R_1^\gamma(t)|\le c\gamma^{d/2}\qquad t\le T\,.$$

By choosing in (4.23) $f(y)=y^2$, (4.22) can be written as

$$F_2^\gamma(t)=4\gamma^d\sum_x c_\gamma(x,\sigma_t)\phi(\gamma x)^2 \tag{4.26}$$

and

$$|F_2^\gamma(t)|\le c\,. \tag{4.27}$$

We use (2.8) to write

$$\left(L^x+\frac{\mathrm{d}}{\mathrm{d}t}\right)\tilde{\sigma}_t(x)=-\tilde{\sigma}_t(x)+[f_x(\sigma_t)-f_x(m_t^\gamma)]\,. \tag{4.28}$$

We then linearize (4.28) around m_t^γ, the solution of (2.8) with initial condition $m_0(\gamma x)$, and we get

$$F_1^\gamma(t) = \gamma^{d/2} \sum_x \phi(\gamma x)(K_t \tilde{\sigma}_t)(x) + \mathcal{R}_2^\gamma(t) \tag{4.29}$$

where

$$\mathcal{R}_2^\gamma(t) = F_1^\gamma(t) - \gamma^{d/2} \sum_x \phi(\gamma x)(K_t \tilde{\sigma}_t)(x) \,. \tag{4.30}$$

By equation (2.11), for any $T > 0$ there are constants c_1 and c_2 so that

$$\hat{\mathbb{E}}^\gamma \left(\left[\gamma^{d/2} \sum_x \phi(\gamma x)(K_t \tilde{\sigma}_t)(x) \right]^2 \right) \le c_1 \qquad t \le T \tag{4.31}$$

$$\hat{\mathbb{E}}^\gamma \big([\mathcal{R}_2^\gamma(t)]^2 \big) \le c_2 \gamma^d \qquad t \le T \,. \tag{4.32}$$

The martingale problem.
To identify the limiting law $\hat{\mathbb{P}}$ along a convergent subsequence $\hat{\mathbb{P}}^\gamma$, we first observe that $\mathbb{P}$ is supported by continuous trajectories. This follows from the fact that the jumps of $Y_t^\gamma(\phi)$ are bounded by the sup norm of ϕ times $\gamma^{d/2}$, see for instance Spohn [13]. We will then identify the limit as the solution of a martingale problem. We start from the martingale characterization of $\hat{\mathbb{P}}^\gamma$. We have that for any function $f \in C^3(\mathbb{R})$

$$f\big(Y_t^\gamma(\phi)\big) - \int_0^t \mathrm{d}s \left(L + \frac{\mathrm{d}}{\mathrm{d}s} \right) f\big(Y_s^\gamma(\phi)\big) \qquad \text{is a } \hat{\mathbb{P}}^\gamma\text{-martingale}\,. \tag{4.33}$$

We use (4.23). For the term (4.26) using (2.11) and arguments similar to those used when proving tightness we have that

$$\sup_{t \le T} \hat{\mathbb{E}}^\gamma \left(\left| F_2^\gamma(t) - 2\gamma^d \sum_x b_\gamma(x,t) \phi(\gamma x)^2 \right|^2 \right) \le c\gamma^d \tag{4.34}$$

where

$$b_\gamma(x,t) = 1 - m_t^\gamma \tanh\{\beta[(J_\gamma \circ m_t^\gamma)(x) + h]\} \,. \tag{4.35}$$

Note that since $m_t^\gamma \to m_t$ solution of (1.1) b_γ is the discretized version of b of (2.51). From this, by taking the limit along the subsequence which converges to $\hat{\mathbb{P}}$, we get that

$$f\big(Y_t(\phi)\big) - \int_0^t \mathrm{d}s\ f'\big(Y_s(\phi)\big) Y_s(\mathcal{L}^+\phi) - \tfrac{1}{2} \int_0^t \mathrm{d}s\ f''\big(Y_s(\phi)\big) \|2b(\cdot,s)\phi^2\|_1 \tag{4.36}$$

is a $\hat{\mathbb{P}}$-martingale, $\mathcal{L}^+$ being the adjoint of $\mathcal{L}$ and $b(r,t)$ is as in (2.51). Since the distribution at time 0 is specified as in theorem 2.8, this information together with the support properties of $\hat{\mathbb{P}}$ and (4.36) (for arbitrary f, $t \le T$ and ϕ) identifies $\hat{\mathbb{P}}$ according to Holley and Stroock's theory, see [10, 13].

It is easy to compute from (4.36) the covariance at different time of $\hat{\mathbb{P}}$ and verify that they are given by (2.44). Less straightforward is the identification of the equal time covariance, which we report below for the sake of completeness. Call $C_t(r, r')$ the expression $C_{t,t}(r, r')$ of (2.45) and recalling (2.43) consider

$$\hat{\mathbb{E}}\big(Y_t(\phi)^2\big) = \int \mathrm{d}r\, \mathrm{d}r'\, C_t(r, r')\phi(r)\phi(r')\,.$$

Then, from (4.36)

$$\begin{aligned} C_t(r, r') = \int_0^t \mathrm{d}s \int \mathrm{d}r'' \left\{\mathcal{L}(r, r'', s)C_s(r'', r') + \mathcal{L}(r', r'', s)C_s(r, r'')\right\} \\ + \int_0^t \mathrm{d}s\, 2b(r, s)\delta(r - r') + \delta(r - r')[1 - m_0(r)^2]\,. \end{aligned} \tag{4.37}$$

Defining $C_t^0(r, r')$ so that (2.45) holds and substituting it into (4.37), we get

$$\begin{aligned} C_t^0(r, r') - \int_0^t \mathrm{d}s \int \mathrm{d}r'' \left\{\mathcal{L}(r, r'', s)C_s^0(r'', r') + \mathcal{L}(r', r'', s)C_s^0(r, r'')\right\} \\ = -\delta(r - r')[1 - m_t(r)^2] + \delta(r - r')[1 - m_0(r)^2] \\ + \int_0^t \mathrm{d}s \int \mathrm{d}r'' \left\{\mathcal{L}(r, r'', s)\delta(r' - r'')[1 - m_s(r')^2]\right. \\ \left. + \mathcal{L}(r', r'', s)\delta(r - r'')[1 - m_s(r)^2]\right\} \\ + \int_0^t \mathrm{d}s\, 2b(r, s)\delta(r - r')\,. \end{aligned} \tag{4.38}$$

The last two integrals can be rewritten, using (2.46), (1.1) and (2.51) as

$$\begin{aligned} \int_0^t \mathrm{d}s \left\{\mathcal{L}^0(r, r', s)[1 - m_s(r')^2] + \mathcal{L}^0(r', r, s)[1 - m_s(r)^2]\right\} \\ - \int_0^t \mathrm{d}s\, 2\delta(r - r')[1 - m_s(r)^2] + 2\int_0^t \mathrm{d}s\, b(r, s)\delta(r - r') \\ = \int_0^t \mathrm{d}s\left\{\mathcal{L}^0(r, r', s)[1 - m_s(r')^2] + \mathcal{L}^0(r', r, s)[1 - m_s(r)^2]\right\} \\ - \int_0^t \mathrm{d}s\, \frac{\mathrm{d}}{\mathrm{d}s} m_s(r)^2\delta(r - r')\,. \end{aligned} \tag{4.39}$$

Using equation (4.39), equation (4.38) shows that C_t^0 defined by (2.45) is the same as in (2.48). Theorem 2.8 is thus proven. □

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