PhD Courses a.y. 2023/24

PhD program in Mathematics and Modeling

a) **Perturbation Methods for the Stability Analysis of Dynamical Systems** (8h)

Lecturer: **Simona di Nino**

The course introduces the basics of the perturbation analysis for weakly nonlinear dynamical systems, with special reference to the multiple scale method for ordinary differential systems. The following topics are addressed: eigenvalue and eigenvector sensitivity analysis; initial value problems: straightforward expansions; the multiple scale method: basic aspects and advanced topics; Duffing oscillator under external excitation: primary, super-harmonic and sub-harmonic resonances; Duffing oscillator under parametric excitation; multi-d.o.f. quasi-Hamiltonian systems under external/parametric/internal resonances.

b) **On the theory of polynomial identities in algebra** (10h)

Lecturer: **Antonio Ioppolo**

The course aims to introduce young researchers to the theory of polynomial identities for associative algebras. A polynomial identity of an algebra A is a polynomial in non-commuting variables vanishing under all evaluations in A. The algebras having at least one such non-trivial relation are called PI-algebras. Our major goal is to characterize them via numerical invariants related to their identities. Along the way we shall emphasize on the computational and combinatorial aspects of the theory, its connection with invariant theory, representation theory, group theory and growth problems. No particular prerequisites are requested to attend this course.

c) **Mathematical models for economic equilibria** (10h)

Lecturer: **Massimiliano Giuli**

In science the term *equilibrium* has been widely used in physics, chemistry, biology, engineering and economics, among others, within different frameworks.

It generally refers to conditions or states of a system in which all competing inuences are balanced.

For instance, the economic equilibrium which studies the dynamics of supply, demand, and prices in an economy within several markets, can be modeled as a variational inequality problem.

In non-cooperative game involving two or more players, Nash proposed an equilibrium solution in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy.

This problem can be reformulated as a fixed point problem.

These mathematical models share an underlying common structure that allows to conveniently formulate them in a unique format of equilibrium.

The course is devoted to describe this format and it focuses on the main mathematical tools which are crucial for studying the existence and the stability of the solutions.

d) **Introduction to the Finite Element Method for Partial Differential Equations** (10h)

Lecturer: **Carmela Scalone**

This is an introductory course on FEM discretizations to selected partial differential equations. The idea is to proceed by classes of problems, identifying the most appropriate numerical solvers and analyzing some relevant accuracy properties. This type of approach is designed for Ph.D. students involved in research areas where the numerical treatment of PDEs is required. Tentative list of topics: FEM for elliptic problems; FEM for parabolic problems; FEM for hyperbolic problems; Spectral methods; Discontinuous Galerkin.

e) **An introduction to geometric inequalities and constant mean curvature hypersurfaces** (16h)

Lecturer: **Mario Santilli**

Constant mean curvature (cmc) hypersurfaces provide natural models for equilibrium configurations of physical models, including soap bubbles, soap films and the shape of crystals. The study of these hypersurfaces often leads to consider sharp geometric inequalities, as the isoperimetric inequality, the Willmore inequality and the Heintze-Karcher inequality.

The aim of this course is to provide a basic and rigorous introduction to some of the fundamental ideas to deal with some of these geometric variational problems, as well as to give an overview of some open problems.

f) **Introduction to quantum computing** (14h)

Lecturer: **Leonardo Guidoni**

The present short course is a joint Ph.D. course between the Ph.D. in Mathematics and Models and the Ph.D. in Informatics. The aim of the short course is to provide to students with background in mathematics and informatics the foundation of quantum computation. The course will consist of theoretical lectures as well as hands-on tutorial lead by the Quantum Computing experts from IBM-Italia.

Topics: General overview on quantum computation. Introduction to Quantum Mechanics and Qubits. Quantum circuits and algorithms. Single and double Qubit gates with examples. Present and future applications. Perspective of quantum computation and practical implementation of algorithms on the IBM-Q quantum computer and simulator.

g) **Variational methods in continuum mechanics** (10h)

Lecturers: **Alessandro Ciallella, Francesco dell'Isola**

1. Principle of Virtual Work as a fundamental postulate for mechanics Second Gradient Continuum Mechanics. Hamilton Rayleigh Principle for dissipative systems

2. Generalisation of the concept of Deformation and Stress: Necessary strong form for Equilibrium Conditions Essential and Natural Boundary Conditions

3. Piola Transformations and contact interactions for Second Gradient Continua

4. Edge and Surface contact interactions in second gradient continua: forces and double forces. Representation of contact interactions in terms of stresses, double stresses and shape of Cauchy cuts Limitations of so called Cauchy postulate

5. Some remarks on relevant aspects of history of mechanics and in particular on the development of the concepts of force, stress and couples.

h) **Fluctuation Relations and Response Theory in Nonequilibrium Statistical Mechanics** (10h)

Lecturers: **Lamberto Rondoni** *(Politecnico di Torino)*, **Matteo Colangeli**

The course aims at shedding light on some important results in nonequilibrium statistical mechanics. A major focus will be on the Fluctuation Relations (such as the celebrated Gallavotti-Cohen formula) which constitute one of the few exact results available for systems far from equilibrium. Other relations, like the Jarzynski and Crooks formulae, will also be reviewed. The course will then explore the formalism of Response Theory, which establishes a firm connection between the spontaneous fluctuations of a manyparticle system and its response to an external perturbation. A concise summary of the course is the following: Green-Kubo linear response, nonequilibrium molecular dynamics, Fluctuation Relations, Jarzynski and Crooks Relations, Exact Response, Tmixing and irreversibility.

i) **Introduction to Hyperbolic systems in several space dimensions** (8h)

Lecturer: **Debora Amadori**

In this course, we will consider hyperbolic systems of balance laws, that is, a class of partial differential equations arising in continuum physics. The Euler equations for isentropic gas flow provide a reference example. We will address two main questions: - the existence (local-in-time) and stability of classical solutions to the Cauchy problem in several space dimensions for systems endowed by a convex entropy; - the role of dissipation mechanisms in preventing the formation of singularities and the existence, global-in-time, of classical solutions.

j) **On the Smoluchowski Coagulation Equation: a stochastic particle system approximation** (10h)

Lecturer: **Alessia Nota**

The phenomenon of coagulation is the mechanism by which particles (clusters) grow, the underlying process being successive mergers. It can be observed in physical systems, e.g., aerosol and raindrop formation, smoke, sprays and galaxies, as well as in biological systems, e.g., hematology, bacteria and animal grouping. The evolution of the particle size distribution is described at mesoscopic level by the Smoluschowki coagulation equation, a fundamental mean-field model of clustering dynamics. Its solutions exhibit rich behavior depending on the structure of the kernel which gives the rate to each coagulation, such as the phase transition termed as gelation (giant particles appear in finite time) or self-similarity (preservation of the shape over time). We will present an overview of the Smoluchowski's equation and investigate the connection between the coagulation equation and a stochastic coalescence model, the Marcus-Lushnikov process. This microscopic model describes the stochastic Markov evolution of a finite particle system of coalescing particles. We will discuss the statistical derivation of Smoluchowski equation and investigate the rate of convergence of the Marcus-Lushnikov process to the solution of the Smoluchowski coagulation equation as the number of particles tends to infinity.

A) **Conservation laws and traffic flow models** (10h)

Lecturer: **Felisia Chiarello**

This course will deal with nonlinear conservation laws in one space dimension and their application to traffic flow. In the first part of the course, we will show the basic theory: relationship with the Hamilton-Jacobi equation, weak solutions, admissibility conditions and shocks, solution to the Riemann problem, and the construction of a nonlinear contractive semigroup of solutions in L^1. Moreover, we will analyze weak solutions to the Cauchy problem for systems with initial data having small total variation. The second part of the course will concern about traffic models based on conservation laws. In particular, we will apply the theory to the study of the Lighthill-Whitham Richards (LWR) model, focusing on the relation with the microscopic Follow-the Leader (FtL) model. After that, we will generalize the LWR to the nonlocal setting.

B) **Geometric structures in incompressible fluids: vortex and magnetic reconnection** (6h)

Lecturer: **Gennaro Ciampa** *(Università di Milano)*

The goal of this course is to construct smooth solutions of the Navier-Stokes (NS) and the Magneto-hydrodynamic (MHD) equations for which the topology of the vortex and the magnetic lines changes under the flow without any loss of regularity.

In the case of the Euler equations, as long as the solution does not blow up, the vorticity is given by the push-forward of the initial vorticity along the flow generated by the velocity field. Thus, there are no changes in the topology of the vortex structures. In presence of viscosity, the vorticity is no longer transported and the topological coherence of the vortex structures may be broken. This phenomenon is known as vortex reconnection: for example, the vortex lines can knot or concatenate differently from what happens for the initial vorticity. There are overwhelming numerical and physical evidences for vortex reconnection but analytical examples were given for the first time in [2]. In the more complex but similar case of the (MHD) equations, magnetic reconnection (that is, the breaking and topological rearrangement of magnetic field lines) is known to occur and has deep physical implications, being responsible for many dynamic phenomena in solar physics, such as flares, the solar wind, and the aurora (see [6]). After introducing the physics of the problem, I will give an overview of the techniques developed in [2] providing a rigorous mechanism of vortex reconnection in viscous incompressible fluids. After that, I will first show how these techniques fit the case of the (MHD) equations in both the 2D and the 3D case, presenting the results obtained in [1]. Then, I will generalize the result of [2] to the case of the full space R^3.

C) **Introduction to differential inclusions of Curl-free and Div-free type** (8h)

Lecturer: **Mariapia Palombaro**

Differential inclusions arise in many problems in the Calculus of Variations where one is interested in minimising an integral functional. At the crudest level, one may say that a certain bound on the minimum of the functional is optimal, or attained, if there exist solutions of an associated differential inclusion, namely, if there exist a matrix field that satisfies a differential constraint and takes values in a prescribed set of matrices K. We will consider differential constraints of Curl-free and Div-free type. In the former case the matrix field is the gradient of a Sobolev map, while in the latter case we require that each row of the matrix field be a divergence-free field. We will study conditions that guarantees the existence of solutions in the case when the given set K is a finite or a compact set of matrices. We will also explore the case when K is unbounded and show some applications to problems in materials science.

D) **An introduction to Mathematical Theory of Control** (8h)

Lecturer: **Vasile Staicu** *(University of Aveiro, Portugal)*

In this short course, we introduce the notions of control systems, of differential inclusions and show that differential inclusions provide a convenient alternative approach for the analysis of control systems.

We start with some mathematical preliminaries including the Banach contraction mapping principle, elements of Lebesgue measure theory and elements of multivalued analysis.

Then we proceed with the basic facts on Carathéodory solutions to ordinary differential equations essential for applications to control problems: existence, uniqueness and dependence on date of Carathéodory solutions and characterization of maximal solutions.

Now we are ready to introduce control systems and optimal control problems. We introduce an equivalent differential inclusion associated to a control system and prove Filippov's result. We proceed with the fundamental properties of trajectories and of reachable sets of a nonlinear control system. Then we introduce the minimum time function, the minimum time problem and the Bellman optimality principle. We introduce then optimal control problems and study the existence of optimal controls for Mayer problems and for the problem of Bolza. Then we derive necessary and sufficient conditions for optimality.

References:

1. J.P. Aubin and A. Cellina, Differential inclusions, Springer-Verlag,1984

2. A. Bressan and B. Piccoli, Introduction to the Mathemtical Theory of Control, AIMS Series in Applied Mathematics, 2007

3. F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski, Nonsmooth analysis and Control Theory, Springer-Verlag, New York, 1998

4. A.F. Filippov, Differential equations with discontinuous right-handside, Kluwer, Doldrecht, 1988

5. L. Cesari, Optimisation - Theory and Applications, Springer, 1983

6. E. B. Lee and L. Marcus, Foundations of Optimal control theory, Willey, 1967

7. P. D. Loewen, Optimal control via nonsmooth analysis, C. R. M. Proceedings & Lecture Notes, Amer. Math. Soc., Providence, 1993

8. J. Macki, A. Strauss, Introduction to optimal control theory, Springer Verlag, New York, 1982

9. L. S. Pontryagin, V. Boltyanskii, R. V. Gamkrelidze, E. F. Mishenko, The Mathematical Theory of Optimal processes, Wiley, 1962

E) **Direct methods in Calculus of Variations** (10h)

Lecturer: **E. Radici**

In this course we consider mathematical models which can be regarded as a variational principle where the optimisation problem is associated to functionals in integral form. We introduce some weak notion of convexity (quasi-convexity, poly-convexity and rank-one convexity) and discuss the solvability of the optimisation problem via Direct Method of calculus of variations. The applicability of this method relies on the coercivity and the lower semicontinuity properties of the functionals. We focus on the existence of minimisers for quasi-convex functionals and examine the lower semicontinuity through the concept (of independent interest) of Young measures.