Compensated Integrability and Applications to Mathematical Physics

Denis SERRE

UMPA, UMR 5669 CNRS
École Normale Supérieure de Lyon
France

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Foreword

This series of lectures is about a new tool of functional analysis and its applications. The central results have a qualitative side, called Compensated Integrability, which is supported by a quantitative side in the form of a sharp Functional Inequality. It is versatile enough that it contains, as particular cases, the isoperimetric inequality and the Gagliardo inequality.

The applications concern a larger array of models :

- Inviscid gases,
- Rarefied gases,
- Hard spheres dynamics,
- Multi-D scalar conservation laws,
- Minkowski's problem for convex bodies, minimal surfaces, ...

To give a sample, I shall establish a new estimate for an inviscid gas evolving in the whole space $\mathbb{R}^d$:

$$\int_0^\infty dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p \, dy \leq c_d M^{\frac{1}{d}} \sqrt{ME} \, ,$$

where ρ is the mass density, p the pressure, M and E the total mass and energy at initial time.

The course is organized as follows.

Lesson #0 collects a few useful facts about Matrix Analysis.
Lesson #1 provides motivations from various models of Mathematical Physics[1]. Lesson #2 provides mathematical motivations, in the spirit of C. B. Morrey's and J. M. Ball's contributions to the Calculus of Variations.

The main theoretical results are stated in Lesson #3, which also displays two important examples of Divergence-free positive symmetric tensors (DPT). The proofs are given in Lesson #4, where a duality appears between DPTs and the "2nd boundary-value problem" for the Monge-Ampère equation. This makes an important link with the theory of optimal transport.

More practical statements, especially in view of applications to evolution problems, are stated and proved in Lesson #5.

1. This L #1 has been taught in L'Aquila on Wednesday March 4th.

The remaining lessons are devoted to the various applications.

Lesson #6 is devoted to gas dynamics, either in a thermodynamical context (Euler equations), or in a kinetic one (Boltzmann equation).

Lesson #7 addresses a rather original side of the theory, that of singular DPTs. On the one hand, we consider homogeneous tensors for which the Functionality Inequality involves Dirac masses, the so-called determinantal masses. On the other hand, we describe rather natural DPTs that are supported by submanifolds or graphs.
The first part of L#7 applies, in Lesson #8, to the hard spheres dynamics, where we show that, even if the collision set may be very large, most of the collisions are actually very weak.

Eventually, Lesson #9 deals with the dynamics of systems of particles that interact through long-range potential forces, like gravity of Coulomb force. The description can be either discrete, or that of a continuum (Vlasov-type equations).

Somehow, it is remarquable that Compensated Integrability is able to say something about every level of description of a gas :

- Microscopic (hard spheres),
- Mesoscopic (kinetic equations, Boltzmann),
- Macroscopic (Euler equation).

Facts from matrix analysis

Notations. The transpose of M is denoted M^T. For two vectors a, b, $a \otimes b$ is the rank-one matrix ab^T of entries $a_i b_j$. The space of $n \times n$ matrices with entries in a field k is $\mathbf{M}_n(k)$, while the group of invertible matrices is $\mathbf{GL}_n(k)$. The cone of positive semi-definite symmetric matrices is $\mathbf{Sym}_n^+$, while $\mathbf{SPD}_n$ is that of positive definite ones.

The main objects of our theory are <u>positive semi-definite symmetric tensors</u>, that is maps $x \mapsto S(x) \in \mathbf{Sym}_n^+$, where the size [2] n is ≥ 2. We shall make use of a few technical tools.

2. The case $n = 2$ is often 'trivial'.

Proposition 1 (Lesson #0 - Schur complement)

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{M}_n(k)$ *be given blockwise with* $A \in \mathbf{GL}_p(k)$.
Then

$$\det M = \det A \cdot \det(D - CA^{-1}B).$$

In particular, M *is invertible if and only if the Schur complement* $D - CA^{-1}B$ *is so.*
Suppose instead that $M \in \mathbf{Sym}_n(\mathbb{R})$. *Then*

$$(M \in \mathbf{SPD}_n) \Longleftrightarrow (A \in \mathbf{SPD}_p \text{ and } D - B^T A^{-1} B \in \mathbf{SPD}_{n-p}).$$

Proof

1) Decompose $M = LU$ blockwise :

$$L = \begin{pmatrix} I_p & 0 \\ CA^{-1} & I_{n-p} \end{pmatrix}, \qquad U = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

2) Here $C = B^T$ and A, D are symmetric. Let us write the quadratic form $q(x) = x^T M x$ in terms of the blocks $y \in \mathbb{R}^p$ and $z \in \mathbb{R}^{n-p}$:

$$q(x) = (y + A^{-1}Bz)^T A(y + A^{-1}Bz) + z^T(D - B^T A^{-1}B)z.$$

Since $x \mapsto (y + A^{-1}Bz, z)$ is a change of variable, q is positive if and only if the forms $u^T A u$ and $z^T(D - B^T A^{-1}B)z$ are positive separately.

∎

Recall that a matrix $A \in \mathbf{Sym}_n^+$ admits a unique square root in $\mathbf{Sym}_n^+$, denoted $A^{1/2}$ or $\sqrt{A}$. Its existence is obvious with an orthonormal diagonalization. The uniqueness is slightly more involved and is a consequence of the following.

Proposition 2

Let $A, B \in \mathbf{Sym}_n^+$ be given. Then the spectrum of AB is real.
If A (resp. B) is $\mathbf{SPD}_n$, then AB is diagonalizable and the signs of its eigenvalues ($+$, $-$ or 0) are the same as the signs of the eigenvalues of B (resp. A).

Mind that AB is not symmetric in general.

Proof

Say that $A \in \mathbf{SPD}_n$. Then AB is similar to $A^{1/2}BA^{1/2}$. The latter is symmetric and represents the same quadratic form as B (in a different basis). Hence the result.
The general case is obtained by continuity of the eigenvalues, and the density of $\mathbf{SPD}_n$ in $\mathbf{Sym}_n^+$.

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Corollary 1

For $A, B \in \mathbf{Sym}_n^+$, *one has*

$$(\det A)^{\frac{1}{n}}(\det B)^{\frac{1}{n}} \leq \frac{1}{n}\, Tr(AB). \tag{1}$$

Proof

According to Proposition **??**, the eigenvalues μ_j of AB are non-negative real numbers. The inequality is thus nothing but the arithmetic-geometric mean inequality :

$$\left(\prod_{j=1}^{n}\mu_j\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{j=1}^{n}\mu_j .$$

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Corollary 2

The map

$$\begin{array}{rcl} \mathbf{Sym}_n^+ & \to & \mathbb{R}_+ \\ A & \mapsto & (\det A)^{\frac{1}{n}} \end{array}$$

is concave.

Proof

Recall that $\mathbf{Sym}_n^+$ is a closed convex cone. It is enough to prove the concavity over its interior $\mathbf{SPD}_n$. Remark that the equality in (**??**) is achieved when $B = A^{-1}$. We thus have

$$(\det A)^{\frac{1}{n}} = \inf \left\{ \frac{1}{n} \operatorname{Tr}(AB) \,|\, B \in \mathbf{SPD}_n \,, \det B = 1 \right\} .$$

Our function appears therefore as the infimum of a collection of linear functions ; as such, it is concave.

Corollary **??** is accurate because the function is positively homogeneous of degree 1. A function $(\det A)^s$ with $s > \frac{1}{n}$ cannot be concave, because it is super-linear.

Remark also that $A \mapsto \log \det A$ is concave (compose with the concave increasing function $\log$) ; which is accurate among statements that are dimension-independent.

Minors & cofactor matrix

Let $M \in \mathbf{M}_n(k)$ be given. If $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$ are multi-indices of equal length, one denotes $M\binom{I}{J}$ the minor

$$\det(m_{i_s j_t})_{1 \le s,t \le r}.$$

If $i \in [\![1, n]\!]$, denote $\hat{\imath}$ the multi-index $(\dots, i-1, i+1, \dots)$ (indices in increasing order, i omitted). The cofactor matrix $\hat{M}$ has entries

$$\hat{m}_{ik} := (-1)^{i+k} M \begin{pmatrix} \hat{\imath} \\ \hat{k} \end{pmatrix}.$$

The following formulæ are classical

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- $\hat{M}^T M = M\hat{M}^T = (\det M) I_n$.
- In particular, if $M \in \mathbf{GL}_n(k)$, then

$$\hat{M} = (\det M) M^{-T}.$$

- This implies

$$\det \hat{M} = (\det M)^{n-1}.$$

- $\frac{\partial}{\partial m_{ij}} \det M = \hat{m}_{ij}$.
- (Sherman–Morrison) If $a, b \in k^n$, then

$$\det(M + a \otimes b) = \det M + a^T \hat{M} b.$$

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We shall also use the following

For $A, B \in \mathbf{M}_n(k)$, we have

$$\det(A+B) = \sum_{r=0}^{n} \sum_{|I|=|J|=r} \epsilon(I, I^c)\epsilon(J, J^c) A\begin{pmatrix} I \\ J \end{pmatrix} B\begin{pmatrix} I^c \\ J^c \end{pmatrix}. \tag{2}$$

In this formula, I and J are (increasingly) ordered r-uplets in $[1, n]$, $A\binom{I}{J}$ is the corresponding minor, I^c is the ordered complement of I and $\epsilon(I, I^c)$ is the signature of the permutation $[1, n] \mapsto (I, I^c)$.

For instance, if $n = 4$ and $I = (1, 3)$, then $I^c = (2, 4)$ and $\epsilon(I, I^c) = \epsilon(\tau_{23}) = -1$, where τ denotes a transposition.

For a proof, see [3] Marvin Marcus, "determinant of sums", College mathematics journal, March 1990.

3. Courtesy of Reimundo Heluani.