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Non-equilibrium Stationary States in the Symmetric Simple Exclusion with Births and Deaths

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Abstract We consider the symmetric simple exclusion process in the interval $\Lambda_N := [-N, N] \cap \mathbb{Z}$ with births and deaths taking place respectively on suitable boundary intervals I_+ and I_-, as introduced in De Masi et al. (J. Stat. Phys. 144:1151–1170, 2011). We study the stationary measure and its macroscopic density profile in the limit $N \to \infty$.

Keywords Interacting particle systems · Hydrodynamic limit

1 Introduction

This paper is a follow-up of the study initiated in [1, 2], where current reservoirs in the context of stochastic interacting particle systems have been proposed as a method to investigate stationary non-equilibrium states with steady currents produced by action at the boundary.

Due to the particular difficulties in implementing this new method, we consider the simplest possible particle system. The bulk dynamics is the symmetric simple exclusion process (SSEP) in the interval $\Lambda_N = [-N, N] \cap \mathbb{Z}$ (N a positive integer and $N \to \infty$ eventually), namely the state space is $\{0, 1\}^{\Lambda_N}$ (at most one particle per site): independently each particle

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tries to jump at rate $N^2/2$ to each one of its nearest neighbor (n.n.) sites, the jump then takes place if and only if the chosen site is empty, jumps outside Λ_N are suppressed. To induce a current we send in particles from the right and take them out from the left, and would like this to happen at rate $Nj/2$, $j > 0$ a fixed parameter independent of N. Due to the restrictions imposed by the configurational space, we have to be more precise when defining this dynamics. For this we fix a parameter $K \geq 1$ (an integer) and two intervals $I_\pm$ of length $K - 1$ at the boundaries: $I_+ \equiv [N - K + 1, N]$ and $I_- \equiv [-N, -N + K - 1]$. At rate $Nj/2$, when I_+ is not totally occupied, we create a particle at its rightmost empty site; with the same rate, unless I_- is empty, we take out a particle from its leftmost occupied site. In case I_+ is already full, or I_- empty, the corresponding mechanism aborts.

In [1, 2] we have proved that at any time $t > 0$ propagation of chaos holds and that in the limit $N \to \infty$ the hydrodynamical equation is the linear heat equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\rho(r,t) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\rho(r,t), \qquad r \in (-1,1),\ t > 0, \\ &\rho(r,0) = \rho_0(r), \qquad \rho(\pm 1, t) = u_\pm(t), \end{aligned} \tag{1}$$

where $\rho_0(\cdot)$ is given but $u_\pm(t)$ are solutions of a nonlinear system of two integral equations, see (6) below.

The goal of this paper is to investigate the limiting density profile (as $N \to \infty$) of the (unique) invariant measure of the process. The main result is Theorem 2.2, which shows that this rescaled limiting profile coincides with the unique stationary solution of (1). In particular, taking into account the validity of the Fourier law, proven as Theorem 2 in [1], we see that the effective current in the stationary regime is strictly smaller than its desired maximum value which is $\min\{j/2, 1/4\}$, but this value is indeed approached by letting $K \to \infty$.

2 Model and Main Results

Particle configurations are elements η of $\{0,1\}^{\Lambda_N}$, $\eta(x) = 0, 1$ being the occupation number at $x \in \Lambda_N$. We consider the Markov process on $\{0,1\}^{\Lambda_N}$ defined via the generator

$$L_N := N^2\left(L_0 + \frac{1}{N}L_b\right),$$

where $L_b = L_{b,+} + L_{b,-}$ and

$$\begin{aligned} L_0 f(\eta) &:= \frac{1}{2}\sum_{x=-N}^{N-1}\left[f\left(\eta^{(x,x+1)}\right) - f(\eta)\right], \\ L_{b,\pm} f(\eta) &:= \frac{j}{2}\sum_{x \in I_\pm} D_\pm\eta(x)\left[f\left(\eta^{(x)}\right) - f(\eta)\right], \end{aligned} \tag{2}$$

$\eta^{(x)}$ being the configuration obtained from η by changing the occupation number at x, $\eta^{(x,x+1)}$ by exchanging the occupation numbers at $x, x+1$; for any $u : \Lambda_N \to [0,1]$

$$\begin{aligned} &D_+u(x) = \left[1 - u(x)\right]u(x+1)u(x+2)\ldots u(N), \qquad x \in I_+ \\ &D_-u(x) = u(x)\left[1 - u(x-1)\right]\left[1 - u(x-2)\right]\ldots\left[1 - u(-N)\right], \quad x \in I_-. \end{aligned} \tag{3}$$

Given $\rho_0 \in C([-1,1],[0,1])$, let $\nu^{(N)}$ be the product probability measure on $\{0,1\}^{\Lambda_N}$ such that $\nu^{(N)}(\eta(x)) = \rho_0(N^{-1}x)$ for all $x \in \Lambda_N$. Let $\mathbb{P}_{\nu^{(N)}}$ denote the law of the process with initial distribution $\nu^{(N)}$ and $\mathbb{E}_{\nu^{(N)}}$ the corresponding expectation.[1]

The following theorem has been proven (in a stronger form) in [1, 2]. The statement below contains all what is needed in the present paper. In the following, for n a positive integer we write $\Lambda_N^{n,\neq}$ for the set of all sequences $(x_1,\dots,x_n)$ in Λ_N^n such that $x_i \neq x_j$ whenever $i \neq j$.

Theorem 2.1 *There exists $\tau > 0$ so that for any ρ_0 as above and any $n \geq 1$,*

$$\lim_{N\to\infty} \sup_{t\leq\tau\log N} \left| \mathbb{E}_{\nu^{(N)}}\left(\prod_{i=1}^n \eta(x_i,t)\right) - \prod_{i=1}^n \mathbb{E}_{\nu^{(N)}}\big(\eta(x_i,t)\big)\right| = 0. \tag{4}$$

Furthermore

$$\lim_{N\to\infty} \sup_{x\in\Lambda_N} \sup_{t\leq\tau\log N} \left|\mathbb{E}_{\nu^{(N)}}\big(\eta(x,t)\big) - \rho\big(N^{-1}x,t\big)\right| = 0, \tag{5}$$

where the function $\rho(r,t)$ solves (1), the pair $(u_+(t), u_-(t))$ being the unique solution of the non linear system

$$\begin{aligned} u_\pm(t) = {} & \int_{[-1,1]} P_t(\pm1,r)\rho_0(r)dr + \frac{j}{2}\int_0^t \big\{P_s(\pm1,1)\big(1-u_+(t-s)^K\big) \\ & - P_s(\pm1,-1)\big(1-\big(1-u_-(t-s)\big)^K\big)\big\}ds, \end{aligned} \tag{6}$$

where $P_t(r,r')$ is the density kernel of the semigroup (also denoted as P_t) with generator $\Delta/2$, Δ the Laplacian in $[-1,1]$ with reflecting, Neumann, boundary conditions.

The function $\rho(r,t)$ satisfies

$$\frac{\partial\rho(r,t)}{\partial r}\bigg|_{r=1} = j\big(1-u_+(t)^K\big), \qquad \frac{\partial\rho(r,t)}{\partial r}\bigg|_{r=-1} = j\big(1-\big(1-u_-(t)\big)^K\big). \tag{7}$$

Remark The following is the integral form of the macroscopic equation:

$$\begin{aligned} \rho(r,t) = {} & \int_{[-1,1]} P_t\big(r,r'\big)\rho\big(r',0\big)dr' + \frac{j}{2}\int_0^t \big\{P_s(r,1)\big(1-\rho(1,t-s)^K\big) \\ & - P_s(r,-1)\big(1-\big(1-\rho(-1,t-s)\big)^K\big)\big\}ds. \end{aligned} \tag{8}$$

It will be convenient to recall the expression for the density kernel $P_t(r,r')$ in terms of the Gaussian kernel

$$G_t\big(r,r'\big) = \frac{e^{-(r-r')^2/(2t)}}{\sqrt{2\pi t}}, \qquad r,r' \in \mathbb{R}, \tag{9}$$

[1] Omitting the initial profile to avoid too heavy notation.

as

$$P_t(r,r') = \sum_{r'':\psi(r'')=r'} G_t(r,r'') \quad \text{for } r' \neq \pm 1,$$
$$P_t(r,\pm 1) = \sum_{r'':\psi(r'')=\pm 1} 2G_t(r,r''), \tag{10}$$

where $\psi : \mathbb{R} \to [-1,1]$ denotes the usual reflection map: $\psi(x) = x$ for $x \in [-1,1]$, $\psi(x) = 2 - x$ for $x \in [1,3]$, ψ extended to the whole line as periodic of period 4.

Notation $P_t g(r) = \int P_t(r,r')g(r')dr'$, for g a bounded continuous function, $t > 0$.

The main result of this paper is about the density profile of the unique invariant measure μ_N.

Theorem 2.2 *For any integer $k \geq 1$ we have*

$$\lim_{N\to\infty} \max_{(x_1,\dots,x_k)\in\Lambda_N^{k,\neq}} \left|\mu_N\big(\eta(x_1)\cdots\eta(x_k)\big) - \rho^*(x_1/N)\cdots\rho^*(x_k/N)\right| = 0 \tag{11}$$

where $\rho^*(r)$ is the unique stationary solution of the macroscopic equation. Namely $\rho^*(r) = Jr + \frac{1}{2}$,

$$J = j\big(1-\alpha^K\big), \quad \text{with } \alpha \text{ the solution of } \alpha\big(1 + j\alpha^{K-1}\big) = j + \frac{1}{2}. \tag{12}$$

By Theorem 2.2 it follows that μ_N concentrates on a L^1-neighborhood of the limit profile ρ^*: let $r \in (0,1)$ and

$$\rho^{(\ell)}(r;\eta) = \frac{1}{2\ell+1} \sum_{x\in\Lambda_N:|x-rN|\leq\ell} \eta(x).$$

Then for any $a \in (0,1)$

$$\lim_{N\to\infty} \mu_N\left(\int_{-1}^{1} \left|\rho^{(N^a)}(r;\eta) - \rho^*(r)\right| dr\right) = 0.$$

Theorem 2.2 will follow from • uniformly on the initial datum ρ_0 the solution $\rho(r,t|\rho_0)$ of the macroscopic equation (8) converges in sup norm to ρ^* exponentially fast, see Theorem 4.1 below; • for any integer $k \geq 1$,

$$\lim_{t\to\infty} \lim_{N\to\infty} \max_{\eta\in\{0,1\}^{\Lambda_N}} \max_{(x_1,\dots,x_k)\in\Lambda_N^{k,\neq}} \left|\mathbb{E}_\eta\left(\prod_{i=1}^{k} \eta(x_i,t)\right) - \prod_{i=1}^{k} \rho^*(x_i/N)\right| = 0. \tag{13}$$

We are also working on an extension of the theorem where we prove exponential convergence in time to μ_N uniformly in N.

3 Monotonicity Properties

We consider the space $\{0,1\}^{\Lambda_N}$ endowed with the usual partial order, namely we say that $\eta \leq \xi$ iff $\eta(x) \leq \xi(x)$ for all $x \in \Lambda_N$. The following proposition is an immediate consequence of general facts on attractive systems, see e.g. [3] (Chaps. II and III).

Proposition 3.1 *Let η_0 and ξ_0 be two particle configurations such that $\eta_0 \le \xi_0$, and let $\mathbb{P}_{\eta_0}$, respectively $\mathbb{P}_{\xi_0}$, be the law of the process starting from η_0, respectively ξ_0. Then there is a coupling $\mathbb{Q}$ of $\mathbb{P}_{\eta_0}$ and $\mathbb{P}_{\xi_0}$ (i.e. $\mathbb{Q}$ is a measure on the product space, with $\mathbb{P}_{\eta_0}$ as its first marginal, and $\mathbb{P}_{\xi_0}$ as the second one) such that*

$$\mathbb{Q}\big\{(\eta,\xi):\, \eta_t \le \xi_t, \forall t\big\} = 1. \tag{14}$$

Proof Being well known that the process corresponding to L_0 is attractive, it suffices to observe that the flip rates $c(x,\eta) := D_\pm \eta(x)$ in $I_\pm$ are attractive in the sense that if $\eta(x) = \xi(x) = 0$ and $\eta \le \xi$ then $c(x,\eta) \le c(x,\xi)$, while if $\eta(x) = \xi(x) = 1$ and $\eta \le \xi$ then $c(x,\xi) \le c(x,\eta)$. □

The analogous monotonicity property holds for the macroscopic equation. Instead of a direct proof we derive the result as a consequence of the monotonicity of the particle system and that it converges to the macroscopic equation.

Theorem 3.2 *Let $\rho_0, \tilde\rho_0$ be bounded measurable functions from $[-1,1]$ to $[0,1]$ such that $\rho_0(r) \le \tilde\rho_0(r)$ for all $r \in [-1,1]$, and let $\rho(r,t)$, respectively $\tilde\rho(r,t)$, be the corresponding solution of (8) with initial datum ρ_0, respectively $\tilde\rho_0$. Then $\rho(r,t) \le \tilde\rho(r,t)$ for all $r \in [-1,1]$ and $t \ge 0$.*

Proof Let $\nu^{(N)}$ and $\tilde\nu^{(N)}$ be the product probability measures on $\{0,1\}^{\Lambda_N}$ such that $\nu^{(N)}(\eta(x)) = \rho_0(N^{-1}x)$ and $\tilde\nu^{(N)}(\eta(x)) = \tilde\rho_0(N^{-1}x)$ for all $x \in \Lambda_N$. It is well known that a coupling $\lambda^{(N)}$ of $\nu^{(N)}$ and $\tilde\nu^{(N)}$ such that $\lambda^{(N)}\{(\eta,\tilde\eta):\, \eta \le \tilde\eta\} = 1$ exists. Using Proposition 3.1 and the notation of Theorem 2.1 we have

$$\mathbb{E}_{\nu^{(N)}}\big(\eta(x,t)\big) \le \mathbb{E}_{\tilde\nu^{(N)}}\big(\eta(x,t)\big), \quad \forall x \in \Lambda_N,\ \forall t \ge 0. \tag{15}$$

From (5) we then have that for all $t \ge 0$ and for all $r \in [-1,1]$ (below $[\cdot]$ denotes the integer part)

$$\rho(r,t) = \lim_{N\to\infty} \mathbb{E}_{\nu^{(N)}}\big(\eta\big([Nr],t\big)\big) \le \lim_{N\to\infty} \mathbb{E}_{\tilde\nu^{(N)}}\big(\eta\big([Nr],t\big)\big) = \tilde\rho(r,t). \tag{16}$$

□

4 The Macroscopic Profile

We first prove that the function ρ^* in the statement of Theorem 2.2 is a stationary solution to the Dirichlet problem (1) with boundary condition (6) or, equivalently, of the integral equation (8). In fact by requiring that a stationary solution is a linear function we get, due to (7), that the values of this function at ± 1, denoted with $u_\pm$, must satisfy

$$j\big(1 - u_+^K\big) = j\big(1 - (1-u_-)^K\big).$$

This implies

$$u_+ = (1 - u_-), \quad \text{and} \quad \frac{2u_+ - 1}{2} = j\big(1 - u_+^K\big), \qquad u_+ = \frac{1}{2} + j\big(1 - u_+^K\big).$$

Solving we get

$$u_+\big(1 + j u_+^{K-1}\big) = j + \frac{1}{2}$$

in agreement with (12).

On the other hand, since $\frac{\partial}{\partial t} P_t(r, r') = \frac{1}{2} \frac{\partial^2}{\partial (r')^2} P_t(r, r')$ and it satisfies Neumann boundary conditions at ± 1 we easily see that

$$\frac{d}{dt} \int_{[-1,1]} P_t\big(r, r'\big) r' dr' = \frac{1}{2}\big(P_t(r, -1) - P_t(r, 1)\big).$$

Recalling (from (12)) that $J = j(1 - (\rho^*(1))^K) = j(1 - (1 - \rho^*(-1))^K)$ we see at once that ρ^* satisfies (8), which in this case can be written as:

$$\rho^*(r) = P_t \rho^*(r) + \frac{j}{2}\big(1 - \big(\rho^*(1)\big)^K\big) \int_0^t \big\{P_s(r, 1) - P_s(r, -1)\big\} ds, \tag{17}$$

for all $t \geq 0$.

We now prove that any solution to the Dirichlet problem converges exponentially fast to ρ^* as $t \to \infty$. In particular, one has uniqueness of the stationary solution.

Theorem 4.1 *There exist positive constants c, c' so that for any function $\rho_0 \in L^\infty([-1, 1], [0, 1])$ the solution $\rho(r, t|\rho_0)$ of the macroscopic equation (8) with initial datum $\rho(r, 0) = \rho_0(r)$ satisfies*

$$\sup_{r \in [-1,1]} \big|\rho(r, t|\rho_0) - \rho^*(r)\big| \leq c' e^{-ct}. \tag{18}$$

Proof Let $\bar{\rho}(r, t)$ denote the solution with initial datum $\rho \equiv 1$, and $\underline{\rho}(r, t)$ that corresponding to initial datum $\rho \equiv 0$. From Theorem 3.2 we know that $\underline{\rho}(r, t) \leq \rho(r, t|\rho_0) \leq \bar{\rho}(r, t)$, for any initial ρ_0. Hence, calling

$$w(r, t) := \bar{\rho}(r, t) - \underline{\rho}(r, t) \geq 0, \qquad w(t) = \sup_{r \in [-1,1]} w(r, t)$$

it suffices to show that $w(t) \leq c' e^{-ct}$ for suitable positive constants c, c' and all $t > 0$.

In the proof below $c, \bar{c}, \tilde{c}$ will denote suitable positive constants (that might depend on the model parameter j) whose value may change from line to line. Let

$$\bar{u}_\pm(t) := \bar{\rho}(\pm 1, t), \qquad \underline{u}_\pm(t) := \underline{\rho}(\pm 1, t), \qquad w_\pm(t) := \bar{u}_\pm(t) - \underline{u}_\pm(t) \geq 0.$$

From (8) we see that for all $r \in [-1, 1]$, and all $t \geq t_0 \geq 0$,

$$w(r, t) = \big(P_{t-t_0} w(\cdot, t_0)\big)(r) - \frac{j}{2} \int_{t_0}^t f(r, s, t - s) ds, \tag{19}$$

where

$$\begin{aligned} f(r, s, t - s) := P_s(r, 1)\big\{\bar{u}_+(t - s)^K - \underline{u}_+(t - s)^K\big\} \\ + P_s(r, -1)\big\{\big(1 - \underline{u}_-(t - s)\big)^K - \big(1 - \bar{u}_-(t - s)\big)^K\big\}. \end{aligned} \tag{20}$$

Interchanging particles and holes, one can couple at once the evolutions starting from the configurations $\bar\eta = \underline{1}$ (*all occupied sites*) and $\underline{\eta} = \underline{0}$ (*all empty sites*) so that $\bar\eta(x,t) = 1 - \underline{\eta}(-x,t)$. Therefore, by the same argument as in the proof of Theorem 3.2 one has $\bar\rho(r,t) = 1 - \underline{\rho}(-r,t)$ for all r and all t. In particular $w(-r,t) = w(r,t)$, $\bar u_\pm(t) = 1 - \underline{u}_\mp(t)$ and $w_+(t) = w_-(t)$ for all t. (Still from Theorem 3.2 we see that $w(r,t)$ and so also $w(t)$ decrease in t.) Of course $w(r,0) = 1$ for all r.

In particular, we may rewrite (19) with $t_0 = 0$ as

$$w(r,t) = 1 - \frac{j}{2}\int_0^t \big[P_s(r,1) + P_s(r,-1)\big] w(1,t-s) h(t-s)\,ds \tag{21}$$

where

$$h(t-s) := \sum_{\ell=0}^{K-1} \bar u_+(t-s)^{K-1-\ell}\,\underline{u}_+(t-s)^\ell \tag{22}$$

and where we have used that for any integer $K \ge 1$,

$$a^K - b^K = (a-b)\sum_{\ell=0}^{K-1} b^\ell a^{K-1-\ell}, \qquad a \ge b \ge 0. \tag{23}$$

Also, from (22) and the monotonicity properties we see that

$$b := \rho^*(1)^{K-1} \le h(t) \le b + K - 1 =: c_K. \tag{24}$$

The proof will use local times. To this end we introduce the kernel operators $K_s^{(\epsilon)}$, $\epsilon > 0$:

$$K_s^{(\epsilon)} f(r) = \frac{1}{\epsilon}\int_{[-1,-1+\epsilon]\cup[1-\epsilon,1]} P_s\big(r,r'\big) f\big(r'\big)\,dr', \qquad f \in C\big([-1,1],\mathbb{R}\big).$$

In particular $K_s^{(\epsilon)} f(r) = K_s^{(\epsilon)} f(-r)$ for all $r \in [-1,1]$. Let $w^{(\epsilon)}$ be the solution to the following integral equation:

$$w^{(\epsilon)}(r,t) = 1 - \frac{j}{2}\int_0^t \big(K_s^{(\epsilon)} w^{(\epsilon)}(\cdot,t-s)\big)(r) h(t-s)\,ds.$$

We shall next prove that for all $T > 0$,

$$\lim_{\epsilon\to 0} \sup_{r\in[-1,1]} \sup_{0\le t\le T} \big|w(r,t) - w^{(\epsilon)}(r,t)\big| = 0. \tag{25}$$

Calling

$$\psi(r,t) = \big|w(r,t) - w^{(\epsilon)}(r,t)\big|, \qquad \Psi(t) = \sup_{r\in[-1,1]} \psi(r,t) \tag{26}$$

and using (24), we can write

$$\left|\int_0^t \Big\{\big(K_s^{(\epsilon)} w^{(\epsilon)}(\cdot,t-s)\big)(r) - \big\{P_s(r,1) + P_s(r,-1)\big\} w(1,t-s)\Big\} h(t-s)\,ds\right|$$

$$\le c\epsilon + c_K \int_\epsilon^t \left\{\frac{1}{\epsilon}\int_{1-\epsilon}^1 \big|P_s(r,y) - P_s(r,1)\right.$$

$$+ P_s(-r,y) - P_s(-r,1)\big|dy\Big\}\big|w^{(\epsilon)}(y,t-s)\big|ds$$

$$+ c_K \int_\epsilon^t \big\{P_s(r,1) + P_s(-r,1)\big\}\Psi(t-s)ds. \tag{27}$$

Using that for all $y \in [1-\epsilon, 1]$, $r \in [-1,1]$

$$\big|P_s(r,y) - P_s(r,1)\big| \le c\frac{1-y}{\sqrt{s^3}}, \quad \forall s \in [\epsilon, t] \tag{28}$$

we see that the second term on the r.h.s. of (27) is bounded above by

$$\tilde{c}\int_\epsilon^t \frac{1}{\sqrt{s^3}}ds\frac{1}{\epsilon}\int_{1-\epsilon}^1 (1-y)dy \le c'\sqrt{\epsilon}$$

for suitable constants $\tilde{c}, c'$. We then easily get

$$\psi(r,t) \le c_1\sqrt{\epsilon} + c_2\int_0^t \Psi(s)ds \tag{29}$$

for suitable constants c_1, c_2. By the Gronwall inequality we conclude (25). □

We now estimate $w^{(\epsilon)}$. Let $\{B_t\}$ be a standard Brownian motion with reflecting b.c. at ± 1, with $\mathbb{P}_r$ denoting its law when $B_0 = r$ (and corresponding expectations denoted by $\mathbb{E}_r$). Then

$$w^{(\epsilon)}(r,t) = \mathbb{E}_r\big(e^{-\int_0^t \varphi_\epsilon(B_s,t-s)ds}w^{(\epsilon)}(B_t,0)\big) \tag{30}$$

where

$$\varphi_\epsilon(B,t-s) = \phi_\epsilon(B)h(t-s), \qquad \phi_\epsilon(r) = \frac{j}{2\epsilon}\mathbf{1}_{[1-\epsilon,1]}\big(|r|\big), \quad r \in [-1,1]. \tag{31}$$

By (24)

$$w^{(\epsilon)}(r,t) \le \mathbb{E}_r\big(e^{-b\int_0^t \phi_\epsilon(B_s)ds}\big). \tag{32}$$

For $0 < \bar{t} < t$ we write

$$w^{(\epsilon)}(r,t) \le \mathbb{E}_r\Big(e^{-b\int_0^{t-\bar{t}}\phi_\epsilon(B_s)ds}\,\mathbb{E}_{B_{t-\bar{t}}}\big(e^{-b\int_{t-\bar{t}}^t \phi_\epsilon(B_s)ds}\big)\Big). \tag{33}$$

We shall prove below that taking $\bar{t}$ sufficiently small, we can take $\alpha < 1$ so that for all $\epsilon > 0$

$$\sup_{r\in[-1,1]} \mathbb{E}_r\big(e^{-b\int_0^{\bar{t}}\phi_\epsilon(B_s)ds}\big) \le 1-\alpha. \tag{34}$$

From (34) and (33) we then get

$$\big|w^{(\epsilon)}(r,t)\big| \le (1-\alpha)^{[t/\bar{t}]} \tag{35}$$

($[a]$ the integer part of a) which then concludes the proof of the theorem.

Proof of (34) Let $T = \inf\{t \ge 0\colon\ |B_t| = 1\}$. We then have

$$\mathbb{E}_r\big(e^{-b\int_0^{\bar{t}}\phi_\epsilon(B_s)ds}\big) \le \mathbb{E}_r\Big(\mathbf{1}_{\{T\le\bar{t}/2\}}e^{-b\int_T^{\bar{t}}\phi_\epsilon(B_s)ds}\Big) + \mathbb{P}_r(T > \bar{t}/2) \tag{36}$$

and write

$$\mathbb{E}_r\left(\mathbf{1}_{\{T\le\bar t/2\}}e^{-b\int_T^{\bar t}\phi_\epsilon(B_s)ds}\right)\le\mathbb{E}_r\left(\mathbf{1}_{\{T\le\bar t/2\}}\mathbb{E}_{B_T}\left(e^{-b\int_0^{\bar t/2}\phi_\epsilon(B_s)ds}\right)\right)$$
$$\le\mathbb{P}_r(T\le\bar t/2)\mathbb{E}_1\left(e^{-b\int_0^{\bar t/2}\phi_\epsilon(B_s)ds}\right)\quad(37)$$

where we also used that $\mathbb{E}_1(e^{-b\int_0^{\bar t/2}\phi_\epsilon(B_s)ds})=\mathbb{E}_{-1}(e^{-b\int_0^{\bar t/2}\phi_\epsilon(B_s)ds})$ by symmetry.

By Taylor expansion

$$\mathbb{E}_1\left(e^{-b\int_0^{\bar t/2}\phi_\epsilon(B_s)ds}\right)\le 1-b\mathbb{E}_1\left(\int_0^{\bar t/2}\phi_\epsilon(B_s)ds\right)+\xi_2\quad(38)$$

where

$$\xi_2=\left(\frac{jb}{2\epsilon}\right)^2\int_0^{\bar t/2}dt_1\int_0^{t_1}dt_2\int_{|y_1|\in[1-\epsilon,1],|y_2|\in[1-\epsilon,1]}P_{t_1}(1,y_1)P_{t_2}(y_1,y_2)dy_1dy_2.\quad(39)$$

But, from (9)–(10) we see that

$$\sup_{x,y\in[-1,1]}P_s(x,y)\le c\frac{1}{\sqrt{s}}\quad(40)$$

so that for $\bar t$ small enough we get

$$\xi_2\le\bar c\bar t/2\quad(41)$$

for suitable constant $\bar c$. Using again (9)–(10), we see at once that a positive constant c can be taken so that for all $\bar t$ small, and all $\epsilon>0$

$$b\mathbb{E}_1\left(\int_0^{\bar t/2}\phi_\epsilon(B_s)ds\right)\ge c\sqrt{\bar t/2}.\quad(42)$$

From (38), (41) and (42) we then get for $\bar t$ small (with possibly different constant c),

$$\mathbb{E}_1\left(e^{-b\int_0^{\bar t/2}\phi_\epsilon(B_s)ds}\right)\le 1-c\sqrt{\bar t/2}.\quad(43)$$

By (36) and (37) we then have

$$\mathbb{E}_r\left(e^{-b\int_0^{\bar t}\phi_\epsilon(B_s)ds}\right)\le[1-c\sqrt{\bar t/2}]\mathbb{P}_r(T\le\bar t/2)+\mathbb{P}_r(T>\bar t/2)\le 1-\alpha\quad(44)$$

with

$$\alpha=\inf_{r\in[-1,1]}\mathbb{P}_r(T\le\bar t/2)c\sqrt{\bar t/2}.\quad(45)$$

□

5 Proof of Theorem 2.2

The proof is a direct consequence of the following three facts.

(i) For any $t > 0$ and any integer $k \geq 1$

$$\lim_{N\to\infty} \max_{\eta\in\{0,1\}^{\Lambda_N}} \max_{(x_1,\ldots,x_k)\in\Lambda_N^{k,\neq}} \left| \mathbb{E}_\eta\left(\prod_{i=1}^k \eta(x_i,t)\right) - \prod_{i=1}^k \mathbb{E}_\eta\big(\eta(x_i,t)\big)\right| = 0. \tag{46}$$

(ii) For any $t > 0$

$$\lim_{N\to\infty} \max_{\eta\in\{0,1\}^{\Lambda_N}} \max_{x\in\Lambda_N} \big|\mathbb{E}_\eta\big(\eta(x,t)\big) - \rho(x/N,t|\eta)\big| = 0 \tag{47}$$

(iii)

$$\lim_{t\to\infty} \sup_{\rho_0\in L^\infty([-1,1];[0,1])} \big\|\rho(\cdot,t|\rho_0) - \rho^*(\cdot)\big\|_\infty = 0 \tag{48}$$

(i) and (ii) are proved in [1, 2], (48) is proved in Theorem 4.1.

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